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Weston, Vaughan H

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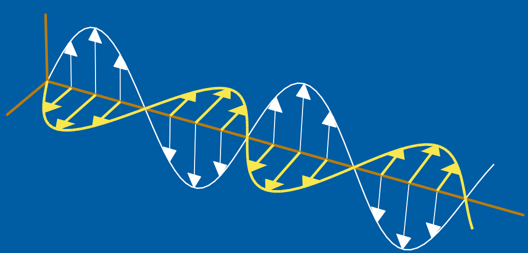
Vaughan H. Weston

Electromagnetic Theory
Department of Electrical and Information Technology
Lund University
Sweden



Vaughan H. Weston

Department of Department of Mathematics
Purdue University
West Lafayette, IN 47907
USA

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Abstract

Wave splitting of the time dependent Maxwell's equations in three dimensions with and without dispersive terms in the constitutive equation is treated. The procedure is similar to the method developed for the scalar wave equation except as follows. The up-and down-going wave condition is expressed in terms of a linear relation between the tangential components of $\boldsymbol{E}$ and $\boldsymbol{H}$. The resulting system of differential-integral equations for the up-and down-going waves is directly obtained from Maxwell's equations. This splitting (arising from the principal part of Maxwell's equations) is applied to the case where there is dispersion. A formal derivation of the imbedding equation for the reflection operator in a medium with no dispersion is obtained.

1 Introduction

Here the concepts that were used in the three-dimensional wave splitting of the scalar wave equation are extended to Maxwell's equations. For the case where the constitutive equations have dispersive terms, etc., the splitting associated with the principal part (corresponding to $\boldsymbol{D} = \epsilon \boldsymbol{E},\ \boldsymbol{B} = \mu \boldsymbol{H}$) is employed, the actual splitting will be restricted to the case where the principal part is isotropic, although it can be generalized to the anisotropic case.

As with the scalar case the splitting will be based upon the development of an up-going and down-going wave condition for solutions of Maxwell's equations. Physically an up-going wave on a plane arises from sources that lie below that plane, with a similar concept for the down-going wave. Mathematically the up- and down-going condition will be represented by a linear relationship involving the field components on a plane. Thus we need to examine solutions of the mixed problems (initial conditions and boundary conditions) for a half-space. In order to specify what components are needed, the conditions for uniqueness of the solution of Maxwell's equations with general (anisotropic) constitutive conditions (including dispersive terms) for the mixed problem is examined. This is investigated in the next section where as is expected it is shown that only tangential components of $\boldsymbol{E}$ or $\boldsymbol{H}$ on the surface need be specified.

In section 3, the physical geometry and the invariant imbedding geometry is presented.

Using the results of the requirements for uniqueness, it follows that the up- and down-going wave condition will be a linear relationship between the tangential components of $\boldsymbol{E}$ and $\boldsymbol{H}$ on a planar surface.

In section 4, an explicit form for the operators involved in these conditions is obtained using fundamental solutions of Maxwell's equations for the invariant imbedding medium where ϵ, μ depend upon the transverse variables x_1 and x_2 only, as well as a parameter α, i.e. $\epsilon = \epsilon(x_1, x_2, \alpha),\ \mu = \mu(x_1, x_2, \alpha)$.

In section 5, the up-and down-going wave splitting is defined for a medium $\epsilon = \epsilon(x_1, x_2, \alpha),\ \mu = \mu(x_1, x_2, \alpha)$ and the differential equation system for up-and down-going electric field components are obtained directly from Maxwell's equations.

In section 6, the same splitting is applied to the case $\epsilon = \epsilon(x_1, x_2, x_3)$, $\mu = \mu(x_1, x_2, x_3)$. From the resulting system of differential equations, the imbedding equation for the reflection operator is obtained.

Finally, in section 7, the splitting is extended to a medium where there is a dispersive term.

Throughout this paper a point will be represented by $x = (x_1, x_2, x_3)$, and unless otherwise noted a repeated index indicates summation from 1 to 3.

We will use the notation that ∂_t is the partial derivative with respect to t, ∂_i is the partial derivative with respect to either x_i or y_i depending upon whether the function involved is a function of x or y. For a function of both x and y we will employ the form ∂_{x_i} or ∂_{y_i}.

2 The Mixed Problem for Maxwell's Equations with Constitutive Relations

We will consider Maxwell's equations

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}, \ \nabla \times \boldsymbol{H} = \partial_t \boldsymbol{D} \tag{1}$$

with constitutive relations of the form

$$D_i = a_{ij} E_j + b_{ij} H_j + (G_{ij} * E_j)(t) + (K_{ij} * H_j)(t) \tag{2}$$

$$B_i = b_{ji} E_j + d_{ij} H_j + (L_{ij} * E_j)(t) + (F_{ij} * H_j)(t), \tag{2'}$$

where the $*$ indicates convolution in the time variable. The coefficients $a_{ij}(x)$, $b_{ij}(x)$, $d_{ij}(x)$ are independent of time, but the coefficients G_{ij}, K_{ij}, L_{ij}, F_{ij} depend upon x and t.

The general form of the constitutive relations expressed in the form $\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{E}, \boldsymbol{B})$, $\boldsymbol{H} = \boldsymbol{H}(\boldsymbol{E}, \boldsymbol{B})$ where $\boldsymbol{E}$ and $\boldsymbol{B}$ are independent variables are developed by Karlsson and Kristensson [1]. Equations (2), (2′) can be obtained from these by suitable inversion. As will be seen, the form given by Eqs. (2), (2′) where $\boldsymbol{E}$ and $\boldsymbol{H}$ are independent variables, is much more suitable for proving uniqueness.

Let Ω be a simply-connected open region in $\mathbb{R}^3$, bounded by a piecewise smooth surface $\delta\Omega$, with unit outward normal $\boldsymbol{n}$.

It will be *assumed* that the matrix

$$A_{ij}(x) = \begin{bmatrix} a_{ij}(x) & b_{ij}(x) \\ b_{ji}(x) & d_{ij}(x) \end{bmatrix} \tag{3}$$

(i) is real and symmetric.

(ii) and as an operator mapping the Hilbert Space $(\mathcal{L}_2(\Omega))^3 \times (\mathcal{L}_2(\Omega))^3$ into itself, is positive definite, i.e;

$$\sum_{i,j=1}^{6} \int_\Omega U_i(x) A_{ij}(x) U_j(x) dx > 0 \tag{4}$$

iff $\boldsymbol{U}(x) \neq \mathbf{0}$. A stronger condition could be imposed, namely that $A_{ij}(x)$ be a positive definite matrix for each $x \in \Omega$.

The assumptions that will be imposed on the functions G_{ij}, K_{ij}, L_{ij} and F_{ij} are that

(i) $G_{ij}(x,0) = K_{ij}(x,0) = L_{ij}(x,0) = F_{ij}(x,0) = 0$.

(ii) the bounded self-adjoint[1] operator mapping the Hilbert space $(\mathcal{L}_2(\Omega_T))^3 \times (\mathcal{L}_2(\Omega_T))^3$ with $\Omega_T = \Omega \times (0,T)$, into itself according to the rule

$$V_i(x,t) = \int_0^T \sum_{j=1}^{6} \phi_{ij}(x,t,s) U_j(x,s) ds, \ i = 1, \dots 6 \tag{5}$$

$$\text{where } \phi_{ij}(x,t,s) = \begin{bmatrix} G'_{ij}(x,t-s) & K'_{ij}(x,t-s) \\ L'_{ij}(x,t-s) & F'_{ij}(x,t-s) \end{bmatrix}, \ 0 \leq s \leq t \leq T \tag{6}$$

$$\phi_{ij}(x,t,s) = \phi_{ji}(x,s,t) \tag{6$'$}$$

is positive definite, i.e.;

$$\sum_{i,j=1}^{6} \int_\Omega \int_0^T \int_0^T U_i(x,t)\phi_{ij}(x,t,s)U_j(x,s) ds dt dx > 0 \tag{7}$$

iff $\boldsymbol{U}(x,t) \neq \mathbf{0}$. Note that the prime in expression (6) indicates that $G'_{ij}(x,u) = \frac{\partial}{\partial u} G_{ij}(x,u)$.

Applying the Poynting Theorem

$$\nabla \cdot \boldsymbol{S} + \boldsymbol{E} \cdot \partial_t \boldsymbol{D} + \boldsymbol{H} \cdot \partial_t \boldsymbol{B} = \mathbf{0} \tag{8}$$

with $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$, to system (1), (2) and (3$'$), and integrating over Ω and with respect to t from 0 to T, it can be shown that

$$-\int_{\delta\Omega} \int_0^T \boldsymbol{S}(t) \cdot \boldsymbol{n} d\sigma dt = \int_\Omega \{\omega_{em}(x,T) + \omega_d(x,T)\} dx \tag{8$'$}$$

From Appendix A, it is shown that

$$\omega_{em}(x,T) = \frac{1}{2} \sum_{i,j=1}^{6} U_i(x,T) A_{ij}(x) U_j(x,T), \tag{9}$$

$$\omega_d(x,T) = \frac{1}{2} \sum_{i,j=1}^{6} \int_0^T \int_0^T U_i(x,t)\phi_{ij}(x,t,s)U_j(x,s) dx ds, \tag{10}$$

where $\boldsymbol{U}(x,t) = [E_i(x,t), H_i(x,t)]$.

It immediately follows from the positive definite condition (4) and (7) that

$$\int_\Omega \{\omega_{em}(x,T) + \omega_d(x,T)\} dx > 0 \tag{11}$$

iff $(\boldsymbol{E}, \boldsymbol{H}) \neq (\mathbf{0}, \mathbf{0})$. From this result one can conclude the following

[1]Note: no special conditions on the matrices G'_{ij}, etc. are required for self-adjointness. Because of condition (6$'$) the operator is automatically self-adjoint.

Lemma 1. *Let $\boldsymbol{E}(x,t), \boldsymbol{H}(x,t)$ belong to the Hilbert Space*

$$\{\boldsymbol{U}(x,t) \in [(\mathcal{L}_2(\Omega_T))^3 |\ curl\, \boldsymbol{U} \in (\mathcal{L}_2(\Omega_T))^3]\};$$

where $\Omega_T = \Omega \times (0,T)$, and be the solution of equations (1), (2), (2′) in the space Ω_T. Let either $\boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{0}$, or $\boldsymbol{H} \times \boldsymbol{n} = \boldsymbol{0}$ on $\delta\Omega$ for $0 < t < T$. If $\boldsymbol{E}(x,0) = \boldsymbol{H}(x,0) = \boldsymbol{0},\ x \in \Omega$, then

$$\boldsymbol{E}(x,t) = \boldsymbol{H}(x,t) = \boldsymbol{0},\ x \in \Omega,\ 0 < t < T.$$

Proof:

Since $\boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{0}$ or $\boldsymbol{H} \times \boldsymbol{n} = \boldsymbol{0}$ on $\delta\Omega$, it follows from Eq. (8′) that

$$\int_\Omega \{\omega_{em}(x,T) + \omega_d(x,T)\} dx = 0$$

which because of condition (11) can only be true if $\boldsymbol{E} = \boldsymbol{H} = \boldsymbol{0}$.

From this, one can get the uniqueness result for the mixed problem. With regard to the existence of the solution of the mixed problem, we have limited results. From Leis [2], it is shown that the mixed problem for the bounded domain Ω,

$$\begin{aligned} \nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B},\ \nabla \times \boldsymbol{H} = \partial_t \boldsymbol{D},\ x \in \Omega,\ 0 < t < T \\ \boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{0},\ x \in \delta\Omega,\ 0 < t < T \\ \boldsymbol{E}(x,0) = \boldsymbol{E}_0(x),\ \boldsymbol{H}(x,0) = \boldsymbol{H}_0(x),\ x \in \Omega \end{aligned}$$

is uniquely solvable in the weak sense if $\boldsymbol{E}_0, \boldsymbol{H}_0 \in (\mathcal{L}_2(\Omega))^3$, where the constitutive relations are given by

$$B_i = \mu_{ij} H_j,\ D_i = \epsilon_{ij} E_j,$$

where

$$\left.\begin{aligned} \xi_i \epsilon_{ij}(x) \xi_j \geq \epsilon_1 |\xi|^2, \\ \xi_i \mu_{ij}(x) \xi_j \geq \mu_1 |\xi|^2 \end{aligned}\right\} \text{ for all } \xi \in \mathbb{R}^3$$

with $\epsilon_1 > 0,\ \mu_1 > 0$, and $\epsilon_{ij} = \epsilon_{ji},\ \mu_{ij} = \mu_{ji}$.

Additional results for the non-homogeneous system

$$\left.\begin{aligned} \nabla \times \boldsymbol{E} = -\mu \partial_t \boldsymbol{H} + \boldsymbol{F}(x,t), \\ \nabla \times \boldsymbol{H} = \epsilon \partial_t \boldsymbol{E} + \boldsymbol{G}(x,t), \end{aligned}\right\} x \in \mathbb{R}^3,\ 0 < t < T$$

$$\boldsymbol{E}(x,0) = \boldsymbol{E}_0(x),\ \boldsymbol{H}(x,0) = \boldsymbol{H}_0(x),\ x \in \mathbb{R}^3,$$

can be obtained by transforming it to the linear hyperbolic system

$$\begin{aligned} \partial_t \boldsymbol{u} + \sum A_k(x) \partial_k \boldsymbol{u} + B\boldsymbol{u} = \boldsymbol{f} \\ \boldsymbol{u}(x,0) = \boldsymbol{v}_0(x), \end{aligned}$$

by setting

$$\boldsymbol{u} = \begin{bmatrix} \sqrt{\epsilon}\boldsymbol{E} \\ \sqrt{\mu}\boldsymbol{H} \end{bmatrix},\ \boldsymbol{v}_0 = \begin{bmatrix} \sqrt{\epsilon}\boldsymbol{E}_0 \\ \sqrt{\mu}\boldsymbol{H}_0 \end{bmatrix},\ \boldsymbol{f} = \begin{bmatrix} -\epsilon^{-1/2} & \boldsymbol{G} \\ \mu^{-1/2} & \boldsymbol{F} \end{bmatrix}$$

The matrices $A_k(x),\ B$ are given by

$$A_k = c\begin{bmatrix} 0 & -D_k \\ D_k & 0 \end{bmatrix},\ B = \frac{c}{2}\begin{bmatrix} 0 & P_\mu \\ -P_\epsilon & 0 \end{bmatrix}$$

where $c = (\epsilon\mu)^{-1/2}$ and

$$D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},\ D_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},\ D_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P_\phi = \frac{1}{\phi}\begin{bmatrix} 0 & -\partial_3\phi & \partial_2\phi \\ \partial_3\phi & 0 & -\partial_1\phi \\ -\partial_2\phi & \partial_1\phi & 0 \end{bmatrix}$$

If $\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{E}_0, \boldsymbol{H}_0$ have compact support in $\mathbb{R}^3$, the field quantities $\boldsymbol{E}, \boldsymbol{H}$ will be zero for $0 \leq t \leq T$ on the surface of a large parallelepiped $\{x : |x| < R\}$. By employing a $2R$-periodic boundary condition, Ladyzhenskaya [3], outlines in the supplement to chapter 5, existence and uniqueness of the hyperbolic system involving the vector $\boldsymbol{u}$. The smoothness conditions require at least that the derivatives of ϵ and μ exist and be bounded. Furthermore, in the supplement to chapter 6, an outline for the numerical solution to the system is given. Both an explicit and an implicit scheme are presented.

Throughout this paper we will be working with solutions of Maxwell equations for a finite time interval $0 < t < T$, where the boundary data has compact support, and the initial conditions are $\boldsymbol{E}(x, 0) = \boldsymbol{H}(x, 0) = \boldsymbol{0}$. Hence effectively we will be working in a bounded domain Ω, even though the original problem may be stated for all of $\mathbb{R}^3$ or for a half-space.

3 Imbedding Geometry

For the physical geometry it will be assumed that the half-space $x_3 \geq 0$ is free space (no scattering medium is present), hence

$$\boldsymbol{B} = \mu_0\boldsymbol{H},\ \boldsymbol{D} = \epsilon_0\boldsymbol{E}, \quad x_3 \geq 0,$$

where ϵ_0, μ_0 are constants. The scattering medium lies in the lower half-space $x_3 < 0$, and the constitutive relations will be given by Eq. (2), (2′) where the coefficients satisfy conditions of the assumptions required for uniqueness given in the previous section. It will be assumed that the coefficients are continuous functions of x, so that there is no jump discontinuity on velocity.

We will consider a one-parameter set of invariant imbedding geometries, taking for the moment the parameter to be α. Since we are splitting with the principal of the constitutive equations, the associated invariant imbedding geometry will be described by the constitutive relations as follows. For $x_3 < \alpha$, the constitutive relations will be given by the principal part of Eq. (2), (2′) (the physical medium).

For $x_3 > \alpha$ the constitutive relations will be described by the principal part of the constitutive relations at $x_3 = \alpha$,

$$\left.\begin{aligned} D_i &= a_{ij}(x_1, x_2, \alpha)E_j + b_{ij}(x_1, x_2, \alpha)H_j \\ B_i &= b_{ji}(x_1, x_2, \alpha)E_j + d_{ij}(x_1, x_2, \alpha)H_j \end{aligned}\right\} x_3 > \alpha \tag{12}$$

where the coefficients depend upon the transverse variables x_1 and x_2 only. Note that in the invariant imbedding geometry a_{ij}, b_{ij}, d_{ij} are continuous across the surface $x_3 = \alpha$, so that there is an impedance match at the surface.

4 Up-and Down-Going Wave Condition in Isotropic Medium with Transverse Dependence

Here, the up-and down-going wave condition will be developed for solutions of Maxwell's equations, where the constitutive equations are given.

$$D_i = \epsilon(x_1, x_2, \alpha)E_i, \; B_i = \mu(x_1, x_2, \alpha)H_i, \quad i = 1, 2, 3 \tag{13}$$

with ϵ, μ being sufficiently smooth functions of x_1 and x_2, and α is a parameter. To obtain the up-and down-going wave condition we will employ the initial value boundary-value problems in the half-spaces $x_3 > 0$ and $x_3 < 0$ respectively. Hence, consider the following mixed problem.

Problem P_1: Solve for $\boldsymbol{E}(x, t), \boldsymbol{H}(x, t)$

(i) $\nabla \times \boldsymbol{E} = -\mu(x_1, x_2, \alpha)\partial_t \boldsymbol{H}, \; \nabla \times \boldsymbol{H} = \epsilon(x_1, x_2, \alpha)\partial_t \boldsymbol{E}, \; x_3 > 0, \; 0 < t < T$

(ii) $\boldsymbol{E}(x, 0) = \boldsymbol{H}(x, 0) = \mathbf{0}, \; x_3 > 0$

(iii) $E_i(x_1, x_2, 0, t) = E_i^0(x_1, x_2, t), \; i = 1, 2, \; x_3 = 0, \; 0 < t < T$

where the given data E_i^0 has compact support, and is sufficiently smooth for existence of solution.

An important result of the condition that the boundary data has compact support (say in a disk $|x_1 - x_1^0|^2 + |y_1 - y_1^0|^2 < p^2$) and a finite time interval T is employed, is that the problem reduces to a mixed problem in a bounded region Ω where Ω can be taken to be the solid hemisphere in $x_3 > 0$, with center $(x_1^0, x_2^0, 0)$ and radius $p + c_M T$ where c_M is the maximum velocity. In this case, the tangential E fields on the hemispherical portion of the bounding surface would be taken to be zero.

Once a solution of Problem P_1 is found, we can take the $\lim_{x_3 \to 0+} H_i(x, t)$ for $i = 1, 2$, and obtain a relation of the form

$$\boldsymbol{H}_T + \frac{1}{\mu} M_\alpha \boldsymbol{E}_T = \mathbf{0}, \; x_3 = 0, \; 0 < t < T \tag{14}$$

where $\boldsymbol{E}_T = (E_1, E_2)$ and $\boldsymbol{H}_T = (H_1, H_2)$. This is the up-going wave condition on the surface $x_3 = 0$. Since ϵ, μ are independent of x_3 the operator M_α is independent of x_3, so that condition (14) holds for any surface $x_3 =$ constant ≥ 0 (when the data is given on $x_3 = 0$, or the sources producing the wave lie below $x_3 = 0$).

By treating the same problem for the lower half-space $x_3 < 0$ we would get the down-going wave condition. It (as will be seen) takes the form

$$\boldsymbol{H}_T - \frac{1}{\mu} M_\alpha \boldsymbol{E}_T = \boldsymbol{0},\ x_3 = 0,\ 0 < t < T \tag{15}$$

We want to obtain an alternative form of the up- and down-going wave equation, so as to show the existence of and get the form of the *inverse* operator to M_α. For this we consider the following problem.

Problem P_2: Solve for $\boldsymbol{E}(x,t), \boldsymbol{H}(x,t)$, for $x_3 \geq 0,\ 0 < t < T$.

(i) $\nabla \times \boldsymbol{E} = -\mu(x_1, x_2, \alpha)\partial_t \boldsymbol{H},\ \nabla \times \boldsymbol{H} = \epsilon(x_1, x_2, \alpha)\partial_t \boldsymbol{E},\ x_3 > 0,\ 0 < t < T$

(ii) $\boldsymbol{E}(x,0) = \boldsymbol{H}(x,0) = \boldsymbol{0},\ x_3 > 0$

(iii) $H_i(x_1, x_2, 0, t) = H_i^0(x_1, x_2, t),\ i = 1, 2,\ x_3 = 0,\ 0 < t < T$

where the boundary data $H_i^0,\ i = 1, 2$ has compact support and is sufficiently smooth for existence of solution.

Once a solution is found, we can then take the $\lim_{x_3 \to 0+} E_i(x,t)$, and obtain a condition of the form

$$\boldsymbol{E}_T - \frac{1}{\epsilon} N_\alpha \boldsymbol{H}_T = \boldsymbol{0},\ x_3 = 0,\ 0 < t < T \tag{16}$$

This is an alternative form of the up-going wave condition. Hence for a suitable class of functions, $-\frac{1}{\epsilon}N_\alpha$ is the inverse of $\frac{1}{\mu}M_\alpha$. One would then expect

$$-\frac{1}{\epsilon} N_\alpha (\frac{1}{\mu} M_\alpha)\boldsymbol{u} = -\frac{1}{\mu} M_\alpha \frac{1}{\epsilon} N_\alpha \boldsymbol{u} = I\boldsymbol{u}$$

where I is the identity operator. The precise domain of the vector function $\boldsymbol{u}$ for which this is valid remains to be investigated.

The corresponding down-going wave condition obtained by treating Problem P_2 in the lower half-space $x_3 < 0$ is

$$\boldsymbol{E}_T + \frac{1}{\epsilon} N_\alpha \boldsymbol{H}_T = \boldsymbol{0} \tag{17}$$

The next step is to get an explicit form of the operators in terms of the fundamental solutions of Maxwell's equations (Dyadic Green's Functions, solutions corresponding to the pulsed electric and magnetic dipoles). See [4, 5] for properties of the Dyadic Green's Function in the c.w. case. From this we can deduce the various properties of the operators. We will work initially with two sets of fundamental solutions. See Appendix B for additional fundamental solutions and the reciprocal and other relations between them.

Let $\mathcal{E}_j^B(x, y, t_0 - t), \mathcal{H}_j^B(x, y, t_0 - t)$ be the free-space fundamental solution satisfying

$$\nabla \times \mathcal{E}_j^B = -\mu(x_1, x_2, \alpha)\partial_t \mathcal{H}_j^B \tag{18}$$

$$\nabla \times \mathcal{H}_j^B = \epsilon(x_1, x_2, \alpha)\partial_t \mathcal{E}_j^B + \boldsymbol{i}_j \delta(x - y)\delta(t - t_0) \tag{18'}$$

$$\text{where } \mathcal{E}_j^B = \mathcal{H}_j^B = 0,\ \text{for } t - t_0 > 0 \tag{18''}$$

where $\boldsymbol{i}_j$ is the unit vector in the direction of the x_j-axis. We shall denote the components of $\mathcal{E}_j^B$ by $\mathcal{E}_{ij}^B$, i.e.

$$\mathcal{E}_j^B = (\mathcal{E}_{1j}^B, \mathcal{E}_{2j}^B, \mathcal{E}_{3j}^B)$$

with similar notation for $\mathcal{H}_j^B$.

In addition to the above, let the pair $\widetilde{\mathcal{E}}_j^B(x, y, t_0 - t), \widetilde{\mathcal{H}}_j^B(x, y, t_0 - t)$ be the free space fundamental solutions satisfying

$$\nabla \times \widetilde{\mathcal{E}}_j^B = -\mu(x_1, x_2, \alpha)\partial_t \widetilde{\mathcal{H}}_j^B + \boldsymbol{i}_j \delta(x - y)\delta(t - t_0) \tag{19}$$

$$\nabla \times \widetilde{\mathcal{H}}_j^B = \epsilon(x_1, x_2, \alpha)\partial_t \widetilde{\mathcal{E}}_j^B \tag{19$'$}$$

$$\text{where } \widetilde{\mathcal{E}}_j^B = \widetilde{\mathcal{H}}_j^B = 0 \text{ for } t - t_0 > 0 \tag{19$''$}$$

Since ϵ and μ are independent of x_3, both pairs of functions $(\mathcal{E}_j^B, \mathcal{H}_j^B)$ and $(\widetilde{\mathcal{E}}_j^B, \widetilde{\mathcal{H}}_j^B)$ will depend upon the x_3 and y_3 variables in the linear combination $(x_3 - y_3)$, (this is due to the invariance under translation in the x_3-direction). It will then be convenient to decompose a point into its transverse components (x_1, x_2) and longitudinal component x_3 by using the notation

$$x = (\boldsymbol{x}, x_3), \;\; \boldsymbol{x} = (x_1, x_2)$$

Since $\delta(x_3 - y_3)$ has even parity with respect to $(x_3 - y_3)$, one can deduce the corresponding parity for the components of $\mathcal{E}^B, \mathcal{H}^B$ etc. from Eqs. (18-19$''$). It can be shown that following components of interest have the following indicated parity
$\mathcal{E}_{ij}^B, \widetilde{\mathcal{H}}_{ij}^B$, $i, j = 1, 2$, have even parity with respect to $(x_3 - y_3)$,
$\mathcal{H}_{ij}^B, \widetilde{\mathcal{E}}_{ij}^B$, $i, j = 1, 2$, have odd parity with respect to $(x_3 - y_3)$.

From this information we will form a set of fundamental solutions which satisfy in the distributional sense [6] certain boundary conditions on the surface $x_3 = 0$ (Method of Images).

For $j = 1, 2$, and $y_3 > 0$ let

$$\mathcal{E}_j^\dagger(\boldsymbol{x}, \boldsymbol{y}, x_3, y_3; t_0 - t) = \mathcal{E}_j^B(\boldsymbol{x}, \boldsymbol{y}, x_3 - y_3; t_0 - t) + \mathcal{E}_j^B(\boldsymbol{x}, \boldsymbol{y}, x_3 + y_3; t_0 - t)$$

with a similar definition for $\mathcal{H}_j^\dagger$, and

$$\widetilde{\mathcal{E}}_j^\dagger(\boldsymbol{x}, \boldsymbol{y}, x_3, y_3; t_0 - t) = \widetilde{\mathcal{E}}_j^B(\boldsymbol{x}, \boldsymbol{y}, x_3 - y_3; t_0 - t) + \widetilde{\mathcal{E}}_j^B(\boldsymbol{x}, \boldsymbol{y}, x_3 + y_3; t_0 - t)$$

with similar definition for $\widetilde{\mathcal{H}_j^\dagger}$.

It then follows from the parity relations given above that for $y_3 > 0$, $i, j = 1, 2$,

$$\mathcal{H}_{ij}^\dagger = \widetilde{\mathcal{E}}_{ij}^\dagger = 0, \text{ when } x_3 = 0 \tag{20}$$

$$\mathcal{E}_{ij}^\dagger(\boldsymbol{x}, \boldsymbol{y}, 0, y_3; t_0 - t) = 2\mathcal{E}_{ij}^B(\boldsymbol{x}, \boldsymbol{y}, |y_3|; t_0 - t) \tag{21}$$

$$\widetilde{\mathcal{H}}_{ij}^\dagger(\boldsymbol{x}, \boldsymbol{y}, 0, y_3; t_0 - t) = 2\widetilde{\mathcal{H}}_{ij}^B(\boldsymbol{x}, \boldsymbol{y}, |y_3|; t_0 - t) \tag{21$'$}$$

where relations (20), (21), (21′) hold in the distributional sense for $0 \le t \le T$, for example

$$\int_0^T \int_{\boldsymbol{R}^2} \mathcal{H}^\dagger_{ij}(x,y,t)\phi(\boldsymbol{x},t)d\boldsymbol{x}dt = 0, \quad x_3 = 0,$$

where ϕ is a suitable test function.

We will now proceed to use these results to get the up-going wave condition.

Let $(\boldsymbol{E},\boldsymbol{H})$ be a solution of Problem P_1, and use the identity

$$-\nabla\cdot(\boldsymbol{E}\times\mathcal{H}^\dagger_j + \mathcal{E}^\dagger_j\times\boldsymbol{H}) = \partial_t(\epsilon\boldsymbol{E}\cdot\mathcal{E}^\dagger_j + \mu\boldsymbol{H}\cdot\mathcal{H}^\dagger_j) + E_j\delta(x-y)\delta(t-t_0),\ x_3>0$$

by integrating it over the half-space $x_3 > 0$, and with respect to t from 0 to T (where $t_0 \le T$). Use the divergence theorem, boundary conditions (20), (21) to obtain

$$E_j(y,t_0) = 2\int_0^{t_0}\int_{\mathbb{R}^2} \boldsymbol{i}_3\cdot\mathcal{E}^B_j\times\boldsymbol{H}|_{x_3=0}d\boldsymbol{x}dt,\ y_3>0,\ j=1,2. \tag{22}$$

In a similar manner, using the solution $(\boldsymbol{E},\boldsymbol{H})$ of Problem P_2, and the fundamental solution $(\widetilde{\mathcal{E}}^\dagger,\widetilde{\mathcal{H}}^\dagger)$ it can be shown that

$$H_j(y,t_0) = -2\int_0^{t_0}\int_{\mathbb{R}^2} \boldsymbol{i}_3\cdot\boldsymbol{E}\times\widetilde{\mathcal{H}}^B_j|_{x_3=0}d\boldsymbol{x}dt,\ y_3>0,\ j=1,2 \tag{22′}$$

Taking the limit as $y_3 \to 0+$, we get the two forms for the up-going wave condition

$$\left.\begin{aligned} E_j(\boldsymbol{y},0,t) = 2\int_0^t\int_{\mathbb{R}^2}\{&-\mathcal{E}^B_{2j}(\boldsymbol{x},\boldsymbol{y},0,t-s)H_1(\boldsymbol{x},0,s)\\ &+\mathcal{E}^B_{1j}(\boldsymbol{x},\boldsymbol{y},0,t-s)H_2(\boldsymbol{x},0,s)\}d\boldsymbol{x}ds\end{aligned}\right. \tag{23}$$

$$\left.\begin{aligned} H_j(\boldsymbol{y},0,t) = 2\int_0^t\int_{\mathbb{R}^2}\{&-\widetilde{\mathcal{H}}^B_{2j}(\boldsymbol{x},\boldsymbol{y},0,t-s)E_1(\boldsymbol{x},0,s)\\ &+\widetilde{\mathcal{H}}^B_{1j}(\boldsymbol{x},\boldsymbol{y},0,t-s)E_2(\boldsymbol{x},0,s)\}d\boldsymbol{x}ds\end{aligned}\right. \tag{24}$$

where $j = 1,2$.

Note that we have to be careful in taking the limit as $y_3 \to 0$, due to the singularity of the fundamental solutions [7] . We will assume that ϵ,μ are sufficiently smooth so that the behaviour of the solution in the region of the singularity of $\boldsymbol{x}=\boldsymbol{y}$ can be given by the case where ϵ,μ are constant. This special case will be examined next but before we do so, we want to identify the elements of the matrix operators M_α and N_α.

Identifying Eq. (24) with Eq. (14) we see that the operator M_α is given by

$$M_\alpha\boldsymbol{u} = \int_0^t\int_{\mathbb{R}^2}\begin{bmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{bmatrix}\begin{bmatrix} u_1(\boldsymbol{x},s)\\ u_2(\boldsymbol{x},s)\end{bmatrix}d\boldsymbol{x}ds \tag{25}$$

where

$$\left.\begin{aligned} m_{i1} &= 2\mu(\boldsymbol{y},\alpha)\widetilde{\mathcal{H}}^B_{2i}(\boldsymbol{x},\boldsymbol{y},0,t-s)\\ m_{i2} &= -2\mu(\boldsymbol{y},\alpha)\widetilde{\mathcal{H}}^B_{1i}(\boldsymbol{x},\boldsymbol{y},0,t-s)\end{aligned}\right\} \tag{26}$$

and identifying Eq. (23) with Eq. (16) we see that

$$N_\alpha \boldsymbol{u} = \int_0^t \int_{\mathbb{R}^2} \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} \begin{bmatrix} u_1(\boldsymbol{x}, s) \\ u_2(\boldsymbol{x}, s) \end{bmatrix} d\boldsymbol{x} ds \tag{27}$$

where

$$\left. \begin{aligned} n_{i1} &= -2\epsilon(\boldsymbol{y}, \alpha)\mathcal{E}^B_{2i}(\boldsymbol{x}, \boldsymbol{y}, 0, t-s) \\ n_{i2} &= 2\epsilon(\boldsymbol{y}, \alpha)\mathcal{E}^B_{1i}(\boldsymbol{x}, \boldsymbol{y}, 0, t-s) \end{aligned} \right\} \tag{28}$$

The down-going wave condition can be obtained in a similar manner as the up-going wave condition, by treating the mixed problems corresponding to problems P_1 and P_2 for the lower half-space $x_3 < 0$. The resulting condition corresponding to Eqs. (23) and (24) is similar except for a difference of sign on the right-hand side of the equation. Thus the down-going wave conditions are given by Eqs. (15) and (17).

For the case where ϵ and μ are constant of position, i.e. $\epsilon = \epsilon(\alpha)$. The fundamental solution $\mathcal{E}^B_j(x, y, t-s), \mathcal{H}^B_j(x, y, t-s)$ is given by (with y, t fixed) [8],

$$\mathcal{H}^B_j = \nabla \times (\psi \boldsymbol{i}_j) \tag{29}$$

$$\epsilon(\alpha)\partial_s \mathcal{E}^B_j = \nabla \partial_j \psi - c^{-2}(\alpha)\partial_s^2 \psi \boldsymbol{i}_j \tag{30}$$

where

$$\psi = \frac{1}{4\pi r}\delta(t - s - r/c), \;\; r = |x - y| \tag{31}$$

is the fundamental solution of the wave equation [9] with velocity $c = c(\alpha)$, i.e.

$$\frac{1}{c^2}\psi_{ss} - \nabla^2 \psi = \delta(x-y)\delta(t-s) \tag{32}$$

$$\psi = 0, \;\; t - s < 0 \tag{32$'$}$$

The limit

$$\begin{aligned} &\lim_{y_3 \to 0+} 2\int_0^t \int_{\mathbb{R}^2} \frac{\delta(t-s-r/c)}{4\pi r}\Big|_{x_3=0} u(\boldsymbol{x}, s) d\boldsymbol{x} ds \\ &= \int_{\mathbb{R}^2} \frac{u(\boldsymbol{x}, t - |\boldsymbol{x} - \boldsymbol{y}|/c)}{2\pi |\boldsymbol{x} - \boldsymbol{y}|} d\boldsymbol{x} = K_\alpha u[\boldsymbol{y}, t] \end{aligned} \tag{33}$$

where u is a scalar function, defines the operator K_α which is used in the wave splitting for the scalar wave equation [10, 11]. Its inverse is given by

$$K_\alpha^{-1} = \Box^T K_\alpha = K_\alpha \Box^T$$

where $\Box^T = (c^{-2}\partial_t^2 - \partial_1^2 - \partial_2^2)$

Noting that if $u(\boldsymbol{x}, t)$ is twice differentiable with compact support, we have the limit

$$\begin{aligned} &\lim_{y_3 \to 0+} 2\int_0^t \int_{\mathbb{R}^2} \frac{\partial^2 \psi}{\partial x_i \partial x_j}\Big|_{x_3=0} u(\boldsymbol{x}, s) d\boldsymbol{x} ds \\ &= \lim_{y_3 \to 0+} 2\int_0^t \int_{\mathbb{R}^2} \psi\big|_{x_3=0} \partial_i \partial_j u(\boldsymbol{x}, s) d\boldsymbol{x} ds = K_\alpha \partial_i \partial_j u \end{aligned} \tag{34}$$

If $u(x,t)$ is Hölder continuous it can be shown that with $r = |\boldsymbol{x} - \boldsymbol{y}|$,

$$\partial_i K_\alpha u = K_\alpha \partial_i u = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{1}{r}\partial_{x_i} r[\frac{1}{c}u_t(\boldsymbol{x}, t - r/c) + \frac{1}{r}u(\boldsymbol{x}, t - r/c)]d\boldsymbol{x}$$

and the second term on the right-hand side can be integrated as the limit

$$\lim_{\delta\to 0}\int_\delta^\infty \int_0^{2\pi} \frac{1}{r}(\cos\theta)u(\boldsymbol{y} + r\hat{\theta}, t - r/c)d\theta dr$$

Here $\hat{\theta}$ is the unit vector in the direction $\boldsymbol{x} - \boldsymbol{y}$, and $r\cos\theta = x_i - y_i$. Thus it follows that at worst $\partial_t^{-1} K_\alpha \partial_i \partial_j u$ is a first order differential operator provided that the derivatives $\partial_i u$ are Hölder continuous. The same will be true for the convolution $\eta * K_\alpha \partial_i \partial_j u$ if $\eta(x,t)$ is differentiable with respect to t.

On noting that $\partial_s E_j = -\partial_t E_j$, Eq. (23) can be expressed in the form

$$E_1(\boldsymbol{y}, 0, t) = \frac{1}{\epsilon}\partial_t^{-1} K_\alpha\{\partial_1\partial_2 H_1 + (\frac{1}{c^2}\partial_s^2 - \partial_1^2)H_2\} \tag{35}$$

$$E_2(\boldsymbol{y}, 0, t) = \frac{1}{\epsilon}\partial_t^{-1} K_\alpha\{(-\frac{1}{c^2}\partial_s^2 + \partial_2^2)H_1 - \partial_1\partial_2 H_2\} \tag{36}$$

and the operator N_α is given explicitly by

$$N_\alpha \boldsymbol{u}[\boldsymbol{y}, t] = \partial_t^{-1} K_\alpha \begin{bmatrix} \partial_{x_1}\partial_{x_2} & (\frac{1}{c^2}\partial_t^2 - \partial_{x_1^2}) \\ (\partial_{x_2}^2 - \frac{1}{c^2}\partial_t^2) & -\partial_{x_1}\partial_{x_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{37}$$

with $c = c(\alpha), -\mu\partial_s \widetilde{\mathcal{H}}^B(x, y, t - s)$ will be given by the right-hand side of Eq. (30), thus we see that $-\mu\widetilde{\mathcal{H}}_{ij}^B(\boldsymbol{x}, \boldsymbol{y}, 0; t)$ has the same form $\epsilon\mathcal{E}_{ij}^B(\boldsymbol{x}, \boldsymbol{y}, 0, t)$, hence from Eqs. (26) and (28) it follows that

$$M_\alpha = N_\alpha, \text{ when } \epsilon = \epsilon(\alpha),\ \mu = \mu(\alpha) \tag{38}$$

5 Splitting in a Medium where $\epsilon = \epsilon(x_1, x_2, \alpha), \mu = \mu(x_1, x_2, \alpha)$

Using the up-and down-going wave conditions developed in the previous section, the decomposition of solutions of Maxwell's equations into up-and down-going waves will be obtained. Recall that the up-going wave condition is given by

$$\boldsymbol{E}_T - \frac{1}{\epsilon}N_\alpha \boldsymbol{H}_T = \boldsymbol{0},$$
$$\boldsymbol{H}_T + \frac{1}{\mu}M_\alpha \boldsymbol{E}_T = \boldsymbol{0},$$

and the down-going wave condition by

$$\boldsymbol{E}_T + \frac{1}{\epsilon}N_\alpha \boldsymbol{H}_T = \boldsymbol{0},$$
$$\boldsymbol{H}_T - \frac{1}{\mu}M_\alpha \boldsymbol{E}_T = \boldsymbol{0}.$$

We shall decompose $\boldsymbol{E}_T$ into components $\boldsymbol{E}_T^+$ and $\boldsymbol{E}_T^-$ as follows

$$\boldsymbol{E}_T = \frac{1}{2}(\boldsymbol{E}_T + \frac{1}{\epsilon}N_\alpha \boldsymbol{H}_T) + \frac{1}{2}(\boldsymbol{E}_T - \frac{1}{\epsilon}N_\alpha \boldsymbol{H}_T) = \boldsymbol{E}_T^+ + \boldsymbol{E}_T^-$$

This splitting will be expressed in matrix form

$$\begin{bmatrix} \boldsymbol{E}_T^+ \\ \boldsymbol{E}_T^- \end{bmatrix} = T_\alpha \begin{bmatrix} \boldsymbol{E}_T \\ \boldsymbol{H}_T \end{bmatrix} = \frac{1}{2} \begin{bmatrix} I & \frac{1}{\epsilon}N_\alpha \\ I & -\frac{1}{\epsilon}N_\alpha \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_T \\ \boldsymbol{H}_T \end{bmatrix} \tag{39}$$

and since $\frac{1}{\mu}M_\alpha \frac{1}{\epsilon}N_\alpha = -I$, the inverse transformation is given by

$$\begin{bmatrix} \boldsymbol{E}_T \\ \boldsymbol{H}_T \end{bmatrix} = T_\alpha^{-1} \begin{bmatrix} \boldsymbol{E}_T^+ \\ \boldsymbol{E}_T^- \end{bmatrix} = \begin{bmatrix} I & I \\ -\frac{1}{\mu}M_\alpha & \frac{1}{\mu}M_\alpha \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_T^+ \\ \boldsymbol{E}_T^- \end{bmatrix} \tag{40}$$

Using the inverse relationship between the operators M_α and N_α it can be easily shown that $\boldsymbol{E}_T^+$ and $\boldsymbol{E}_T^-$ satisfy the up-and down going wave conditions (14), (15) respectively. Next we will verify that $\boldsymbol{E}_T^+$, $\boldsymbol{E}_T^-$ are uncoupled waves in the medium where $\epsilon = \epsilon(x_1, x_2, \alpha)$, $\mu = \mu(x_1, x_2, \alpha)$. Let $(\boldsymbol{E}, \boldsymbol{H})$ be solutions of Maxwell's Equations with *compact support* in the slab $a < x_3 < b$, for $0 < t < T$, i.e.

$$\nabla \times \boldsymbol{E} = -\mu(x_1, x_2, \alpha)\partial_t \boldsymbol{H}, \ \nabla \times \boldsymbol{H} = \epsilon(x_1, x_2, \alpha)\partial_t \boldsymbol{E}$$

with initial conditions

$$\boldsymbol{E}(x, 0) = \boldsymbol{H}(x, 0) = \mathbf{0}$$

We shall first express E_3 and H_3 in terms of the tangential components, as follows. The third component of Maxwell's equations gives

$$H_3 = -\frac{1}{\mu}\partial_t^{-1}[\partial_1 E_2 - \partial_2 E_1], \ E_3 = \frac{1}{\epsilon}\partial_t^{-1}[\partial_1 H_2 - \partial_2 H_1]$$

which yields

$$\begin{bmatrix} \partial_1 H_3 \\ \partial_2 H_3 \end{bmatrix} = -\partial_t^{-1} J_\mu \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \quad \begin{bmatrix} \partial_1 E_3 \\ \partial_2 E_3 \end{bmatrix} = \partial_t^{-1} J_\epsilon \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \tag{41}$$

$$\text{where } J_\mu = \begin{bmatrix} -\partial_1(\frac{1}{\mu}\partial_2) & \partial_1(\frac{1}{\mu}\partial_1) \\ -\partial_2(\frac{1}{\mu}\partial_2) & \partial_2(\frac{1}{\mu}\partial_1) \end{bmatrix} \tag{42}$$

with a similar definition for matrix operator J_ϵ

The first two components of Maxwell's equation can be written in the form

$$\partial_3 \begin{bmatrix} \boldsymbol{E}_T \\ \boldsymbol{H}_T \end{bmatrix} = \begin{bmatrix} 0 & -\mu\partial_t\sigma \\ \epsilon\partial_t\sigma & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_T \\ \boldsymbol{H}_T \end{bmatrix} + \begin{bmatrix} \nabla^T E_3 \\ \nabla^T H_3 \end{bmatrix} \tag{43}$$

where the terms $\nabla^T E_3,\ \nabla^T H_3$ are given by Eq. (41), and σ is the matrix

$$\sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{44}$$

Insert expression (41) into (43) to yield

$$\partial_3 \begin{bmatrix} \boldsymbol{E}_T \\ \boldsymbol{H}_T \end{bmatrix} = \begin{bmatrix} 0 & (-\mu\partial_t\sigma + \partial_t^{-1}J_\epsilon) \\ (\epsilon\partial_t\sigma - \partial_t^{-1}J_\mu) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_T \\ \boldsymbol{H}_T \end{bmatrix} \tag{45}$$

Introduce the wave splitting given by Eq. (39) into (45)

$$\begin{bmatrix} \boldsymbol{E}_T \\ \boldsymbol{H}_T \end{bmatrix} = T_\alpha^{-1} \begin{bmatrix} \boldsymbol{E}_T^+ \\ \boldsymbol{E}_T^- \end{bmatrix},$$

differentiate, and then premultiply the expression by T_α to yield

$$\partial_3 \begin{bmatrix} \boldsymbol{E}_T^+ \\ \boldsymbol{E}_T^- \end{bmatrix} = W \begin{bmatrix} \boldsymbol{E}_T^+ \\ \boldsymbol{E}_T^- \end{bmatrix} \tag{46}$$

$$\begin{aligned} \text{where } W &= T_\alpha \begin{bmatrix} 0 & (-\mu\partial_t\sigma + \partial_t^{-1}J_\epsilon) \\ (\epsilon\partial_t\sigma - \partial_t^{-1}J_\mu) & 0 \end{bmatrix} T_\alpha^{-1} \\ &= \frac{1}{2} \begin{bmatrix} A^+ & A^- \\ -A^- & -A^+ \end{bmatrix} \end{aligned} \tag{47}$$

where $A^\pm = \pm(\mu\partial_t\sigma - \partial_t^{-1}J_\epsilon)\frac{1}{\mu}M_\alpha + \frac{1}{\epsilon}N_\alpha(\epsilon\partial_t\sigma - \partial_t^{-1}J_\mu)$
It is shown in Appendix C that

$$\frac{1}{\epsilon}N_\alpha(\epsilon\partial_t\sigma - \partial_t^{-1}J_\mu) = (\mu\partial_t\sigma - \partial_t^{-1}J_\epsilon)\frac{1}{\mu}M_\alpha = -\Lambda_\alpha \tag{48}$$

$$\text{where } \Lambda_\alpha \boldsymbol{u}(\boldsymbol{y}, t) = \lim_{y_3 \to 0+} 2 \int_0^t \int_{\mathbb{R}} \begin{bmatrix} \frac{\partial \mathcal{H}_{21}^B}{\partial x_3} & -\frac{\partial \mathcal{H}_{11}^B}{\partial x_3} \\ \frac{\partial \mathcal{H}_{22}^B}{\partial x_3} & -\frac{\partial \mathcal{H}_{12}^B}{\partial x_3} \end{bmatrix}_{x_3=0} \begin{bmatrix} u_1(\boldsymbol{x}, s) \\ u_1(\boldsymbol{x}, s) \end{bmatrix} d\boldsymbol{x}ds \tag{49}$$

where $\mathcal{H}_{ij}^B = \mathcal{H}_{ij}^B(\boldsymbol{x}, \boldsymbol{y}, x_3 - y_3; t - s)$.
Hence it follows that

$$W = \begin{bmatrix} -\Lambda_\alpha & 0 \\ 0 & \Lambda_\alpha \end{bmatrix} \tag{50}$$

and system (46) becomes an *uncoupled* system the up-and down-going waves

$$\partial_3 \begin{bmatrix} \boldsymbol{E}_T^+ \\ \boldsymbol{E}_T^- \end{bmatrix} = \begin{bmatrix} -\Lambda_\alpha(\boldsymbol{x}, t) & 0 \\ 0 & \Lambda_\alpha(\boldsymbol{x}, t) \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_T^+ \\ \boldsymbol{E}_T^- \end{bmatrix} \tag{51}$$

For the special case where $\mu = \mu(\alpha)$, $\epsilon = \epsilon(\alpha)$, i.e. they are independent of x_1 and x_2,

$$\Lambda_\alpha \boldsymbol{u}(\boldsymbol{y}, t) = \lim_{y_3 \to 0+} 2 \int_0^t \int_{\mathbb{R}^2} \frac{\partial^2 \psi}{\partial x_3^2} \Big|_{x_3=0} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} d\boldsymbol{x} ds$$

where ψ is a solution of Eq. (32). The expression in the integral yields

$$2 \int_0^t \int_{\mathbb{R}^2} (\Box^T \psi) \boldsymbol{u} d\boldsymbol{x} ds = 2 \int_0^t \int_{\mathbb{R}^2} \psi \Box^T \boldsymbol{u} d\boldsymbol{x} ds$$

and on taking the limit as $y_3 \to 0$, one obtains for the vector function $\boldsymbol{u}$

$$\Lambda_\alpha \boldsymbol{u}(\boldsymbol{y}, t) = K_\alpha \Box^T \boldsymbol{u} = K_\alpha^{-1} \boldsymbol{u} \tag{52}$$

Thus Eq. (51) becomes

$$\partial_3 \begin{bmatrix} \boldsymbol{E}_T^+ \\ \boldsymbol{E}_T^- \end{bmatrix} = \begin{bmatrix} -K_\alpha^{-1} I & 0 \\ 0 & K_\alpha^{-1} I \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_T^+ \\ \boldsymbol{E}_T^- \end{bmatrix} \tag{51$'$}$$

which is expected.

6 Wave Splitting in Medium where $\epsilon = \epsilon(x_1, x_2, x_3)$ and $\mu = \mu(x_1, x_2, x_3)$, and the Imbedding Equation for the Reflection operator

We are interested in the case where the physical geometry is described by the constitutive equations

$$\boldsymbol{D} = \epsilon(x_1, x_2, x_3) \boldsymbol{E}, \quad \boldsymbol{B} = \mu(x_1, x_2, x_3) \boldsymbol{H}$$

with ϵ, μ sufficiently smooth functions.

The splitting designed for the case of transverse dependence upon x_1, and x_2 and dependence upon the parameter α, ($\epsilon = \epsilon(x_1, x_2, \alpha)$, etc.) will be used in a similar fashion as well done for the scalar wave equation [10, 11] to obtain the generalization of the differential system (51) for up-and down-going waves. From this the imbedding equations for the reflection operator $\boldsymbol{R}$ where $\boldsymbol{E}_T^+ = \boldsymbol{R}\boldsymbol{E}_T^-$ is obtained.

The up-and down-going wave decomposition will be defined as follows. On each plane $x_3 = x_3^0$, we will define the up-and down-going wave component by Eq. (39) with the parameter α taken to be x_3^0,

$$\begin{bmatrix} \boldsymbol{E}_T^+ \\ \boldsymbol{E}_T^- \end{bmatrix} = T_{x_3^0} \begin{bmatrix} \boldsymbol{E}_T \\ \boldsymbol{H}_T \end{bmatrix} \tag{53}$$

This means that the parameter α in the definition of the operators N_α and M_α is replaced by x_3^0. On the plane $x_3 = x_3^0$, Maxwell's equations are expressed in the form

$$\partial_3 \begin{bmatrix} \boldsymbol{E}_T \\ \boldsymbol{H}_T \end{bmatrix} = \begin{bmatrix} 0 & (-\mu \partial_t \sigma + \partial_t^{-1} J_\epsilon) \\ (\epsilon \partial_t \sigma - \partial_t^{-1} J_\mu) & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_T \\ \boldsymbol{H}_T \end{bmatrix}, x = x_3^0 \tag{54}$$

as follows from Eq (45). Note that in expression (54) the parameter α is replaced by x_3^0 in the terms involving ϵ and μ, i.e. $\epsilon = \epsilon(x_1, x_2, x_3^0)$, etc. Then as in the previous section, the splitting given by Eq. (53) is inserted into Eq. (54), the appropriate terms differentiated with respect to x_3^0 and finally the result premultiplied by $T_{x_3^0}^{-1}$. The results will be the same as in the previous section, except there is an additional term due to the derivative of $T_{x_3^0}^{-1}$ (essentially the derivative with respect to the parameter). This term appearing on the right-hand side of the equation is given by

$$-T_{x_3^0}\frac{\partial}{\partial x_3^0}T_{x_3^0}^{-1} = (\frac{\partial T_{x_3^0}}{\partial x_3^0})T_{x_3^0}^{-1} = \frac{1}{2}\begin{bmatrix} -\Gamma_{x_3^0} & \Gamma_{x_3^0} \\ \Gamma_{x_3^0} & -\Gamma_{x_3^0}, \end{bmatrix} \tag{55}$$

where the matrix operator $\Gamma_{x_3^0}$ is given by

$$\Gamma_{x_3^0} = [\frac{\partial}{\partial \alpha}(\epsilon^{-1}(x_1, x_2, \alpha)N_\alpha)]\mu^{-1}(x_1, x_2, \alpha)M_\alpha, \;\; \alpha = x_3^0 \tag{56}$$

At this point (as with the scalar case [10, 11] we will drop the superscript 0 on x_3^0, and for convenience define

$$M = M_{x_3^0}, \;\; N = N_{x_3^0}, \;\; \Gamma = \Gamma_{x_3^0}, \Lambda = \Lambda_{x_3^0}, K = K_{x_3^0},$$

The resulting system thus becomes

$$\partial_3\begin{bmatrix} \boldsymbol{E}_T^+ \\ \boldsymbol{E}_T^- \end{bmatrix} = \begin{bmatrix} -\Lambda & 0 \\ 0 & \Lambda \end{bmatrix}\begin{bmatrix} \boldsymbol{E}_T^+ \\ \boldsymbol{E}_T^- \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -\Gamma & \Gamma \\ \Gamma & -\Gamma \end{bmatrix}\begin{bmatrix} \boldsymbol{E}_T^+ \\ \boldsymbol{E}_T^- \end{bmatrix} \tag{57}$$

where the operator Λ is given by Eq. (49), and Γ by Eq. (56) with the parameter α replaced by x_3.

For the special case where the physical medium is plane stratified, i.e. $\epsilon = \epsilon(x_3)$, $\mu = \mu(x_3)$, then from Eq. (52) and Appendix D,

$$\Lambda = K^{-1}I \tag{58}$$

$$\Gamma = (\frac{\epsilon'}{\epsilon} + \frac{2c'}{c} - K'K^{-1})I + \frac{2c'}{c}K^2\begin{bmatrix} \partial_1^2 & \partial_1\partial_2 \\ \partial_1\partial_2 & \partial_2^2 \end{bmatrix} \tag{59}$$

where

$$K'u[\boldsymbol{x}, t] = \frac{c'(x_3)}{2\pi c^2(x_3)}\int_{\mathbb{R}^2} u_t(\boldsymbol{y}, t - |\boldsymbol{x} - \boldsymbol{y}|/c)d\boldsymbol{y} \tag{59$'$}$$

and I is the identity matrix, and c' is the derivative of c with respect to x_3.

The imbedding equation for the reflection operator can be easily obtained. Returning to the invariant imbedding geometry discussed earlier, on the plane $x_3 = \alpha$, the up-going wave is related to the down-going wave by a reflection operator $\boldsymbol{R}(\alpha)$ that depends upon the parameter α. To get the differential equation for $\boldsymbol{R}(\alpha)$ we replace the parameter α by x_3^0, and hence just x_3. For simplification we denote the resulting operator by $\boldsymbol{R}$, yielding the relation

$$\boldsymbol{E}_T^+ = \boldsymbol{R}\boldsymbol{E}_T^- \tag{60}$$

The imbedding equation for $\boldsymbol{R}$ follows from the usual procedure [10–12]. Set $\boldsymbol{E}_T^+ = \boldsymbol{R}\boldsymbol{E}_T^-$ in system (57) to give

$$(\frac{\partial \boldsymbol{R}}{\partial x_3})\boldsymbol{E}_T^- + \boldsymbol{R}\frac{\partial \boldsymbol{E}_T^-}{\partial x_3} = -\Lambda \boldsymbol{R}\boldsymbol{E}_T^- - \frac{1}{2}\Gamma \boldsymbol{R}\boldsymbol{E}_T^- + \frac{1}{2}\Gamma \boldsymbol{E}_T^- \tag{61}$$

$$\frac{\partial \boldsymbol{E}_T^-}{\partial x_3} = \Lambda \boldsymbol{E}_T^- - \frac{1}{2}\Gamma \boldsymbol{E}_T^- + \frac{1}{2}\Gamma \boldsymbol{R}\boldsymbol{E}_T^- \tag{62}$$

Use Eq. (62) to eliminate the derivative of $\boldsymbol{E}_T^-$ from Eq. (61) giving

$$\left\{\frac{\partial \boldsymbol{R}}{\partial x_3} + (\boldsymbol{R}\Lambda + \Lambda \boldsymbol{R}) + \frac{1}{2}(\Gamma \boldsymbol{R} - \boldsymbol{R}\Gamma) + \frac{1}{2}\boldsymbol{R}\Gamma \boldsymbol{R}\right\}\boldsymbol{E}_T^- = \frac{1}{2}\Gamma \boldsymbol{E}_T^- \tag{63}$$

7 Wave-Splitting in a Medium with Dispersion

For the physical medium we will take the constitutive equations

$$\begin{aligned} \mu &= \mu(x_1, x_2, x_3) \\ \boldsymbol{D} &= \epsilon(x_1, x_2, x_3)\boldsymbol{E} + G(x_1, x_2, x_3, t) * \boldsymbol{E} \end{aligned} \tag{64}$$

where $G = 0$ at $t = 0$, and $G = 0$ for $x_3 \geq 0$.

The procedure here is the same as in the previous section (we are splitting with the principal part), except that we will get an extra term in Eq. (54) and Eq. (57) due to the dispersion term. In particular we need to invert the following component of Maxwell's equation

$$\epsilon \partial_t E_3 + G * \partial_t E_3 = (\partial_1 H_2 - \partial_2 H_1) \tag{65}$$

for $\partial_t E_3$. Let $\eta(x, t)$ be the solution of the Volterra integral equation

$$\epsilon \eta + \epsilon^{-1} G + G * \eta = 0 \tag{66}$$

Then Eq. (65) can be inverted to give

$$\partial_t E_3 = (\frac{1}{\epsilon} + \eta *)(\partial_1 H_2 - \partial_2 H_1).$$

Hence it follows that

$$\begin{bmatrix} \partial_1 E_3 \\ \partial_2 E_3 \end{bmatrix} = [\partial_t^{-1} J_\epsilon + \partial_t^{-1} J_{\frac{1}{\eta}} *] \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \tag{67}$$

where $J_{\frac{1}{\eta}}$ is given by Eq. (42) with μ replaced by $\frac{1}{\eta}$.

Also we get an extra term in the following system

$$\partial_3 \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = (\epsilon \partial_t \sigma - \partial_t^{-1} J_\mu)\boldsymbol{E}_T + G_t \sigma * \boldsymbol{E}_T$$

Thus the effect of the dispersive term is to produce the extra term

$$\begin{bmatrix} 0 & \partial_t^{-1} J_{\frac{1}{\eta}} \\ G_t \sigma & 0 \end{bmatrix} * \begin{bmatrix} \boldsymbol{E}_T \\ \boldsymbol{H}_T \end{bmatrix}$$

on the right-hand side of Eq. (54). As a consequence, the following extra term

$$T \begin{bmatrix} 0 & \partial_t^{-1} J_{\frac{1}{\eta}} \\ G_t \sigma & 0 \end{bmatrix} * T^{-1} = \frac{1}{2} \begin{bmatrix} (-P+Q) & (P+Q) \\ (-P-Q) & (P-Q) \end{bmatrix} \tag{68}$$

arises on the right-hand side of Eq. (57), when the operators P and Q are given by

$$P\boldsymbol{u} = \partial_t^{-1}[J_{\frac{1}{\eta}} * (\frac{1}{\mu} M \boldsymbol{u})] \tag{69}$$

$$Q\boldsymbol{u} = \frac{1}{\epsilon} N[G_t \sigma * \boldsymbol{u}] \tag{70}$$

Thus the resulting system for the up-and down-going waves is

$$\partial_3 \begin{bmatrix} \boldsymbol{E}_T^+ \\ \boldsymbol{E}_T^- \end{bmatrix} = \begin{bmatrix} -\Lambda & 0 \\ 0 & \Lambda \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_T^+ \\ \boldsymbol{E}_T^- \end{bmatrix} + \frac{1}{2} \begin{bmatrix} (-\Gamma - P + Q) & (\Gamma + P + Q) \\ (\Gamma - P - Q) & (-\Gamma + P - Q) \end{bmatrix} \begin{bmatrix} \boldsymbol{E}_T^+ \\ \boldsymbol{E}_T^- \end{bmatrix}$$

For the case where $\mu = \mu(x_3)$, $\epsilon = \epsilon(x_3)$, Λ and Γ are given by Eqs. (58), (59), (59′). Also in this case $N = M$ with N given by Eq. (37) (with α replaced by x_3).

Note that is a generalization of the work of Beezley and Krueger [13].

8 Conclusion

It is a straightforward process to generalize the splitting in the last section to the case where one has the general anisotropic dispersion term as given by Eqs. (2), (2′), but with isotropic principal part where $a_{ij} = \epsilon\delta_{ij}$, $d_{ij} = \mu\delta_{ij}$, $b_{ij} = 0$ with δ_{ij} being the Kronecker delta. However for the case where the principal part is anisotropic, the concepts can be applied but with much more work requiring a thorough investigation of the associated fundamental solutions (Dyadic Green's functions).

A number of assumptions need to be examined in more detail, among them the requirements on the smoothness of ϵ and μ so that the singular part as $x \to y$ of the fundamental solution is described by the case where ϵ and μ are constant (these being the local values). Also, the conditions on ϵ, μ and the appropriate function space for the domain of the operators M_α, N_α so that the $-\frac{1}{\epsilon} N_\alpha \frac{1}{\mu} M_\alpha = I$ holds, need to be examined.

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Appendix A Expressions for $\omega_{em}(T),\ \omega_d(T)$

From Eq. (8) we have

$$\omega_{em}(x,T)+\omega_d(x,T)=\int_0^T(\boldsymbol{E}\cdot\partial_t\boldsymbol{D}+\boldsymbol{H}\cdot\partial_t\boldsymbol{B})dt$$

Using the constitutive Eqs. (2), (2′), we will set

$$\omega_{em}(x,T)=\int_0^T\{E_i(t)\partial_t[a_{ij}E_j+b_{ij}H_j]+H_i(t)\partial_t[b_{ji}E_j+d_{ij}H_j]\}dt \tag{A.1}$$

$$\begin{aligned}\omega_d(x,T)=&\int_0^T E_i(t)\partial_t[G_{ij}*E_j(t)+K_{ij}*H_j(t)]dt\\&+\int_0^T H_i(t)\partial_t[L_{ij}*E_j(t)+F_{ij}*H_j(t)]dt\end{aligned} \tag{A.2}$$

Since $\boldsymbol{E}=\boldsymbol{H}=\mathbf{0}$ at $t=0$, it follows (using the fact that the matrix $A_{ij}(x)$ given by Eq. (3) is symmetric).

$$\begin{aligned}\omega_{em}(x,T)=&\frac{1}{2}E_i(T)a_{ij}E_j(T)+\frac{1}{2}H_i(T)d_{ij}H_j(T)\\&+\frac{1}{2}E_i(T)b_{ij}H_j(T)+\frac{1}{2}E_j(T)b_{ji}H_i(T)\end{aligned} \tag{A.3}$$

$$\omega_{em}(x,T)=\frac{1}{2}\sum_{i,j=1}^{6}U_iA_{ij}(x)U_j$$

where $\boldsymbol{U}=(E_i(x,T),H_i(x,T))$

To reduce $\omega_d(x,T)$ we need to prove an identity involving a convolution term $P*b$, where we may take $P(0)=0$.

$$\begin{aligned}\int_0^T a(t)\partial_t[P*b(t)]dt&=\int_0^T a(t)\int_0^t P'(t-s)b(s)dsdt\\&=\frac{1}{2}\int_0^T a(t)\int_0^t P'(t-s)b(s)dsdt+\frac{1}{2}\int_0^T b(s)\int_s^T P'(t-s)a(t)dtds\\&=\frac{1}{2}\int_0^T\{a(t)\int_0^t P'(t-s)b(s)ds+b(t)\int_t^T P'(s-t)a(s)ds\}dt\end{aligned}$$

where $P'(u)=\frac{dP(u)}{du}$. Applying identity we obtain (using the fact $G_{ij}(0)=0$)

$$\begin{aligned}\int_0^T E_i(t)\partial_t(G_{ij}*E_j)dt=&\frac{1}{2}\int_0^T\{E_i(t)\int_0^t G'_{ij}(t-s)E_j(s)ds\\&+E_j(t)\int_t^T G'_{ij}(s-t)E_i(s)ds\}dt\\=&\frac{1}{2}\int_0^T\int_0^T E_i(t)g_{ij}(t,s)E_j(s)dsdt\end{aligned}$$

where

$$g_{ij}(t,s) = \begin{cases} G'_{ij}(t-s) & , \text{ for } 0 \le s \le t, \\ G'_{ji}(s-t) & , \text{ for } t \le s \le T \end{cases} \tag{A.4}$$

In a similar manner it follows

$$\int_0^T H_i(t)\partial_t(F_{ij} * H_j)dt = \frac{1}{2}\int_0^T \int_0^T H_i(t) f_{ij}(t,s) H_j(s) ds dt$$

$$\text{where } f_{ij}(t,s) = \begin{cases} F'_{ij}(t-s) & , \text{ for } 0 \le s \le t \\ F'_{ji}(s-t) & , \text{ for } t \le s \le T \end{cases} \tag{A.5}$$

$$\begin{aligned}
&\int_0^T \{E_i(t)\partial_t(K_{ij} * H_j) + H_i(t)\partial_t(L_{ij} * E_{ij})\}dt \\
&= \frac{1}{2}\int_0^T \{E_i(t)\int_0^t K'_{ij}(t-s)H_j(s)ds + H_j(t)\int_t^T K'_{ij}(s-t)E_i(s)ds\}dt \\
&+ \frac{1}{2}\int_0^T \{H_i(t)\int_0^t L'_{ij}(t-s)E_j(s)ds + E_j(t)\int_t^T L'_{ij}(s-t)H_i(s)ds\}dt \\
&= \frac{1}{2}\int_0^T \int_0^T \{E_i(t)k_{ij}(t,s)H_j(s) + H_i(t)\ell_{ij}(t,s)E_j(s)\}dsdt
\end{aligned}$$

where

$$k_{ij}(t,s) = \begin{cases} K'_{ij}(t-s) & , \text{ for } 0 \le s \le t \\ L'_{ji}(s-t) & , \text{ for } t \le s \le T \end{cases} \tag{A.6}$$

$$\ell_{ij}(t,s) = \begin{cases} L'_{ij}(t-s) & , \text{ for } 0 \le s \le t \\ K'_{ji}(s-t) & , \text{ for } t \le s \le T \end{cases} \tag{A.7}$$

Combining the results we obtain

$$\omega_d(x,T) = \frac{1}{2}\int_0^T \int_0^T [E_i(x,t), H_i(x,t)] \begin{bmatrix} g_{ij}(t,s) & k_{ij}(t,s) \\ \ell_{ij}(t,s) & f_{ij}(t,s) \end{bmatrix} \begin{bmatrix} E_j(x,s) \\ H_j(x,s) \end{bmatrix} dsdt \tag{A.8}$$

Appendix B Reciprocal Relations Between Fundamental Solutions

The pair $(\mathcal{E}_j^F, \mathcal{H}_j^F)$, $(\mathcal{E}_j^B, \mathcal{H}_j^B)$ are fundamental solutions of the system $(y, t_o$ fixed)

$$\nabla \times \mathcal{E}_j = -\mu\partial_t \mathcal{H}_j, \ \nabla \times \mathcal{H}_j = \epsilon\partial_t \mathcal{E}_j + \boldsymbol{i}_j \delta(x-y)\delta(t-t_0) \tag{B.1}$$

$$\left.\begin{aligned} \text{where } \mathcal{E}_j^F &= \mathcal{H}_j^F = \mathbf{0}, \text{ for } t - t_0 < 0 \\ \mathcal{E}_j^B &= \mathcal{H}_j^B = \mathbf{0}, \text{ for } t - t_0 > 0 \end{aligned}\right\} \tag{B.2}$$

They will be expressed in terms of the variables $x, y, t - t_0$ as follows.

$$\mathcal{E}_j^F = \mathcal{E}_j^F(x,y,t-t_0), \ \mathcal{E}_j^B = \mathcal{E}_j^B(x,y,t_0-t) \tag{B.3}$$

The pair $(\widetilde{\mathcal{E}}_j^F, \widetilde{\mathcal{H}}_j^F)$, $(\widetilde{\mathcal{E}}_j^B, \widetilde{\mathcal{H}}_j^B)$ are fundamental solutions of the system (y, t_o fixed)

$$\nabla \times \widetilde{\mathcal{E}}_j = -\mu \partial_t \widetilde{\mathcal{H}}_j + \boldsymbol{i}_j \delta(x-y)\delta(t-t_0), \ \nabla \times \widetilde{\mathcal{H}}_j = \epsilon \partial_t \widetilde{\mathcal{E}}_j \tag{B.4}$$

$$\left.\begin{array}{r} \text{where } \widetilde{\mathcal{E}}_j^F = \widetilde{\mathcal{H}}_j^F = 0 \text{ for } t - t_0 < 0 \\ \widetilde{\mathcal{E}}_j^B = \widetilde{\mathcal{H}}_j^B = 0 \text{ for } t - t_0 > 0 \end{array}\right\} \tag{B.5}$$

with notation

$$\widetilde{\mathcal{E}}_j^F = \widetilde{\mathcal{E}}_j^F(x, y, t - t_0) \text{ and } \widetilde{\mathcal{E}}_j^B = \widetilde{\mathcal{E}}_j^B(x, y, t_0 - t) \tag{B.6}$$

If $(\boldsymbol{E}^F, \boldsymbol{H}^F)$ and $(\boldsymbol{E}^B, \boldsymbol{H}^B)$ are solutions of the systems

$$\begin{aligned} \nabla \times \boldsymbol{E}^{(F,B)} &= -\mu \partial_t \boldsymbol{H}^{(F,B)} + \boldsymbol{M}^{(F,B)} \\ \nabla \times \boldsymbol{H}^{(F,B)} &= \epsilon \partial_t \boldsymbol{E}^{(F,B)} + \boldsymbol{J}^{(F,B)} \end{aligned}$$

Then we have the identity

$$\begin{aligned} &\partial_t \{\epsilon \boldsymbol{E}^F \cdot \boldsymbol{E}^B + \mu \boldsymbol{H}^F \cdot \boldsymbol{H}^B\} + \boldsymbol{E}^F \cdot \boldsymbol{J}^B + \boldsymbol{E}^B \cdot \boldsymbol{J}^F - \boldsymbol{H}^B \cdot \boldsymbol{M}^F - \boldsymbol{H}^F \cdot \boldsymbol{M}^B \\ &\qquad = \nabla \cdot (\boldsymbol{H}^F \times \boldsymbol{E}^B + \boldsymbol{H}^B \times \boldsymbol{E}^F) \end{aligned}$$

Integrate this expression with respect to x over $\mathbb{R}^3$ and with respect to t from t_0 to t_1, where it is assumed that $(\boldsymbol{E}^F, \boldsymbol{H}^F)$ vanishes for $t < t_0$, and $(\boldsymbol{E}^B, \boldsymbol{H}^B)$ vanishes for $t > t_1$ giving

$$\begin{aligned} &\int_{t_0}^{t_1} \int_{\mathbb{R}^3} \{\boldsymbol{E}^F(x, y, t - t_0) \cdot \boldsymbol{J}^B(x, z, t_1 - t) + \boldsymbol{E}^B(x, z, t_1 - t) \boldsymbol{J}^F(x, y, t - t_0)\} dx dt \\ &\quad = \int_{t_0}^{t_1} \int_{\mathbb{R}^3} \Big\{ \boldsymbol{H}^F(x, y, t - t_0) \cdot \boldsymbol{M}^B(x, z, t_1 - t) \\ &\qquad\qquad + \boldsymbol{H}^B(x, z, t_1 - t) \cdot \boldsymbol{M}^F(x, y, t - t_0) \Big\} dx dt \end{aligned}$$

We will now replace $(\boldsymbol{E}^F, \boldsymbol{H}^F)$ and $(\boldsymbol{E}^B, \boldsymbol{H}^B)$ with the various fundamental solutions

Recalling that we denote the components of $\mathcal{E}_j^F$ etc. by $(\mathcal{E}_{1j}^F, \mathcal{E}_{2j}^F, \mathcal{E}_{3j}^F)$, we can obtain the following relations

$$\mathcal{E}_{ij}^F(z, y, t_1 - t_0) + \mathcal{E}_{ji}^B(y, z, t_1 - t_0) = 0 \tag{B.7}$$

$$\widetilde{\mathcal{H}}_{ij}^F(z, y, t_1 - t_0) + \widetilde{\mathcal{H}}_{ji}^B(y, z, t_1 - t_0) = 0 \tag{B.8}$$

$$\widetilde{\mathcal{E}}_{ji}^B(y, z, t_1 - t_0) = \mathcal{H}_{ij}^F(z, y, t_1 - t_0) \tag{B.9}$$

$$\widetilde{\mathcal{E}}_{ij}^F(z, y, t_1 - t_0) = \mathcal{H}_{ji}^B(y, z, t_1 - t_0) \tag{B.10}$$

Appendix C Verification of Relation

$$\frac{1}{\epsilon}N_\alpha(\epsilon\partial_t\sigma - \partial_t^{-1}J_\mu) = (\mu\partial_t\sigma - \partial_t^{-1}J_\epsilon)\frac{1}{\mu}M_\alpha \tag{C.1}$$

We will first simplify the expression on the left-hand side of Eq. (C.1). Note that

$$\frac{1}{\epsilon}N_\alpha\partial_t^{-1}J_\mu\boldsymbol{u} = \epsilon^{-1}(\boldsymbol{y})\int_0^t\int_{\mathbb{R}^2}\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22}\end{bmatrix}\begin{bmatrix} -\partial_{x_1}(\frac{1}{\mu}\partial_{x_2}) & \partial_{x_1}(\frac{1}{\mu}\partial_{x_1}) \\ -\partial_{x_2}(\frac{1}{\mu}\partial_{x_2}) & \partial_{x_2}(\frac{1}{\mu}\partial_{x_1})\end{bmatrix}\begin{bmatrix}\partial_s^{-1}u_1 \\ \partial_s^{-1}u_2\end{bmatrix} d\boldsymbol{x}ds \tag{C.2}$$

where $n_i = n_i(\boldsymbol{x}, \boldsymbol{y}, t-s)$, and $u_i = u_i(\boldsymbol{x}, s)$.

It follows on integrating by parts with respect to x_1 and x_2 that the right-hand side of (C.2) reduces to

$$\int_0^t\int_{\mathbb{R}^2}\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2\end{bmatrix}\begin{bmatrix}\partial_s^{-1}u_1 \\ \partial_s^{-1}u_2\end{bmatrix} d\boldsymbol{x}ds$$

where for $i = 1, 2$

$$\begin{aligned} a_i &= -\epsilon^{-1}\partial_{x_2}(\frac{1}{\mu}\sum_{j=1}^{2}\partial_{x_j}n_{ij}) = 2\partial_{x_2}(\frac{1}{\mu}[\partial_{x_1}\mathcal{E}_{2i}^B - \partial_{x_2}\mathcal{E}_{1i}^B]) \\ &= -2\partial_{x_2}\partial_s\mathcal{H}_{3i}^B(\boldsymbol{x}, \boldsymbol{y}, 0, t-s) \\ b_i &= \epsilon^{-1}\partial_{x_1}(\frac{1}{\mu}\sum_{j=1}^{2}\partial_{x_j}n_{ij}) = 2\partial_{x_1}\partial_s\mathcal{H}_{3i}^B(\boldsymbol{x}, \boldsymbol{y}, 0, t-s)\end{aligned}$$

Hence on integrating by parts with respect to s, it follows that

$$\frac{1}{\epsilon}N_\alpha\partial_t^{-1}J_\mu\boldsymbol{u} = 2\int_0^t\int_{\mathbb{R}^2}\begin{bmatrix}\partial_{x_2}H_{31}^B & -\partial_{x_1}\mathcal{H}_{31}^B \\ \partial_{x_2}\mathcal{H}_{32}^B & -\partial_{x_1}\mathcal{H}_{32}^B\end{bmatrix}\begin{bmatrix}u_1(\boldsymbol{x}, s) \\ u_2(\boldsymbol{x}, s)\end{bmatrix} d\boldsymbol{x}ds \tag{C.3}$$

where $\mathcal{H}_{3i}^B = \mathcal{H}_{3i}^B(\boldsymbol{x}, \boldsymbol{y}, 0, t-s)$.
The term

$$\begin{aligned}\frac{1}{\epsilon}N_\alpha\epsilon\partial_t\sigma\boldsymbol{u} &= 2\int_0^t\int_{\mathbb{R}^2}\epsilon(\boldsymbol{x})\begin{bmatrix}-\mathcal{E}_{21}^B & \mathcal{E}_{11}^B \\ -\mathcal{E}_{22}^B & \mathcal{E}_{12}^B\end{bmatrix}\begin{bmatrix}\partial_s u_2(\boldsymbol{x}, s) \\ -\partial_s u_1(\boldsymbol{x}, s)\end{bmatrix} d\boldsymbol{x}ds \\ &= 2\int_0^t\int_{\mathbb{R}^2}\epsilon(\boldsymbol{x})\partial_s\begin{bmatrix}\mathcal{E}_{11}^B & \mathcal{E}_{21}^B \\ \mathcal{E}_{12}^B & \mathcal{E}_{22}^B\end{bmatrix}\begin{bmatrix}u_1(\boldsymbol{x}, s) \\ u_2(\boldsymbol{x}, s)\end{bmatrix} d\boldsymbol{x}ds\end{aligned} \tag{C.4}$$

From (C.3) and (C.4) we have the relation

$$\frac{1}{\epsilon}N_\alpha(\epsilon\partial_t\sigma - \partial_t^{-1}J_\mu)\boldsymbol{u} = -2\int_0^t\int_{\mathbb{R}^2}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}u_1(\boldsymbol{x}, s) \\ u_2(\boldsymbol{x}, s)\end{bmatrix} d\boldsymbol{x}ds$$

$$\text{where } a_{i1} = +\partial_{x_2}\mathcal{H}_{3i}^B - \epsilon(\boldsymbol{x})\partial_s\mathcal{E}_{1i}^B$$
$$a_{i2} = -\partial_{x_1}\mathcal{H}_{3i}^B - \epsilon(\boldsymbol{x})\partial_s\mathcal{E}_{2i}^B$$

Since $\mathcal{H}_{3i}^B$ and $\mathcal{E}_{ji}^B$ are evaluated on the plane $x_3 = y_3 = 0$, we will for comparison of the terms of Eq. (C.1), express this result as the limit as $y_3 \to 0+$. Thus from (18′) we have

$$\frac{1}{\epsilon}N_\alpha(\epsilon\partial_t\sigma - \partial_t^{-1}J_\mu)\boldsymbol{u} = -\Lambda_\alpha\boldsymbol{u} \tag{C.5}$$

where

$$\Lambda_\alpha\boldsymbol{u}[y,t] = 2\lim_{y_3\to 0+}\int_0^t\int_{\mathbb{R}^2}\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}u_1(\boldsymbol{x},s)\\ u_2(\boldsymbol{x},s)\end{bmatrix}d\boldsymbol{x}ds \tag{C.6}$$

where

$$a_{i1} = +\frac{\partial}{\partial x_3}\mathcal{H}_{2i}^B(\boldsymbol{x},\boldsymbol{y},x_3-y_3,t-s)\big|_{x_3=0} \tag{C.7}$$

$$a_{i2} = -\frac{\partial}{\partial x_3}\mathcal{H}_{1i}^B(\boldsymbol{x},\boldsymbol{y},x_3-y_3,t-s)\big|_{x_3=0} \tag{C.7′}$$

Note that we assume that ϵ,μ are sufficiently smooth so that the singular behaviour of the fundamental solution is essentially given by the case ϵ,μ constant. Next we will consider the term

$$\partial_t^{-1}J_\epsilon\frac{1}{\mu}M_\alpha\boldsymbol{u} = \begin{bmatrix}\partial_{y_1} & 0\\ 0 & \partial_{y_2}\end{bmatrix}\partial_t^{-1}\int_0^t\int_{\mathbb{R}^2}\begin{bmatrix}\beta_1 & \beta_2\\ \beta_1 & \beta_2\end{bmatrix}\begin{bmatrix}u_1(\boldsymbol{x},s)\\ u_2(\boldsymbol{x},s)\end{bmatrix}d\boldsymbol{x}ds$$

where

$$\epsilon\beta_i = -\partial_{y_2}(\frac{m_{1i}}{\mu}) + \partial_{y_1}(\frac{m_{2i}}{\mu})$$

From Eq. (26), (B.4), (B.6) and (B.8) we obtain (with $\boldsymbol{x}, s$ fixed variables)

$$\begin{aligned}\epsilon(\boldsymbol{y})\beta_1 &= 2[-\partial_{y_2}\widetilde{\mathcal{H}}_{21}^B(\boldsymbol{x},\boldsymbol{y},0,t-s) + \partial_{y_1}\widetilde{\mathcal{H}}_{22}^B(\boldsymbol{x},\boldsymbol{y},0,t-s)]\\ &= 2[\partial_{y_2}\widetilde{\mathcal{H}}_{12}^F(\boldsymbol{y},\boldsymbol{x},0,t-s) - \partial_{y_1}\widetilde{\mathcal{H}}_{22}^F(\boldsymbol{y},\boldsymbol{x},0,t-s)],\\ \beta_1 &= -2\partial_t\widetilde{\mathcal{E}}_{32}^F(\boldsymbol{y},\boldsymbol{x},0,t-s)\end{aligned}$$

Similarly it can be shown

$$\begin{aligned}\epsilon(\boldsymbol{y})\beta_2 &= 2[\partial_{y_2}\widetilde{\mathcal{H}}_{11}^B(\boldsymbol{x},\boldsymbol{y},0,t-s) - \partial_{y_1}\widetilde{\mathcal{H}}_{12}^B(\boldsymbol{x},\boldsymbol{y},0,t-s)]\\ \beta_2 &= 2\partial_t\widetilde{\mathcal{E}}_{31}^F(\boldsymbol{y},\boldsymbol{x},0,t-s)\end{aligned}$$

Hence it follows that

$$\partial_t^{-1}J_\epsilon\frac{1}{\mu}M_\alpha\boldsymbol{u} = 2\begin{bmatrix}\partial_{y_1} & 0\\ 0 & \partial_{y_2}\end{bmatrix}\int_0^t\int_{\mathbb{R}^2}\begin{bmatrix}-\widetilde{\mathcal{E}}_{32}^F & \widetilde{\mathcal{E}}_{31}^F\\ -\widetilde{\mathcal{E}}_{32}^F & \widetilde{\mathcal{E}}_{31}^F\end{bmatrix}\begin{bmatrix}u_1(\boldsymbol{x},s)\\ u_2(\boldsymbol{x},s)\end{bmatrix}d\boldsymbol{x}ds \tag{C.8}$$

The term

$$\mu\partial_t\sigma\frac{1}{\mu}M_\alpha\boldsymbol{u} = 2\mu\int_0^t\int_{\mathbb{R}^2}\partial_t\begin{bmatrix}\widetilde{\mathcal{H}}_{22}^B & -\widetilde{\mathcal{H}}_{12}^B\\ -\widetilde{\mathcal{H}}_{21}^B & +\widetilde{\mathcal{H}}_{11}^B\end{bmatrix}\begin{bmatrix}u_1(\boldsymbol{x},s)\\ u_2(\boldsymbol{x},s)\end{bmatrix}d\boldsymbol{x}ds$$

where $\widetilde{\mathcal{H}}^B_{ij} = \widetilde{\mathcal{H}}^B_{ij}(\boldsymbol{x}, \boldsymbol{y}, 0, t-s)$, reduces on using Eq. (B.8).

$$2\mu \int_0^t \int_{\mathbb{R}^2} \partial_t \begin{bmatrix} -\widetilde{\mathcal{H}}^F_{22} & +\widetilde{\mathcal{H}}^F_{21} \\ \widetilde{\mathcal{H}}^F_{12} & -\widetilde{\mathcal{H}}^F_{11} \end{bmatrix} \begin{bmatrix} u_1(\boldsymbol{x}, s) \\ u_2(\boldsymbol{x}, s) \end{bmatrix} d\boldsymbol{x} ds \tag{C.9}$$

where $\widetilde{\mathcal{H}}^F_{ij} = \widetilde{\mathcal{H}}^F_{ij}(\boldsymbol{y}, \boldsymbol{x}, 0, t-s)$.

Combining Relation (C.8) and (C.9) and expressing the results as a limit $y_3 \to 0+$, we have

$$(\mu \partial_t \sigma - \partial_t^{-1} J_\epsilon) \frac{1}{\mu} M_\alpha \boldsymbol{u} = 2 \lim_{y_3 \to 0+} \int_0^t \int_{\mathbb{R}^2} \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \begin{bmatrix} u_1(\boldsymbol{x}, s) \\ u_2(\boldsymbol{x}, s) \end{bmatrix} d\boldsymbol{x} ds \tag{C.10}$$

where for $y_3 > 0$, on using Eq. (B.4) with x replaced by y

$$\begin{aligned}
\beta_{11} &= -\mu \partial_t \widetilde{\mathcal{H}}^F_{22} + \partial_{y_1} \widetilde{\mathcal{E}}^F_{32} = \partial_{y_3} \widetilde{\mathcal{E}}^F_{12}(\boldsymbol{y}, \boldsymbol{x}, x_3 - y_3, t-s)\big|_{x_3=0} \\
\beta_{12} &= \mu \partial_t \widetilde{\mathcal{H}}^F_{21} - \partial_{y_1} \widetilde{\mathcal{E}}^F_{31} = -\partial_{y_3} \widetilde{\mathcal{E}}^F_{11}(\boldsymbol{y}, \boldsymbol{x}, x_3 - y_3, t-s)\big|_{x_3=0} \\
\beta_{21} &= \mu \partial_t \widetilde{\mathcal{H}}^F_{12} + \partial_{y_2} \widetilde{\mathcal{E}}^F_{32} = \partial_{y_3} \widetilde{\mathcal{E}}^F_{22}(\boldsymbol{y}, \boldsymbol{x}, x_3 - y_3, t-s)\big|_{x_3=0} \\
\beta_{22} &= -\mu \partial_t \widetilde{\mathcal{H}}^F_{11} - \partial_{y_2} \widetilde{\mathcal{E}}^F_{31} = -\partial_{y_3} \widetilde{\mathcal{E}}^F_{21}(\boldsymbol{y}, \boldsymbol{x}, x_3 - y_3, t-s)\big|_{x_3=0}
\end{aligned}$$

Using the fact that the dependence of $\widetilde{\mathcal{E}}^F_{ij}$ on x_3 and y_3 is of the form $(x_3 - y_3)$, it follows that on using Eq. (B.10)

$$\partial_{y_3} \widetilde{\mathcal{E}}^F_{ij} = -\partial_{x_3} \widetilde{\mathcal{E}}^F_{ij} = -\partial_{x_3} \mathcal{H}^B_{ji}(\boldsymbol{x}, \boldsymbol{y}, x_3 - y_3, t-s)\big|_{x_3=0}$$

Thus expression (C.10) reduces to

$$(\mu \partial_t \sigma - \partial_t^{-1} J_\epsilon) \frac{1}{\mu} M_\alpha \boldsymbol{u} = -\Lambda_\alpha \boldsymbol{u}[\boldsymbol{y}, t] \tag{C.11}$$

Thus we see that expressions (C.5) and (C.11) are the same.

Appendix D Evaluation of Γ_α when $\epsilon = \epsilon(\alpha)$, $\mu = \mu(\alpha)$

From Eq. (56) the matrix operator Γ_α is defined by

$$\Gamma_\alpha = [\frac{\partial}{\partial \alpha}(\epsilon^{-1}(\alpha) N_\alpha)] \mu^{-1}(\alpha) N_\alpha$$

When ϵ, μ depend upon the parameter α only, we have from Eqs. (37), (38)

$$M_\alpha = N_\alpha = K_\alpha \partial_t^{-1} \begin{bmatrix} \partial_1 \partial_2 & (\frac{1}{c^2}\partial_t^2 - \partial_1^2) \\ (\partial_2^2 - \frac{1}{c^2}\partial_t^2) & -\partial_1 \partial_2 \end{bmatrix}.$$

Hence it follows that

$$\frac{\partial}{\partial \alpha}(\epsilon^{-1}(\alpha) N_\alpha) = [-\frac{\epsilon'}{\epsilon} + K'_\alpha K_\alpha^{-1}] \frac{1}{\epsilon} N_\alpha - \frac{2c'}{\epsilon c^3} K_\alpha \partial_t \sigma$$

where the matrix σ is given by Eq. (44), ϵ', c' are the derivatives of ϵ, c with respect to the parameter α, and K'_α is the derivative of the operator K_α with respect to the parameter α, [10,11]. Using the inverse relation between the matrix operators M_α and N_α, it follows that

$$\Gamma_\alpha = (\frac{\epsilon'}{\epsilon} - K'_\alpha K_\alpha^{-1})I + \frac{2c'}{c} K_\alpha^2 \begin{bmatrix} (\Box^T + \partial_1^2) & \partial_1\partial_2 \\ \partial_1\partial_2 & (\Box^T + \partial_2^2) \end{bmatrix}$$

Hence using relationship given by Eq. (52), it follows that

$$\Gamma_\alpha = (\frac{\epsilon'}{\epsilon} + \frac{2c'}{c} - K'_\alpha K_\alpha^{-1})I + \frac{2c'}{c} K_\alpha^2 \begin{bmatrix} \partial_1^2 & \partial_1\partial_2 \\ \partial_1\partial_2 & \partial_2^2 \end{bmatrix}$$

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