Physical limitations on G and B for antennas

Sohl, Christian; Gustafsson, Mats; Kristensson, Gerhard

2007

PHYSICAL LIMITATIONS ON G AND B FOR ANTENNAS

Christian Sohl, Mats Gustafsson, and Gerhard Kristensson

Dept. of Electroscience, Lund University, Box 118, S-211 00 Lund, Sweden
{Christian.Sohl, Mats.Gustafsson, Gerhard.Kristensson}@es.lth.se

Abstract: In this paper, physical limitations on the partial gain G and relative bandwidth B are derived for antennas of arbitrary shape based on holomorphic properties of the forward scattering dyadic. The limitation on the performance of G and B is shown to be bounded from above by the long wavelength response of the antenna in terms of the electric and magnetic polarizability dyadics. The special case of ellipsoidal geometries are discussed.

INTRODUCTION

The physical limitations introduced in this paper generalize in many aspects the classical results on the directivity and the Q-factor in Chu[1]. The advantages of the new limitations are at least fourfold: 1) they are no longer restricted to the sphere, but holds for arbitrary antenna geometries; 2) they are formulated in the antenna gain and bandwidth rather than the directivity and Q-factor; 3) they permits studies of polarization effects such as diversity in MIMO applications; 4) they successfully separate electric and magnetic antenna properties in terms of the intrinsic material parameters. The present analysis is based on Gustafsson *et al.*[2] and is an application of the forward dispersion relation for broadband scattering introduced in Sohl *et al.*[3].

SCATTERING AND ABSORBTION OF ANTENNAS

Consider an antenna of arbitrary shape surrounded by free space and subject to a plane-wave excitation with Fourier amplitude $\boldsymbol{E}_0$ impinging in the $\hat{\boldsymbol{k}}$-direction, see Figure 1. The antenna is modeled by the anisotropic and dispersive electric and magnetic susceptibility dyadics. The bounding volume of the antenna is assumed to be of arbitrary shape but with the restriction that the complete power absorption of the incident wave is contained within this volume.

Based on the physical principles of linearity, time-translational invariance, and causality, there is no fundamental difference between antennas and properly modeled scatterers. In fact, for a large class of antennas, the forward scattering dyadic $\mathbf{S}$ is holomorphic in the upper half plane $\operatorname{Im} k > 0$ and Cauchy's integral theorem can be applied to the properly chosen function $\varrho(k) = \hat{\boldsymbol{p}}_{\mathrm{e}}^* \cdot \mathbf{S}(k, \hat{\boldsymbol{k}}) \cdot \hat{\boldsymbol{p}}_{\mathrm{e}}/k^2$, where $\hat{\boldsymbol{p}}_{\mathrm{e}} = \boldsymbol{E}_0/|\boldsymbol{E}_0|$ denotes the electric polarization. Based on the optical theorem and the long wavelength limit of ϱ, a dispersion relation for the extinction cross section σ_{ext} can be derived in terms of the electric and magnetic polarizability dyadics, $\boldsymbol{\gamma}_{\mathrm{e}}$ and $\boldsymbol{\gamma}_{\mathrm{m}}$, respectively, *viz.*,

$$\int_0^\infty \sigma_{\mathrm{ext}}(\lambda)\ \mathrm{d}\lambda = \pi^2(\hat{\boldsymbol{p}}_{\mathrm{e}}^* \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{\boldsymbol{p}}_{\mathrm{e}} + \hat{\boldsymbol{p}}_{\mathrm{m}}^* \cdot \boldsymbol{\gamma}_{\mathrm{m}} \cdot \hat{\boldsymbol{p}}_{\mathrm{m}}), \tag{1}$$

where $\hat{\boldsymbol{p}}_{\mathrm{m}} = \hat{\boldsymbol{k}} \times \hat{\boldsymbol{p}}_{\mathrm{e}}$ denotes the associated magnetic polarization. Note that the right hand side of (1) only depends on the long wavelength response of the antenna. For details on the derivation, see Gustafsson *et al.*[2] and Sohl *et al.*[3].

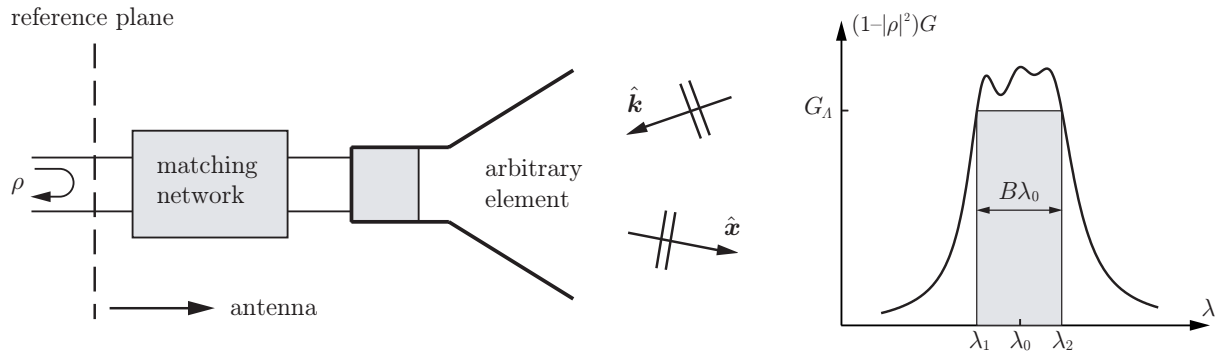

Figure 1: Illustration of a hypothetic antenna subject to an incident plane-wave (left figure) and the generic physical limitation addressed in this paper (right figure).

LIMITATIONS ON GAIN AND BANDWIDTH

For an unmatched antenna, the absorption cross section $\sigma_{\rm a}$ is given by $\sigma_{\rm a} = (1-|\rho|^2)\sigma_{\rm a0}$, where $\sigma_{\rm a0}$ is the absorption cross section for the corresponding perfectly matched antenna. Thus, for any wavelength $\lambda \in [0,\infty)$,

$$\sigma_{\rm ext} \geq \sigma_{\rm a} = (1-|\rho|^2)\sigma_{\rm a0} = \frac{1}{4\pi}(1-|\rho|^2)\lambda^2 G, \tag{2}$$

where the partial gain $G = 4\pi\sigma_{\rm a0}/\lambda^2$ have been introduced. The upper bound in (2) is in general not isoperimetric but can be sharpened with a priori knowledge of the absorption efficiency $\eta = \sigma_{\rm a}/\sigma_{\rm ext}$. Recall that G, $\sigma_{\rm a}$, and, $\sigma_{\rm ext}$ depend on the incident direction $\hat{\boldsymbol{k}}$ as well as the electric polarization $\hat{\boldsymbol{p}}_{\rm e}$.

Introduce the wavelength interval $\Lambda = [\lambda_1, \lambda_2]$ with center wavelength λ_0 and associated relative bandwidth B. Then, (2) implies

$$\int_0^\infty \sigma_{\rm ext}(\lambda)\ {\rm d}\lambda \geq \frac{1}{4\pi}\int_\Lambda (1-|\rho|^2)\lambda^2 G(\lambda)\ {\rm d}\lambda \geq \lambda_0^3 G_\Lambda B\left(1+\frac{1}{12}B^2\right) \tag{3}$$

where $G_\Lambda = \min_{\lambda\in\Lambda}(1-|\rho|^2)G$ denotes the minimum partial realized gain over Λ. The factor $1+B^2/12$ can be estimated from below by unity since $B>0$. Recall that $B \ll 1$ in many antenna applications. Based on this fact, (1) inserted into (3) yields the following limitation on the product $G_\Lambda B$ valid for any linear, time-translational and causal antenna:

$$G_\Lambda B \leq \frac{4\pi^3}{\lambda_0^3}(\hat{\boldsymbol{p}}_{\rm e}^* \cdot \boldsymbol{\gamma}_{\rm e} \cdot \hat{\boldsymbol{p}}_{\rm e} + \hat{\boldsymbol{p}}_{\rm m}^* \cdot \boldsymbol{\gamma}_{\rm m} \cdot \hat{\boldsymbol{p}}_{\rm m}). \tag{4}$$

Since both $\boldsymbol{\gamma}_{\rm e}$ and $\boldsymbol{\gamma}_{\rm m}$ are proportional to the volume of the antenna, it follows that the upper bound in (4) scales as $(k_0 a)^3$, where a denotes the radius of, say, the volume-equivalent sphere.

In many antenna applications it is desirable to bound $G_\Lambda B$ independently of the material properties. For this purpose, introduce the high-contrast polarizability dyadic $\boldsymbol{\gamma}_\infty$ as the limit of either $\boldsymbol{\gamma}_{\rm e}$ or $\boldsymbol{\gamma}_{\rm m}$ when the elements of the electric and magnetic susceptibility dyadics simultaneously become infinitely large. From the variational properties of $\boldsymbol{\gamma}_{\rm e}$ and $\boldsymbol{\gamma}_{\rm m}$ discussed in Sohl *et al.*[3], it then follows that

$$\sup_{\hat{\boldsymbol{p}}_{\rm e}\cdot\hat{\boldsymbol{p}}_{\rm m}=0} G_\Lambda B \leq \frac{4\pi^3}{\lambda_0^3}(\gamma_1+\gamma_2), \tag{5}$$

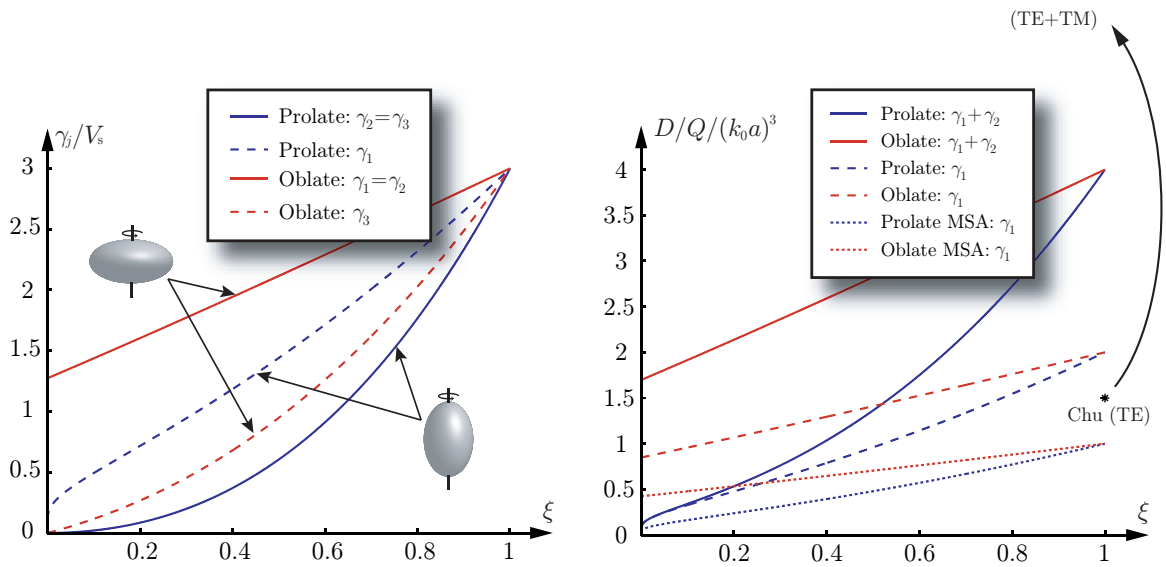

Figure 2: The eigenvalues γ_1, γ_2 and γ_3 (left figure) and the the ratio D/Q (right figure) for the prolate and oblate spheroids as function of the semi-axis ratio ξ. Note the normalization with the volume $V_s = 4\pi a^3/3$ of the smallest circumscribing sphere.

where γ_1 and γ_2 denote the largest and second largest eigenvalue of $\boldsymbol{\gamma}_\infty$, respectively. For non-magnetic material parameters, $\gamma_2 = 0$, and the right hand side of (5) can be improved by at most a factor of two. Recall that γ_1 and γ_2 are easily calculated for arbitrary geometries using either the finite element method (FEM) or the method of moments (MoM).

THE ELLIPSOIDAL GEOMETRIES

Closed-form expressions of γ_1, γ_2 and γ_3 exist for the ellipsoidal geometries, see Gustafsson *et al.*[2]. These eigenvalues are depicted in Figure 2 for the prolate and oblate spheroids together with the bounds on D/Q in Gustafsson *et al.*[2] corresponding to $G_\Lambda B$ in (5). The eigenvalues γ_1, γ_2 and γ_3 are seen to degenerate for the sphere ($\xi = 1$). In the case of a minimum scattering antenna (MSA) with non-magnetic material parameters, (5) is, for any semi-axis ratio $\xi \in [0, 1]$, sharper than the limitation in Chu[1] for the TE-polarization. For the generalization of Chu[1] to both TE- and TM-polarizations, the present theory provides even sharper physical limitations, see Figure 2.

REFERENCES

[1] L. J. Chu. "Physical limitations of omni-directional antennas". *Appl. Phys.*, vol. 19, pp. 1163–1175, 1948.

[2] M. Gustafsson, C. Sohl, and G. Kristensson. "Physical limitations on antennas of arbitrary shape". Submitted to IEEE Transactions on Antennas and Propagation.

[3] C. Sohl, M. Gustafsson, and G. Kristensson. "Physical limitations on broadband scattering by heterogeneous obstacles". Submitted to Journal of Mathematical Physics.