Algebraic Properties of Ore Extensions and Their Commutative Subrings

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Preface

This thesis is based on six papers (A–F). In Part I of the thesis we give an introduction to the subject and present a summary of the results found in the six papers. Part II consists of the papers themselves, which are the following:

A. J. Richter, S. D. Silvestrov, *On algebraic curves for commuting elements in q-Heisenberg algebras*, J. Gen. Lie. T. Appl. **3** (2009), no. 4, 321–328.

B. J. Richter, S.D. Silvestrov, *Burchnall-Chaundy annihilating polynomials for commuting elements in Ore extension rings*, J. Phys.: Conf. Ser. **342** (2012)

C. J. Öinert, J. Richter, S.D. Silvestrov, *Maximal commutative subrings and simplicity of Ore extensions*, J. Algebra Apppl. **12**, 1250192 (2013), arXiv:1111.1292 (2011)

D. J. Richter, *Burchnall-Chaundy theory for Ore extensions*, in Springer Proceedings in Mathematics & Statistics, Vol. 85, ed: Abdenacer Makhlouf et al: Algebra, Geometry and Mathematical Physics, arXiv:1309.4415

E. J. Richter, *A Note on "A Combinatorial Proof of Associativity of Ore Extensions"*, Discrete Mathematics, Volumes 315–316, 6 February 2014, Pages 156–157

F. J. Richter, S.D. Silvestrov, *Centralizers in Ore extensions of polynomial rings*, International Electronic Journal of Algebra, Volume 15 (2014), 196–207, arXiv:1308.3430

Contents

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Part I

Introduction and summary

Chapter 1
Introduction

The first chapter will be devoted to presenting background and motivation from the areas of ring theory and operator algebras. We start by introducing the notation and conventions that we have adopted.

1.1 Notation and conventions

By a ring we will always mean an associative and unital ring. All morphisms between rings will be assumed to respect the multiplicative identity.

By an ideal we shall mean a two-sided ideal.

$\mathbb{R}$ will denote the field of real numbers, $\mathbb{C}$ the field of complex numbers. $\mathbb{Z}$ and $\mathbb{N}$ will denote the integers and the non-negative integers respectively.

If R is a ring then $R[x_1, x_2, \ldots x_n]$ denotes the ring of polynomials over R in central indeterminates $x_1, x_2, \ldots, x_n$.

If R is a ring we can regard it as a module (indeed algebra) over $\mathbb{Z}$ by defining $0r = 0$, $nr = \sum_{i=1}^{n} r$ if $n > 0$ and $nr = -(-n)r$ if $n < 0$. If there is a positive integer n such that $n1_R = 0$, where 1_R is the multiplicative identity in R, we call the least such positive integer the characteristic of R and denote it by $\mathrm{char}(R)$. If no such integer exists we set $\mathrm{char}(R) = 0$.

If R is a ring, then $Z(R)$ denotes the center of R, i.e. the set of elements in R that commute with everything in R. If A is a subset of a ring R, then the *centralizer* of A, denoted $C_R(A)$, is the set of elements in R that commute with everything in A. We write $C_R(a)$ for $C_R(\{a\})$, if $a \in R$.

Let R be a commutative ring and S an algebra over R. Two commuting elements, $p, q \in S$, are said to be algebraically dependent (over R) if there is a non-zero polynomial, $f(s,t) \in R[s,t]$, such that $f(p,q) = 0$.

If X is a topological space then $C(X)$ denotes the continuous complex-valued functions on X. This is as an algebra over $\mathbb{C}$, under the operations of pointwise addition and multiplication.

1.2 Ore extensions

All the papers in this thesis deal with Ore extensions, in one way or another. In this section we define Ore extensions is and give some examples of the definition.

We first introduce a special type of Ore extensions, the *differential operator rings.*

1.2.1 Differential operator rings

Let R be a ring. A *derivation*, δ, of R is a map from R into itself satisfying, for all $a, b \in R$,

$$\delta(a+b) = \delta(a) + \delta(b)$$

and

$$\delta(ab) = a\delta(b) + \delta(a)b.$$

This is a straightforward generalization of two properties of the usual derivative, namely its additivity and the Leibniz rule. A ring with a derivation will sometimes be called a *differential ring*. A *differential field* is a field with a derivation that makes it into a differential ring.

If δ is a derivation on R and $a \in R$ is such that $\delta(a) = 0$, then a is said to be a *constant* (for δ.)

Proposition 1.2.1. *Let R be a ring and $\delta : R \to R$ a derivation. Let C be the set of constants of δ. Then*

(i) $1 \in C$;

(ii) *C is a subring of R, called the ring of constants;*

(iii) *for any $c \in C$ and $r \in R$ we have*

$$\delta(cr) = c\delta(r),$$
$$\delta(rc) = \delta(r)c.$$

Proof. (i)

$$\delta(1) = \delta(1 \cdot 1) = 1\delta(1) + \delta(1)1 = \delta(1) + \delta(1) \Rightarrow \delta(1) = 0.$$

(ii) If $a, b \in C$ it follows from additivity that $a + b$ is a constant as well. To prove that $ab \in C$ we use the Leibniz rule as follows:

$$\delta(ab) = a\delta(b) + \delta(a)b = 0 + 0 = 0.$$

Since 1 and 0 are constants it is clear that C is a subring.

(iii) Let c be a constant and r any element of R. A short calculation gives

$$\delta(cr) = c\delta(r) + \delta(c)r = c\delta(r).$$

The other claim is proved analogously.

□

Any derivation satisfies a version of the quotient rule.

Proposition 1.2.2. *Let R be a ring with a derivation, δ, and let a be any invertible element of R. Then*

$$\delta(a^{-1}) = -a^{-1}\delta(a)a^{-1}.$$

Proof.

$$0 = \delta(1) = \delta(a^{-1}a) = a^{-1}\delta(a) + \delta(a^{-1})a \Rightarrow \delta(a^{-1}) = -a^{-1}\delta(a)a^{-1}.$$

□

Corollary 1.2.3. *Let R be a differential ring and C its ring of constants. If a is an invertible element that lies in C, then so does a^{-1}. If R is a field, then C is a sub-field of R.*

Example 1.2.4. As the ring R we can take $C^\infty(\mathbb{R},\mathbb{C})$, the ring of all infinitely many times differentiable complex-valued functions on the real line. For δ we can take the usual derivative. The ring of constants in this case will consist of the constant functions.

The ring $R = C^\infty(\mathbb{R},\mathbb{C})$ can be seen as a vector space over $\mathbb{C}$, with operations defined pointwise. So we can consider the ring $\mathrm{End}_\mathbb{C}(R)$ of all vector space endomorphisms of R. (Note that we are not requiring the endomorphisms to be multiplicative.) The space $\mathrm{End}_\mathbb{C}(R)$ is in turn an algebra over R. One of the operators in $\mathrm{End}_\mathbb{C}(R)$ is the derivation operator, which we denote by D. Further, for any $f \in R$ there is the multiplication operator M_f that maps any function $g \in R$ to fg. The operator D and all the M_f together generate a subalgebra of $\mathrm{End}_\mathbb{C}(R)$, which we denote by T.

It is clear that the set of all M_f, for $f \in R$, is a subalgebra of T, isomorphic to R. Thus we abuse notation and identify M_f with f. If we do this we can write any element of T as a finite sum, $\sum_{i=0}^n a_i D^i$, where all the a_i:s are functions in $C^\infty(\mathbb{R},\mathbb{C})$. Further such a decomposition is unique, or in other words: the powers of D form a basis for T as a free module over R.

We now compute the commutator of D and f for any $f \in R$. We temporarily revert to writing M_f for the element in T to make our calculations easier to understand. Let g be an arbitrary function in R. We find that

$$\begin{aligned}(DM_f - M_f D)(g) = DM_f(g) - M_f D(g) = D(fg) - M_f(g') = \\ f'g + fg' - fg' = f'g = M_{\delta(f)}(g).\end{aligned}$$

Hence

$$DM_f - M_f D = M_{\delta(f)}.$$

Relapsing into our abuse of notation we write this as $Df - fD = \delta(f)$ or equivalently as $Df = fD + \delta(f)$.

Denote the identity function on the real line by y. Then $Dy - yD = 1$, a relation known as the Heisenberg relation. The elements y and D together generate a subalgebra of T known as the Weyl algebra or the Heisenberg algebra, which is of interest in quantum mechanics, among other areas.

Any element, P, of T can be written as $P = \sum_{i=0}^{n} p_i D^i$, for some non-negative integer n and some $p_i \in C^{\infty}(\mathbb{R}, \mathbb{C})$. If $p_n \neq 0$ we say that P has degree n and leading coefficient p_n. The degree and leading coefficient are clearly uniquely defined for any non-zero element of T. We say that the zero element has 0 as its leading coefficient and degree $-\infty$.

Generalizing the Weyl algebra is an important motivation for the next definition.

Definition 1.2.5. Let R be any differential ring with derivation δ. We form the *ring of formal differential operators* over R, denoted by $R[x; \mathrm{id}_R, \delta]$, by equipping the ring of polynomials over R, $R[x]$ (with coefficients written on the left), with a new multiplication satisfying

$$xr = rx + \delta(r)$$

for all r in R. There is exactly one such multiplication making $R[x; \mathrm{id}_R, \delta]$ into a ring. (See Subsection 1.2.2 for a proof of this fact.)

Suppose that $R[x]$ is equipped with a multiplication satisfying $xr = rx + \alpha(r)$ for all $r \in R$, where α is some function $R \to R$, and that $R[x]$ with this multiplication satisfies all the ring axioms. By distributivity we have that $x(r+s) = xr + xs$, for any $r, s \in R$. This implies that $\alpha(r+s) = \alpha(r) + \alpha(s)$.

If $r, s \in R$ we have that $x(rs) = rsx + \alpha(rs)$ by assumption. On the other hand, associativity tells us that

$$x(rs) = (xr)s = (rx + \delta(r))s = rxs + \alpha(r)s = rsx + r\alpha(s) + \alpha(r)s. \tag{1.1}$$

Equating the constant coefficient we see that $\alpha(rs) = r\alpha(s) + \alpha(r)s$. Thus α must be a derivation for the rule $xr = rx + \alpha(r)$ to be compatible with a ring structure on $R[x]$.

We define the *degree* of an element in $R[x; \mathrm{id}_R, \delta]$ to coincide with the degree in $R[x]$. In particular the zero element has degree $-\infty$.

1.2.2 General Ore extensions

In this subsection we describe a generalization of differential operator rings, the so called Ore extensions. They were introduced by Norwegian mathematician Øystein Ore in the 1933 paper [Ore33].

Ore's paper [Ore33] is entitled "Theory of Non–Commutative Polynomials", which gives some idea of the motivation. As with the ring of formal differential operators, we equip an ordinary polynomial ring (with coefficients written on the left) with a new multiplication. We want it to remain true that $\deg(ab) = \deg(a) + \deg(b)$ with our new multiplication. Of course, even for the ring of ordinary polynomials this relation is not always true, unless the ring of coefficients is a domain. So we insist rather that $\deg(ab) \leq \deg(a) + \deg(b)$. We also want the multiplicative identity element in the coefficient ring, R, to remain an identity element.

We consider in particular what this implies for the product $x \cdot r$. For any $r \in R$ we must have the relation

$$xr = \sigma(r)x + \delta(r), \tag{1.2}$$

hold.

We thus get two functions, σ and δ, from R to itself. We now proceed to investigate what conditions they should satisfy.

Since $x \cdot 1 = x$, by hypothesis, we find that $\sigma(1) = 1$ and $\delta(1) = 0$.

Distributivity implies that

$$x(r+s) = xr + xs,$$

for any $r, s \in R$. Using Equation (1.2) we get the equality

$$\sigma(r+s)x + \delta(r+s) = (\sigma(r) + \sigma(s))x + \delta(r) + \delta(s).$$

By identifying coefficients we find that both σ and δ must be additive functions.

Using associativity we can derive further conditions on σ and δ. Let r and s be any two elements of R. By associativity we have

$$\begin{gathered} x(rs) = (xr)s \\ \Longleftrightarrow \\ \sigma(rs)x + \delta(rs) = (\sigma(r)x + \delta(r))s \\ \Longleftrightarrow \\ \sigma(rs)x + \delta(rs) = \sigma(r)\sigma(s)x + \sigma(r)\delta(s) + \delta(r)s. \end{gathered}$$

We find that σ and δ must satisfy the following relations, for all $r, s \in R$,

$$\begin{aligned} \sigma(rs) &= \sigma(r)\sigma(s) \\ \delta(rs) &= \sigma(r)\delta(s)x + \delta(r)s. \end{aligned}$$

The relations we have derived imply that σ must be an endomorphism, if the Ore extension is to be well-defined. The conditions on δ are similar, but more general, to the definition of a derivation. This inspires the next definition.

Definition 1.2.6. Let R be a ring and $\sigma : R \to R$ a ring endomorphism. (Recall that all morphisms are unital.) A *σ-derivation* on R is an additive function δ from R to R such that

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b,$$

for all $a, b \in R$. If σ is the identity map, id_R, we recover the usual definition of a derivation.

Remark 1.2.7. We note that if δ is a σ-derivation, then

$$\delta(1) = \delta(1 \cdot 1) = \sigma(1)\delta(1) + \delta(1)1 = 2 \cdot \delta(1),$$

which implies that $\delta(1) = 0$.

Definition 1.2.8. For any σ and any $a \in R$ define δ by $\delta(r) = ar - \sigma(r)a$. Then δ is a σ-derivation. All derivations of this form are called *inner derivations*. Those derivations that are not inner are called *outer*.

We proceed by actually defining what an Ore extension is.

Definition 1.2.9. Let R be a ring, σ an endomorphism of R and δ a σ-derivation. The Ore extension $R[x; \sigma, \delta]$ is the ring of polynomials over R equipped with a new multiplication, satisfying

$$xr = \sigma(r)x + \delta(r)$$

for all r in R. This defines the multiplication uniquely.

Definition 1.2.10. An element $a \in R$ is a scalar of an Ore extension $R[x; \sigma, \delta]$ if $\sigma(a)a$ and $\delta(a) = 0$.

Remark 1.2.11. The scalars form a subring of any Ore extension.

We have showed above that the conditions on σ and δ in the definition are necessary if we hope to get a ring with the desired properties. They are also sufficient, as we will shortly show. Due to the importance of the construction in our work we take the time to explain the construction in more detail and give a detailed proof that it works. We follow the treatment in [GW04, Chapter 2]. One can show, using an argument similar to the one we presented above in the case of differential operator rings, that the conditions that σ is an endomorphism and δ is a σ-derivation, are necessary for $R[x; \sigma, \delta]$ to be an associative, distributive and unital ring. To prove that the multiplication is uniquely defined it suffices to prove that the multiplication of monomials is uniquely defined, which can be proved by an induction argument. We skip the details and proceed to show that there is some multiplication with the desired properties on $R[x]$.

As a set $S = R[x;\sigma,\delta]$ consists of sequences $(r_0, r_1, r_2, \ldots)$ of elements in R taking non-zero values only for finitely many indices. An alternative way of phrasing this is that S consist of functions from the non-negative integers to R with finite support. This is made into a left R-module in the standard way.

We identify an element r of R with $(r, 0, 0, 0\ldots)$, the sequence with r in the first place and zero everywhere else. We denote the element $(0, 1, 0, 0, \ldots)$ by x. As we have said we want to define an associative, distributive multiplication that coincides with the multiplication we started with on the subset R and that satisfies $xr = \sigma(r)x + \delta(r)$. We note that this would imply that 1_R is an identity for S as well and that x^n is the sequence with a 1 at index n and zeroes elsewhere.

We could derive formulas for what the product of two elements of S should be and then try to show that the operation defined by these formulas is associative, distributes over addition, $(1, 0, 0, \ldots)$ is an identity and that the subset $(r, 0, 0, \ldots)$ really can be identified with R. (Ie, that it forms a subring isomorphic to R with the isomorphism sending r to $(r, 0, 0, \ldots)$.)

This is the approach followed by Nystedt in [Nys13]. In Paper E we give an alternative approach to part of Nystedt's paper. We will follow a different route here and instead we construct a different ring that will be seen to be isomorphic to S and which manifestly satisfies the desired properties.

Consider an ordinary polynomial ring $R[z]$ and let A be the ring of all additive functions from $R[z]$ to $R[z]$, with multiplication given by composition. There is an injection from R to A sending r to the operation of multiplication by r. Identify R with its image in A under this map. Let X be the operator sending $\sum r_i z^i$ to $\sum(\sigma(r_i)z^{i+1} + \delta(r_i)z^i)$. Clearly X belongs to A. R and X together generate a subring of A that we will denote by $\bar{S}$.

A straightforward calculation (omitted here) shows that $Xr = \sigma(r)X + \delta(r)$. Form the set $\hat{S}$, consisting of R-linear combinations of powers of X, with coefficients on the left. Clearly $\hat{S} \subseteq \bar{S}$. Further since $XR \subset \hat{S}$ it follows by induction that $X^k R \subset \hat{S}$ for all k and thus that $\hat{S}$ is a subring of A. Accordingly $\bar{S} = \hat{S}$. We now show that the powers of X are linearly independent over R.

Suppose $r_0 + r_1 X + \ldots + r_m X^m$ is the zero operator. We apply it to the constant polynomial 1 and get $0 = r_0 + r_1 z + \ldots + r_m z^m$ since $X^j(1) = z^j$ (by induction). But the powers of z are linearly independent in $R[z]$ so $r_0 = r_1 = \ldots = r_m = 0$.

Let e_n, for $n \geq 0$, be the functions, $\mathbb{N} \to R$, such that $e_n(n) = 1$ and $e_n(m) = 0$ if $n \neq m$. Clearly $e_n \in S$ and $e_1 = x$. The R-linear function sending e_n to X^n is an isomorphism of S and $\bar{S}$ as R-modules and maps x to X. We can define a multiplication on S from the multiplication on $\bar{S}$. Clearly this multiplication will satisfy $xr = \sigma(r)x + \delta(r)$. That it is associative, distributive and that 1_R is the multiplicative identity element is now clear. We note that $1, x, x^2, \ldots$ form a basis for S as a free left R-module.

Example 1.2.12. Let R be a ring with a derivation, δ. δ is a σ-derivation for

$\sigma = \mathrm{id}_R$ so we can form $R[x; \mathrm{id}, \delta]$. This is the ring of formal differential operators. We have simultaneously showed that formal differential operator rings are well-defined and explained our choice of notation for them.

Example 1.2.13. If R is any ring and σ is an endomorphism of R, then $\delta \equiv 0$ is a σ-derivation. The ring $R[x; \sigma, 0]$ is called a skew polynomial ring. (The reader is warned that other authors may use this term to denote other classes of rings.) If $\sigma = \mathrm{id}_R$ we recover the ordinary polynomial ring in one variable over R.

Example 1.2.14. Let $R = k[y]$ for some field k and let q be a non-zero element of k. We define σ to be the identity on k and $\sigma(y) = qy$. This extends uniquely to an endomorphism of R. We also define a σ-derivation, δ, by setting $\delta(y) = 1$ and $\delta(a) = 0$ for all $a \in k$. One can check that σ and δ extend to R in the required way. The Ore-extension $R[x; \sigma, \delta]$ is known as the q-Weyl algebra. Note that $R[x; \sigma, \delta]$ is generated as an algebra over k by the two elements x and y satisfying the commutation relation $xy = qyx + 1$.

There are several other ways of showing that Ore extensions are well-defined objects. One way is by using generators and relations. $R[x; \sigma, \delta]$ would be defined as a ring with generators consisting of all $r \in R$ and an additional element x. The relations would consist of all relations between elements in R and the new relations $xr = \sigma(r)x + \delta(r)$ for every $r \in R$. Clearly this gives a well-defined ring where every element can be written $\sum_i a_i x^i$. To show that the coefficients a_i are uniquely defined, one can use the obvious homomorphism into the ring $\bar{S}$.

We have already mentioned Nystedt's construction of Ore extensions and our variation on it. We next give an overview of Nystedt's approach.

1.2.3 Nystedt's proof

We will now sketch Nystedt's proof, in order to give a background for Paper E. Recall that we want to define a new multiplication of $R[x]$, such that Equation (1.2) holds.

Introduce functions π_i^m, for $m, i \in \mathbb{Z}$, defined as the sum over all possible composition of i copies of σ and $m - i$ copies of δ. If $i < 0$ or $i > m$, then we set $\pi_i^m = 0$.

For example

$$\pi_2^3 = \sigma \circ \sigma \circ \delta + \sigma \circ \delta \circ \sigma + \delta \circ \sigma \circ \sigma.$$

If $a, b \in R$ one can show that $ax^n bx^m = \sum_{i=0}^n a\pi_i^n(b)x^{i+m}$, assuming (1.2) holds. We define the multiplication of monomials in this way, and define the multiplication of general elements by bilinearity. We want to show that this gives rise to an associative operation. It is enough to prove that the multiplication of monomials is associative.

So one needs to show, for all $b, b', b'' \in R$ and $m, n, p \in \mathbb{N}$, that

$$bx^m(b'x^n b''x^p) = (bx^m b'x^n)b''x^p.$$

By (1.2) and distributivity, one gets that

$$bx^m(b'x^n b''x^p) = bx^m \sum_{j=0}^{\infty} b'\pi_j^n(b'')x^{j+p} = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} b\pi_i^m(b'\pi_j^n(b''))x^{i+j+p}$$

and

$$(bx^m b'x^n)b''x^p = \left(\sum_{i=0}^{\infty} b\pi_i^m(b')x^{i+n}\right) b''x^p = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b\pi_i^m(b')\pi_j^{i+n}(b'')x^{j+p}.$$

By identifying coefficients one sees that that we want the following equality to hold,

$$\sum_{i=0}^{\infty} \pi_i^m(b')\pi_j^{i+n}(b'') = \sum_{i=0}^{\infty} \pi_i^m(b'\pi_{j-i}^n(b'')) \tag{1.3}$$

Nystedt now shows that the number of terms, when expanded into products of compositions of σ and δ, are the same on both sides of (1.3). Nystedt then finishes the proof by showing that every term on the right-hand side of Equation (1.3) also occurs on the left-hand side.

In our alternative proof we show directly that every term on the left-hand side of (1.3) occurs also on the right. Together with the second part of Nystedt's proof, this shows that Ore extensions are well-defined associative rings.

1.3 Centralizers in Ore extensions

An Ore extension $R[x;\sigma,\delta]$ will be commutative if and only if R is commutative, $\sigma = \mathrm{id}$ and $\delta = 0$. It is thus interesting to investigate centralizers of elements in Ore extensions.

Amitsur [Ami58] studied this question for differential operator rings. (Earlier results in special cases had been obtained by Flanders [Fla55] and Schur [Sch05]). Amitsur obtained the following theorem.

Theorem 1.3.1. *Let k be a field of characteristic zero with a derivation δ. Let F denote the subfield of constants. Form the differential operator ring $S = k[x;\mathrm{id},\delta]$, and let P be an element of S of degree n. Denote by by $F[P]$ the ring of polynomials in P with constant coefficients, $F[P] = \{\sum_{j=0}^{m} b_j P^j \mid b_j \in F\}$. Then $C_S(P)$ is a commutative subring of S and a free $F[P]$-module of rank at most n.*

Several authors have extended the results of Amitsur to more general settings. The following theorem, by Goodearl, is contained in [Goo83, Theorem 1.2]. Goodearl proves his result by adapting the proof of Amitsur.

Theorem 1.3.2. *Let R be a semiprime commutative ring with derivation δ and assume that its ring of constants is a field, F. If P is an operator in $R[x;\mathrm{id}_R,\delta]$ of positive degree n, where n is invertible in F, and has an invertible leading coefficient, then $C_S(P)$ is a free $F[P]$-module of rank at most n.*

We recall that a commutative ring is semiprime if and only if it has no nonzero nilpotent elements.

Goodearl notes that if R is a semiprime ring of positive characteristic such that the ring of constants is a field, then R must be a field. In that case he proves the following theorem [Goo83, Theorem 1.11].

Theorem 1.3.3. *Let R be a field, with a derivation δ, and let F be its subfield of constants. If P is an element of $S = R[x;\mathrm{id}_R,\delta]$ of positive degree, n, and with invertible leading coefficient, then $C_S(P)$ is a free $F[P]$-module of rank at most n^2.*

Another author who has shown how to apply Amitsur's proof to other types of rings is Bavula. In [Bav92] he proves an an analogue of Theorem 1.3.1 for so-called *Generalized Weyl Algebras*.

Bell and Small [BS04] study centralizers in domains of Gelfand-Kirillov (GK) dimension 2. The Weyl algebra is an example of such a domain, and their results can thus be applied there. Sharifi in the PhD-thesis [Sha13] considers centralizers in, among other rings, the second Weyl algebra. The second Weyl algebra has four generators x_1, x_2, y_1, y_2 that satsify

$$x_1x_2 - x_2x_1 = y_1y_2 - y_2y_1 = x_1y_2 - y_2x_1 = x_2y_1 - y_1x_2 = 0$$

and

$$x_1y_1 - y_1x_1 = x_2y_2 - y_2x_2 = 1.$$

Sharifi proves that centralizers in the second Weyl algebra must have GK-dimension 1, 2 or 3.

Hellström and Silvestrov [HS07] have generalized Amitsur's proof to a wide class of graded algebras.

Papers E and F in this thesis generalize Amitsur's proof to Ore extensions of the form $K[y][x;\sigma,\delta]$, where K is a field and $\deg_y(\sigma(y)) > 1$, a generalization that is not included in any earlier result we have found. The method of proof is still very close to the original one by Amitsur, however.

1.4 Burchnall-Chaundy theory

Papers A,B,D and F in this thesis deal with algebraic dependence of commuting elements in Ore extensions. In this section we describe previous results of this type.

The name of this section comes from two British mathematicians, Burchnall and Chaundy, who studied, in a series of papers in the 1920s and 30s [BC23, BC28, BC31], the properties of commuting pairs of ordinary differential operators. In our terminology they may be said to investigate the properties of pairs of commuting elements of the Ore extension $T = C^{\infty}(\mathbb{R},\mathbb{C})[D;\mathrm{id},\delta]$, where δ is the ordinary derivation. (At least that is a possible interpretation, their paper is not quite up to modern standards of rigour.) The following theorem is essentially found in their papers.

Theorem 1.4.1. *Let $P = \sum_{i=0}^{n} p_i D^i$ and $Q = \sum_{j=0}^{m} q_j D^j$ be two commuting elements of T with constant leading coefficients. Then there is a non-zero polynomial $f(s,t)$ in two commuting variables over $\mathbb{C}$ such that $f(P,Q) = 0$. Note that the fact that P and Q commute guarantees that $f(P,Q)$ is well-defined.*

Proof. Suppose P has degree n. If μ is any complex number there are n linearly independent solutions of the eigenvalue problem $Py = \mu y$. Denote the space of all such solutions by V_μ. Since $PQy = QPy = \mu Qy$ for any $y \in V_\mu$, Q maps V_μ into itself. As a basis for V_μ we can take the functions y_i with $\frac{d^j y_i}{dx^j}(0) = [i = j]$. (Here $[i = j]$ evaluates to 1 if $i = j$ and to zero otherwise.) I claim that the matrix of Q, when expressed in this basis, has elements that are polynomials in μ.

To see this consider the operator T mapping y_i to the solution of $Py = \mu y$, such that $(Ty)^{(i)}(0) = y^{(i+1)}(0)$ for $i = 0,1,\dots n-1$. Let $B = (b_{i,j})$ denote the matrix of T in the chosen basis. It is clear that $b_{i,j} = [i = j+1]$ if $i \leq n-1$. To determine the last row we compute $(Ty_i)^{(n-1)}(0) = y_i^{(n)}(0)$.

$$y_i^n(0) = \frac{1}{p_n}\left(Py_i - \sum_{j=0}^{n-1} p_j y_j\right)(0) = \frac{1}{p_n}\left(\mu y_i(0) - \sum_{j=0}^{n-1} p_j(0) y_i^{(j)}\right). \tag{1.4}$$

If $i = 0$ this equals $\frac{\mu - a_0(0)}{p_n}$. If $0 < i < n$ it equals $\frac{p_i(0)}{p_n}$. In any case, we see that all the elements in B are polynomials in μ. Since the action of Q on V_μ coincides with $\sum_{j=0}^{m} q_j T^j$, it follows that the matrix of Q has elements that are polynomial in μ.

We define $f(s,\mu) = \det(sI - \bar{Q}_\mu)$ where $\bar{Q}_\mu$ is the matrix representation of Q on V_μ with the chosen basis. This is a polynomial in two variables over $\mathbb{C}$. We need only show that $f(P,Q) = 0$. But for every complex number μ, Q must have an eigenvector (eigenfunction) in V_μ. If z is an eigenfunction of Q in V_μ with eigenvalue ν then $f(P,Q)z = f(\nu,\mu)z$. But $f(\nu,\mu) = 0$ by the definition of f. Thus the operator $f(P,Q)$ has an infinite-dimensional kernel and must be identically zero. $\square$

This proof probably corresponds to the proof that Burchnall and Chaundy intended to give, though it seems there are some gaps in their version. They do not require the operators to have constant leading coefficient. One can modify the proof of Theorem 1.4.1 to only require invertible leading coefficient without much difficulty. If the coefficients come from a differential field, such as the field of meromorphic functions, the leading coefficient is of course always invertible.

The result of Burchnall and Chaundy was rediscovered independently during the 70s by researchers in the area of PDEs. It turns out that several important equations can be equivalently formulated as a condition that a pair of differential operators commute. These differential equations are completely integrable as a result, which roughly means that they possess an infinite number of conservation laws. In fact the proof we gave of Theorem 1.4.1 is taken from the article [Kri77] by Krichever on integrable systems, originally written without knowledge of the work of Burchnall and Chaundy.

In subsection 1.4.1 we describe, in a more general context, an algorithm for computing the annihilating polynomial that is found already in the works of Burchnall and Chaundy .

Burchnall's and Chaundy's work rely on analytical facts, such as the existence theorem for solutions of linear ordinary differential equations. However it is possible to give algebraic proofs for the existence of the annihilating polynomial.

In fact, Burchnall's and Chaundy's result follows from the results in Section 1.3.

Theorem 1.4.2. *Let S be a K-algebra and let a be an element of S such that $\mathbb{C}_S(a)$ is a free $K[a]$-module of finite rank. If b is an element that commutes with a then there exists a nonzero $f(s,t) \in K[s,t]$ such that $f(a,b) = 0$.*

Proof. Consider the sequence of elements $1, b, b^2, \ldots$. They all belong to $C_S(a)$ and must therefore be linearly dependent over $K[a]$. So for some $n \in \mathbb{N}$ we can find $f_0(a), f_1(a), \ldots, f_n(a)$, not all zero, such that $\sum_{i=0}^{n} f_i(a) b^i = 0$. Then $f(s,t) = \sum_{i=0}^{n} f_i(s) t^i$ is the desired polynomial. $\square$

1.4.1 Algorithmic Burchnall-Chaundy theory

Silvestrov and collaborators [dJSS09, HS00, LS03] have extended the Burchnall-Chaundy theory to the q-Weyl algebra. In particular, in [dJSS09] they manage to extend the algorithmic method in Burchnall's and Chaundy's work for the computation of the annihilating curve. We describe that method now.

Let $P = \sum_{i=0}^{n} p_i(y)x^i$ and $Q = \sum_{j=0}^{m} q_j(y)x^j$ be commuting elements in a q-Weyl algebra over some field K. For $e = 0, 1, \dots m-1$ compute

$$x^e(P-s) = \sum p_{i,e}(y,s))x^i$$

and similarly, for $l = 0, 1, \dots n-1$ compute

$$x^l(Q-t) = \sum q_{j,l}(y,t)x^j.$$

Here the computation is done in the ring $K[y][x;\sigma,\delta][s,t]$, the polynomial ring in two central indeterminates over $K[y][x;\sigma,\delta]$. Form a square matrix of size $n+m$ with $p_{i,e}$ as the element in row $e+1$ and column $i+1$. Let $q_{j,l}$ be the matrix element in row $j+m+1$ and column $l+1$. The determinant of this matrix will be called the eliminant (of P and Q) and frequently denoted $\Delta_{P,Q}$.

De Jeu, Svensson and Silvestrov [dJSS09] prove the following theorem.

Theorem 1.4.3. *Let k be a field, and q an element of K such that $\sum_{i=0}^{N} q^i \neq 0$ for all natural numbers N. (Note that such a q only exists if K is an infinite field.) Let $\Delta_{P,Q}$ denote the eliminant constructed above. (A polynomial in y, s and t.) Write $\Delta_{P,Q} = \sum f_i(s,t)y^i$. Then*

(i) *at least one of the f_i are non-zero;*

(ii) *$f_i(P,Q) = 0$ for all i.*

In the case when $K = \mathbb{R}$ and $q = 1$, this is the same method as Burchnall and Chaundy describe.

Example 1.4.4. That a condition on q is needed in the theorem can be seen as follows: if q is a primitive nth root of unity, where $n > 1$, then x^n and y^n both belong to the center of $K[y][x;\sigma,\delta]$. But there is no non-zero polynomial over K that annihilates x^n and y^n.

Example 1.4.5. We describe an example of the eliminant when $q = 1$. Let $P = yx$ and $Q = y^2x^2$. Then

$$\Delta_{P,Q} = \begin{vmatrix} 0 & y & yD - s \\ y & (1-s) & yD^2 + (1-s)D \\ y^2 & 0 & y^2D^2 - t \end{vmatrix}.$$

Two of the papers in this thesis deal with the eliminant construction of Burchnall and Chaundy theory. Paper A deals with the case of q-Weyl algebras and proves some properties of the eliminant when P and Q have a special form. Paper B studies the eliminant construction in a general Ore extension and shows that if P and Q

are commuting elements of $R[x;\sigma,\delta]$, where R is an integral domain and σ is an injective endomorphism then we can use the eliminant construction to compute a non-zero polynomial in $R[s,t]$ such that $f(P,Q)=0$.

Note that if we specialize to the Weyl case our conclusion is weaker than the results of Burchnall and Chaundy. Our results imply the existence of an annihilating polynomial in $K[y][s,t]$ whereas their result guarantee the existence of a polynomial in $K[s,t]$. The difference is unsurprising in light of Example 1.4.4. Our results may instead be seen as a generalization of the fact that any pair of elements in $K[x]$ are algebraically dependent [ER93], but with a method of proof inspired by Burchnall-Chaundy theory.

1.5 Simplicity and maximal commutativity

1.5.1 Motivation from operator algebras and dynamical systems

The third paper in this thesis, Paper C, studies the question of when an Ore extension is simple, and how this is related to properties of R as a subring of $R[x;\sigma,\delta]$. Specifically we will study when R coincides with its centralizer in $R[x;\sigma,\delta]$ and whether every ideal of $R[x;\sigma,\delta]$ must intersect the centralizer of R in $R[x;\sigma,\delta]$. We formulate a definition before we proceed.

Definition 1.5.1. Let T be a subring of a ring R. If $T\cap I\neq\{0\}$ for every ideal I in R we say that T has the *ideal intersection property*. (Recall that by ideal we always mean a two-sided ideal.)

The notions of maximal commutativity and the ideal intersection property were first studied in the field of operator algebras, where they have implications for topological dynamical systems. We proceed to describe this background briefly. See the textbook in several volumes by Kadison and Ringrose [KR97a, KR97b] for a detailed discussion of operator algebras, and the textbooks [Tom87, Dav96] for a discussion of crossed product C^*-algebras.

Recall that a *Banach algebra* is a Banach space equipped with a multiplication that satisfies $||xy||\leq ||x||\,||y||$ for all elements x,y of the algebra. Our Banach algebras will be algebras over the complex numbers. A *Banach *-algebra* is a Banach algebra, A, equipped with a map *: $A\to A$ satisfying $(x+y)^*=x^*+y^*$, $(xy)^*=y^*x^*$, $(\lambda x)^*=\bar{\lambda}x^*$, $(x^*)^*=x$ and $||x^*||=||x||$ for all $x,y\in A$ and all $\lambda\in\mathbb{C}$. If in addition A satisfies $||x^*x||=||x||^2$ for all x then A is a C*-algebra.

Every C*-algebra is isomorphic to a subalgebra of $B(H)$, the algebra of bounded linear operators on the Hilbert space H, with the involution given by the taking the adjoint of an operator.

A *topological dynamical system* is a compact Hausdorff space, X, equipped with a homeomorphism α. α induces an automorphism, σ, of $C(X)$ defined by $\sigma(f)(x)=$

$f(\alpha^{-1}(x))$, for $f \in C(X)$ and $x \in X$. In turn this induces an action of $\mathbb{Z}$ on $C(X)$ by letting the integer n act as σ^n on $C(X)$. With the norm equal to the supremum and involution equal to pointwise conjugation $C(X)$ becomes a C*-algebra.

Let F be the set of functions from $\mathbb{Z}$ to $C(X)$ with finite support. Let u_n be the element of F satisfying $u_n(n) = \mathrm{id}$ and $u_n(m) = 0$ if $m \neq n$. Then any element, c, of F can be written uniquely as $\sum_n c(n)u_n$ where the sum is over all integers n. We make F into a vector space over $\mathbb{C}$ in the obvious way. We define a norm, involution and multiplication operation on F by

$$\begin{aligned} c^*(n) &= \sigma^n(c(-n)^*) \\ cd(n) &= \sum_k c(k)\sigma^k(d(n-k)) \\ ||c||_1 &= \sum ||c(n)||_{C(X)}, \end{aligned}$$

for any elements $c, d \in F$.

We note that, for any integers m, n, $u_m u_n = u_{m+n}$ and $u_m^* = u_{-m}$. If we identify $f \in C(X)$ with the element $c \in F$ such that $c(0) = f$ and $c(n) = 0$ if $n \neq 0$, then we can regard $C(X)$ as a subalgebra of F. We note further that if $f \in C(X)$, then $u_n f = \sigma^n(f)u_n$.

We can complete F, with respect to the $||\cdot||_1$-norm, to a Banach *-algebra which we denote by $\ell^1(\mathbb{Z}, C(X))$. A *-representation of $A = \ell^1(\mathbb{Z}, C(X))$ is a *-homomorphism from A to $B(H)$, the bounded linear operators on H, for some Hilbert space H. It turns out that if ϕ is a *-representation of A and x is any element of A then $||\phi(x)||_{B(H)} \leq ||x||_A$. We can define a new norm by

$$||x|| = \sup_\phi ||\phi(x)||_{B(H)}$$

where the supremum is taken over all possible values of $||\phi(x)||_{B(H)}$. (It turns out that such representations exists. Thus we are taking a supremum over a bounded set of real numbers and it is clear $||x||$ is a well-defined number.) This turns out to be not only a norm but a C*-norm. We take the completion of $\ell^1(\mathbb{Z}, C(X))$ in this norm to finally get a C*-algebra which we denote by $C(X) \rtimes_\sigma \mathbb{Z}$. This construction is a special case of a more general construction of crossed products for C*-algebras acted upon by more general groups.

$C(X)$ sits as a subalgebra inside $C(X) \rtimes_\sigma \mathbb{Z}$. As we said its properties as a subalgebra are related to the properties of the dynamical system. In particular we have the following

Theorem 1.5.2. *Let (X, α) be a topological dynamical system. Then the following three assertions are equivalent*

(i) *the set $\{x \in X \mid \alpha^n(x) \neq x \; \forall n \in Z\}$ is dense in X;*

(ii) *every closed non-zero ideal, I, in $C(X) \rtimes_\sigma \mathbb{Z}$ intersects $C(X)$;*

(iii) *$C(X)$ is a maximal commutative C*-subalgebra of $C(X) \rtimes_\sigma \mathbb{Z}$.*

See the book by Jun Tomiyama [Tom87, Theorem 4.3.5] for a proof of this result. We note that all closed ideals in C*-algebras are automatically self-adjoint.

In a series of articles [dST09, SSdJ07, SSdJ09a, SSdJ09b, ST09], de Jeu, Silvestrov, Svensson and Tomiyama (in various constellations), have studied the algebras F and $\ell^1(\mathbb{Z}, C(X))$. They have also investigated analogues of the crossed product construction for subalgebras of F. We mention only a few representative results. In [SSdJ07] de Jeu, Svensson and Silvestrov prove that $C(X)$ is maximal commutative in F precisely when the aperiodic points are dense in X. In [SSdJ09a] de Jeu, Silvestrov and Svensson show that if A is a complex commutative semi-simple regular Banach algebra and σ is an automorphism then every non-zero ideal in (the algebraic crossed product) $A \rtimes_\sigma \mathbb{Z}$ has non-zero intersection with the centralizer of A. A more general result can be found in an article by Öinert and Silvestrov [ÖS08] where they show that if A is any commutative ring then any non-zero ideal of an algebraic crossed product $A \rtimes_\sigma^\alpha G$ has non-zero intersection with the centralizer of A. Building upon these works Svensson and Tomiyama [ST09] partially answer a question posed in [SSdJ09a], by proving that the centralizer of $C(X)$ in the C^*-crossed product $C(X) \rtimes_\sigma \mathbb{Z}$ always has the ideal intersection property. The cited articles with Svensson as a co-author have been collected as part of his PhD thesis [Sve09].

In Öinert's PhD thesis [Öin09a] one finds other algebraic analogues of these questions studied. The thesis is based on the articles [Öin09b, ÖL10, ÖS08, ÖS09a, ÖS09b, ÖSTAV09] by Öinert and his co-authors Lundström, Silvestrov, Theohari-Apostolidi and Vavatsoulas.

There are also similar results in the field of von Neumann algebras. Von Neumann algebras turn out to have a connection with measurable dynamical systems that is similar to the one C^*-algebras have with topological dynamical systems. See e.g. [BR92, Chapter 7.3] for a discussion of these results.

1.5.2 Simplicity of Ore extensions

We will be interested in the question of when an Ore extension $R[x;\sigma,\delta]$, and in particular a differential polynomial ring $R[x;\mathrm{id}_R,\delta]$, is simple. We will in this section give a background to the third paper of this thesis, where simplicity of Ore extensions play a major role.

An important concept in this study will be the found in the next definition.

Definition 1.5.3. If I is an ideal in a ring R and f is a map from R to R we say that I is f-invariant, or an f-ideal, if $f(i) \in I$ for all $i \in I$. If the only f-invariant ideals in a ring R are $\{0\}$ and R, then R is said to be f-simple. Our most important cases of this definition will be when $f = \sigma$ or $f = \delta$. We say I is a σ-δ-ideal if it is both σ-invariant and δ-invariant and similarly define a σ-δ-simple ring.

An elementary result is contained in the next proposition.

Proposition 1.5.4. *Let $S = R[x;\sigma,\delta]$ be a simple Ore extension. Then R is σ-δ-simple.*

Proof. Suppose that J is a non-trivial σ-δ-invariant ideal of R. Then JS is a non-trivial ideal of S. □

A skew polynomial ring $R[x;\sigma,0]$ is never simple since x generates a proper ideal. It turns out that if δ is an inner derivation, then $R[x;\sigma,\delta]$ is isomorphic to a skew polynomial ring [RS08] and is thus not simple.

A deeper result is found in [LL92]. Their Theorem 5.8 says that $S = R[x;\sigma,\delta]$ is non-simple if and only if there is some $R[y;\sigma',0]$ that can be embedded in S. See also [JLL09, Lemma 4.1] for necessary and sufficient conditions for $R[x;\sigma,\delta]$ to be simple.

It is not trivial to construct simple Ore extensions with $\sigma \neq \mathrm{id}_R$ but in [CF75, Chapter 3] one finds an example. The coefficient ring there is a non-commutative division ring.

Jordan proves in his PhD-thesis [Jor75] the following theorem. (Cozzens and Faith have independently discovered part of this result in [CF75].)

Theorem 1.5.5. *If $S = R[x;\mathrm{id}_R,\delta]$ is a simple ring, then the characteristic of R is either zero or prime. If R has characteristic zero, then S is simple if and only if δ is an outer derivation and R is δ-simple. If R has prime characteristic p, then $R[x;\mathrm{id}_R,\delta]$ is simple if and only if R is δ-simple and no sum $\sum_{i=0}^{m} a_i\delta^{p^i}$, with all a_i central constants, is an inner derivation induced by a constant.*

Note that if R is a δ-simple ring of characteristic p, then $pr = 0$ for any $r \in R$ and it follows that δ^p is a derivation as well.

If one has a family of commuting derivations, $\delta_1,\dots,\delta_n$, one can form a differential polynomial ring in several variables. The articles [Mal88, Pos60, Vos85] consider the question when such rings are simple. In [Hau77] Hauger studies a class of rings similar, but not identical to, the differential polynomial rings defined here, and a characterization of when they are simple is obtained.

After the writing of Paper C, Jordan and Wells have published an article [JW13] where they study simplicity criteria for a class of iterated Ore extensions that they call *ambiskew polynomial rings*. Their results are of a rather different form than the ones obtained in Paper C.

A direct generalization of the results on simplicity of differential polynomial rings in Paper C appears in [NÖ14], also written after Paper C, by Nystedt and Öinert (one of the co-authors of Paper C).

Chapter 2

Summary of the thesis

2.1 Overview of Paper A

In this paper we study certain pairs of commuting elements of the q-Weyl algebra $S = k[y][x;\sigma,\delta]$, where k is a field of characteristic zero, $\sigma(y) = qy$ and $\delta(y) = q$. Here q is a element of k that is non-zero, and not a root of unity.

We say that an element P of S is a 0-chain element if it can be written as $P = \sum_{i=0}^{n} a_i(yx)^i$, where the a_i:s are elements of k. The set of all 0-chain elements form a commutative k-subalgebra of S. In the paper we first show (the presumably known fact) that this is a maximal commutative subalgebra of S. We then study what happens when we compute the annihilating polynomial using the methods of Section 1.4.1.

Suppose that P and Q are two non-zero 0-chain elements, of degrees n and m respectively. Theorem 1.4.3 tells us that there is a non-zero polynomial $f \in k[s,t]$ such that $f(P,Q) = 0$ and gives us a method for computing such a polynomial. It in fact gives finite set of annihilating polynomials, f_i, and guarantees that at least one of them is non-zero. In Paper A we show that, when P and Q are 0-chain elements, that of all the f_i only f_{nm} will be non-zero.

2.2 Overview of Paper B

In Paper B we turn our attention to general Ore extensions $R[x;\sigma,\delta]$ over a commutative ring R.

We start by describing work by Larsson, see [Lar08], and Li, see [Li98], which allows one to extend the determinant construction in Theorem 1.4.3 to general Ore extensions. In the general Ore case we get a polynomial $f \in R[s,t]$ as the annihilating polynomial.

The contribution of Paper B is to analyze this determinant algorithm. We give formulas for the coefficients of f as sums of certain determinants. These formulas have intrinsic interest, as well as allowing us to prove that if R is an integral domain and σ is injective, then the computed polynomial is non-zero. This thus gives a partial extension of the Burchnall-Chaundy theory to a more general class of Ore extensions.

If R is not assumed to be an integral domain, or if σ is not injective, then we demonstrate that the polynomial computed with the determinant construction can be trivial.

We finally analyze the determinant algorithm in a special case when the Ore extension has the form $k[y][x;\sigma,0]$, for some field k. We consider commuting elements $P=\sum_i a_i(yx)^i$ and $Q=\sum_j b_j(yx)^j$, where the a_i:s and b_j:s are elements of k, and describe the annihilating polynomial computed. Our results allow us to show that in this special case P and Q become algebraically dependent over k.

2.3 Overview of Paper C

In Paper C we consider several related questions concerning Ore extensions. We start by describing the centralizer of R in $R[x;\sigma,\delta]$, and noting that if R is commutative, then its centralizer is a maximal commutative subring of $R[x;\sigma,\delta]$.

We continue by giving conditions for when R coincides with its centralizer. We start by considering the general case $R[x;\sigma,\delta]$, for some commutative R, and give a sufficient condition for R to be maximal commutative. If R is an integral domain this condition is that σ has infinite order, i.e. $\sigma^n \neq \mathrm{id}$ for all positive integers n. If $\delta = 0$ and R is still an integral domain we show that the infinite order of σ is also a necessary condition.

When we have a differential polynomial ring $R[x;\mathrm{id}_R,\delta]$, σ will of course not have infinite order. Instead we prove that if R is an integral domain of characteristic zero and δ is non-zero, then R is a maximal commutative subring.

In the next section we describe the center of $R[x;\sigma,\delta]$. We treat both the general case and various special cases when the description of the center simplifies.

We then come to the main topic of the paper which is to investigate when Ore extensions, and in particular differential polynomial rings, are simple. We start with the afore-mentioned Proposition 1.5.4.

If R is an integral domain and $R[x;\sigma,\delta]$ is simple we cite results showing that σ must be the identity on R. This gives some justification for focusing on simple differential polynomial rings.

We prove the following two theorems on simple differential polynomial rings:

Theorem. *Let R be an integral domain of characteristic zero. Then the following assertions are equivalent:*

(i) $R[x;\sigma,\delta]$ *is a simple ring;*

(ii) $\sigma=\mathrm{id}_R$*, R is δ-simple, and R is a maximal commutative subring of* $R[x;\sigma,\delta]$.

Theorem. *Let R be a ring and δ a derivation of R. The differential polynomial ring $R[x;\mathrm{id}_R,\delta]$ is simple if and only if R is δ-simple and $Z(R[x;\mathrm{id}_R,\delta])$ is a field.*

2.4 Overview of Paper D

In Paper D we show how the method introduced by Amitsur can be used to prove the following theorem.

Theorem. *Let R be an integral domain, σ an injective endomorphism of R and δ a σ-derivation on R. Suppose that the ring of constants, F, is a field. Let a be an element of $S = R[x;\sigma,\delta]$ of degree n and assume that if b and c are two elements in $C_S(a)$ such that* $\deg(b) = \deg(c) = m$*, then $b_m = \alpha c_m$, where b_m and c_m are the leading coefficients of b and c respectively, and α is some constant.*

Then $C_S(a)$ is a free $F[a]$-module of rank at most n.

We also show that Ore extensions of the form $K[y][x;\sigma,\delta]$, where K is a field and σ is a K-algebra endomorphism such that $\deg_y(\sigma(y)) > 1$, satisfy the conditions of the preceding theorem.

2.5 Overview of Paper E

This paper is a response to an article by Nystedt. Nystedt defines an Ore extension by giving the multiplication law for two arbitrary elements and then proceeds to show that this gives rise to an associative ring through a direct combinatorial argument.

Nystedt shows that the associativity of an Ore extension $R[x;\sigma,\delta]$ is equivalent to that the fact that following equation,

$$\sum_{i=0}^{\infty} \pi_i^m(b')\pi_j^{i+n}(b'') = \sum_{i=0}^{\infty} \pi_i^m(b'\pi_{j-i}^n(b'')), \tag{2.1}$$

holds for all non-negative integers m, n, j and all elements $b', b'' \in R$. Here the π_a^b are certain sums of compositions of σ and δ.

Nystedt shows that every term in the expansion of $\pi_i^m(b'\pi_{j-i}^n(b''))$ corresponds to a term in $\pi_{i+v}^m(b')\pi_j^{v+n+i}$, for some non-negative integer v. He then shows that the correspondence is injective and uses a counting argument to show that the number of terms on both sides of Equation (2.1) are equal.

In Paper E we dispense with the counting argument and prove directly that every term in the expansion of the left side of Equation (2.1) occurs on the right side as well. Together with the other results by Nystedt this proves that Ore extensions are associative rings.

2.6 Overview of Paper F

In Paper F we continue the study we initiated in Paper D. Two theorems (F.3.1 and F.4.1) can be combined to state the following result.

Theorem. *Let K be a field and set $R = K[y]$. Let σ be a K-algebra endomorphism of R such that* $\deg_y(\sigma(y)) > 1$ *and let δ be a σ-derivation. Let a be any non-scalar of $S = R[x;\sigma,\delta]$. Then $C_S(a)$ is a free $K[a]$-module of finite rank and commutative.*

This theorem is partially included in Paper D but we give a complete, and slightly different, proof. The proof once again follows Amitsur's method.

We proceed to describe some corollaries of this result. One result we find is that if A is an arbitrary subset of S then the centralizer of A equals either S, K or $C_S(P)$ for an element $a \in S \setminus K$. We also remark that the maximal commutative subsets of S are precisely the sets of the form $C_S(a)$, with $a \in S \setminus K$.

We also show in a number of cases that $C_s(a)$ is singly generated as a subalgebra of S. For example we prove the following proposition.

Proposition. *Let a be an element of S of degree n, where n is a prime. Let a_n be the leading coefficient of P and let ρ be the degree of a_n as a polynomial in y. Let s be the degree of $\sigma(y)$, also as a polynomial in y. Then if $\sum_{i=0}^{n-1} s^i$ does not divide ρ it follows that $C_S(a) = \{\sum c_i a^i \mid c_i \in K\} =: K[a]$.*

We also prove

Proposition. *Let a be an element of S of degree $n > 0$ in x and suppose that a_n (the leading coefficient of a) has degree greater than zero but not greater than n as a polynomial in y. Then $C_S(a) = K[a]$.*

We manage to prove that $C_S(a)$ is a singly generated algebra in some further cases, by considering only the leading coefficient of elements that commute with P. We suspect that it is true in greater generality that $C_S(a)$ is a a singly generated algebra but this will require other techniques to prove.

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Part II

Scientific papers

A

Paper A

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Paper A

On algebraic curves for commuting elements in q-Heisenberg algebras

Johan Richter and Sergei D. Silvestrov

Abstract. *In the present paper we continue the investigation of the algebraic dependence of commuting elements in q-deformed Heisenberg algebras. We provide a simple proof that the* 0*-chain subalgebra is a maximal commutative subalgebra when q is of free type and that it coincides with the centralizer (commutant) of any one of its elements different from a scalar multiple of the unity. We review the Burchnall-Chaundy type construction for proving algebraic dependence and obtaining corresponding algebraic curves for commuting elements in the q-deformed Heisenberg algebra by computing a certain determinant of a matrix with entries depending on two commuting variables and one of the generators. The coefficients in front of the powers of the generator in the expansion of the determinant are polynomials in the two variables defining some algebraic curves and annihilating the two commuting elements. We show that for the elements from the* 0*-chain subalgebra exactly one algebraic curve arises in the expansion of the determinant. Finally, we present several examples of computations of such algebraic curves and also make some observations on the properties of these curves.*

A.1 Introduction

In 1994, one of the authors of the present paper, S. Silvestrov, based on consideration of the previous literature and a series of trial computations, made the following three part conjecture.

- The first part of the conjecture stated that the Burchnall–Chaundy type result on algebraic dependence of commuting elements can be proved in greater generality, that is for much more general classes of non-commutative algebras and rings than the Heisenberg algebra and related algebras of differential operators treated by Burchnall and Chaundy and in subsequent literature [BC23, BC28, BC31, Kri77a, Kri77b, Mum78].

- The second part stated that the Burchnall–Chaundy eliminant construction of annihilating algebraic curves formulated in determinant (resultant) form works after some appropriate modifications for most or possibly all classes of algebras where the Burchnall–Chaundy type result on algebraic dependence of commuting elements can be proved.

- Finally, the third part of the conjecture stated that the proof of the vanishing of the corresponding determinant algebraic curves on the commuting elements can be performed in a purely algebraic way for all classes of algebras or rings where this fact is true, that is using only the internal structure and calculations with the elements in the corresponding algebras or rings and the algebraic combinatorial expansion formulas for the corresponding determinants, that is, without any need of passing to operator representations and use of analytic methods as in the Burchnall–Chaundy type proofs.

This third part of the conjecture remains widely open with no general such proofs available for any classes of algebras or rings, even in the case of the usual Heisenberg algebra and differential operators, and with only a series of examples calculated for the Heisenberg algebra, q-Heisenberg algebra and some more general algebras, all supporting the conjecture. In the first and the second part of the conjecture progress has been made. In [HS00], the key Burchnall–Chaundy type theorem on algebraic dependence of commuting elements in q-deformed Heisenberg algebras (and thus as a corollary for q-difference operators as operators representing q-deformed Heisenberg algebras) was obtained. The result and the methods have been extended to more general algebras and rings generalizing q-deformed Heisenberg algebras (generalized Weyl structures and graded rings) in [HS07]. The proof in [HS00] is totally different from the Burchnall–Chaundy type proof. It is an existence argument based only on the intrinsic properties of the elements and internal structure of q-deformed Heisenberg algebras, thus supporting the first part of the conjecture. It can be used successfully for an algorithmic implementation for computing the corresponding algebraic curves for given commuting elements. However, it does not give any specific information on the structure or properties of such algebraic curves or any general formulae. It is thus important to have a way of describing such algebraic curves by some explicit formulae, as for example those obtained using the Burchnall–Chaundy eliminant construction for the $q = 1$ case, i.e., for the classical Heisenberg algebra. In [LS03], a step in that direction was taken by offering a number of examples, all supporting the claim that the eliminant determinant method should work in the general case. However, no general proof for this was provided. The complete proof following the Burchnall-Chaundy approach in the case of q not a root of unity has been recently obtained [dJSS09], by showing that the determinant eliminant construction, properly adjusted for the q-deformed Heisenberg algebras, gives annihilating curves for commuting elements in the q-deformed Heisenberg algebra when q is not a root of unity, thus confirming the second part of the conjecture for these algebras. That proof was obtained by adapting the Burchnall-Chaundy eliminant determinant method of the case $q = 1$ of differential operators to the q-deformed case, after passing to a specific faithful representation of the q-deformed Heisenberg algebra on Laurent series and then performing a detailed analysis of the kernels of arbitrary operators in the image of this representation. While exploring the determinant eliminant construction of the annihilating curves, we also obtained some further information on such curves and some other results on dimensions and bases in the eigenspaces of the q-difference operators in the image of the chosen representation of the q-deformed Heisenberg algebra. In the case of q being a root of unity the algebraic dependence of commuting elements holds only over the center of the q-deformed Heisenberg algebra [HS00], and it is still unknown how to modify the eliminant determinant construction to yield annihilating curves for this case.

In the present paper we continue the investigation of the algebraic dependence of commuting elements in q-deformed Heisenberg algebras within the context of [HS00], [LS03] and [dJSS09]. In Section A.2, following [HS00], we recall some preliminaries on q-deformed Heisenberg algebra, including degree function, decomposition into the direct sum of the "chain" subspaces indexed by the integers and corresponding to this decomposition the upper and lower chain functions. In Section A.3, we consider in more detail the 0-chain subspace (indexed by zero). This subspace is a commutative subalgebra in the q-deformed Heisenberg algebra playing a pivotal role for the structure of this algebra [HS00]. We provide a simple proof that this subalgebra is the maximal commutative subalgebra when q is of free type, and that it coincides with the centralizer (commutant) of any one of its elements different from the scalar multiple of the unity. In Section A.4, we review the Burchnall-Chaundy type construction for proving algebraic dependence in the q-deformed Heisenberg algebra, following [dJSS09]. We work directly in an abstract q-deformed Heisenberg algebra, rather then passing to a specific representation. The construction is based on computing a certain determinant of a matrix with entries depending on two commuting variables and containing one of the generators of the q-deformed Heisenberg algebra. This matrix is constructed from commuting elements. The coefficients in front of the powers of the generator in the expansion of the determinant are polynomials in the two variables defining some algebraic curves. The commuting elements satisfy the equations of these algebraic curves [dJSS09]. In Section A.5, we show that for the elements from the 0-chain subalgebra exactly one algebraic curve arises via this construction in the expansion of the determinant and then present several examples of computations of such algebraic curves and also make some observations on the properties of these curves based on these examples and further computer experiments.

A.2 Preliminaries

Let K be a field of characteristic 0, and q a non-zero element of K. We say that q is of free type if it is 1 or not a root of unity. If q is a root of unity we say it is of torsion type. We define the q-deformed Heisenberg algebra over K as

$$H(q) = K\langle A, B\rangle / (AB - qBA - I)$$

The identity element will be denoted by I. For $q = 1$ we recover the classical Heisenberg algebra (called also Weyl algebra). One can define degree functions $\deg_A$ and $\deg_B$ with respect to A and B on $H(q)$ just as on the commutative algebra of polynomials. One can evaluate these functions by inspection just as one would in a commutative algebra.

Thus for example $\deg_A(A^2B + B^3) = 2$ and $\deg_B(AB + B^3A) = 3$.

That the functions are well-defined and does not depend on how the elements are written is proved in [HS00, Chapter 4]. We also define the total degree function $\deg(\alpha) = \deg_A(\alpha) + \deg_B(\alpha)$. In [HS00, Chapter 4] the following theorem is proved

Theorem A.2.1. *Let $\alpha, \beta \in H(q)$ for some $q \neq 0$ and let $V \in \{A, B\}$. Then*

$$\deg_V(\alpha\beta) = \deg_V(\alpha) + \deg_V(\beta).$$

We define the sets R_n for all integers n by

$$R_n = \{ \sum_{j \geq \max(0,-n)} a_j B^{j+n} A^j \mid a_j \in K, a_j \neq 0 \text{ for at most finitely many } j \}.$$

If the element $\alpha \in H(q)$ belongs to some R_n we say that it is homogeneous. We also define a function

$$\chi : \{\alpha \in H(q) \mid \alpha \text{ is non-zero and homogeneous}\} \longrightarrow \mathbb{Z}$$

by defining $\chi(\alpha)$ to be the unique integer such that $\alpha \in R_{\chi(\alpha)}$. This function is called the chain function.

All R_i are vector spaces over K. Further $H(q)$ is the direct sum of all the R_i. We can use this to define a projection operation. Let α be an element of $H(q)$. We can write $\alpha = \sum_j \alpha_j$, where $\alpha_j \in R_j$. This decomposition is unique. We then define the projection of α on R_n by $\alpha \sqcap R_n = \alpha_n$. The notation is intended to recall the notation for intersection. At this point we define two new functions. They are defined for all non-zero elements of $H(q)$:

$$\overline{\chi}(\alpha) = \max\{n \in \mathbb{Z} \mid \alpha \sqcap R_n \neq 0\}, \quad \underline{\chi}(\alpha) = \min\{n \in \mathbb{Z} \mid \alpha \sqcap R_n \neq 0\}.$$

These functions are known as the upper and lower chain functions respectively.

We denote the commutator, $ab - ba$, of two elements a, b of some ring by $[a, b]$.

A.3 R_0 is maximal commutative

We begin by noting that all elements of R_0 commute with each other [HS00]. Furthermore, the products of two elements $\alpha, \beta \in R_0$ are in R_0. So R_0 is a commutative subalgebra. We want to show that it is in fact a maximal commutative subalgebra.

For an element $\alpha \in H(q)$ we define $Cen(\alpha) = \{\beta \in H(q) \mid [\alpha, \beta] = 0\}$.

In [HS00, Chapter 6] the following theorem is proved (as a part of Theorem 6.10)

Theorem A.3.1. *Let q be of free type. Let α, β be two commuting elements in $H(q)$. Then the following assertions hold:*

- *If $\overline{\chi}(\alpha) = 0$ then $\overline{\chi}(\beta) = 0$ or $\alpha \sqcap R_0 = cI$ for some $c \in K$.*
- *If $\underline{\chi}(\alpha) = 0$ then $\underline{\chi}(\beta) = 0$ or $\alpha \sqcap R_0 = cI$ for some $c \in K$.*

We now describe the centralizer of an element in R_0.

Theorem A.3.2. *Let q be of free type and $\alpha \in R_0 \subset H(q)$. Assume further that $\alpha \neq cI$ for all $c \in K$. Then $Cen(\alpha) = R_0$.*

Proof. As we noted above $R_0 \subseteq Cen(\alpha)$. It remains to show the other inclusion. Let β be an arbitrary non-zero element of $Cen(\alpha)$. By Theorem A.3.1 we must have $\overline{\chi}(\beta) = 0$, since $\alpha \sqcap R_0 \neq cI$. Similarly we must have that $\underline{\chi}(\beta) = 0$. So in the direct sum decomposition only elements in R_0 occurs. Thus $\beta \in R_0$. □

Corollary A.3.3. *If q is of free type, then R_0 is maximal commutative subring of $H(q)$.*

Proof. Let β be an element that commutes with everything in R_0. Then in particular it must commute with BA. But $Cen(BA) = R_0$ by the preceding theorem. Thus $\beta \in R_0$. □

A.4 Annihilating polynomials

As mentioned in the introduction any two commuting elements in $H(q)$ must be algebraically dependent when q is of free type. This is Theorem 7.4 in [HS00], which we state here:

Theorem A.4.1. *Let $q \in K$ be of free type. If $\alpha, \beta \in H(q)$ commute then there exists a non-zero $P \in K[x, y]$ such that $P(\alpha, \beta) = 0$.*

We now describe an explicit construction of this polynomial. We let s and t be variables that take values in the base field K. We write the commuting elements α, β as

$$\alpha = \sum_{i=0}^{n} p_i(B)A^i, \quad \beta = \sum_{i=0}^{m} r_i(B)A^i$$

where the p_i and r_i are polynomials. We will form an $n+m$ determinant that will give us the annihilating polynomial.

Consider the expressions obtained by reordering all A to the right of B in

$$A^k(\alpha - sI) = \sum_{i=0}^{n+k} \theta_{i,k}A^i \ k = 0, 1, \ldots, m-1$$
$$A^k(\beta - tI) = \sum_{i=0}^{m+k} \omega_{i,k}A^i \ k = 0, 1, \ldots, n-1$$

where $\theta_{i,k}$ and $\omega_{i,k}$ are functions of B, s, t arising after reordering. The coefficients of the powers of A will be the elements in the determinant that we compute. $\theta_{i,k}$ will be placed as the element in row $k+1$ and column i. $\omega_{i,k}$ will be placed in row $k+m+1$ and column i. The determinant will thus be a polynomial in s, t and B. This polynomial, which we will call the eliminant of α and β can be written as $\sum_i \delta_i(s,t)B^i$. Every such δ_i will satisfy $\delta_i(\alpha, \beta) = 0$ and at least one of them will not be identically zero. This is proven in [dJSS09].

A more precise formulation of the result in [dJSS09] can be found in the following

Theorem A.4.2. *Let*

$$\alpha = \sum_{j=0}^{m} p_j(B)A^j, \quad \beta = \sum_{j=0}^{n} r_j(B)A^j$$

be two commuting elements, the p_j:s and r_j:s being polynomials, and denote their eliminant by $\Delta_{\alpha,\beta}(B, s, t)$. Then $\Delta_{\alpha,\beta} \neq 0$. Furthermore $\Delta_{\alpha,\beta}$ has degree n seen as polynomial in s. If $r_n(B) = \sum_i a_i B^i$ ($a_i \in K$) then $\Delta_{\alpha,\beta}$ has leading coefficient

$$(-1)^n \prod_{k=0}^{m-1} \left(\sum_i a_i q^{ki} B^i \right), \tag{A.1}$$

once again seen as a polynomial in s. Symmetrically, $\Delta_{\alpha,\beta}$ will have degree m seen as a polynomial in t. The coefficient of t^m will be

$$(-1)^m \prod_{k=0}^{n-1} \left(\sum_i b_i q^{ki} B^i \right) \tag{A.2}$$

if $p_m(B) = \sum_i b_i B^i$. *Let* $g = n \max_j deg(p_j) + m \max_j deg(r_j)$. *We can write*

$$\Delta_{\alpha,\beta}(B,s,t) = \sum_{i=0}^{g} \delta_i(s,t)B^i.$$

Then there exists j *such that* $\delta_j(s,t) \neq 0$ *and* $\delta_i(\alpha,\beta) = 0$ *for all* i.

A.5 The eliminant when the elements belong to R_0

In the general case the theorem does not rule out that one can get several non-zero δ_i:s in the expansion of the eliminant, $\Delta_{\alpha,\beta}$. This does not however occur when α and β belong to R_0.

Theorem A.5.1. *Let* $\alpha = \sum_{k=0}^{n} p_k B^k A^k$ *and* $\beta = \sum_{l=0}^{m} r_k B^l A^l$. *Then, with the same notation as before, there will be only one non-zero* δ_i *when the eliminant is computed and this* i *will equal* nm.

Proof. We begin by noting that $A^n(\alpha - sI)$ will be of the form $\sum_{i=n}^{n+k} a_i B^{i-k} A^i$, where the a_i:s belong to K.

We use this result to describe the structure of the eliminant. Denote the element in row u and column v by $e_{u,v}$. Then we will have $e_{u,v} = \phi_{i,j}(s,t)B^{v-u}$ if $u \leq m$ (that is in the first m rows) and $e_{u,v} = \phi_{i,j}B^{v-u+m}$ otherwise (in the last n rows), where the $\phi_{i,j}(s,t)$ are polynomials over K. Many of them will of course be zero, in particular those where B would otherwise occur with a negative exponent.

We know, from ordinary linear algebra, that

$$\Delta_{\alpha,\beta}(B,s,t) = \sum_{\sigma} sign(\sigma) \prod_{g=1}^{m+n} e_{g,\sigma(g)},$$

where σ denotes a permutation. But looking at an arbitrary term of the sum we find that it can be written as

$$sign(\sigma) \prod_{g=1}^{g=m} (\phi_{g,\sigma(g)}(s,t)B^{\sigma(g)-g}) * \prod_{g=m+1}^{m+n} \phi_{g,\sigma(g)}(s,t)B^{\sigma(g)-g+m} =$$
$$= \Phi(s,t) * B^{\sum_{g=1}^{m+n}(\sigma(g)) - \sum_{g=1}^{n+m}(g) + mn},$$

for a polynomial $\Phi(s,t)$. But the two sums in the exponent cancel, since they have the same terms in different order, and we conclude that we get the exponent mn. Since we picked an arbitrary term we are done. ☐

A.5.1 Examples

We will include some examples here to give a feeling for the construction of the eliminant and our result. Let $\alpha = BA$ and $\beta = B^2A^2$. Then

$$\Delta_{\alpha,\beta}(B,s,t) = \begin{vmatrix} -s & B & 0 \\ 0 & 1-s & qB \\ -t & 0 & B^2 \end{vmatrix}$$

On computing the determinant we find that the annihilating polynomial is

$$s^2 - s - tq$$

This is only a slight modification of the classical case when $q = 1$. (We note that it makes no difference whether we set $q = 1$ at the beginning of the calculation or at the end.)

For our next example let α be as before and let $\beta = B^3A^3$. Then we find that

$$\Delta_{\alpha,\beta}(B,s,t) = \begin{vmatrix} -s & B & 0 & 0 \\ 0 & 1-s & qB & 0 \\ 0 & 0 & 1+q-s & q^2B \\ -t & 0 & 0 & B^3 \end{vmatrix}$$

We get the annihilating polynomial

$$s^3 - (2+q)s^2 + (1+q)s - q^3t$$

Once again no essential simplification occurs if we let q approach 1.

Now set $\beta = B^2A^2$ and $\alpha = B^3A^3$. The determinant becomes

$$\begin{vmatrix} -s & 0 & 0 & B^3 & 0 \\ 0 & -s & 0 & (1+q+q^2)B^2 & q^3B^3 \\ -t & 0 & B^2 & 0 & 0 \\ 0 & -t & (1+q)B & q^2B^2 & 0 \\ 0 & 0 & 1+q-t & q(1+2q+q^2)B & q^4B^2 \end{vmatrix}$$

and we get the annihilating polynomial

$$q^3s^2 + (q+2q^2)st + (1+q)t^2 - t^3.$$

In the classical case this polynomial becomes

$$s^2 + 3st + 2t^2 - t^3$$

As a final example we can take $\alpha = B^2A^2$ and $\beta = B^4A^4$. The eliminant is

$$\begin{vmatrix} -s & 0 & B^2 & 0 & 0 & 0 \\ 0 & -s & (1+q)B & q^2B^2 & 0 & 0 \\ 0 & 0 & 1+q-s & q(1+2q+q^2)B & q^4B^2 & 0 \\ 0 & 0 & 0 & 1+2q+2q^2+q^3-s & q^2(1+2q+2q^2+q^3)B & q^6B^2 \\ -t & 0 & 0 & 0 & B^4 & 0 \\ 0 & -t & 0 & 0 & (1+q+q^2+q^3)B^3 & q^4B^4 \end{vmatrix}.$$

We then get the annihilating polynomial

$$\begin{gathered} q^8t^2 - 2q^6ts - 3q^5ts - 2q^4ts^2 \\ -2q^4ts - q^3ts + q^4s^2 - q^3s^3 + 3q^3s^2 \\ -2q^2s^3 + 4q^2s^2 - 3qs^3 + 3qs^2 + s^2 \\ +s^4 - 2s^3 \end{gathered}$$

The limit when q goes towards 1 is

$$t^2 - 8ts - 2s^2t + 12s^2 - 8s^3 + s^4$$

This is a simpler expression but only because the coefficients are simpler. No coefficient has become zero.

This illustrates that the complexity of the resulting polynomial grows pretty fast.

Computer experiments indicate that Theorem A.5.1 can be generalized substantially. We would be interested to know whether the annihilating polynomials always have genus 0, a conjecture we have been unable to find any counterexamples to [1].

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[1] In fact this is false even in the case of the classical Weyl algebra. See [Mok12] for counterexamples.

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B

Paper B

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Paper B

Burchnall-Chaundy annihilating polynomials for commuting elements in Ore extension rings

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Abstract. *In this paper further progress is made in extending the Burchnall-Chaundy type determinant construction of annihilating polynomial for commuting elements to broader classes of rings and algebras by deducing an explicit general formula for the coefficients of the annihilating polynomial obtained by the Burchnall-Chaundy type determinant construction in Ore extension rings. It is also demonstrated how this formula can be used to compute the annihilating polynomials in several examples of commuting elements in Ore extensions. Also it is demonstrated that additional properties which may be possessed by the endomorphism, such as for example injectivity, may influence strongly the annihilating polynomial.*

B.1 Introduction

It is a classical result, going back to [BC23, BC28, BC31], that all pairs of commuting elements in the Heisenberg (Weyl) algebra are algebraiclly dependent over $\mathbb{C}$. This result was later rediscovered and applied to the study of non-linear partial differential equations [Kri77a, Kri77b, Mum78].

In 1994, one of the authors of the present paper, S. Silvestrov, based on consideration of the previous literature and a series of trial computations, made the following three part conjecture.

- The first part of the conjecture stated that the Burchnall–Chaundy type result on algebraic dependence of commuting elements can be proved in greater generality, that is for much more general classes of non-commutative algebras and rings than the Heisenberg algebra and related algebras of differential operators treated by Burchnall and Chaundy and in subsequent literature [BC23, BC28, BC31, Kri77a, Kri77b, Mum78].

- The second part stated that the Burchnall–Chaundy eliminant construction of annihilating polynomials formulated in determinant (resultant) form works after some appropriate modifications for most or possibly all classes of algebras where the Burchnall–Chaundy type result on algebraic dependence of commuting elements can be proved.

- Finally, the third part of the conjecture stated that the proof of the vanishing of the corresponding determinant polynomial on the commuting elements can be performed in a purely algebraic way for all classes of algebras or rings where this fact is true, that is using only the internal structure and calculations with the elements in the corresponding algebras or rings and the algebraic combinatorial expansion formulas for the corresponding determinants, that is, without any need of passing to operator representations and use of analytic methods as in the Burchnall–Chaundy type proofs.

In the first and the second part of the conjecture more progress has been made. In [HS00], the key Burchnall–Chaundy type theorem on algebraic dependence of commuting elements in q-deformed Heisenberg algebras (and thus as a corollary for q-difference operators as operators representing q-deformed Heisenberg algebras) was obtained. The result and the methods have been extended to more general algebras and rings generalizing q-deformed Heisenberg algebras (generalized Weyl structures and graded rings) in [HS07]. The proof in [HS00] is totally different from the Burchnall–Chaundy type proof. It is an argument based only on the intrinsic properties of the elements and internal structure of q-deformed Heisenberg algebras, thus supporting the first part of the conjecture. It can be used successfully for an algorithmic implementation for computing the corresponding annihilating polynomial for given commuting elements. However, it does not give any specific information on the structure or properties of such polynomials or any general formulae. It is thus important to have a way of describing such annihilating polynomials by some explicit formulae, as for example those obtained using the Burchnall–Chaundy eliminant construction for the $q = 1$ case, i.e., for the classical Heisenberg algebra. In [LS03], a step in that direction was taken by offering a number of examples, all supporting the claim that the eliminant determinant method should work in the general case. However, no general proof for this was provided. The complete proof following the Burchnall–Chaundy approach in the case of q not a root of unity has been recently obtained [dJSS09], by showing that the determinant eliminant construction, properly adjusted for the q-deformed Heisenberg algebras, gives annihilating polynomials for commuting elements in the q-deformed Heisenberg algebra when q is not a root of unity, thus confirming the second part of the conjecture for these algebras. That proof was obtained by adapting the Burchnall–Chaundy eliminant determinant method of the case $q = 1$ of differential operators to the q-deformed case, after passing to a specific faithful representation of the q-deformed Heisenberg algebra on Laurent series and then performing a detailed analysis of the kernels of arbitrary operators in the image of this representation. While exploring the determinant eliminant construction of the annihilating polynomials, we also obtained some further information on such polynomials and some other results on dimensions and bases in the eigenspaces of the q-difference operators in the image of the chosen representation of the q-deformed Heisenberg algebra. In the case of q being a root of unity the algebraic dependence of commuting elements holds only over the center of the q-deformed Heisenberg algebra [HS00], and it is still unknown how to modify the eliminant determinant construction to yield annihilating polynomials for this case.

This third part of the conjecture remains widely open for most classes of non-commutative algebras and rings, even in the case of the usual Heisenberg algebra and differential operators, and with only a series of examples calculated for the Heisenberg algebra, q-Heisenberg algebra and some more general algebras, all supporting the conjecture. An interesting partial progress in this direction has been made in the recent work by Daniel Larsson on extension of Burchnall–Chaundy theory to Ore extensions [Lar08]. An algebraic proof has been given in this paper in case of Ore extension algebras. However it uses some general properties of resultants in Ore extension rings in order to deduce that Burchnall–Chaundy type determinant polynomial for commuting elements is annihilating. This proof does not involve explicit combinatorics of computations in Ore extensions based on their defining commutation relations, and thus still does not reveal explicit formulas for annihilating polynomials and does not add to understanding of why it again works for such broader class of algebras defined by this more general class of commutation relations.

In the present paper we make further progress in this third part of the conjecture by deducing an explicit general formula for the coefficients of annihilating polynomial obtained by the Burchnall–Chaundy type determinant construction in Ore extensions, and we also demonstrate how this formula can be used to compute the annihilating polynomials in several examples of commuting elements.

B.2 Extension of Burchnall–Chaundy theory to Ore extensions

In this section we describe an extension of Burchnall–Chaundy theory to Ore extensions developed by Daniel Larsson in [Lar08]. See also the paper by Li [Li98] for background.

Given a commutative ring $\mathbf{R}$ with an endomorphism σ and another additive map δ that satisfies the σ-Leibniz' rule:

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b, \ \forall a, b \in R,$$

we define a multiplication on $\mathbf{R}[x]$ by

$$xa = \sigma(a)x + \delta(a)$$

for all $a \in R$. As usual we write multiplication as juxtaposition. This is known as an Ore polynomial ring or an Ore extension of $\mathbf{R}$. The multiplication in an Ore polynomial ring is associative and distrubitive, but typically not commutative. There is a normal form in $\mathbf{R}[x]$ where we can write all elements uniquely as $\sum_{j\in\mathbb{N}} a_j x^j$, where $\mathbb{N} = \{n \in \mathbb{Z} \colon n \geq 0\}$. We can also write explicitly how the multiplication operation acts in this normal form. For this purpose we introduce, following [Lar08], the functions π_i^n defined as the sum of all possible compositions of i copies of σ and $n-i$ copies of δ. Thus for example $\pi_0^1(a) = \sigma(a)$ and $\pi_1^2 = \sigma(\delta(a)) + \delta(\sigma(a))$. Also it is convenient to define that $\pi_i^j(a) = 0$ if $i < 0$ or $i > j$ or $a = 0$. Using these functions we get the following expression for the multiplication

$$\left(\sum_{i=0}^{n} a_i x^i\right) \cdot \left(\sum_{j=0}^{m} b_j x^j\right) = \sum_{z=0}^{n+m} \sum_{i=0}^{n} \sum_{j=0}^{m} a_i \pi_{z-j}^{i}(b_j) x^z.$$

B.2.1 Determinant polynomial

Let M be an $r \times c$ matrix with entries in $\mathbf{R}$ where $r \leq c$. Then we define the determinant polynomial of M, denoted by $|M|$, as follows

$$|M| = \sum_{i=0}^{c-r} \det(M_i) x^i$$

where M_i is the $r \times r$ matrix whose first $r-1$ columns coincide with M and whose last column equals the $(c-i)th$ column of M.

Larsson and Li show how to rewrite the determinant polynomial as a determinant of a $r \times r$ matrix, where the entries in the last column are elements of $\mathbf{R}[x]$ and all other entries lie in $\mathbf{R}$. The expansion of such a determinant is defined by the convention that the element of $\mathbf{R}[x]$ is placed on the right-hand side of the entries in $\mathbf{R}$, in the usual expansion that defines the determinant.

The following two useful propositions easily follow from basic properties of the determinant.

Proposition B.2.1. *Let M be an $r \times c$ matrix, $r \leq c$, with determinant polynomial $|M|$. Let H_i denote the polynomial*

$$m_{i,1} x^{c-1} + \ldots + m_{i,r} x^{c-r} + \ldots + \ldots + m_{i,c}.$$

Then

$$|M| = \det \begin{pmatrix} m_{1,1} & \dots & m_{1,r-1} & H_1 \\ m_{2,1} & \dots & m_{2,r-1} & H_2 \\ . & \dots & . & . \\ m_{r,1} & \dots & m_{r,r-1} & H_r \end{pmatrix}$$

Now assume we have a sequence $A_1, A_2, \dots, A_r$ of polynomials in $\mathbf{R}[x]$ and let d be the maximum degree of the polynomials. We assume that $d \geq r$. We form an $r \times (d+1)$ matrix, denoted mat(A), whose entry in the ith row and jth column is the coefficient of x^{d+1-j} in A_i. The determinant polynomial of A is defined as $|\operatorname{mat}(A)|$ and denoted $|A|$.

Proposition B.2.2. *Let $A_1, A_2, \dots, A_r$ be a sequence of elements of $\mathbf{R}[x]$ of maximum degree d. Then*

$$|A| = \det \begin{pmatrix} a_{1,d} & \dots & a_{1,d-r+1} & A_1 \\ a_{2,d} & \dots & a_{2,d-r+1} & A_2 \\ \vdots & \ddots & \vdots & \vdots \\ a_{r,d} & \dots & a_{r,d-r+1} & A_r \end{pmatrix}$$

where $a_{i,j}$ is the coefficient of x^j in A_i.

B.2.2 The resultant

Let P and Q be two elements of $\mathbf{R}[x]$, of degree m and n respectively. We define their resultant, Res(P, Q), as the determinant polynomial of the sequence

$$P, xP, \dots, x^{n-1}P, Q, xQ, \dots, x^{m-1}Q.$$

It is easy to see that this will be a determinant of size $m + n$.

Proposition B.2.3. *For all P and Q in $\mathbf{R}[x]$ there exist element S, T in $\mathbf{R}[x]$ such that*

$$\operatorname{Res}(P, Q) = SP + TQ.$$

Proof. By the definition and the previous proposition we know that for some $m_{i,j} \in \mathbf{R}$

$$\operatorname{Res}(P, Q) = \det \begin{pmatrix} m_{n,1} & \dots & m_{n,n+m-1} & P \\ \vdots & \ddots & \vdots & \vdots \\ m_{2,1} & \dots & m_{2,n+m-1} & x^{n-2}P \\ m_{1,1} & \dots & m_{1,n+m-1} & x^{n-1}P \\ m_{n+m} & \dots & m_{n+m,n+m-1} & Q \\ \vdots & \ddots & \vdots & \vdots \\ m_{n+1} & \dots & m_{n+1,n+m-1} & x^{m-1}Q \end{pmatrix}.$$

If we expand this using the definition of the determinant we see that the theorem is true. □

Now let P and Q be commuting elements, and let s and t be variables that commute with each other and everything in $\mathbf{R}[x]$. (So they can be seen as elements in the larger algebra $\mathbf{R}[x][s,t]$.) Then $\text{Res}(P-s,Q-t) = S(P-s)+T(Q-t)$ for some S,T in $\mathbf{R}[x][s,t]$. We note that $\text{Res}(P-s,Q-t)$ is an element of $\mathbf{R}[s,t]$, a fact that follows from the definition. It is also true that $\text{Res}(P-s,Q-t)$ will depend polynomially on s and t, once again by the definition of the resultant. Finally if we formally replace s by P and t by Q the resultant becomes zero. Putting it all together we have proven

Theorem B.2.4. *If P and Q are commuting elements of $\mathbf{R}[x]$ then*

$$f(s,t) = \text{Res}(P-s,Q-t)$$

is a polynomial in two commuting variables such that $f(P,Q)=0$.

B.3 Recursive construction of the matrix of the resultant

We can also construct the matrix used for computing the resultant in a recursive way. For the first row we simply take the coefficients of $P-s$. Let $d_{i,j}$ denote the element in row i and column j of the resultant. If i or j is non-positive we agree to set $d_{i,j}=0$. Further we set $\sigma(s)=s$ and $\delta(s)=0$. This means that $xs = \sigma(s)x+\delta(s)$ in accordance with the σ-Leibniz rule. For $1<i\leq n$ and $n+1<i\leq n+m$ we then have the recursive formula

$$d_{i,j} = \delta(d_{i-1,j}) + \sigma(d_{i-1,j-1}).$$

This formula simply expresses the definition of the resultant and the σ-Leinbiz rule.

B.3.1 The Heisenberg algebra case

We illustrate the result in the classical Heisenberg (Weyl) algebra case.

Assume that P and Q are commuting elements in the Heisenberg algebra, of degree m and n respectively. We change notation and write $P=\sum_j a_j D^j$ and $Q=\sum_i b_i D^i$, where the a_i and b_i are polynomials over $\mathbb{C}$ in one variable, which we denote by y.

We want to give a more explicit expression for the resultant of P and Q. To this end we use Leibniz' rule. In the context of Heisenberg algebras this takes the form

$$D^n p = \sum_{k=0}^{n} \binom{n}{k} p^{(k)} D^{n-k}$$

for any polynomial p in y.

Thus

$$D^e P = D^e \sum_{j=0}^{m} a_j D^j = \sum_{j=0}^{m} (\sum_{k=0}^{n} \binom{e}{k} a_j^{(k)} D^{e-k}) D^j =$$

$$\sum_{j=0}^{m} (\sum_{l=0}^{n+m} \binom{e}{n+j-l} a_j^{(n+j-l)} D^l) = \sum_{l=0}^{n+m} \sum_{j=0}^{m} \binom{e}{n+j-l} a_j^{(n+j-l)} D^l.$$

We can now write down the expression for $\mathrm{Res}(P,Q)$. At place (e,f), when $e \leq n$, we get

$$\sum_{j=0}^{m} \binom{e-1}{e-1+j-n-m+f} a_j^{(e-1+j-n-m+f)},$$

and similarly for the last m rows.

The expression for the resultant becomes

$$\mathrm{Res}(P,Q) = \begin{vmatrix} a_{n+m} & \cdots & a_1 & P \\ a'_{n+m} + a_{n+m-1} & \cdots & a'_1 + a_0 & DP \\ \vdots & \ddots & \vdots & \vdots \\ a_m & \cdots & \sum_{j=0}^{m} \binom{n-1}{n+j-2} a_j^{(n+j-2)} & D^{n-1}P \\ b_{n+m} & \cdots & b_1 & Q \\ b'_{n+m} + b_{n+m-1} & \cdots & b'_1 + b_0 & DQ \\ \vdots & \ddots & \vdots & \vdots \\ b_n & \cdots & \sum_{i=0}^{n} \binom{m-1}{m+j-2} b_i^{(m+i-2)} & D^{m-1}Q \end{vmatrix}.$$

As an example take $P = yD$ and $Q = y^2D^2$. Then

$$\mathrm{Res}(P-s, Q-t) = \begin{vmatrix} 0 & y & yD-s \\ y & (1-s) & yD^2+(1-s)D \\ y^2 & 0 & y^2D^2-t \end{vmatrix}.$$

Expanding this we get

$$\mathrm{Res}(P-s,Q-t) = y^3(yD^2+(1-s)D) - y^2(y^2D^2-t) - y^2(1-s)(yD-s) =$$
$$y^4D^2 + y^3D - sy^3D - y^4D^2 + y^2t + sy^2 - y^3D - y^2s^2 + sy^3D = (t+s-s^2)y^2.$$

We note that $Q+P-P^2=0$.

B.4 Necessity of injectivity

We give an example to show what can happen if we do not require σ to be injective. Let k be a field and set $\mathbf{R}=k[y]$, the polynomials over k in the variable y.

Set $\delta(f)=0$ and $\sigma(f)=f(0)$, for all polynomials f. Then σ is an endomorphism and δ is a σ-derivation. So $\mathbf{R}[x]$ is well-defined.

Now set $P=yx^2$ and $Q=P$. Then P and Q commute. But the resultant we get is

$$\operatorname{Res}(P-s,Q-t)=\begin{vmatrix} 0 & y & 0 & -s \\ 0 & 0 & -s & 0 \\ 0 & y & 0 & -t \\ 0 & 0 & -t & 0 \end{vmatrix}.$$

This resultant is zero. We note that as long as the leading coefficient of both P and Q belong to the kernel of σ the resultant will be zero.

Theorem B.4.1. *Let P,Q be commuting elements in some Ore extension $\mathbf{R}[x;\sigma,\delta]$ of degrees m and n respectively in x. Suppose the highest coefficients a_m and b_n both belong to the kernel of σ. Then* $\operatorname{Res}(P-s,Q-t)=0$.

Proof. Form the matrix of the resultant according to the definition. Consider the first column of the matrix. The only potentially non-zero elements in it are the elements in row n and $n+m$ where the elements $\sigma^{n-1}(a_m)$ respectively $\sigma^{m-1}(b_n)$ appear. But since a_m and b_n belong to the kernel of σ these elements must also be zero and thus the determinant is zero. $\square$

B.5 The case $\delta=0$

When we set $\delta(r)=0$, for all $r\in\mathbf{R}$ the formulae for the elements in the determinant simplify. We transpose the columns of the resultant to simplify the formulae further.

Let $D_{i,j}$ denote the element in place i,j of $\operatorname{Res}(P-s,Q-t)$, where $P=\sum_{i=0}^{m}a_ix^i$ has degree m and $Q=\sum_{j=0}^{n}b_jx^j$ has degree n. Multiplying by x repeatedly and using the commutation rule $xa=\sigma(a)x$ we find that if $i\leq n$ then

$$D_{i,j}=\sigma^{i-1}(a_{j-i})-\Delta_{i,j}s$$

where Δ is the Kronecker delta-function. If $i>n$ then

$$D_{i,j}=\sigma^{i-n-1}(b_{j-i-n})-\Delta_{i-n,j}t.$$

The formula is in some sense simpler than it looks. The first row simply has the same coefficients as $P - s$. Then you apply σ to the coefficients and shift them one step to the left to get the second row. (Where we use the rule $\sigma(s) = s$.) This gives us the first n rows. For the last m rows we do the same with $Q - t$. We note that it is a generalization of the classical resultant for polynomials which we recover if we set $\sigma \equiv 1$.

We can expand the determinant along the last column and find that $\mathrm{Res}(P - s, Q - t)$ equals

$$(-1)^{m+n+1} \begin{vmatrix} \sigma(a_{n+m-1}) & \dots & \sigma(a_1) & \sigma(a_0) - s \\ \sigma^2(a_{n+m-2}) & \dots & \sigma^2(a_0) - s & 0 \\ \vdots & \ddots & \vdots & \vdots \\ b_{n+m} & \dots & b_2 & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ \sigma^{m-1}(b_n) & \sigma^{m-1}(b_{n-1}) & \dots & 0 \end{vmatrix} (a_0 - s) +$$

$$+ (-1)^{m+2n+1} \begin{vmatrix} a_{n+m} & \dots & a_2 & a_1 \\ \sigma(a_{n+m-1}) & \dots & \sigma(a_1) & \sigma(a_0) - s \\ \vdots & \ddots & \vdots & \vdots \\ \sigma(b_{n+m-1}) & \dots & \sigma(b_1) & \sigma(b_0) - t \\ \vdots & \ddots & \vdots & \vdots \\ \sigma^{m-1}(b_n) & \sigma^{m-1}(b_{n-1}) & \dots & 0 \end{vmatrix} (b_0 - t).$$

Now in every row there is only one element that contains s or t. This helps us to determine the coefficient of s^n and t^m which are the highest possible powers of s and t. Start with s. In the preceding expression for $\mathrm{Res}(P - s, Q - t)$ it is easy to see that we need only expand the first determinant. Further, when we have expanded this determinant we see that there is only one relevant subdeterminant and so on. It help with the presentation of the result to make a new definition. Set p_k to be the coefficient of x^k in $P - s$. (So p_k equals b_k plus possibly a $(-s)$-term.) We set $\sigma(b_k - s) = \sigma(b_k) - s$.

Doing this we get the cofficient of s^n as

$$(-1)^{m+n+1} \begin{vmatrix} 0 & \dots & p_n \\ \vdots & \ddots & \vdots \\ \sigma^{m-1}(p_n) & \dots & \sigma^{m-1}(p_{n-m}) \end{vmatrix}.$$

Similarly we get (with q_k the coefficient of x^k in $Q - t$) that the coefficient of

t^m is

$$(-1)^{m+2n+1}\begin{vmatrix} 0 & \dots & q_m \\ \vdots & \ddots & \vdots \\ \sigma^{n-1}(q_m) & \dots & \sigma^{n-1}(q_{m-n}) \end{vmatrix}.$$

These matrices are of rather special form ("triangular with respect to the anti-diagonal") so we can compute them explicitly. The coefficient of s^n is equal to

$$(-1)^{n+1} b_n \sigma(b_n) \dots \sigma^{m-1}(b_n)$$

and similarily the coefficient of t^m equals

$$(-1)^{m+n+1} a_m \sigma(a_m) \dots \sigma^{n-1}(a_m).$$

B.5.1 The leading coefficients in general

We can extend our computation of the leading coefficients to the general case. So let P and Q be two Ore-polynomials of degree m and n respectively. If we form the determinant $\operatorname{Res}(P-s, Q-t)$ we notice that all elements containing s lie on the anti-diagonal. We also note that there are n such elements so the highest possible power of s that can occur is s^n.

We have the following expression for the resultant

$$\operatorname{Res}(P-s, Q-t) = \sum_{\alpha} p(\alpha) \prod_{i=1}^{m+n} d_{i,\alpha(i)}$$

where α runs over all the permutations of $m+n$ elements, $p(\alpha)$ denotes the sign of the permutation and $d_{i,j}$ denotes the element in place (i,j) of the resultant.

Now if $i > n$ and $i < m+n-j$ then $d_{i,j} = 0$. This helps us compute the coefficient of s^n. The only element of the sum given earlier that contains s^n is the one that comes from multiplication of all the elements on the anti-diagonal. Thus up to some sign we get that s^n has the coefficient $\prod_{k=0}^{m-1} \sigma^k(b_n)$, where the exponent denotes functional iteration. Similarly we get that the coefficient of t^m is $\prod_{k=0}^{n-1} \sigma^k(a_m)$. As a corollary we obtain

Theorem B.5.1. $\operatorname{Res}(P-s, Q-t)$ *is non-zero if R is an integral domain and σ is injective.*

Proof. We have just shown that s^n has the coefficient $\prod_{k=0}^{m-1} \sigma^k(b_n)$. b_n is non-zero and if σ is injective so must all the $\sigma^k(b_n)$ be. Finally, if R is an integral domain the product of non-zero elements is non-zero. $\square$

The reason that the formulae for the leading coefficients remain the same in the general case as when $\delta = 0$ is that the highest term when you multiply a Ore-polynomial by x from the left is determined by σ alone.

B.6 The lower-order coefficients

One of our main goals in this paper is to obtain formula for calculating the annihilating polynomial. In other words we want to derive a formula not only for the highest coefficient but also for the lower-order coefficients of Res$(P-s, Q-t)$. To do this we start with formulating a formula for the k-derivative of a determinant in our general context of Ore extension rings. This formula has been obtained in the special case of determinants of ordinary enough times differentiable real valued functions in [CH64].

Theorem B.6.1. *Let A be a square matrix of size m with entries in $R[t]$ for some commutative ring R. We denote the determinant of A by $|A|$. Then*

$$\frac{\partial^k |A|}{\partial t^k} = \sum_{k_1+k_2+\ldots+k_m=k} \frac{k!}{k_1!k_2!\ldots k_m!} \begin{vmatrix} a_{11}^{(k_1)} & a_{12}^{(k_1)} & \cdots & a_{1m}^{(k_1)} \\ a_{21}^{(k_2)} & a_{22}^{(k_2)} & \cdots & a_{2m}^{(k_2)} \\ \ddots & \ddots & \ddots & \ddots \\ a_{m1}^{(k_m)} & a_{m2}^{(k_m)} & \cdots & a_{mm}^{(k_m)} \end{vmatrix}$$

with the sum taken over all combinations of non-negative integers $k_1, k_2 \ldots k_m$ that sum to k. Multiplication by a positive integer here denotes the obvious and usual way that $\mathbb{N}$ acts on $R[t]$.

Proof. We give a proof by induction on both k and m. The formula is clearly true when $k=0$ for any m. It is also easy to check that it is true when $k=1$ and $m=2$. So we assume it is true for $k=1$ and m and try to prove it is true for $m+1$ and $k=1$. Let A be any $m+1$-sized square matrix. Then

$$\frac{\partial |A|}{\partial t} = \frac{\partial}{\partial t}\Big(\sum_{i=1}^{m+1}(-1)^{i+1} a_{1i}|A_{1i}|\Big) = \sum_{i=1}^{m+1} a_{1i}^{'}|A_{1i}| + \sum_{i=1}^{m+1} a_{1i}|A_{1i}^{'}| =$$

$$\sum_{i=1}^{m+1} \begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m+1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m+1} \\ \ddots & \ddots & \ddots & \ddots \\ a_{i,1}^{'} & a_{i,2}^{'} & \cdots & a_{i,m+1}^{'} \\ \ddots & \ddots & \ddots & \ddots \\ a_{m+1,1} & a_{m+1,2} & \cdots & a_{m+1,m+1} \end{vmatrix}.$$

This proves that the claim is valid for $k=1$ and m arbitrary. So we now assume the theorem is true for k and try to prove it for $k+1$. If A is a square matrix of size

m, then

$$\frac{\partial^{k+1}|A|}{\partial t^{k+1}} = \frac{\partial}{\partial t}(\frac{\partial^k |A|}{\partial t^k}) =$$

$$\frac{\partial}{\partial t}\sum_{k_1+k_2+\ldots+k_m=k}\frac{k!}{k_1!k_2!\ldots k_m!}\begin{vmatrix} a_{1,1}^{(k_1)} & a_{1,2}^{(k_1)} & \ldots & a_{1,m}^{(k_1)} \\ a_{2,1}^{(k_2)} & a_{2,2}^{(k_2)} & \ldots & a_{2,m}^{(k_2)} \\ \ddots & \ddots & \ddots & \ddots \\ a_{m,1}^{(k_m)} & a_{m,2}^{(k_m)} & \ldots & a_{m,m}^{(k_m)} \end{vmatrix} =$$

$$= \sum_{k_1+k_2+\ldots+k_m=k}\frac{k!}{k_1!k_2!\ldots k_m!}\sum_{i=1}^{m}\begin{vmatrix} a_{1,1}^{(k_1)} & a_{1,2}^{(k_1)} & \ldots & a_{1,m}^{(k_1)} \\ a_{2,1}^{(k_2)} & a_{2,2}^{(k_2)} & \ldots & a_{2,m}^{(k_2)} \\ \ddots & \ddots & \ddots & \ddots \\ a_{i,1}^{(k_i+1)} & a_{i,2}^{(k_i+1)} & \ldots & a_{i,m}^{(k_i+1)} \\ \ddots & \ddots & \ddots & \ddots \\ a_{m,1}^{(k_m)} & a_{m,2}^{(k_m)} & \ldots & a_{m,m}^{(k_m)} \end{vmatrix} =$$

$$= \sum_{k_1+k_2+\ldots+k_m=k+1}\frac{(k+1)!}{k_1!k_2!\ldots k_m!}\begin{vmatrix} a_{1,1}^{(k_1)} & a_{1,2}^{(k_1)} & \ldots & a_{1,m}^{(k_1)} \\ a_{2,1}^{(k_2)} & a_{2,2}^{(k_2)} & \ldots & a_{2,m}^{(k_2)} \\ \ddots & \ddots & \ddots & \ddots \\ a_{m,1}^{(k_m)} & a_{m,2}^{(k_m)} & \ldots & a_{m,m}^{(k_m)} \end{vmatrix}$$

where we have used a well-known property of multinomial coefficients. □

We now apply Proposition B.6.1 to get a formula for the lower order coefficients of Res$(P-s, Q-t)$. We note that any time we differentiate a row of the determinant twice we end up with zero and that only the last m rows contains any t-terms. Set $A = \text{Res}(P-s, Q-t)$ and let $\mathbf{a}_i$ denote the vector consisting of the elements in row i of A.

We find that

$$\frac{\partial^k |A|}{\partial t^k} = \sum_{\substack{k_1+k_2+\ldots+k_m=k \\ 0\le k_i\le 1}} k! \begin{vmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \\ \frac{\partial^{k_1}\mathbf{a}_{n+1}}{\partial t^{k_1}} \\ \frac{\partial^{k_2}\mathbf{a}_{n+2}}{\partial t^{k_2}} \\ \vdots \\ \frac{\partial^{k_m}\mathbf{a}_{n+m}}{\partial t^{k_m}} \end{vmatrix}.$$

We differentiate this expression with respect to s, and get for non-negative integers k, q,

$$\frac{\partial^{k+q}|A|}{\partial s^q \partial t^k} = \sum_{\substack{k_1+k_2+\ldots+k_m=k \\ q_1+q_2+\ldots+q_n=q \\ 0\le k_i \le 1 \\ 0 \le q_i \le 1}} k!q! \begin{vmatrix} \frac{\partial^{q_1}\mathbf{a}_1}{\partial s^{q_1}} \\ \vdots \\ \frac{\partial^{q_n}\mathbf{a}_n}{\partial s^{q_n}} \\ \frac{\partial^{k_1}\mathbf{a}_{n+1}}{\partial t^{k_1}} \\ \frac{\partial^{k_2}\mathbf{a}_{n+2}}{\partial t^{k_2}} \\ \vdots \\ \frac{\partial^{k_m}\mathbf{a}_{n+m}}{\partial t^{k_m}} \end{vmatrix}.$$

This gives almost immediately an expression for the coefficient of $s^q t^k$. (At least if $\mathbf{R}$ is torsion-free, seen as a module over $\mathbb{Z}$.)

The coefficient in front of $s^q t^k$ is the same as the derivative of $\operatorname{Res}(P-s, Q-t)$ with respect to s and t taken respectively q and k times and then evaluated at $t=0$ and $s=0$. Thus we get the following theorem describing the formula for all coefficients of $\operatorname{Res}(P-s, Q-t)$.

Theorem B.6.2. *Let $\mathbf{R}$ be torsion-free as a module over $\mathbb{Z}$ with the natural action of $\mathbb{Z}$. Let P and Q be two commuting elements of $\mathbf{R}[x]$. Denote the coefficient of $s^q t^k$ in $\operatorname{Res}(P-s, Q-t)$ by $c_{q,k}$. Denote row number i in the resultant matrix by $\mathbf{a}_i$.*

Then

$$c_{q,k} = \sum_{\substack{k_1+k_2+\ldots+k_m=k \\ q_1+q_2+\ldots+q_n=q \\ 0\le k_i \le 1 \\ 0 \le q_i \le 1}} \begin{vmatrix} \frac{\partial^{q_1}\mathbf{a}_1}{\partial s^{q_1}}\big|_{s=0,t=0} \\ \vdots \\ \frac{\partial^{q_n}\mathbf{a}_n}{\partial s^{q_n}}\big|_{s=0,t=0} \\ \frac{\partial^{k_1}\mathbf{a}_{n+1}}{\partial t^{k_1}}\big|_{s=0,t=0} \\ \frac{\partial^{k_2}\mathbf{a}_{n+2}}{\partial t^{k_2}}\big|_{s=0,t=0} \\ \vdots \\ \frac{\partial^{k_m}\mathbf{a}_{n+m}}{\partial t^{k_m}}\big|_{s=0,t=0} \end{vmatrix}.$$

We will illustrate now the preceding theorem with several examples.

Example 1. We choose the Heisenberg algebra setting we considered earlier for our example. So we set $\mathbf{R} = \mathbb{C}[y]$ and denote the Ore extension by $\mathbf{R}[D]$.

Set $P = yD$ and $Q = y^2D^2$. We have already computed that $\operatorname{Res}(P-s, Q-t) = (t+s-s^2)y^2$.

We want to check that we get the same result using the theorem. We compute

$$c_{0,0} = \begin{vmatrix} 0 & y & 0 \\ y & 1 & 0 \\ y^2 & 0 & 0 \end{vmatrix} = 0,$$

$$c_{0,1} = \begin{vmatrix} 0 & y & 0 \\ y & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = y^2,$$

$$c_{1,0} = \begin{vmatrix} 0 & 0 & -1 \\ y & 1 & 0 \\ y^2 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & y & 0 \\ 0 & -1 & 0 \\ y^2 & 0 & 0 \end{vmatrix} = y^2,$$

$$c_{1,1} = \begin{vmatrix} 0 & 0 & -1 \\ y & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} + \begin{vmatrix} 0 & y & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 0,$$

$$c_{2,0} = \begin{vmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ y^2 & 0 & 0 \end{vmatrix} = -y^2.$$

Thus we get the same result as before.

Example 2. We now give an example using the quantum plane. This can be defined by setting $\mathbf{R} = \mathbb{C}[y]$, setting $\sigma(y) = qy$ for some constant q and $\sigma(a) = a$ for all $a \in \mathbb{C}$ and finally defining $\delta \equiv 0$. This defines an Ore extension that is known as the quantum plane.

In this algebra $P = yx$ and $Q = (yx)^2 = qy^2x^2$ commute. We compute $\operatorname{Res}(P - s, Q - t)$ in two ways. Using the definition we get

$$\operatorname{Res}(P - s, Q - t) = \begin{vmatrix} 0 & y & -s \\ qy & -s & 0 \\ qy^2 & 0 & -t \end{vmatrix} = (t - s^2)qy^2.$$

We now compute the same thing using Theorem B.6.2:

$$c_{0,0} = \begin{vmatrix} 0 & y & 0 \\ qy & 0 & 0 \\ qy^2 & 0 & 0 \end{vmatrix} = 0,$$

$$c_{0,1} = \begin{vmatrix} 0 & y & 0 \\ qy & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} = qy^2,$$

$$c_{1,1} = \begin{vmatrix} 0 & 0 & -1 \\ qy & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} + \begin{vmatrix} 0 & y & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 0,$$

$$c_{1,0} = \begin{vmatrix} 0 & 0 & -1 \\ qy & 0 & 0 \\ qy^2 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & y & 0 \\ 0 & -1 & 0 \\ qy^2 & 0 & 0 \end{vmatrix} = 0,$$

$$c_{2,0} = \begin{vmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ qy^2 & 0 & -0 \end{vmatrix} = -qy^2,$$

$$c_{2,1} = \begin{vmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 0.$$

Example 3. We now describe a larger example, using the q-Heisenberg algebra. We generate this as an Ore extension by setting $\mathbf{R} = \mathbb{C}[y]$, $\sigma(y) = qy$ and $\delta(y) = 1$ for some $q \in \mathbb{C}$.

Set $P = Q = (yx)^2 = qy^2x^2 + yx$. Then

$$\operatorname{Res}(P-s, Q-t) = \begin{vmatrix} 0 & qy^2 & y & -s \\ q^3y^2 & (2q+q^2)y & 1-s & 0 \\ 0 & qy^2 & y & -t \\ q^3y^2 & (2q+q^2)y & 1-t & 0 \end{vmatrix}.$$

On computing the determinant we find that

$$\operatorname{Res}(P-s, Q-t) = q^4y^4t^2 - 2q^4y^4st + q^4y^4s^2.$$

We now wish to compute the resultant using our formula. With the same notation as in the previous examples we get

$$c_{0,0} = \begin{vmatrix} 0 & qy^2 & y & 0 \\ q^3y^2 & (2q+q^2)y & 1 & 0 \\ 0 & qy^2 & y & 0 \\ q^3y^2 & (2q+q^2)y & 1 & 0 \end{vmatrix} = 0,$$

$$c_{0,1} = \begin{vmatrix} 0 & qy^2 & y & 0 \\ q^3y^2 & (2q+q^2)y & 1 & 0 \\ 0 & 0 & 0 & -1 \\ q^3y^2 & (2q+q^2)y & 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & qy^2 & y & 0 \\ q^3y^2 & (2q+q^2)y & 1 & 0 \\ 0 & qy^2 & y & 0 \\ 0 & 0 & -1 & 0 \end{vmatrix} = 0+0 = 0,$$

$$c_{1,0}=\begin{vmatrix}0&0&0&-1\\q^3y^2&(2q+q^2)y&1&0\\0&qy^2&y&0\\q^3y^2&(2q+q^2)y&1&0\end{vmatrix}+\begin{vmatrix}0&qy^2&y&0\\0&0&-1&0\\0&qy^2&y&0\\q^3y^2&(2q+q^2)y&1&0\end{vmatrix}=0+0=0,$$

$$c_{1,1}=\begin{vmatrix}0&0&0&-1\\q^3y^2&(2q+q^2)y&1&0\\0&0&0&-1\\q^3y^2&(2q+q^2)y&1&0\end{vmatrix}+\begin{vmatrix}0&0&0&-1\\q^3y^2&(2q+q^2)y&1&0\\0&qy^2&y&0\\0&0&-1&0\end{vmatrix}+$$
$$\begin{vmatrix}0&qy^2&y&0\\0&0&-1&0\\0&0&0&-1\\q^3y^2&(2q+q^2)y&1&0\end{vmatrix}+\begin{vmatrix}0&qy^2&y&0\\0&0&-1&0\\0&qy^2&y&0\\0&0&-1&0\end{vmatrix}=$$
$$=0-q^4y^4-q^4y^4+0=-2q^4y^4,$$

$$c_{2,0}=\begin{vmatrix}0&0&0&-1\\0&0&-1&0\\0&qy^2&y&0\\q^3y^2&(2q+q^2)y&1&0\end{vmatrix}=q^4y^4,$$

$$c_{0,2}=\begin{vmatrix}0&qy^2&1&0\\q^3y^2&(2q+q^2)y&1&0\\0&0&0&-1\\0&0&-1&0\end{vmatrix}=q^4y^4,$$

$$c_{2,1}=\begin{vmatrix}0&0&0&-1\\0&0&-1&0\\0&0&0&-1\\q^3y^2&(2q+q^2)y&1&0\end{vmatrix}+\begin{vmatrix}0&0&0&-1\\0&0&-1&0\\0&qy^2&y&0\\0&0&-1&0\end{vmatrix}=0+0=0,$$

$$c_{1,2}=\begin{vmatrix}0&0&0&-1\\q^3y^2&(2q+q^2)y&1&0\\0&0&0&-1\\0&0&-1&0\end{vmatrix}+\begin{vmatrix}0&qy^2&y&0\\0&0&-1&0\\0&0&0&-1\\0&0&-1&0\end{vmatrix}=0+0=0,$$

$$c_{2,2}=\begin{vmatrix}0&0&0&-1\\0&0&-1&0\\0&0&0&-1\\0&0&-1&0\end{vmatrix}=0.$$

As expected we get the same result using both calculations.

Example 4. We give a second example in the q-Heisenberg algebra. Take $P = (y^2x)^2 + y^2x$ and $Q = y^2x$. Then $P = q^2y^4x^2 + (y^3 + qy^3 + y^2)x$ and

$$\operatorname{Res}(P-s, Q-t) = \begin{vmatrix} y^4 & qy^3+y^3+y^2 & -s \\ 0 & y^2 & -t \\ q^2y^2 & qy+y-t & 0 \end{vmatrix} = -q^2y^4t^2 - q^2y^4t + q^2y^4s.$$

As before we also compute the coefficients directly using our formula:

$$c_{0,1} = \begin{vmatrix} y^4 & qy^3+y^3+y^2 & 0 \\ 0 & y^2 & 0 \\ 0 & -1 & 0 \end{vmatrix} + \begin{vmatrix} y^4 & qy^3+y^3+y^2 & 0 \\ 0 & 0 & -1 \\ q^2y^2 & qy+y & 0 \end{vmatrix} = -q^2y^4,$$

$$c_{0,2} = \begin{vmatrix} y^4 & qy^3+y^3+y^2 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{vmatrix} = -q^2y^4,$$

$$c_{1,0} = \begin{vmatrix} 0 & 0 & -1 \\ 0 & y^2 & 0 \\ q^2y^2 & (1+q)y & 0 \end{vmatrix} = q^2y^4,$$

$$c_{1,1} = \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ q^2y^2 & (1+q)y & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & -1 \\ 0 & y^2 & 0 \\ 0 & -1 & 0 \end{vmatrix} = 0.$$

B.7 Annihilating polynomials for elements in a specific commutative subalgebra

Suppose that $\mathbf{R} = S[y]$ for some commutative ring S, by which we mean simply the ordinary polynomial ring over S. In this case it is natural to suppose that σ and δ are S-linear functions. The important case of the Heisenberg algebra is one such example as are the more general q-deformed Heisenberg algebras. Under the preceding assumptions we can specify σ and δ completely by just giving $\sigma(y)$, respectively $\delta(y)$.

The elements of the form $(yx)^n$, for natural numbers n, generate a subalgebra of $S[y][x;\sigma,\delta]$. It will be a commutative subalgebra, as is easy to see. If we assume that P and Q come from this subalgebra we can give a formula for the elements in the determinant, using the function π_i^n introduced in section B.2. We first have that

$$x^n(yx)^m = \sum_{i_1=1}^{n+m} \sum_{i_2=1}^{n+m-1} \cdots \sum_{i_m=1}^{n+1} \pi^n_{i_m-1}(y)\pi^{i_m}_{i_{m-1}-1}(y)\ldots\pi^{i_2}_{i_1}(y)x^{i_1},$$

which is proven by induction. If $P = \sum_{i=0}^{m} a_i(yx)^m$ and $Q = \sum_{j=0}^{n} b_j(yx)^n$ we can from this get an expression for the elements of the matrix whose determinat is $\mathrm{Res}(P-s, Q-t)$. Setting $e_{i,j}$ to be the element of row i and column $n+m-j$ and letting $d_{i,j}$ denote the Kronecker delta-function we have that

$$d_{i,j}s + e_{i,j} = \sum_{k=0}^{m} a_k \sum_{i_2=1}^{k+i-2} \cdots \sum_{i_k=1}^{i} \pi_{i_k-1}^{i-1}(y)\ldots\pi_{j-2}^{i_2}(y)$$

if $j \le n$, and similarily if $j > n$

$$d_{i,j+n}t + e_{i,j} = \sum_{k=0}^{n} b_k \sum_{i_2=1}^{k+i-n-2} \cdots \sum_{i_k=1}^{i-n} \pi_{i_k-1}^{i-n-1}(y)\ldots\pi_{j-2}^{i_2}(y).$$

Assume now that $\mathbf{R} = k[y]$ for some field k, and that $\delta \equiv 0$.

Set $P = \sum_{i=0}^{m} a_i(yx)^m$ and $Q = \sum_{j=0}^{n} b_j(yx)^n$. In this special case we can prove the following

Theorem B.7.1. *Let $\mathbf{R} = k[y]$ for some field k. Let σ be any k-endomorphism of R and assume that δ is identically zero. If $P = \sum_{i=0}^{m} a_i(yx)^m$ and $Q = \sum_{j=0}^{n} b_j(yx)^n$ then P and Q commute and $\mathrm{Res}(P-s, Q-t) = G(s,t)\prod_{j=0}^{n-1}\prod_{i=0}^{j+m-1}\sigma^i(y)$ where $G(s,t)$ does not contain any non-zero power of y.*

Proof. Denote the element in row i and column $n+m-j+1$ by $r_{i,j}$. Then it can be seen that

$$r_{i,j} = C_{i,j}\prod_{k=1}^{j-1}(\sigma^{i+k-2}(y))$$

if $i \le n$, where $C_{i,j}$ does not depend on y and we interpret the empty product as 1. If $i > n$ we get that

$$r_{i,j} = C_{i,j}\prod_{k=1}^{j+n-1}\sigma^{i+k-2-n}(y).$$

If $i \le n$ and $i > j$ we must have that $C_{i,j} = 0$. Similarly if $i > n$ and $i - n > j$ we have that $C_{i,j} = 0$.

We know that

$$\mathrm{Res}(P-s, Q-t) = \sum_{\alpha \in S_{n+m}} \mathrm{sign}(\alpha)\prod_{i=1}^{m+n} r_{i,\alpha(n+m-i+1)}$$

where the sum is over all possible permutations of $n+m$ elements. We let γ denote the permutation that maps $n+m-j+1$ to j for all $1 \le j \le n+m$. Then we can write

$$\mathrm{Res}(P-s,Q-t)=\sum_{\alpha\in S_{n+m}}\mathrm{sign}(\alpha\circ\gamma)\prod_{i=1}^{m+n}r_{i,\alpha((\gamma(n+m-i+1))}=\sum_{\alpha\in S_{n+m}}\mathrm{sign}(\alpha\circ\gamma)\prod_{i=1}^{m+n}r_{i,\alpha(i)}.$$

Now we compute one term of this expansion, ignoring the sign:

$$\begin{aligned}\prod_{i=1}^{n+m}r_{i,\alpha(i)}&=\prod_{i=1}^{n}C_{i,\alpha(i)}\left(\prod_{k=1}^{\alpha(i)-i}\sigma^{k+i-2}(y)\right)\prod_{i=n+1}^{n+m}C_{i,\alpha(i)}\left(\prod_{k=1}^{\alpha(i)+n-i}\sigma^{i+k-2-n}(y)\right)\\&=\left(\prod_{i=1}^{n+m}C_{i,\alpha(i)}\right)\prod_{i=1}^{n}\left(\prod_{k=1}^{\alpha(i)-i}\sigma^{k+i-2}(y)\right)\prod_{i=n+1}^{n+m}\left(\prod_{k=1}^{\alpha(i)+n-i}\sigma^{i+k-2-n}(y)\right).\end{aligned}$$

We now count the number of times a factor $\sigma^p(y)$ appears in this expression. It can be seen that for any natural number p, $\sigma^p(y)$ occurs

$$S=|A_1\cap C_1|+|A_2\cap C_1|$$

number of times in the factorization, where

$$\begin{aligned}A_1&=\{i|\ 1\le i\le n \text{ and } i-1\le p\},\\A_2&=\{i|\ n+1\le i\le n+m \text{ and } i\le p+n+1\},\\C_1&=\{i|\ \alpha(i)\ge p+2\}\end{aligned}$$

and $|F|$ denotes the number of elements of set F. We are not interested in those α that make any $C_{i,\alpha(i)}$ So we assume that $\alpha(i)\ge i$ if $i\le n$ and $\alpha(i)\ge i-n$ otherwise.

Define

$$\begin{aligned}B_1&=\{i|\ 1\le i\le n \text{ and } p\le i-2\},\\B_2&=\{i|\ n+1\le i\le n+m \text{ and } p+n+2\le i\},\\C_2&=\{i|\ \alpha(i)\le p+1\}.\end{aligned}$$

We see that the A_i and B_i form one partition of the set $\{1,\dots n+m\}$ and the C_i another. We further note that if $i\in C_2\cap B_1$ we must have that $i>\alpha(i)$ and $i\le n$ which we have assumed can not happen. Similarly $C_2\cap B_2$ is empty.

We use this to rewrite S as

$$S=|A_1\cap C_1|+|A_2\cap C_1|-|B_1\cap C_2|-|B_2\cap C_2|.$$

Setting $A=A_1\cup A_2$ and $B=B_1\cup B_2$ we then compute S as follows

$$\begin{aligned}S=|A\cap C_1|-|B\cap C_2|=|A\cap C_1|-|A^c\cap C_1^c|=|A\cap C_1|-|(A\cup C_1)^c|=\\|A\cap C_1|-(m+n-|A\cup C_1|)=|A\cap C_1|+|A\cup C_1|-m-n=\\|A\cap C_1|+|A|+|C_1|-|A\cap C_1|-m-n=|A|+|C_1|-m-n\end{aligned}$$

where we have used the principle of inclusion-exlusion in the last step. This result is independent of the permuation α which almost proves the proposition. Simply see that if we multiply all the elements along the anti-diagonal we get the claimed formula to finish the proof.

$\square$

As a special case of the theorem we get the following

Corollary B.7.2. *Assume that* $\mathbf{R} = k[y]$. *If P,Q are linear combinations of elements of the form $(yx)^i$,where P is of degree m and Q of degree n in x, δ is identically zero and $\sigma(y) = qy^p$, where q is an element of the field k and p is a positive integer, then P and Q commute and*

$$\mathrm{Res}(P-s, Q-t) = G(s,t) \cdot y^{(\sum_{i=0}^{n-1} p^i)(\sum_{i=0}^{m-1} p^i)}.$$

For general commuting P and Q we can write the resultant as

$$\mathrm{Res}(P-s, Q-t) = \sum G_i(s,t) y^i,$$

where the G_i do not contain any power of y. It would be interesting to find conditions guaranteeing that $G_i(P,Q) = 0$ for all i. This will not be true in general but if $\mathbf{R}[x] = S[y][x;\sigma,0]$ where S is an integral domain and σ is injective and P and Q are of the form considered in this section then $G_i(P,Q) = 0$ for all i, since all the G_i will in fact be equal.

Example 5. We include an example to illustrate Corollary B.7.2. Set $P = (yx)^2$ and $Q = yx$. Then $P = qy^{k+1}x^2$ and $xQ = qy^k x^2$ so

$$\mathrm{Res}(P-s, Q-t) = \begin{vmatrix} qy^{k+1} & 0 & -s \\ 0 & y & -t \\ qy^k & -t & 0 \end{vmatrix} = q(s^2 - t) \cdot y^{k+1}.$$

With the notation of the theorem $G(s,t) = q(s - t^2)$ in this case.

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Paper C

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Paper C

Maximal commutative subrings and simplicity of Ore extensions

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Abstract. *The aim of this paper is to describe necessary and sufficient conditions for simplicity of Ore extension rings, with an emphasis on differential polynomial rings. We show that a differential polynomial ring, $R[x; \mathrm{id}_R, \delta]$, is simple if and only if its center is a field and R is δ-simple. When R is commutative we note that the centralizer of R in $R[x; \sigma, \delta]$ is a maximal commutative subring containing R and, in the case when $\sigma = \mathrm{id}_R$, we show that it intersects every non-zero ideal of $R[x; \mathrm{id}_R, \delta]$ non-trivially. Using this we show that if R is δ-simple and maximal commutative in $R[x; \mathrm{id}_R, \delta]$, then $R[x; \mathrm{id}_R, \delta]$ is simple. We also show that under some conditions on R the converse holds.*

C.1 Introduction

A topic of interest in the field of operator algebras is the connection between properties of dynamical systems and algebraic properties of crossed products associated to them. More specifically the question when a certain canonical subalgebra is maximal commutative and has the ideal intersection property, i.e. each non-zero ideal of the algebra intersects the subalgebra non-trivially. For a topological dynamical systems (X, α) one may define a crossed product C*-algebra $C(X) \rtimes_{\tilde{\alpha}} \mathbb{Z}$ where $\tilde{\alpha}$ is an automorphism of $C(X)$ induced by α. It turns out that the property known as topological freeness of the dynamical system is equivalent to $C(X)$ being a maximal commutative subalgebra of $C(X) \rtimes_{\tilde{\alpha}} \mathbb{Z}$ and also equivalent to the condition that every non-trivial closed ideal has a non-zero intersection with $C(X)$. An excellent reference for this correspondence is [Tom87]. For analogues, extensions and applications of this theory in the study of dynamical systems, harmonic analysis, quantum field theory, string theory, integrable systems, fractals and wavelets see [AJ94, CS09, DJ07, DJS12, EV06, Mac, OS99, Pop82, SS08, Tom87].

For any class of graded rings, including gradings given by semigroups or even filtered rings (e.g. Ore extensions), it makes sense to ask whether the ideal intersection property is related to maximal commutativity of the degree zero component. For crossed product-like structures, where one has a natural action, it further makes sense to ask how the above mentioned properties of the degree zero component are related to properties of the action.

These questions have been considered recently for algebraic crossed products and Banach algebra crossed products, both in the traditional context of crossed products by groups as well as generalizations to graded rings, crossed products by groupoids and general categories in [dJST12, LÖ12, Öin09, ÖL10, ÖL12, ÖS08a, ÖS08b, ÖS09a, ÖS09b, ÖSTV09, SSdJ07, SSdJ09a, SSdJ09b, ST09].

Ore extensions constitute an important class of rings, appearing in extensions of differential calculus, in non-commutative geometry, in quantum groups and algebras and as a uniting framework for many algebras appearing in physics and engineering models. An Ore extension of R is an overring with a generator x satisfying $xr = \sigma(r)x + \delta(r)$ for $r \in R$ for some endomorphism σ and a σ-derivation δ.

This paper aims at studying the centralizer of the coefficient subring for an Ore extension, investigating conditions for the simplicity of Ore extensions and demonstrating the connections between these two topics.

Necessary and sufficient conditions for a differential polynomial ring (an Ore extension with $\sigma = \mathrm{id}_R$) to be simple have been studied before. An early paper by Jacobson [Jac37] studies the case when R is a division ring of characteristic zero. His results are generalized in the textbook [CF75, Chapter 3] in which Cozzens and Faith prove that if R is a $\mathbb{Q}$-algebra and δ a derivation on R, then $R[x;\mathrm{id}_R,\delta]$ is simple if and only if δ is a so called outer derivation and the only ideals invariant under δ are $\{0\}$ and R itself. In his PhD thesis [Jor75] Jordan shows that if R is a ring of characteristic zero and with a derivation δ, then $R[x;\mathrm{id}_R,\delta]$ is simple if and only if R has no non-trivial δ-invariant ideals and δ is an outer derivation. In [Jor75] Jordan also shows that if $R[x;\mathrm{id}_R,\delta]$ is simple, then R has zero or prime characteristic and gives necessary and sufficient conditions for $R[x;\mathrm{id}_R,\delta]$ to be simple when R has prime characteristic. (See also [Jor77].)

In [CF75] Cozzens and Faith also prove that if R is an integral domain, then $R[x;\mathrm{id}_R,\delta]$ is simple if and only if the subring of constants, K, is a field (the constants are the elements in the kernel of the derivation) and R is infinite-dimensional as a vector space over K. In [GW82, Theorem 2.3] Goodearl and Warfield prove that if R is a commutative ring and δ a derivation on R, then $R[x;\mathrm{id}_R,\delta]$ is simple if and only if there are no non-trivial δ-invariant ideals (implying that the ring of constants, K, is a field) and R is infinite-dimensional as a vector space over K.

McConnell and Sweedler [MS71] study simplicity criteria for smash products, a generalization of differential polynomial rings.

Conditions for a general Ore extension to be simple have been studied in [LL90] by Lam and Leroy. Their Theorem 5.8 says that $S = R[x;\sigma,\delta]$ is non-simple if and only if there is some $R[y;\sigma',0]$ that can be embedded in S. See also [LL90, Theorem 4.5] and [JLL09, Lemma 4.1] for necessary and sufficient conditions for $R[x;\sigma,\delta]$ to be simple. In [CF75, Chapter 3] a simple Ore extension $R[x;\sigma,\delta]$ is constructed, with σ a non-trivial endomorphism.

If one has a family of commuting derivations, $\delta_1,\ldots,\delta_n$, one can form a differential polynomial ring in several variables. The papers [Mal88, Pos60, Vos85] consider the question when such rings are simple. In [Hau77] a class of rings with a definition similar, but not identical to, the definition of differential polynomial rings of this paper, are studied and a characterization of when they are simple is obtained.

None of the papers cited have studied the simplicity of Ore extensions from the perspective pursued in this paper. In particular for differential polynomial rings the connection between maximal commutativity of the coefficient subring and simplicity of the differential polynomial ring (Theorem C.5.24) appears to be new, as well as the result that the centralizer of the center of the coefficient subring has the ideal intersection property (Proposition C.5.10). We also show that a differential polynomial ring is simple if and only if its center is a field and the coefficient subring has no non-trivial ideals invariant under the derivation (Theorem C.5.14). In Theorem C.5.4 we note that simple Ore extensions over integral domains are necessarily differential polynomial rings, and hence can be treated by the preceding characterization.

In Section C.2, we recall some notation and basic facts about Ore extension rings used throughout the rest of the paper. In Section C.3, we describe the centralizer of the coefficient subring in general Ore extension rings and then use this description to provide conditions for maximal commutativity of the coefficient subring. These conditions of maximal commutativity of the coefficient subring are further detailed for two important classes of Ore extensions, the skew polynomial rings and differential polynomial rings in Subsections C.3.1 and C.3.2. In Section C.4, we describe the center for Ore extension rings. In Section C.5, we investigate when an Ore extension ring is simple and demonstrate how this is connected to maximal commutativity of the coefficient subring for differential polynomial rings (Subsection C.5.1).

C.2 Ore extensions. Definitions and notations

Throughout this paper all rings are assumed to be unital and associative, and ring morphisms are assumed to respect multiplicative identity elements.

For general references on Ore extensions, see e.g. [GW04, MR87, Row88]. For the convenience of the reader, we recall the definition. Let R be a ring , $\sigma : R \to R$

a ring endomorphism (not necessarily injective) and $\delta : R \to R$ a σ-derivation, i.e.

$$\delta(a+b) = \delta(a) + \delta(b) \quad \text{and} \quad \delta(ab) = \sigma(a)\delta(b) + \delta(a)b$$

for all $a, b \in R$.

Definition C.2.1. The Ore extension $R[x;\sigma,\delta]$ is defined as the ring generated by R and an element $x \notin R$ such that $1, x, x^2, \ldots$ form a basis for $R[x;\sigma,\delta]$ as a left R-module and all $r \in R$ satisfy

$$xr = \sigma(r)x + \delta(r). \tag{C.1}$$

Such a ring always exists and is unique up to isomorphism (see [GW04]). From $\delta(1 \cdot 1) = \sigma(1) \cdot 1 + \delta(1) \cdot 1$ we get that $\delta(1) = 0$, and since $\sigma(1) = 1$ we see that 1_R will be a multiplicative identity for $R[x;\sigma,\delta]$ as well.

If $\sigma = \mathrm{id}_R$, then we say that $R[x;\mathrm{id}_R,\delta]$ is a *differential polynomial ring*. If instead $\delta \equiv 0$, then we say that $R[x;\sigma,0]$ is a *skew polynomial ring*. The reader should be aware that throughout the literature on Ore extensions the terminology varies.

An arbitrary non-zero element $P \in R[x;\sigma,\delta]$ can be written uniquely as $P = \sum_{i=0}^{n} a_i x^i$ for some $n \in \mathbb{Z}_{\geq 0}$, with $a_i \in R$ for $i \in \{0, 1, \ldots, n\}$ and $a_n \neq 0$. The *degree* of P will be defined as $\deg(P) := n$. We set $\deg(0) := -\infty$.

By an *integral domain* we mean a commutative ring with no zero-divisors.

Definition C.2.2. A σ-derivation δ is said to be *inner* if there exists some $a \in R$ such that $\delta(r) = ar - \sigma(r)a$ for all $r \in R$. A σ-derivation that is not inner is called *outer*.

The *centralizer* of a subset $T \subseteq S$ is defined as the set of elements of S that commute with every element of T. If T is a subring of S and the centralizer of T in S coincides with T, then T is said to be a *maximal commutative* subring of S. We define $Z(S)$, the *center* of the ring S, to be the centralizer of S in itself. We denote the characteristic of a ring S by $\mathrm{char}(S)$.

C.3 The centralizer and maximal commutativity of R in $R[x;\sigma,\delta]$

In this section we shall describe the centralizer of R in the Ore extension $R[x;\sigma,\delta]$ and give conditions for when R is a maximal commutative subring of $R[x;\sigma,\delta]$. We start by giving a general description of the centralizer and then derive some consequences in particular cases.

In order to proceed we shall need to introduce some notation. We will define functions $\pi_k^l : R \to R$ for $k,l \in \mathbb{Z}$. We define $\pi_0^0 = \mathrm{id}_R$. If m,n are non-negative integers such that $m > n$, or if at least one of m,n is negative, then we define $\pi_m^n \equiv 0$. The remaining cases are defined by induction through the formula

$$\pi_m^n = \sigma \circ \pi_{m-1}^{n-1} + \delta \circ \pi_m^{n-1}.$$

These maps turn out to be useful when it comes to writing expressions in a compact form. We find by a straightforward induction that for all $n \in \mathbb{Z}_{\geq 0}$ and $r \in R$ we may write

$$x^n r = \sum_{m=0}^{n} \pi_m^n(r) x^m.$$

Proposition C.3.1. *$\sum_{i=0}^n a_i x^i \in R[x;\sigma,\delta]$ belongs to the centralizer of R in $R[x;\sigma,\delta]$ if and only if*

$$r a_i = \sum_{j=i}^{n} a_j \pi_i^j(r)$$

holds for all $i \in \{0,\ldots,n\}$ and all $r \in R$.

Proof. For an arbitrary $r \in R$ we have $r \sum_{i=0}^n a_i x^i = \sum_{i=0}^n r a_i x^i$ and

$$\sum_{i=0}^{n} a_i x^i r = \sum_{i=0}^{n} a_i \sum_{j=0}^{i} \pi_j^i(r) x^j = \sum_{i=0}^{n}\sum_{j=0}^{n} a_i \pi_j^i(r) x^j =$$

$$\sum_{j=0}^{n}\sum_{i=0}^{n} a_i \pi_j^i(r) x^j = \sum_{i=0}^{n}\sum_{j=0}^{n} a_j \pi_i^j(r) x^i.$$

By equating the expressions for the coefficient in front of x^i, for $i \in \{0,\ldots,n\}$, the desired conclusion follows. ☐

The above description of the centralizer of R holds in a completely general setting. We shall now use it to obtain conditions for when R is a maximal commutative subring of the Ore extension ring.

Remark C.3.2. Note that if R is commutative, then the centralizer of R in $R[x;\sigma,\delta]$ is also commutative, hence a maximal commutative subring of $R[x;\sigma,\delta]$. Indeed, take two arbitrary elements $\sum_{i=0}^n c_i x^i$ and $\sum_{j=0}^m d_j x^j$ in the centralizer of R and compute

$$\left(\sum_{i=0}^{n} c_i x^i\right)\left(\sum_{j=0}^{m} d_j x^j\right) = \sum_{j=0}^{m} d_j \left(\sum_{i=0}^{n} c_i x^i\right) x^j = \sum_{j=0}^{m}\sum_{i=0}^{n} d_j c_i x^{i+j} =$$

$$\sum_{i=0}^{n}\sum_{j=0}^{m} c_i d_j x^j x^i = \sum_{i=0}^{n} c_i \left(\sum_{j=0}^{m} d_j x^j\right) x^i = \left(\sum_{j=0}^{m} d_j x^j\right)\left(\sum_{i=0}^{n} c_i x^i\right).$$

Proposition C.3.3. *Let R be a commutative ring. If for every $n \in \mathbb{Z}_{>0}$ there is some $r \in R$ such that $\sigma^n(r) - r$ is a regular element, then R is a maximal commutative subring of $R[x;\sigma,\delta]$. In particular, if R is an integral domain and σ is of infinite order, then R is maximal commutative.*

Proof. Suppose that $P = \sum_{k=0}^{n} a_k x^k$ is an element of degree $n > 0$ which commutes with every element of R. Let r be an element of R such that $\sigma^n(r) - r$ is regular. By Proposition C.3.1 and the commutativity of R, we get that $r a_n = \sigma^n(r) a_n$ or equivalently $(\sigma^n(r) - r)a_n = 0$. Since $\sigma^n(r) - r$ is regular this implies $a_n = 0$, which is a contradiction. This shows that R is a maximal commutative subring of $R[x;\sigma,\delta]$. □

Example C.3.4 (The quantum Weyl algebra)**.** Let k be an arbitrary field of characteristic zero and let $R := k[y]$ be the polynomial ring in one indeterminate over k.

Define $\sigma(y) = qy$ for some $q \in k \setminus \{0,1\}$. Then for any $p(y) \in k[y]$ we have $\sigma(p(y)) = p(qy)$ and σ is an automorphism of R. Define a map $\delta : R \to R$ by

$$\delta(p(y)) = \frac{\sigma(p(y)) - p(y)}{\sigma(y) - y} = \frac{p(qy) - p(y)}{qy - y}$$

for $p(y) \in k[y]$. One easily checks that δ is a well-defined σ-derivation of R. The ring $k[y][x;\sigma,\delta]$ is known as the *q-Weyl-algebra* or the *q-deformed Heisenberg algebra* [HS00]. If q is not a root of unity, then by Proposition C.3.3, $k[y]$ is maximal commutative. If q is a root of unity of order n, then x^n and y^n are central and in particular R is not maximal commutative.

Remark C.3.5. Example C.5.17 demonstrates that infiniteness of the order of σ in Proposition C.3.3 is not a necessary condition for R to be a maximal commutative subring.

C.3.1 Skew polynomial rings

Many of the formulas simplify considerably if we take $\delta \equiv 0$, and as a consequence we can say more about maximal commutativity of R in $R[x;\sigma,0]$.

Proposition C.3.6. *Let R be an integral domain and $R[x;\sigma,0]$ a skew polynomial ring. R is a maximal commutative subring of $R[x;\sigma,0]$ if and only if σ is of infinite order.*

Proof. One direction is just a special case of Proposition C.3.3. If $n \in \mathbb{Z}_{>0}$ is such that $\sigma^n = \mathrm{id}_R$, then the element x^n commutes with each $r \in R$ since $x^n r = \sigma^n(r)x^n = rx^n$. □

Example C.3.7 (The quantum plane)**.** With the same notation as in Example C.3.4 form the ring $k[y][x;\sigma,0]$. It is known as the *quantum plane*. By Proposition C.3.6 $k[y]$ is a maximal commutative subring if and only if σ is of infinite order , which is the same as saying that q is not a root of unity. If q is a root of unity of order n then it is easy to see that x^n and y^n will belong to the center, hence R is not a maximal commutative subring.

The following example shows that the conclusion of Proposition C.3.6 is no longer valid if one removes the assumption that R is an integral domain.

Example C.3.8. Let R be the ring $\mathbb{Q}^{\mathbb{N}}$ of functions from the non-negative integers to the rationals. Define $\sigma : R \to R$ such that, for any $f \in R$, we have $\sigma(f)(0) = f(0)$ and $\sigma(f)(n) = f(n-1)$ if $n > 0$. Then σ is an injective endomorphism. But d_0, the characteristic function of $\{0\}$, satisfies $d_0(n)(\sigma(f)(n) - f(n)) = 0$ for all $f \in R$ and $n \in \mathbb{N}$. Thus it follows as in the proof of Proposition C.3.6 that the element $d_O x$ of $R[x;\sigma,0]$ commutes with everything in R.

C.3.2 Differential polynomial rings

We shall now direct our attention to the case when $\sigma = \mathrm{id}_R$. We omit the proof of the following useful lemma.

Lemma C.3.9. *In $R[x;\mathrm{id}_R,\delta]$ we have*

$$x^n r = \sum_{i=0}^{n} \binom{n}{i} \delta^{n-i}(r) x^i$$

for any non-negative integer n and any $r \in R$

We will make frequent reference to the following lemma.

Lemma C.3.10. *Let $q = \sum_{i=0}^{n} q_i x^i \in R[x;\mathrm{id}_R,\delta]$ and $r \in Z(R)$. The following assertions hold:*

(i) *if $n = 0$, then $rq - qr = 0$;*

(ii) *if $n \geq 1$, then $rq - qr$ has degree at most $n-1$ and $(n-1)$:th coefficient $-nq_n\delta(r)$;*

(iii)

$$xq - qx = \sum_{i=0}^{n} \delta(q_i) x^i.$$

Proof. (i): This is trivial.

(ii):
$$\begin{aligned} rq-qr =& (rq_nx^n + rq_{n-1}x^{n-1} + [\text{lower terms}]) \\ &- (q_nrx^n + nq_n\delta(r)x^{n-1} + q_{n-1}rx^{n-1} + [\text{lower terms}]) \\ =& (-nq_n\delta(r))x^{n-1} + [\text{lower terms}]. \end{aligned}$$

(iii):
$$x\left(\sum_{i=0}^{n} q_ix^i\right) - \left(\sum_{i=0}^{n} q_ix^i\right)x = \sum_{i=0}^{n}(q_ix+\delta(q_i))x^i - \sum_{i=0}^{n} q_ix^{i+1} = \sum_{i=0}^{n}\delta(q_i)x^i.$$

□

The following proposition gives some sufficient conditions for R to be a maximal commutative subring of $R[x;\mathrm{id}_R,\delta]$. Note that in the special case when R is commutative and $\sigma = \mathrm{id}_R$, an outer derivation is the same as a non-zero derivation.

Proposition C.3.11. *Let R be an integral domain of characteristic zero. If the derivation δ is non-zero, then R is a maximal commutative subring of $R[x;\mathrm{id}_R,\delta]$.*

Proof. Suppose that R is not a maximal commutative subring of $R[x;\mathrm{id}_R,\delta]$. We want to show that δ is zero. By our assumption, there is some $n \in \mathbb{Z}_{>0}$ and some $q = bx^n + ax^{n-1} + [\text{lower terms}]$ with $a, b \in R$ and $b \neq 0$ such that $rq - qr = 0$ for all $r \in R$. By Lemma C.3.10 and the commutativity of R, we get $rq - qr = (-nb\delta(r))x^{n-1} + [\text{lower terms}]$.

Hence $nb\delta(r) = 0$ which yields $n\delta(r) = 0$ since R is an integral domain and $\delta(r) = 0$ since R is of characteristic zero. Since $\delta(r) = 0$ for all $r \in R$, we conclude that δ is zero. □

Example C.3.12. Let k be a field of characteristic $p > 0$ and let $R = k[y]$. If we take δ to be the usual formal derivative, then we note that x^p is a central element in $R[x;\mathrm{id}_R,\delta]$. This shows that the assumption on the characteristic of R in Proposition C.3.11 can not be relaxed.

C.4 The center of $R[x;\sigma,\delta]$

We shall now describe the center of $R[x;\sigma,\delta]$.

Recall that for $n \in \mathbb{Z}_{\geq 0}$ and $r \in R$ we have

$$x^nr = \sigma^n(r)x^n + b_{n-1}x^{n-1} + \ldots + b_1x + \delta^n(r)$$

for some $b_{n-1}, \ldots, b_1 \in R$. (In fact $b_i = \pi_i^n(r)$, using the functions from Section C.3, but we will not need that in this section.)

The next proposition follows from a straightforward calculation.

Proposition C.4.1. *$\sum_{i=0}^{n} a_i x^i \in R[x;\sigma,\delta]$ commutes with x if and only if the following three assertions hold:*

(i) $\delta(a_0) = 0$;

(ii) $\sigma(a_n) = a_n$;

(iii) $a_i = \sigma(a_i) + \delta(a_{i+1})$ *for all* $i \in \{0,\dots,n-1\}$.

From Proposition C.3.1 and Proposition C.4.1 we get the following corollary.

Corollary C.4.2. *Let R be a commutative ring. For $a \in R$ and a non-negative integer n, ax^n belongs to $Z(R[x;\sigma,0])$ if and only if the following two assertions hold:*

(i) $\sigma(a) = a$;

(ii) $a(r - \sigma^n(r)) = 0$ *for all* $r \in R$.

We can give some fairly concrete necessary conditions for an element to belong to $Z(R[x;\sigma,\delta])$.

Corollary C.4.3. *If $\sum_{i=0}^{n} a_i x^i$ is an element of $Z(R[x;\sigma,\delta])$, then the following holds:*

(i) $\delta(a_0) = 0$;

(ii) $\sigma(a_n) = a_n$;

(iii) $a_i = \sigma(a_i) + \delta(a_{i+1})$ *for* $i \in \{0,\dots,n-1\}$;

(iv) $ra_n = a_n\sigma^n(r)$ *for all* $r \in R$.

We can also describe the intersection of the center of $R[x;\sigma,\delta]$ with R in a nice way. In [Bha09] a similar but more general result is claimed using the same method of proof. However, there appears to be an error in the assertion that the proof actually works for the more general case.

Proposition C.4.4. *An element $r \in R$ belongs to $Z(R[x;\sigma,\delta])$ if and only if the following three assertions hold:*

(i) $\sigma(r) = r$;

(ii) $\delta(r) = 0$;

(iii) $r \in Z(R)$.

Corollary C.4.5. *If R is a domain and δ is non-zero, then $r \in R$ belongs to $Z(R[x;\sigma,\delta])$ if and only if the following two assertions hold:*

(i) $\delta(r) = 0$;

(ii) $r \in Z(R)$.

Proof. By Proposition C.4.4 we know that the conditions are necessary. We also see that they are sufficient if they imply that $\sigma(r) = r$. Suppose that (i) and (ii) hold.

Since δ is non-zero there is some b such that $\delta(b) \neq 0$. We compute $\delta(rb)$ and $\delta(br)$ which must be equal since $r \in Z(R)$. A calculation yields

$$\begin{aligned}\delta(br) =& \sigma(b)\delta(r) + \delta(b)r = r\delta(b),\\ \delta(rb) =& \sigma(r)\delta(b) + \delta(r)b = \sigma(r)\delta(b).\end{aligned}$$

So $(\sigma(r) - r)\delta(b) = 0$. This implies that $\sigma(r) = r$. $\square$

C.5 Simplicity conditions for $R[x;\sigma,\delta]$

Now we proceed to the main topic of this paper. We investigate when $R[x;\sigma,\delta]$ is simple and demonstrate how this is related to maximal commutativity of R in $R[x;\sigma,\delta]$.

In any skew polynomial ring $R[x;\sigma,0]$, the ideal generated by x is proper and hence skew polynomial rings can never be simple. In contrast, there exist simple skew Laurent rings (see e.g. [Jor84]).

Remark C.5.1. If δ is an inner derivation, then $R[x;\sigma,\delta]$ is isomorphic to a skew polynomial ring and hence not simple (see [Goo92, Lemma 1.5]).

We are very interested in finding an answer to the following question.

Question 1. Let $R[x;\sigma,\delta]$ be a general Ore extension ring where σ is, a priori, not necessarily injective. Does the following implication always hold?

$$R[x;\sigma,\delta] \text{ is a simple ring.} \Longrightarrow \sigma \text{ is injective.}$$

So far, we have not been able to find an answer in the general situation. However, it is clear that the implication holds in the particular case when $\delta(\ker\sigma) \subseteq \ker\sigma$, for example when σ and δ commute.

The following unpublished partial answer to the question has been communicated by Steven Deprez (see [Mat]).

Proposition C.5.2. *Let R be a commutative and reduced ring. If $R[x;\sigma,\delta]$ is simple, then σ is injective.*

Proof. Suppose that σ is not injective. Take $a \in \ker(\sigma)\setminus\{0\}$. By assumption $a^k \neq 0$ for all $k \in \mathbb{Z}_{>0}$. Define $I = \{p \in R[x;\sigma,\delta] \mid \exists k \in \mathbb{Z}_{>0} : pa^k = 0\}$. It is clear that I is a left ideal of $R[x;\sigma,\delta]$, and a right R-module. It is non-zero since it contains $ax-\delta(a)$. Since a is not nilpotent I does not contain 1. If we show that I is closed under right multiplication by x then we have shown that it is a non-trivial ideal of $R[x;\sigma,\delta]$. Take any $p \in I$ and k such that $pa^k = 0$. We compute

$$(px)a^{k+1} = p(xa)a^k = p(\sigma(a)x + \delta(a))a^k = p\delta(a)a^k = 0.$$

This shows that $px \in I$. $\square$

Lemma 1.3 in [Goo92] implies as a special case the following.

Lemma C.5.3. *If R is an integral domain, k its field of fractions, σ an injective endomorphism of R and δ a σ-derivation of R, then σ and δ extends uniquely to k as an injective endomorphism, respectively a σ-derivation.*

Using Proposition C.5.2 we are able to generalize a result proved by Bavula in [Bav99]. Our proof uses the same technique as in [Bav99].

Theorem C.5.4. *If R is an integral domain and $R[x;\sigma,\delta]$ is a simple ring, then $\sigma = \mathrm{id}_R$.*

Proof. By Proposition C.5.2 σ must be injective.

Let k be the field of fractions of R. By Lemma C.5.3, σ and δ extend uniquely to k. $R[x;\sigma,\delta]$ can be seen as a subring of $k[x;\sigma,\delta]$. If $\sigma \neq \mathrm{id}_R$, then there is some $\alpha \in R$ such that $\sigma(\alpha)-\alpha \neq 0$. For every $\beta \in k$ we have $\delta(\alpha\beta) = \delta(\beta\alpha)$. Hence, for every $\beta \in k$ the following three equivalent identities hold.

$$\sigma(\alpha)\delta(\beta)+\delta(\alpha)\beta = \sigma(\beta)\delta(\alpha)+\delta(\beta)\alpha \Leftrightarrow (\sigma(\alpha)-\alpha)\delta(\beta) = (\sigma(\beta)-\beta)\delta(\alpha)$$
$$\Leftrightarrow \delta(\beta) = \frac{\delta(\alpha)}{\sigma(\alpha)-\alpha}(\sigma(\beta)-\beta).$$

Hence δ is an inner σ-derivation. This implies that $k[x;\sigma,\delta]$ is not simple since it is isomorphic to a skew polynomial ring. Letting I be a proper ideal of $k[x;\sigma,\delta]$ one can easily check that $I \cap R[x;\sigma,\delta]$ is a proper ideal of $R[x;\sigma,\delta]$, which is a contradiction. $\square$

An example in [CF75, Chapter 3] shows that Theorem C.5.4 need not hold if R is only assumed to be a domain, not necessarily commutative.

Definition C.5.5. An ideal J of R is said to be *σ-δ-invariant* if $\sigma(J) \subseteq J$ and $\delta(J) \subseteq J$. If $\{0\}$ and R are the only σ-δ-invariant ideals of R, then R is said to be *σ-δ-simple*.

The following necessary condition for $R[x;\sigma,\delta]$ to be simple is presumably well-known but we have not been able to find it in the existing literature. For the convenience of the reader, we provide a proof.

Proposition C.5.6. *If $R[x;\sigma,\delta]$ is simple, then R is σ-δ-simple.*

Proof. Suppose that R is not σ-δ-simple and let J be a non-trivial σ-δ-invariant ideal of R. Let $A = R[x;\sigma,\delta]$. Consider the set $I = JA$ consisting of finite sums of elements of the form ja where $j \in J$ and $a \in A$. We claim that I is a non-trivial ideal of A, and therefore $R[x;\sigma,\delta]$ is not simple;

Indeed, I is clearly a right ideal of A, but it is also a left ideal of A. To see this, note that for any $r \in R$, $j \in J$ and $a \in A$ we have $rja \in I$ and by the σ-δ-invariance of J we conclude that $xja = \sigma(j)xa + \delta(j)a \in I$. By repeating this argument we conclude that I is a two-sided ideal of A. Furthermore, I is non-zero, since A is unital and J is non-zero, and it is proper; otherwise we would have $1 = \sum_{i=0}^{n} j_i a_i$ for some $n \in \mathbb{Z}_{\geq 0}$, $j_i \in J$ and $a_i \in R$ for $i \in \{0,\ldots,n\}$, which implies that $1 \in J$ and this is a contradiction. $\square$

While it is possible for $R[x;\sigma,\delta]$ to be simple it always contains non-trivial left ideals as illustrated in the following example.

Example C.5.7. One can always find a (non-zero) left ideal I of $R[x;\sigma,\delta]$ such that $I \cap R = \{0\}$. Take some $n \in \mathbb{Z}_{>0}$ and let I be the left ideal generated by $1 - x^n$. This left ideal clearly has the desired property.

Remark C.5.8. Recall that the center of a simple ring is a field.

Proposition C.5.9. *Let R be a domain and σ injective. The following holds:*

(i) *$R[x;\sigma,\delta] \setminus R$ contains no invertible element;*

(ii) *if $R[x;\sigma,\delta]$ is simple, then the center of $R[x;\sigma,\delta]$ is contained in R and consists of those $r \in Z(R)$ such that $\delta(r) = 0$.*

Proof. (i) Let $A = \sum_{i=0}^{n} a_i x^i$ be an arbitrary element of degree $n > 0$. Suppose that there exists some $B = \sum_{j=0}^{m} b_j x^j$ of degree m such that $AB = 1$. The highest degree coefficient of AB is $a_n \sigma^n(b_m)$ since

$$a_n x^n b_m x^m = a_n \sigma^n(b_m) x^{n+m} + [\text{lower terms}] \tag{C.2}$$

But $AB = 1$ yields $a_n(\sigma^n(b_m)) = 0$ which is a contradiction since R is a domain and σ is injective. (ii) This follows from (i), Remark C.5.8 and Corollary C.4.5 since δ must be non-zero. $\square$

A similar argument as in part (i) of the above proof can be found in [MR87, Theorem 1.2.9(i)].

C.5.1 Differential polynomial rings

We shall now focus on the case when $\sigma = \mathrm{id}_R$.

Note that for a derivation δ on R we have the *Leibniz rule*:

$$\delta^n(rs) = \sum_{i=0}^{n} \binom{n}{i} \delta^{n-i}(r)\delta^i(s)$$

for $n \in \mathbb{Z}_{\geq 0}$ and $r,s \in R$.

Proposition C.5.10. *$I \cap Z(R)' \neq \{0\}$ holds for any non-zero ideal I of $R[x;\mathrm{id}_R,\delta]$, where $Z(R)'$ denotes the centralizer of $Z(R)$ in $R[x;\mathrm{id}_R,\delta]$.*

Proof. Let I be an arbitrary non-zero ideal of $R[x;\mathrm{id}_R,\delta]$. Take $a \in I \setminus \{0\}$ such that $n := \deg(a)$ is minimal. If $n = 0$ then we are done. Otherwise, if a is of degree $n > 0$ it follows from Lemma C.3.10(ii) that $\deg(ra - ar) < \deg(a)$. Since $ra - ar \in I$ we conclude by the minimality of $\deg(a)$ that $ra - ar = 0$. Hence $I \cap Z(R)' \neq \{0\}$. ☐

Corollary C.5.11. *If R is a maximal commutative subring of $R[x;\mathrm{id}_R,\delta]$, then $I \cap R \neq \{0\}$ holds for any non-zero ideal I of $R[x;\mathrm{id}_R,\delta]$.*

We have seen that if $R[x;\mathrm{id}_R,\delta]$ is a simple ring, then its center is a field and R is δ-simple. These necessary conditions are well-known, see e.g. [GW04]. We will now show that they are also sufficient and begin with the following lemma.

Lemma C.5.12. *Let $S = R[x;\mathrm{id}_R,\delta]$ be a differential polynomial ring where R is δ-simple. For every element $b \in S \setminus \{0\}$ we can find an element $b' \in S$ such that:*

(i) *$b' \in SbS$;*

(ii) *$\deg(b') = \deg(b)$;*

(iii) *b' has 1 as its highest degree coefficient.*

Proof. Let J be an arbitrary ideal of $R[x;\mathrm{id}_R,\delta]$ and n an arbitrary non-negative integer. Define the following set

$$H_n(J) = \{a \in R \mid \exists\, c_0, c_1, \cdots c_{n-1} \in R : \ ax^n + \sum_{i=0}^{n-1} c_i x^i \in J\},$$

consisting of the n:th degree coefficients of all elements in J of degree *at most n*.

Clearly, $H_n(J)$ is an additive subgroup of R. Take any $r \in R$. If $ax^n + \sum_{i=0}^{n-1} c_i x^i$ belongs to J, then so does $rax^n + \sum_{i=0}^{n-1} rc_i x^i$. Thus, $H_n(J)$ is a left ideal of R. Furthermore, if $c = ax^n + \sum_{i=0}^{n-1} c_i x^i$ is an element of J then so is cr, and it is not difficult to see that cr has degree at most n and that its n:th degree coefficient is ar. Thus, $H_n(J)$ is also a right ideal of R and hence an ideal.

We claim that $H_n(J)$ is a δ-invariant ideal. Indeed, take any $a \in H_n(J)$ and a corresponding element $ax^n + \sum_{i=0}^{n-1} c_i x^i \in J$. Then we get

$$x(ax^n + \sum_{i=0}^{n-1} c_i x^i) - (ax^n + \sum_{i=0}^{n-1} c_i x^i)x = \delta(a)x^n + \sum_{i=0}^{n-1} \delta(c_i)x^i \in J.$$

This implies that $\delta(a) \in H_n(J)$ and that $H_n(J)$ is δ-invariant.

Now, take any $b \in S \setminus \{0\}$ and put $n = \deg(b)$. Let b_n denote the n:th degree coefficient of b. Put $J = SbS$ and note that $b_n \in H_n(SbS)$. Since R is δ-simple and $H_n(SbS)$ is non-zero we conclude that $H_n(SbS) = R$. Thus, $1 \in H_n(SbS)$ and the proof is finished. □

We now show that the assumption that R is δ-simple allows us to reach a stronger conclusion than in Proposition C.5.10.

Proposition C.5.13. *Let $S = R[x; \mathrm{id}_R, \delta]$ be a differential polynomial ring where R is δ-simple. Then $I \cap Z(S) \neq \{0\}$ holds for every non-zero ideal I of S.*

Proof. Let I be any non-zero ideal of S and choose a non-zero element $b \in I$ of minimal degree n. By Lemma C.5.12 we may assume that its highest degree coefficient is 1. Let us write $b = x^n + \sum_{i=0}^{n-1} c_i x^i$. Since $b_n = 1$ we have that $\deg(rb - br) < \deg(b)$ for all r in R. Since $rb - br \in I$ it follows from the minimality of $\deg(b)$ that $rb - br = 0$, i.e. b commutes with each element in R.

Similarly note that $xb - bx \in I$. By Lemma C.3.10(iii) and the fact that $\delta(1) = 0$ we get $\deg(xb - bx) < \deg(b)$ which implies that $xb - bx = 0$. Since R and x generate S, b must lie in the center of S. □

We now obtain the promised characterization of when $R[x; \mathrm{id}_R, \delta]$ is simple. In [Hau77] Hauger obtains a similar result for a class of rings that are similar to, but distinct from, the ones studied in the present paper. Hauger's method of proof is also different from ours.

Theorem C.5.14. *Let R be a ring and δ a derivation of R. The differential polynomial ring $R[x; \mathrm{id}_R, \delta]$ is simple if and only if R is δ-simple and $Z(R[x; \mathrm{id}_R, \delta])$ is a field.*

Proof. If R is δ-simple and $Z(R[x; \mathrm{id}_R, \delta])$ is a field then, by Proposition C.5.13, $R[x; \mathrm{id}_R, \delta]$ is simple . The converse follows from Proposition C.5.6 and Remark C.5.8. □

A different sufficient condition for $R[x; \mathrm{id}_R, \delta]$ to be simple is given by the following.

Proposition C.5.15. *If R is δ-simple and a maximal commutative subring of $R[x;\mathrm{id}_R,\delta]$, then $R[x;\mathrm{id}_R,\delta]$ is a simple ring.*

Proof. Let J be an arbitrary non-zero ideal of $R[x;\mathrm{id}_R,\delta]$. Using the notation of the proof of Lemma C.5.12 we see that $H_0(J) = J \cap R$. By Corollary C.5.11 and the proof of Lemma C.5.12, it follows that $H_0(J)$ is a non-zero δ-invariant ideal of R. By the assumptions we get $H_0(J) = R$, which shows that $1_{R[x;\mathrm{id}_R,\delta]} \in J$. Thus, $J = R[x;\mathrm{id}_R,\delta]$. □

Remark C.5.16. By Proposition C.4.4 we know that the center of $R[x;\mathrm{id}_R,\delta]$ in this case consists of the constants in R.

In the following example we verify the well-known fact that the Weyl algebra is simple as an application of Proposition C.5.15.

Example C.5.17 (The Weyl algebra)**.** Take $R = k[y]$ for some field k with characteristic zero. Let $\sigma = \mathrm{id}_R$ and define δ to be the usual formal derivative of polynomials. Then $R[x;\sigma,\delta]$ is the Weyl algebra. It is easy to see that $k[y]$ is δ-simple and a maximal commutative subring of the Weyl algebra and thus $R[x;\sigma,\delta]$ is simple by Proposition C.5.15.

Maximal commutativity of R in $R[x;\mathrm{id}_R,\delta]$ does not imply δ-simplicity of R, as demonstrated in the following example.

Example C.5.18. Let k be a field of characteristic zero and take $R = k[y]$. Define δ to be the unique derivation on R satisfying $\delta(y) = y$ and $\delta(c) = 0$ for $c \in k$. No element outside of R commutes with y and thus R is a maximal commutative subring of $R[x;\mathrm{id}_R,\delta]$. However, R is not δ-simple since Ry is a proper δ-invariant ideal of R.

One can also give a counter-example in characteristic p.

Example C.5.19. Let k be field of positive characteristic p. Let R be the polynomial ring in countably many indeterminate $y_1, y_2, \ldots$ over k. Then R is an integral domain of characteristic p. Define a derivation on R by $\delta(a) = 0$ for all $a \in k$, $\delta(y_1) = 0$ and $\delta(y_i) = y_{i-1}$ if $i > 1$. δ is a locally nilpotent derivation, i.e. for every element $r \in R$ there is some n such that $\delta^n(r) = 0$. We also note that for every positive integer m there is some $r \in R$ with $\delta^m(r) \neq 0$ and $\delta^{m+1}(r) = 0$.

Consider $R[x;\mathrm{id}_R,\delta]$. We claim that R is a maximal commutative subring of $R[x;\mathrm{id}_R,\delta]$. To see this, suppose that $q = \sum_{i=0}^n a_i x^i$ belongs to $R' \setminus R$, where R' is the centralizer of R in $R[x;\mathrm{id}_R,\delta]$. Without loss of generality we may assume that $a_0 = 0$ and $a_n \neq 0$.

The degree zero element of qr is $\sum_{i=1}^n a_i \delta^i(r)$. Choose an r such that $\delta(r) \neq 0$ and $\delta^2(r) = 0$. Then since $\sum_{i=1}^n a_i \delta^i(r) = 0$ we get that $a_1 = 0$. With similar arguments we can prove that $a_i = 0$ for all i, which is a contradiction.

It is clear that the proper ideal generated by y_i^p is δ-invariant for any index i.

The following lemma appears as [Jor75, Lemma 4.1.3] and also follows from Proposition C.5.6 and Remark C.5.1.

Lemma C.5.20. *If $R[x;\mathrm{id}_R,\delta]$ is simple, then R is δ-simple and δ is outer.*

Corollary C.5.21. *Let R be an integral domain of characteristic zero. If $R[x;\mathrm{id}_R,\delta]$ is simple, then R is a maximal commutative subring of $R[x;\mathrm{id}_R,\delta]$.*

Proof. This follows from Lemma C.5.20 and Proposition C.3.11. □

The following proposition follows from [Jor75, Theorem 4.1.4]. For the convenience of the reader we include a proof.

Proposition C.5.22. *Let R be a commutative ring that is torsion-free as a module over $\mathbb{Z}$. The following assertions are equivalent:*

(i) *$R[x;\mathrm{id}_R,\delta]$ is a simple ring;*

(ii) *R is δ-simple and δ is non-zero.*

Proof. (i)⇒(ii): This follows from Lemma C.5.20.
(ii)⇒(i): Suppose that R is δ-simple and δ is non-zero. Let J be an arbitrary non-zero ideal of $R[x;\mathrm{id}_R,\delta]$. Choose some $q \in J\setminus\{0\}$ of lowest possible degree, which we denote by n. Seeking a contradiction, suppose that $n > 0$. By Lemma C.5.12 we may assume that q has 1 as its highest degree coefficient.

Let $r \in R$ be arbitrary. Lemma C.3.10(ii) yields $rq - qr = -n\delta(r)x^{n-1} +$ [lower terms]. By minimality of n and the fact that $rq-qr \in I$, we get $rq-qr = 0$. Since R is torsion-free, we conclude that $\delta(r) = 0$. This is a contradiction and hence $n = 0$. Thus, $q = 1$ and hence $J = R[x;\mathrm{id}_R,\delta]$. □

Example C.5.23 demonstrates that assertion (ii) in Proposition C.5.22 does not imply assertion (i) for a general commutative ring R.

Example C.5.23. Let $\mathbb{F}_2$ be the field with two elements and put $R = \mathbb{F}_2[y]/\langle y^2\rangle$. The ideal of $\mathbb{F}_2[y]$ generated by y^2 is invariant under $\frac{d}{dy}$. From this it follows that $\frac{d}{dy}$ induces a derivation δ on R such that $\delta(y) = 1$.

R is clearly δ-simple but $R[x;\mathrm{id}_R,\delta]$ is not simple. To see this note that x^2 is a central element. From that it is easy to see that the ideal generated by x^2 is proper.

We are now ready to state and prove one of the main results of this paper. Note that by Theorem C.5.4 all simple Ore extensions over integral domains of characteristic zero are differential polynomial rings. So the next result in fact classifies simple Ore extension over such rings.

Theorem C.5.24. *Let R be an integral domain of characteristic zero. The following assertions are equivalent:*

(i) *$R[x;\mathrm{id}_R,\delta]$ is a simple ring;*

(ii) *R is δ-simple and a maximal commutative subring of $R[x;\mathrm{id}_R,\delta]$.*

Proof. (i)⇒(ii): By Lemma C.5.21 δ is non-zero and R is δ-simple. The result now follows from Proposition C.3.11.
(ii)⇒(i): This follows from Proposition C.5.15. □

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Paper D

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D

Paper D

Burchnall-Chaundy theory for Ore extensions

Johan Richter

Abstract. *We begin by reviewing a classical result on the algebraic dependence of commuting elements in the Weyl algebra. We proceed by describing generalizations of this result to various classes of Ore extensions, including both results that are already known and one new result.*

D.1 Introduction

Let R be a commutative ring and S an R-algebra. Let a, b be two commuting elements of S. We are interested in the question whether they are algebraically dependent over R. I.e., does there exist a non-zero polynomial $f(s,t) \in R[s,t]$ such that $f(a,b) = 0$? Furthermore, can we find a proper subring F of R such that a, b are algebraically dependent over F?

In this paper S will typically be an Ore extension of R. We start by introducing the notations and conventions we will use in this paper and define what an Ore extension is. After that we review without giving proofs results obtained by other authors for the case that S is a differential operator ring (a special case of Ore extensions). We then proceed to describe results obtained by the present author and his collaborators and we finish by describing a strengthening of these results we recently obtained.

D.1.1 Notation and conventions

$\mathbb{R}$ will denote the field of real numbers, $\mathbb{C}$ the field of complex numbers. $\mathbb{Z}$ will denote the integers.

If R is a ring then $R[x_1, x_2, \ldots x_n]$ denotes the ring of polynomials over R in central indeterminates $x_1, x_2, \ldots, x_n$.

By a ring we will always mean an associative and unital ring. All morphisms between rings are assumed to map the multiplicative identity element to the multiplicative identity element.

By an ideal we shall mean a two-sided ideal.

If R is a ring we can regard it as a module (indeed algebra) over $\mathbb{Z}$ by defining $0r = 0$, $nr = \sum_{i=1}^{n} r$ if $n > 0$ and $nr = -(-n)r$ if n is a negative integer. If there is a positive integer n such that $n1_R = 0$, we call the smallest such positive integer the *characteristic* of R. If no such integer exists we set say that the characteristic is zero.

Let R be a commutative ring and S an R-algebra. Two commuting elements, $p, q \in S$, are said to be *algebraically dependent* (over R) if there is a non-zero polynomial, $f(s,t) \in R[s,t]$, such that $f(p,q) = 0$, in which case f is called an annihilating polynomial.

If S is a ring and a is an element in S, the *centralizer* of a, denoted $C_S(a)$, is the set of all elements in S that commute with a.

This paper studies a class of rings called Ore extensions. For general references on Ore extensions, see e.g. [9, 14]. We shall briefly recall the definition. If R is a ring and σ is an endomorphism of R, then an additive map $\delta : R \to R$ is said to be a *σ-derivation* if

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$$

holds for all $a, b \in R$.

Definition D.1.1. Let R be a ring, σ an endomorphism of R and δ a σ-derivation. The *Ore extension* $R[x;\sigma,\delta]$ is defined as the ring generated by R and an element $x \notin R$ such that $1, x, x^2, \ldots$ form a basis for $R[x;\sigma,\delta]$ as a left R-module and all $r \in R$ satisfy

$$xr = \sigma(r)x + \delta(r). \tag{D.1}$$

Such a ring always exists and is unique up to isomorphism (see [9]). From $\delta(1 \cdot 1) = \sigma(1) \cdot 1 + \delta(1) \cdot 1$ we get that $\delta(1) = 0$, and since $\sigma(1) = 1$ we see that 1_R will be a multiplicative identity for $R[x;\sigma,\delta]$ as well.

Any element r of R such that $\sigma(r) = r$ and $\delta(r) = 0$ will be called a *constant*. In any ring with an endomorphism σ and a σ-derivation δ the constants form a subring.

If $\sigma = \mathrm{id}_R$, then a σ-derivation is simply called a derivation and $R[x;\mathrm{id}_R,\delta]$ is called a *differential operator ring*.

An arbitrary non-zero element $P \in R[x;\sigma,\delta]$ can be written uniquely as $P = \sum_{i=0}^{n} a_i x^i$ for some $n \in \mathbb{Z}_{\geq 0}$, with $a_i \in R$ for $i \in \{0,1,\ldots,n\}$ and $a_n \neq 0$. The *degree* of P will be defined as $\deg(P) := n$. We set $\deg(0) := -\infty$.

D.2 Burchnall-Chaundy theory for differential operator rings

We shall begin by describing some results on the algebraic dependence of commuting elements in differential operator rings. As the title of this subsection suggests, this sort of question has its origin in a series of papers by the British mathematicians Joseph Burchnall and Theodore Chaundy [2, 3, 4].

Proposition D.2.1. *Let R be a ring and $\delta : R \to R$ a derivation. Let C be the set of constants of δ. Then*

(i) $1 \in C$;

(ii) *C is a subring of R, called the ring of constants;*

(iii) *for any $c \in C$ and $r \in R$ we have*

$$\delta(cr) = c\delta(r),$$
$$\delta(rc) = \delta(r)c.$$

Proof. We skip the simple calculational proof. □

As expected any derivation satisfies a version of the quotient rule.

Proposition D.2.2. *Let R be a ring with a derivation, δ, and let a be any invertible element of R. Then*

$$\delta(a^{-1}) = -a^{-1}\delta(a)a^{-1}.$$

Proof.

$$0 = \delta(1) = \delta(a^{-1}a) = a^{-1}\delta(a) + \delta(a^{-1})a \Rightarrow \delta(a^{-1}) = -a^{-1}\delta(a)a^{-1}.$$

□

Corollary D.2.3. *Let R be a ring with a derivation δ and C its ring of constants. If a is an invertible element that lies in C, then so does a^{-1}. If R is a field, then C is a subfield of R.*

Example 1. As the ring R we can take $C^\infty(\mathbb{R}, \mathbb{C})$, the ring of all infinitely many times differentiable complex-valued functions on the real line. For δ we can take the usual derivative. The ring of constants in this case will consist of the constant functions.

With R and δ as in Example 1 we can form the differential operator ring $R[x;\mathrm{id}_R,\delta]$. We will show that the name "differential operator ring" is apt by constructing a ring of concrete differential operators that is isomorphic to $R[x;\mathrm{id}_R,\delta]$.

The ring $R = C^\infty(\mathbb{R},\mathbb{C})$ can be seen as a vector space over $\mathbb{C}$, with operations defined pointwise. So we can consider the ring $\mathrm{End}_{\mathbb{C}}(R)$ of all linear endomorphisms of R. (Note that the endomorphisms are not required to be multiplicative.) $\mathrm{End}_{\mathbb{C}}(R)$ is in turn an algebra over R. One of the operators in $\mathrm{End}_{\mathbb{C}}(R)$ is the derivation operator, which we denote by D. Furthermore, for any $f \in R$ there is the multiplication operator M_f that maps any function $g \in R$ to fg. The operator D and all the M_f together generate a subalgebra of $\mathrm{End}_{\mathbb{C}}(R)$, which we denote by T.

It is clear that the set of all M_f, for $f \in R$, is a subalgebra of T, isomorphic to R. Thus we abuse notation and identify M_f with f. By doing this we can write any element of T as a finite sum, $\sum_{i=0}^n a_i D^i$, where each a_i is a function in $C^\infty(\mathbb{R},\mathbb{C})$. Furthermore such a decomposition is unique, or in other words: the powers of D form a basis for T as a free module over R.

We now compute the commutator of D and f for any $f \in R$. We temporarily revert to writing M_f for the element in T to make our calculations easier to understand. Let g be an arbitrary function in R. We find that

$$\begin{aligned}(DM_f - M_f D)(g) = DM_f(g) - M_f D(g) = D(fg) - M_f(g') = \\ f'g + fg' - fg' = f'g = M_{\delta(f)}(g).\end{aligned}$$

Hence

$$DM_f - M_f D = M_{\delta(f)}.$$

Relapsing into our abuse of notation we write this as $Df - fD = \delta(f)$ or equivalently as $Df = fD + \delta(f)$.

Denote the identity function on the real line by y. Then $Dy - yD = 1$, a relation known as the Heisenberg relation. The elements y and D together generate a subalgebra of T known as the Weyl algebra or the Heisenberg algebra, which is of interest in quantum mechanics, among other areas.

Any element, P, of T can be written as $P = \sum_{i=0}^n p_i D^i$, for some non-negative integer n and some $p_i \in C^\infty(\mathbb{R},\mathbb{C})$. Conversely every such sum is an element of T. Thus T is isomorphic to $R[x;\mathrm{id}_R,\delta]$ with R and δ defined as in Example 1.

In a series of papers in the 1920s and 30s [2, 3, 4], Burchnall and Chaundy studied the properties of commuting pairs of ordinary differential operators. In our terminology they may be said to study the properties of pairs of commuting elements of T. (They do not specify what function space their differential operators are supposed to act on.) The following theorem is essentially found in their papers.

Theorem D.2.4. *Let $P = \sum_{i=0}^n p_i D^i$ and $Q = \sum_{j=0}^m q_j D^j$ be two commuting elements of T with constant leading coefficients. Then there is a non-zero polynomial $f(s,t)$*

in two commuting variables over $\mathbb{C}$ such that $f(P,Q) = 0$. Note that the fact that P and Q commute guarantees that $f(P,Q)$ is well-defined.

The result of Burchnall and Chaundy was rediscovered independently during the 1970s by researchers in the area of PDEs. It turns out that several important PDEs are equivalent to the condition that a pair of differential operators commute. These differential equations are completely integrable as a result, which roughly means that they possess an infinite number of conservation laws.

Burchnall's and Chaundy's work rely on analytical facts, such as the existence theorem for solutions of linear ordinary differential equations. However, it is possible to give algebraic proofs for the existence of the annihilating polynomial. This was done later by authors such as Amitsur [1] and Goodearl [8, 5]. Once one casts Burchnall's and Chaundy's results in an algebraic form one can also generalize them to a broader class of rings.

More specifically, one can prove Burchnall's and Chaundy's result for certain differential operator rings.

Amitsur [1, Theorem 1] (following work of Flanders [7]) studied the case when R is a field of characteristic zero and δ is an arbitrary derivation on R. He obtained the following theorem.

Theorem D.2.5. *Let k be a field of characteristic zero with a derivation δ. Let F denote the subfield of constants. Form the differential operator ring $S = k[x;\mathrm{id},\delta]$, and let P be an element of S of degree n. Denote by by $F[P]$ the ring of polynomials in P with constant coefficients, $F[P] = \{\sum_{j=0}^{m} b_j P^j \mid b_j \in F\}$. Then $C_S(P)$ is a commutative subring of S and a free $F[P]$-module of rank at most n.*

The next corollary can be found in [1, Corollary 2].

Corollary D.2.6. *Let P and Q be two commuting elements of $k[x;\mathrm{id},\delta]$, where k is a field of characteristic zero. Then there is a nonzero polynomial $f(s,t)$, with coefficients in F, such that $f(P,Q) = 0$.*

Proof. Let P have degree n. Since Q belongs to $C_S(P)$ we know that $1, Q, \dots, Q^n$ are linearly dependent over $F[P]$ by Theorem D.2.5. But this tells us that there are elements $\phi_0(P), \phi_1(P), \dots \phi_n(P)$, in $F[P]$, of which not all are zero, such that

$$\phi_0(P) + \phi_1(P)Q + \ldots + \phi_n(P)Q^n = 0.$$

Setting $f(s,t) = \sum_{i=0}^{n} \phi_i(s)t^i$ the corollary is proved. □

Remark D.2.7. Note that F, the field of constants, equals the center of $R[x;\mathrm{id}_R,\delta]$.

In [8] Goodearl has extended the results of Amitsur to a more general setting. The following theorem is contained in [8, Theorem 1.2].

Theorem D.2.8. *Let R be a semiprime commutative ring with derivation δ and assume that its ring of constants is a field, F. If P is an operator in $R[x;\mathrm{id}_R,\delta]$ of positive degree n, where n is invertible in F, and has an invertible leading coefficient, then $C_S(P)$ is a free $F[P]$-module of rank at most n.*

We recall that a commutative ring is semiprime if and only if it has no nonzero nilpotent elements.

Goodearl notes that if R is a semiprime ring of positive characteristic such that the ring of constants is a field, then R must be a field. In this case he proves the following theorem [8, Theorem 1.11].

Theorem D.2.9. *Let R be a field, with a derivation δ, and let F be its subfield of constants. If P is an element of $S = R[x;\mathrm{id}_R,\delta]$ of positive degree n and with invertible leading coefficient, then $C_S(P)$ is a free $F[P]$-module of rank at most n^2.*

As before we get the following corollary (of both Theorem D.2.8 and Theorem D.2.9), which is found in [8, Theorem 1.13].

Corollary D.2.10. *Let P and Q be commuting elements of $R[x;\mathrm{id}_R,\delta]$, where R is a semiprime commutative ring, with a derivation δ such that the subring of constants is a field. Suppose that the leading coefficient of P is invertible. Then there exists a non-zero polynomial $f(s,t) \in F[s,t]$ such that $f(P,Q) = 0$.*

Note that Amitsur's work does not quite generalize Burchnall's and Chaundy's results since $C^{\infty}(\mathbb{R},\mathbb{C})$ is not a field. Theorem D.2.8 does however imply their results since $C^{\infty}(\mathbb{R},\mathbb{C})$ is certainly commutative, does not have any nonzero nilpotent elements and its ring of constants is a field (isomorphic to $\mathbb{C}$). The only point to notice is that Theorem D.2.8 requires P to have positive degree. If P is an element of degree zero and with constant leading coefficient however, it is itself a constant. Then $f(s,t) = s - P$ will be an annihilating polynomial for P and any Q.

An earlier paper by Carlson and Goodearl, [5], contains results similar to Theorems D.2.8 and D.2.9, in a different setting. Part of the theorem labelled Theorem 1 in [5] can be formulated as follows.

Theorem D.2.11. *Let R be a commutative ring, with a derivation δ such that the ring of constants is a field, F, of characteristic zero. Assume that, for all $a \in R$, if the set $\{r \in R \mid \delta(r) = ar\}$ contains a nonzero element, then it contains an invertible element. Let P be an element of $R[x;\mathrm{id}_R,\delta]$ of positive degree n with invertible leading coefficient. Then $C_S(P)$ is a free $F[P]$-module of rank at most n. As before, this implies that if Q commutes with P, there exists a nonzero polynomial $f(s,t) \in F[s,t]$ such that $f(P,Q) = 0$.*

Note that the ring R in Example 1 satisfies the conditions of the theorem.

D.3 Burchnall-Chaundy theory for Ore extensions

Let k be a field and q a nonzero element of that field, not a root of unity. Set $R = k[y]$, a polynomial ring in one variable over k. There is an endomorphism σ of R such that $\sigma(y) = qy$ and the restriction of σ to k is the identity. For this σ there exists a unique σ-derivation δ such that $\delta(y) = 1$ and $\delta(\alpha) = 0$ for all $\alpha \in k$. The Ore extension $R[x;\sigma,\delta]$ for this choice of R, σ and δ is known as the (first) q-Weyl algebra. (An alternative name is the q-Heisenberg algebra.)

Silvestrov and collaborators [6, 10, 12] have extended the result of Burchnall and Chaundy to the q-Weyl algebra. The cited references contain two different proofs of the fact that any pair of commuting elements of $R[x;\sigma,\delta]$ are algebraically dependent over k. In [6] an algorithm is given to compute an annihilating polynomial explicitly.

The algorithm is a variation of one presented by Burchnall and Chaundy in their original papers and consists of forming a certain determinant that when evaluated gives the annihilating polynomial.

Mazorchuk [13] has presented an alternative approach to showing the algebraic dependence of commuting elements in q-Weyl algebras. He proves the following theorem.

Theorem D.3.1. *Let k be a field and q an element of k. Set $R = k[y]$ and suppose that $\sum_{i=0}^{N} q^i \neq 0$ for any natural number N. Let P be an element of $S = R[x;\sigma,\delta]$ of degree at least 1. Then $C_S(P)$ is a free $k[P]$-module of finite rank.*

If P is as in the theorem and Q is any element of $R[x;\sigma,\delta]$ that commutes with P, then there is an annihilating polynomial $f(s,t)$ with coefficients in k. This is proven in the same way as Corollary D.2.6. The methods used to obtain Theorem D.3.1 have been generalized by Hellström and Silvestrov in [11].

In [6, Theorem 3] Silvestrov and the present author extend the algorithmic method of [6] to more general Ore extensions.

Theorem D.3.2. *Let R be an integral domain with an injective endomorphism σ and a σ-derivation δ. Let a, b be two commuting elements of $R[x;\sigma,\delta]$. Then there exists a nonzero polynomial $f(s,t) \in R[s,t]$ such that $f(a,b) = 0$.*

Note that if we apply this theorem to the q-Weyl algebra with $R = k[y]$ we get a weaker result than the one stated above. We would like to be able to conclude that if a, b are commuting elements of $k[y][x;\sigma,\delta]$ then there is a polynomial $f(s,t)$ in $k[s,t]$ such that $f(a,b) = 0$.

Under certain assumptions on σ we have been able to prove this and we now proceed to describe how. We begin with a general theorem that we use as a lemma.

Theorem D.3.3. *Let R be an integral domain, σ an injective endomorphism of R and δ a σ-derivation on R. Suppose that the ring of constants, F, is a field. Let a be an* *element of $S = R[x;\sigma,\delta]$ of degree n and assume that if b and c are two elements in $C_S(a)$ such that* $\deg(b) = \deg(c) = m$*, then $b_m = \alpha c_m$, where b_m and c_m are the leading coefficients of b and c respectively, and α is some constant.*

Then $C_S(a)$ is a free $F[a]$-module of rank at most n.

The proof we give is the same as used in [8] to prove Theorem D.2.8.

Proof. Denote by M the subset of elements of $\{0,1,\dots,n-1\}$ such that an integer $0 \le i < n$ is in M if and only if $C_S(a)$ contains an element of degree equivalent to i modulo n. For $i \in M$ let p_i be an element in $C_S(a)$ such that $\deg(p_i) \equiv i \pmod n$ and p_i has minimal degree for this property. Take $p_0 = 1$.

We will show that $\{p_i | i \in M\}$ is a basis for $C_S(a)$ as a $F[a]$-module.

Since R is an integral domain and σ is injective, the degree of a product of two elements in $R[x;\sigma,\delta]$ is the sum of the degrees of the two elements.

We start by showing that the p_i are linearly independent over $F[a]$. Suppose $\sum_{i\in M} f_i p_i = 0$ for some $f_i \in F[a]$. If $f_i \neq 0$ then $\deg(f_i)$ is divisible by n, in which case

$$\deg(f_i p_i) = \deg(f_i) + \deg(p_i) \equiv \deg(p_i) \equiv i \pmod n. \tag{D.2}$$

If $\sum_{i\in M} f_i p_i = 0$ but not all f_i are zero, we must have two nonzero terms, $f_i p_i$ and $f_j p_j$, that have the same degree despite $i, j \in M$ being distinct. But this is impossible since $i \not\equiv j \pmod n$.

We now proceed to show that the p_i span $C_S(a)$. Let W denote the submodule they do span. We use induction on the degree to show that all elements of $C_S(a)$ belong to W. If e is an element of degree 0 in $C_S(a)$ we find by the hypothesis on a applied to e and $p_0 = 1$ that $e = \alpha$ for some $\alpha \in F$. Thus $e \in W$.

Now assume that W contains all elements in $C_S(a)$ of degree less than j. Let e be an element in $C_S(a)$ of degree j. There is some i in M such that $j \equiv i \pmod n$. Let m be the degree of p_i. By the choice of p_i we now that $m \equiv j \pmod n$ and $m \le j$. Thus $j = m + qn$ for some non-negative integer q. The element $a^q p_i$ lies in W and has degree j. By hypothesis, the leading coefficient of e equals the leading coefficient of $a^q p_i$ times some constant α. The element $e - \alpha a^q p_i$ then lies in $C_S(a)$ and has degree less than j. By the induction hypothesis it also lies in W, and hence so does e. □

We aim to use Theorem D.3.3 when $R = k[y]$. To that end we have obtained the following proposition.

Proposition D.3.4. *Let k be a field and set $R = k[y]$. Let σ be an endomorphism of R such that $\sigma(\alpha) = \alpha$ for all $\alpha \in k$ and $\sigma(y) = p(y)$, where $p(y)$ is a polynomial of degree (in y) greater than* 1*. Let δ be a σ-derivation such that $\delta(\alpha) = 0$ for all $\alpha \in k$. Form the Ore extension $S = R[x;\sigma,\delta]$. We note that its ring of constants is*

k. Let $a \notin k$ be an element of $R[x;\sigma,\delta]$. Assume that b, c are elements of S such that $\deg(b) = \deg(c) = m$ *(here the degree is taken with respect to x) and b, c both belong to $C_S(a)$. Then $b_m = \alpha c_m$, where b_m, c_m are the leading coefficients of b and c respectively, and α is some constant.*

The author wishes to thank Fredrik Ekström for contributing a crucial idea to the following proof.

Proof. Let a_n be the leading coefficient of a. By comparing the leading coefficient of ab and ba we see that

$$a_n\sigma^n(b_m) = b_m\sigma^m(a_n). \tag{D.3}$$

Similarly

$$a_n\sigma^n(c_m) = c_m\sigma^m(a_n). \tag{D.4}$$

By dividing Equation D.3 by Equation D.4 we see that

$$\frac{\sigma^n(b_m)}{\sigma^n(c_m)} = \frac{b_m}{c_m}. \tag{D.5}$$

We can perform such a division by passing to the quotient field of $k[y]$.

It thus suffices to prove that if f, g, p are polynomials in $k[y]$, with $\deg(p) > 1$, and

$$f(y)g(p(y)) = f(p(y))g(y), \tag{D.6}$$

then $f(y) = \alpha g(y)$ for some $\alpha \in k$.

So suppose that such f, g and p are given. We will also assume that k is algebraically closed, which can be done without loss of generality. If f and g have a common factor h we write $f(y) = h(y)\hat{f}(y)$ and similarly for g. We find that

$$\hat{f}(y)h(y)h(p(y))\hat{g}(p(y)) = \hat{f}(p(y))h(p(y))h(y)\hat{g}(y) \tag{D.7}$$

$$\Rightarrow \hat{f}(y)\hat{g}(p(y)) = \hat{f}(p(y))\hat{g}(y). \tag{D.8}$$

So we can assume without loss of generality that f and g are co-prime. It follows that the composite polynomials $f \circ p$ and $g \circ p$ are also co-prime. For if $f \circ p$ and $g \circ p$ had the common factor $l(y)$ it would follow that $f \circ p$ and $g \circ p$ had a common zero since k is algebraically closed. This would imply that f and g had a common zero, contradicting their co-primeness.

From Equation D.6 we see that f must divide $f \circ p$ and g must divide $g \circ p$. So write $f(p(y)) = e(y)f(y)$ and $g(p(y)) = \hat{e}(y)g(y)$. From D.6 we see that $e = \hat{e}$. But this implies that e is a constant polynomial, since otherwise $f \circ p$ and $g \circ p$ would be co-prime. On the other hand $\deg(f \circ p) = \deg(p) \cdot \deg(f)$, which is a contradiction unless $\deg(f) = 0$. The proposition follows. ∎

Proposition D.3.5. *Let k, σ, δ, a be as in Proposition D.3.4. Then $C_S(a)$ is a free $k[a]$-module of finite rank.*

Proof. This follows directly from Theorem D.3.3. ☐

The following theorem, which as far as the author knows is a new result, follows from what we proved above.

Theorem D.3.6. *Let k be a field. Let σ be an endomorphism of $k[y]$ such that $\sigma(y) = p(y)$, where $\deg(p) > 1$, and let δ be a σ-derivation. Suppose that $\sigma(\alpha) = \alpha$ and $\delta(\alpha) = 0$ for all $\alpha \in k$. Let a, b be two commuting elements of $k[y][x; \sigma, \delta]$. Then there is a nonzero polynomial $f(s,t) \in k[s,t]$ such that $f(a,b) = 0$.*

Proof. Using the reasoning in the proof of Corollary D.2.6 this follows from Theorem D.3.3 and Proposition D.3.4. ☐

Note that the center of $k[y][x; \sigma, \delta]$ coincides with k and thus we have a parallel with, for example, Corollary D.2.6. We would like to generalize Theorem D.3.6 to obtain general conditions under which two commuting elements of $S = R[x; \sigma, \delta]$ are algebraically dependent over the center of S. An example of a result in that direction can be found in [10] where Hellström and Silvestrov prove the following theorem.

Theorem D.3.7 ([10], Theorem 7.5)**.** *Let $R = k[y]$, $\sigma(y) = qy$ and $\delta(y) = 1$, where $q \in k$ and q is a root of unity. Form $S = R[x; \sigma, \delta]$ and let C be the center of S. If a, b are commuting elements of S then there is a nonzero polynomial $f(s,t) \in C[s,t]$ such that $f(a,b) = 0$.*

This theorem can not be strengthened to give algebraic dependence over k. Indeed, suppose that $q^n = 1$. One can check that x^n and y^n commute (in fact they both belong to the center) but they are not algebraically dependent over k.

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Paper E

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E

Paper E

A Note on "A Combinatorial Proof of Associativity of Ore Extensions"

Johan Richter

Abstract. *Nystedt recently gave a combinatorial proof of the fact that Ore extensions are associative. As part of it he proves that two sets of strings are identical, using a counting argument. We show that the equality of the sets can be proven directly.*

E.1 Introduction

In a recent article [Nys13], Nystedt gives a combinatorial proof of the fact that Ore extensions are associative. Nystedt proves this result by first constructing an injection between two sets of strings, and then proving that the injection is a bijection through a counting argument. In this note we prove the bijectivity in a more direct way. We assume that the reader is familiar with Nystedt's article.

Recall that an Ore extension, introduced by Ore in [Or33], is defined over some unital and associative ring B. We are given an endomorphism σ of B. We also have an additive function $\delta : B \to B$ such that

$$\delta(bb') = \sigma(b)\delta(b') + \delta(b)b', \tag{E.1}$$

holds for all $b, b' \in B$.

An Ore extension of B is the additive group of polynomials $B[x]$ equipped with a new multiplication, where the product of two monomials is given by

$$(bx^m)(b'x^n) = \sum_{i=0}^{m} b\pi_i^m(b')x^{i+n},$$

where π_i^m denotes the sum of all possible compositions of i copies of σ and $m-i$ copies of δ.

For example,

$$\pi_2^3 = \sigma \circ \sigma \circ \delta + \sigma \circ \delta \circ \sigma + \delta \circ \sigma \circ \sigma.$$

The multiplication rule is extended to more general elements by bilinearity.

See also e.g. the textbook by Goodearl and Warfield [GW04], for a definition of Ore extensions and a discussion of their properties. They give an algebraic proof of the fact that Ore extensions are associative rings.

E.2 The proof

Extend the definition of π_i^m by setting $\pi_i^m \equiv 0$ if $i < 0$ or $i > m$. As shown in [Nys13], the Ore extension being associative is then equivalent to

$$\sum_{i=0}^{\infty} \pi_i^m(b')\pi_j^{i+n}(b'') = \sum_{i=0}^{\infty} \pi_i^m(b'\pi_{j-i}^n(b'')),$$

for all non-negative integers m, n, j and all elements $b', b'' \in B$.

In [Nys13] it is shown that every term in the expansion of $\pi_i^m(b'\pi_{j-i}^n(b''))$ corresponds to a term in $\pi_{i+\nu}^m(b')\pi_j^{\nu+n+i}$, for some non-negative integer ν. Nystedt proves that this correspondence is injective.

In this note we want to show that this is a bijective correspondence. In other words: we want to show that each term in the sum which you get when expanding $\pi_i^m(b')\pi_j^{i+n}(b'')$ coincides with a term in the expansion of $\pi_{i-\nu}^m(b'\pi_{j+\nu-i}^n(b''))$, for some ν.

We note that a term in $\pi_i^m(b')\pi_j^{i+n}(b'')$ has the form

$$f_1 \circ \ldots \circ f_m(b') \cdot g_1 \circ \ldots \circ g_{i+n}(b''),$$

where each f_α and g_α is either σ or δ.

Now we consider the expansion of $S_2 := \pi_{i-\nu}^m(b'\pi_{j+\nu-i}^n(b''))$. Let x be a sequence of length m, consisting of the symbols σ and δ, with σ occurring $i-\nu$ times. This corresponds to a term in the sum defining $\pi_{i-\nu}^m$. Let y be a sequence of length n, consisting of the symbols σ and δ, with σ occurring $j+\nu-i$ times. This corresponds to a term in the sum defining $\pi_{j+\nu-i}^n$. A particular choice of x and y gives rise, via the repeated use of Equation (E.1) and fact that σ is an endomorphism, to a number of terms in the expansion of S_2 of the form

$$\hat{x}_1 \circ \ldots \circ \hat{x}_m(b') \cdot \hat{y}_1 \circ \ldots \circ \hat{y}_m \circ y(b''),$$

where $\hat{x}_k = \hat{y}_k = \sigma$ if $x_k = \sigma$, and if $x_k = \delta$ either $(\hat{x}_k, \hat{y}_k) = (\sigma, \delta)$ or $(\hat{x}_k, \hat{y}_k) = (\delta, \mathrm{id})$. In fact, S_2 is precisely the sum over all such terms, for choices of $x, y, \hat{x}$ and $\hat{y}$ satisfying the conditions we just stated. Here we follow the article by Nystedt.

The problem can now be stated in combinatorial terms as follows: we are given sequences f and g of lengths m and $i+n$ respectively, consisting of the symbols σ and δ, with σ occurring i times in f and j times in g. We have to show that we can find sequences $x, y, \hat{x}, \hat{y}$ and a non-negative integer v which together satisfy all the conditions stated in the previous paragraph. Furthermore, they should satisfy that $\hat{x} = f$ and $\hat{z} = g$, where $\hat{z}$ is the subsequence of $\hat{y}$ formed by deleting all occurrences of the symbol id.

Let the sequences f and g be given and assume that f contains the symbol σ precisely i times, and that g contains the symbol σ precisely j times.

Set $y = (g_{i+1}, \ldots, g_{i+n})$. Let k be the number of occurrences of the symbol σ in y. We take $v = i + k - j$. Since $k \geq j - i$, we see that $v \geq 0$. We also get that σ occurs $j + v - i$ times in y, as required.

The symbols $\hat{y}_1, \ldots, \hat{y}_m$ have to correspond to the symbols $g_1, \ldots, g_i$, but with the symbol id inserted in $m - i$ places. That is precisely the number of occurrences of σ in f. For each l such that $f_l = \sigma$, set $\hat{y}_l = \text{id}$. The rest of the $\hat{y}_l$ are uniquely fixed by the requirement that $\hat{z} = g$. Set $\hat{x} = f$.

By construction, the sequences $\hat{x}$, $\hat{y}$, y and the integer v satisfy the required conditions on them. Next we show that one can find a sequence x, compatible with the other choices.

If $\hat{x}_k = \delta$, then set $x_k = \delta$. In that case we know that $\hat{y}_k = \text{id}$. If $\hat{x}_k = \sigma$ and $\hat{y}_k = \sigma$, then set $x_k = \sigma$. If $\hat{x}_k = \sigma$ and $\hat{y}_k = \delta$, then set $x_k = \delta$.

It remains only to check that the number of σ's in the sequence x is $i - v$. By checking the construction, we see that the number of σ's in x is the same as the number of σ's in $g_1, \ldots, g_i$. We can compute that number to be

$$j - k = i + k - v - k = i - v.$$

□

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Paper F

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F

Paper F

Centralizers in Ore extensions of polynomial rings

Johan Öinert, Johan Richter and Sergei D. Silvestrov

Abstract. *In this paper we consider centralizers of single elements in certain Ore extensions, with a non-invertible endomorphism, of the ring of polynomials in one variable over a field. We show that they are commutative and finitely generated as an algebra. We also show that for certain classes of elements their centralizer is singly generated as an algebra.*

F.1 Introduction

This paperis concerned with centralizers of elements in Ore extensions of the form $K[y][x;\sigma,\delta]$, where K is a field, σ is an K-algebra endomorphism such that $\deg(\sigma(y)) > 1$ and δ is a K-linear σ-derivation.

We now remind the reader what an Ore extension is. An Ore extension of a ring R is the additive group of polynomials $R[x]$, equipped with a new multiplication, such that $xr = \sigma(r)x + \delta(r)$ for all $r \in R$, for some functions σ and δ, on R. This is well-defined if and only σ is an endomorphism and δ is an additive function such that

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$$

for all a and b in R. We denote the Ore extension by $R[x;\sigma,\delta]$. The elements, $r \in R$, satisfying $\sigma(r) = r$ and $\delta(r) = 0$ are called the *constants* of the Ore extension. In our cases R is an algebra over a field K and we assume that σ and δ are K-linear. See e.g. [GW04] for the definition and basic properties of Ore extensions.

There is a series of results concerning centralizers in rings of the form $R[x;\mathrm{id}_R,\delta]$ in the literature, that has inspired this article. The method of proof we use goes back to an article by Amitsur [Ami58], where he proves the following theorem.

Theorem F.1.1. *Let K be a field of characteristic zero with a derivation δ. Let F denote the subfield of constants. Form the differential operator ring $S = k[x;\mathrm{id},\delta]$, and let P be an element of S of degree $n > 0$. Set $F[P] = \{\sum_{j=0}^{m} b_j P^j \mid b_j \in F\}$,* *the ring of polynomials in P with constant coefficients. Then the centralizer of P is a commutative subring of S and a free $F[P]$-module of rank at most n.*

Generalizations of this result can be found in an article by Goodearl and Carlson [CG80] and in an article by Goodearl alone [Goo83]. Both articles deal with the case that $\sigma = \mathrm{id}$, however. Makar-Limanov, in [ML06], studies centralizer in the quantum plane, ie the ring $K[y][x;\sigma,0]$, with $\sigma(y) = qy$. The results in [ML06] also follows from results in [BS04]. This article, by Bell and Small, describes centralizers of elements in domains of Gelfand-Kirillov dimension 2. Some of our results are similar to theirs but are logically independent, since the algebras in this paperhave infinite Gelfand-Kirillov dimension.

The paper that comes closest in approach to our paper, that we have been able to find, is an unpublished preprint by Tang [Tan05]. Tang also studies Ore extensions over $K[y]$, but with σ an automorphism. Like us, Tang describes the structure of maximal commutative subalgebras of the algebras he studies. He cites [AP74, Bav92, Dix68] by Arnal and Pinczon, Bavula respectively Dixmier, as previous articles obtaining similar results on maximal commutative subalgebras. The article by Dixmier contains many results, including similar descriptions of centralizers to the one we give, but it deals exclusively with the Weyl algebra. Bavula's article studies Generalized Weyl algebras and obtains many results, a few of which have analogues in this article. The class of Generalized Weyl algebras does not include our class of Ore extension however. We have not had access to Arnal's and Pinczon's article, but it appears to deal with a completely different class of algebras from those we study.

In [HS07], Hellström and the second author generalizes Amitsur's method of proof. Among other results, they show that Amitsur's argument works in a large class of graded algebras, provided a condition on the dimension of certain subsets of centralizers is met. We have not found a way to apply their results to the algebras in this article, however.

This paperis a continuation of the paper[Ric] (Paper D in this thesis), by the first author. Theorem F.3.1 can be found in that paper. The arrangement of the proof is somewhat different however. Theorem F.3.1 complements our other results that describe the centralizer.

In the next section we will introduce some notation and lemmas that we will use throughout this article. In the third section we prove that centralizers of a non-constant element, P, are free modules of finite rank over the ring of polynomials in P with constant coefficients (Theorem F.3.1). In the fourth section we prove that centralizers of non-constant elements are commutative (Theorem F.4.1) and describe centralizers of any set (Proposition F.4.2). In the fifth section we try to determine when centralizers are isomorphic to the ring of polynomials in one variable. We manage to prove that this is true in many cases (Propositions F.5.3 and F.5.6), with the sufficient conditions given depending only on the leading coefficient. In Propositions F.5.9 and F.5.10 we restrict the class of Ore extension we are considering and obtain results showing that centralizers of certain elements are isomorphic to the ring of polynomials in one variable.

F.2 Preliminaries

We will adopt the following standing conventions and notations in this article. K is a field and $R = K[y]$ is the polynomial ring in one variable over that field. By σ we denote an K-algebra endomorphism of R such that $\deg_y(\sigma(y)) > 1$. By δ we denote a σ-derivation on R, i.e. a K-linear and additive function $R \to R$ such that

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b,$$

for all a and b in R. Our object of study will be the *Ore extension* $S = R[x;\sigma,\delta]$. We note that the constants are precisely the elements of K.

We define the notion of the degree of an element in S w.r.t. x in the obvious way. We set $\deg(0) := -\infty$. As for the ordinary degree it is true that $\deg(ab) = \deg(a) + \deg(b)$. It is important not to confuse this degree function with the degree of an element of R as a polynomial in y. We will always mean degree w.r.t. x when we write degree, unless we explicitly indicates otherwise.

If A is a subset of a ring B, then by $C_B(A)$ we denote the *centralizer* of A, the set of all elements in B that commute with every element in A. If a is a single element we write $C_B(a)$ instead of $C_B(\{a\})$.

We start with two lemmas that will be important in what follows.

Lemma F.2.1. *Suppose that P is any element in S, and that $Q \in C_S(P)$ has degree m. Let q_m be the leading coefficient of Q. Then*

$$p_n\sigma^n(q_m) = q_m\sigma^m(p_n). \tag{F.1}$$

The solution space of this equation (as an equation for q_m) is at most one-dimensional as a K-sub vector space of $K[y]$.

Proof. The equation follows by equating the highest order coefficients in PQ and QP. To show that the solution space is one-dimensional we begin by noting that if $\rho = \deg_y(p_n)$ and $k = \deg_y(q_m)$ then

$$\rho + s^n k = k + s^m \rho.$$

Thus k is determined uniquely. Now suppose that a, b are two solution of Equation (F.1). Then we can find $\alpha \in K$, such that $\deg(a - \alpha b) < k$. But since $a - \alpha b$ is another solution of (F.1) it follows that $a = \alpha b$.

□

Lemma F.2.2. *For any $P \in S$ of degree larger than 0 it is true that*

$$C_S(P) \cap R = K.$$

Proof. This follows from Lemma F.2.1. □

F.3 Centralizers are free $K[P]$-modules

Theorem F.3.1. *Let P be any element of S that is not constant. Then $C_S(P)$ is a is a free $K[P]$-module of rank at most $n := \deg(P)$.*

The proof we give is similar to one in [Ami58]. As noted above, the theorem can also be found in [Ric].

Proof. Denote by M the subset of elements of $\{0,1,\ldots,n-1\}$ such that an integer $0 \le i < n$ is in M if and only if $C_S(P)$ contains an element of degree equivalent to i modulo n. For $i \in M$ let p_i be an element in $C_S(P)$ such that $\deg(p_i) \equiv i \pmod n$ and p_i has minimal degree for this property. Take $p_0 = 1$.

We will show that $\{p_i | i \in M\}$ is a basis for $C_S(P)$ as a $K[P]$-module.

We start by showing that the p_i are linearly independent over $K[P]$. Suppose $\sum_{i\in M} f_i p_i = 0$ for some $f_i \in K[P]$. If $f_i \neq 0$, for a particular i, then $\deg(f_i)$ is divisible by n, in which case

$$\deg(f_i p_i) = \deg(f_i) + \deg(p_i) \equiv \deg(p_i) \equiv i \pmod n. \tag{F.2}$$

If $\sum_{i\in M} f_i p_i = 0$ but not all f_i are zero, we must have two nonzero terms, $f_i p_i$ and $f_j p_j$, that have the same degree despite $i, j \in M$ being distinct. But this is impossible since $i \not\equiv j \pmod n$.

We now proceed to show that the p_i span $C_S(P)$. Let W denote the submodule they do span. We see that W contains all elements of degree 0 in $C_S(P)$.

Now assume that W contains all elements in $C_S(P)$ of degree less than j. Let Q be an element in $C_S(P)$ of degree j. There is some i in M such that $j \equiv i \pmod n$. Let m be the degree of p_i. By the choice of p_i we now that $m \equiv j \pmod n$ and $m \le j$. Thus $j = m + qn$ for some non-negative integer q. The element $P^q p_i$ lies in W and has degree j. By Lemma F.2.1 the leading coefficient of Q equals the leading coefficient of $P^q p_i$ times some constant α. The element $Q - \alpha P^q p_i$ then lies in $C_S(P)$ and has degree less than j. By the induction hypothesis it also lies in W, and hence so does Q. □

F.4 Centralizers are commutative

We now prove that the centralizer of any non-constant element of S is commutative. For the proof of this we once again follow closely the presentation in [Ami58].

Theorem F.4.1. *Let P be an element of S that is not a constant. Then $C_S(P)$ is commutative.*

Proof. If P is an element of $R \setminus K$ it follows that $C_S(P) = R$ which is commutative. Thus suppose that $n = \deg(P) \geq 1$. Let D be the set of degrees of non-zero elements of Cen(P). Since $C_S(P)$ is a subring, and $\deg(ab) = \deg(a) + \deg(b)$ for any non-zero a, b, it follows that D is closed under addition. Map D into $\mathbb{Z}_n$ in the natural way and denote the image by D_n. Since D_n is finite, closed under addition and contains 0 it is a subgroup of Z_n. So it is a cyclic group.

Let $Q \in C_S(P)$ be an element such that $\deg(Q) \mod n$ generates D_n. Let J be the set of elements of the form

$$H(P,Q) = \phi_0 + \phi_1 Q + \dots \phi_l Q^l, \ \phi_i \in C_S(P), \ i = 0 \dots l$$

and let $E = \{\deg(H(P,Q)) \mid H(P,Q) \in J\}$. Suppose that $t \in \mathbb{N}$ is such that if $m \geq t$ and $m \in D$ then $m \in E$. Such a t must clearly exist. Suppose now that U is any element of $C_S(P)$. If $\deg(U) \geq t$, then, by Observation F.2.1, there is a $H_1(P,Q) \in J$ such that $\deg(U - H_1) < \deg(U)$. By repeating this process if necessary, we find that we can write $U = H(P,Q) + U_0$ where $\deg(U_0) < t$. We note that the set of elements in $C_S(P)$ of degree less than t form a finite-dimensional vector space over K of dimension at most t.

If V is an element of $C_S(P)$ we can write $VP^i = H_i(P,Q) + V_i$, where $\deg(V_i) < t$, for $i = 0, 1, \dots t$. Then the V_i are linearly dependent so there are $c_i \in K$ such that $\sum_{i=0}^{t} c_i V_i = 0$ which implies that

$$V \sum_{i=0}^{t} c_i P^i = \sum_{i=0}^{t} c_i H_i.$$

So for any $V \in C_S(P)$, there are non-zero $f \in K[P]$ and $H(P,Q) \in J$ such that $Vf(P) = H(P,Q)$. The elements in J commute with each other and the elements of $K[P]$ commutes with everything in $C_S(P)$. Thus if V_1, V_2 are two elements in $C_S(P)$, with $V_i f_i(P) = H_i(P,Q)$, we get that

$$\begin{aligned} V_1 V_2 f_1(P) f_2(P) = V_1 f_1(P) V_2 f_2(P) = H_1(P,Q) H_2(P,Q) = \\ = H_2(P,Q) H_1(P,Q) = V_2 f_2(P) V_1 f_1(P) = V_2 V_1 f_1(P) f_2(P). \quad \text{(F.3)} \end{aligned}$$

Since S is a domain this implies that $V_1 V_2 = V_2 V_1$.

□

It is clear that if A is any set containing a non-constant element then A is commutative as well. But we can say more than so as the next proposition illustrates.

Proposition F.4.2. *Let A be any subset of S. Then $C_S(A)$ equals either S, K or $C_S(P)$, where P is a non-constant element in A.*

Proof. Suppose A contains two elements P and Q (necessarily non-constant), that do not commute with each other. Then

$$C_S(A) \subseteq C_S(P) \cap C_S(Q).$$

But if R is some non-constant element in $C_S(P) \cap C_S(Q)$, then $P, Q \in C_S(R)$ and by Theorem F.4.1 it would follow that P and Q commute. Thus $C_S(A) = K$.

Now suppose P contains a non-constant P and everything in A commutes with P. Clearly $C_S(A) \subseteq C_S(P)$. But, since $C_S(P)$ is commutative and $A \in C_S(P)$, every element in $C_S(P)$ commutes with every element in A. Thus $C_S(P) = C_S(A)$.

If, finally, A contains only constants, then $C_S(A) = S$. □

Remark F.4.3. We note that the maximal commutative subrings of S are sets of the form $C_S(P)$, for nonconstant P.

F.5 Singly generated centralizers

We note that we can give a bound on the number of generators needed to generate a centralizer as an algebra.

Corollary F.5.1. *Let $P \in R[x;\sigma,\delta]$ satisfy $n = \deg(P) > 0$. Then we can find n elements that generate $C_S(P)$ as a K-algebra.*

Proof. Follows from Theorem F.3.1 and its proof. □

In some cases we have been able to prove that the centralizer of an element is in fact generated by a single element, not just a finite number of them. To do so we have relied on the the equation stated in Observation F.2.1.

We begin with a lemma which we will use frequently.

Lemma F.5.2. *Let P be a non-constant element of S of degree n. Suppose all elements of $C_S(P)$ have degree divisible by n. Then*

$$C_S(P) = K[P] := \{\sum c_i P^i \mid c_i \in K\}.$$

Proof. We know that $K[P] \subseteq C_S(P)$. We also know that all elements of degree zero in $C_S(P)$ lie in $K[P]$. We give a proof by induction.

Suppose that elements in $C_S(P)$ of degree less than k lie in $K[P]$. We want to show that all elements of degree k in $C_S(P)$ also lie in $K[P]$. If k is not divisible by n this is vacuously true. So suppose $k = pn$ for some integer p and let Q be any element in $C_S(P)$ of degree k. The element P^p lies in $C_S(P)$ and has degree k. By Lemma F.2.1 there is an $\alpha \in K$ such that Q and αP^p have the same leading coefficient. Thus we have that $\deg(Q - \alpha P^p) < k$ which implies that $Q - \alpha P^p \in K[P]$, by the induction assumption. Hence it follows that $Q \in K[P]$. □

Our first result showing a centralizer to be singly generated is in the case when our non-constant element has prime degree.

Proposition F.5.3. *Let P be an element of S of degree n, where n is a prime. Let p_n be the leading coefficient of P and let ρ be the degree of p_n as a polynomial in y. Let s be the degree of $\sigma(y)$, also as a polynomial in y. Then if $\sum_{i=0}^{n-1} s^i$ does not divide ρ it follows that $C_S(P) = \{\sum c_i P^i \mid c_i \in K\}$.*

We use the following lemma in our proof.

Lemma F.5.4. *Let m and n be positive integers and suppose that $\gcd(n,m) = 1$. Let s be a positive integer. Then $\gcd(\sum_{i=0}^{n-1} s^i, \sum_{j=0}^{m-1} s^j) = 1$.*

Proof. This is clearly (vacuously) true for $n = 1$ and it is a simple exercise to prove it is true for $n = 2$. We use induction on n to prove the lemma in general. So suppose it is true if $n < k$ and we want to show it is true for $n = k$. So let $m > k$ be such that $\gcd(k,m) = 1$.

$$\gcd(\sum_{i=0}^{k-1} s^i, \sum_{j=0}^{m-1} s^j) = \gcd(\sum_{i=0}^{k-1} s^i, \sum_{j=0}^{k-1} s^j + \sum_{j=k}^{m-1} s^j) = \gcd(\sum_{i=0}^{k-1} s^i, s^k \sum_{j=0}^{m-1-k} s^j) =$$
$$= \gcd(\sum_{i=0}^{k-1} s^i, \sum_{j=0}^{m-1-k} s^j).$$

Now it is clearly true that $\gcd(k, m-k) = 1$. If $m-k < k$ we can use the induction assumption. If $m-k > k$ set $m' = m-k$ and repeat the previous calculation. Sooner or later we will reduce to a case where we can use the induction assumption. □

Proof of proposition. Let Q be an element of S that commutes with P. Let Q have degree m and suppose that $\gcd(m,n) = 1$. Let q_m be the leading coefficient of Q. Equating the leading coefficients in PQ and QP we find that

$$p_n \sigma^n(q_m) = q_m \sigma^m(p_n).$$

If k denotes the degree of q_m, we find that

$$k = \rho \frac{s^m - 1}{s^n - 1} = \rho \frac{\sum_{i=0}^{m-1} s^i}{\sum_{i=0}^{n-1} s^i}.$$

Now it would follow from the lemma that k is a non-integer which is impossible.

Thus $\gcd(m,n) = n$, since n is prime, and the result follows by Lemma F.5.2. □

We can generalize Lemma F.5.4 to the following lemma.

Lemma F.5.5. *Let m and n be positive integers. Let s be a positive integer greater than* 1. *Set $r = \gcd(m,n)$ Then*

$$\gcd(\sum_{i=0}^{m-1} s^i, \sum_{j=0}^{n-1} s^j) = \sum_{i=0}^{r-1} s^i.$$

Proof. If $n = qm + r$ then

$$\gcd(\sum_{i=0}^{m-1} s^i, \sum_{j=0}^{n-1} s^j) = \gcd(\sum_{i=0}^{m-1} s^i, \sum_{j=0}^{qm-1} s^j + s^{qm} \sum_{j=0}^{r-1} s^j) = \tag{F.4}$$

$$= \gcd(\sum_{i=0}^{m-1} s^i, \sum_{j=0}^{r-1} s^j) = \sum_{i=0}^{r-1} s^i. \tag{F.5}$$

□

We use this lemma in the next proposition.

Proposition F.5.6. *Let P be an element of S of degree $n > 0$ in x and suppose that p_n (the leading coefficient of P) has degree greater than zero but not greater than n as a polynomial in y. Then $C_S(P) = K[P]$.*

Proof. When $n = 1$ this is true by Corollary F.5.1. When $n = 2$ or $n = 3$ this is true by Proposition F.5.3. So suppose that $n \geq 4$.

It will be enough to prove that the degrees of all elements of $C_S(P)$ are divisible by n by Lemma F.5.2.

Let Q be an element of $C_S(P)$. Suppose that Q has degree m. Let q_m be the leading coefficient of Q. By comparing the leading coefficient of PQ and QP we get the equation

$$p_n \sigma^n(q_m) = q_m \sigma^m(p_n).$$

Let k denote the degree of q_m and ρ the degree of p_n. (Both degrees are measured as polynomials in y.) We get the following equation for k.

$$k = \rho \frac{\sum_{i=0}^{m-1} s^i}{\sum_{i=0}^{n-1} s^i}.$$

Set $r = \gcd(m,n)$. What we want to prove is that $r = n$. So suppose that it does not equal n. Then $r \leq \frac{n}{2}$. Write $n = rn'$. Then

$$\sum_{i=0}^{n-1} s^i = \left(\sum_{i=0}^{r-1} s^i\right)\left(\sum_{i=0}^{n'-1} s^{ri}\right).$$

From Lemma F.5.5 we conclude that $\sum_{i=0}^{n'-1} s^{ri}$ must divide ρ if k is to be an integer. However,

$$\sum_{i=0}^{n'-1} s^{ri} > s^{r(n'-1)} \geq 2^{r(n'-1)} = \frac{2^n}{2^r}.$$

Since $r \leq \frac{n}{2}$ we find that

$$\frac{2^n}{2^r} \geq 2^{\frac{n}{2}}.$$

Since $2^{\frac{n}{2}} \geq n$ for all $n \geq 4$ we find, to summarize our calculations, that

$$\sum_{i=0}^{n'-1} s^{ri} > 2^{\frac{n}{2}} \geq n \geq \rho.$$

But this is a contradiction to the fact that the sum had to divide ρ.

□

Corollary F.5.7. *Let n be any positive integer. Then $C_S(y^n x^n) = K[y^n x^n]$.*

Proposition F.5.8. *Let n be any positive integer. Then $C_S(x^n y^n) = K[x^n y^n]$.*

Proof. Set $P = x^n y^n$. P has degree n as an element of S and its leading coefficient is $\sigma^n(y^n)$. The degree of the leading coefficient as a polynomial in y is ns^n.

The proposition is true when $n = 1$ by Corollary F.5.1. It is true when $n = 2$ and when $n = 3$ by Proposition F.5.3.

So suppose that $n \geq 4$. Let Q be an element of degree m. As before it suffices to prove that $\gcd(m, n) = n$. We will use a proof by contradiction, so set $r = \gcd(m, n)$ and suppose that $r < n$. Letting k denote the degree in y of the leading coefficient of Q we get, as before,

$$k = ns^n \frac{\sum_{i=0}^{m-1} s^i}{\sum_{i=0}^{n-1} s^i}.$$

We cancel common factors in the fraction, and by Lemma F.5.5 we get

$$k = ns^n \frac{A}{\sum_{i=0}^{n'-1} s^{ri}},$$

where $n' = \frac{n}{r}$. Since $\gcd(As^n, \sum_{i=0}^{n'-1} s^{ri}) = 1$ we see that we must have that $\sum_{i=0}^{n'-1} s^{ri} | n$. But, as in the proof of Proposition F.5.6, this is not the case.

□

For the next proposition we consider only special σ.

Proposition F.5.9. *Let $R = K[y]$ and suppose that $\sigma(y) = y^k$ for some positive integer $k > 1$. Let P be an element of $S = R[x;\sigma,\delta]$ of degree n and let p_n be its leading coefficient. Suppose that p_n has the following property: there does not exist an $a \in \bar{K}$ and distinct positive integers i, j, such that a^i and a^j both are roots of p_n. (Here $\bar{K}$ is the algebraic closure of K.) Then $C_S(P) = K[P]$.*

Proof. Let Q be an element of $C_S(P)$. As before it suffices to prove that deg(Q) is divisible by n. So suppose $m = \deg(Q)$ is not. Let q_m be the leading coefficient of Q. We get the following equation

$$p_n\sigma^n(q_m) = q_m\sigma^m(p_n).$$

Due to the special form of σ this can be written

$$p_n(y)q_m(y^{k^n}) = q_m(y)p_n(y^{k^m}).$$

Consider $\gcd(p_n(y), p_n(y^{k^m}))$. If this equals a nonzero polynomial h, then h has a root, a, in $\bar{K}$. But then both a and a^{k^m} would be roots of p_n, contradicting the assumption we made. Thus $\gcd(p_n(y), p_n(y^{k^m})) = 1$. So $p_n(y)$ must divide $q_m(y)$. Set $q_m(y) = p_n(y)\hat{q}(y)$ and simplify.

The simplified equation becomes

$$p_n(y^{k^n})\hat{q}(y^{k^n}) = \hat{q}(y)p_n(y^{k^m}).$$

Now we have that $\gcd(p_n(y^{k^n}), p_n(y^{k^m})) = 1$. Thus $\hat{q} = q'(y)p_n(y^{k^n})$ for some q'. Inserting this into our equation and simplifying we get

$$p_n(y^{k^{2n}})q'(y^{k^n}) = q'(y)p_n(y^{k^m}).$$

Since n does not divide m we must have that $2n \neq m$. Thus

$$\gcd(p_n(y^{k^{2n}}), p_n(y^{k^m})) = 1.$$

We trust that the pattern is obvious now. It is clear that we can continue this process for ever and conclude that $q_m(y)$ is divisible by an infinite sequence of polynomials with strictly increasing degrees. Thus our assumption that m was not divisible by n leads to a contradiction.

□

Specialising the definition of S even further we get the following proposition.

Proposition F.5.10. *Let $\sigma(y) = y^s$ and $\delta(y) = 0$. Set $P = y^i x^j$, where $i + j > 0$. Then $C_S(P)$ is singly generated.*

Proof. The result is clear when $j = 0, i > 0$ so suppose that $j > 0$.

Suppose that Q belongs to $C_S(P)$. Write $Q = \sum a_{l,k} y^l x^k$. We can compute that

$$y^i x^j y^l x^k = y^{i+ls^j} x^{j+k}.$$

Since $C_S(P)$ is graded by the powers of x it follows that $\sum_l a_{l,k} y^l x^k \in C_S(P)$ for every k. Since the product of monomials is a new monomial it follows, by induction downwards on the degree in y, that every term $a_{l,k} y^k x^l$ must commute with P.

Suppose $a_{l,k} \neq 0$. Then we must have that

$$y^i x^j y^l x^k = y^l x^k y^i x^j,$$

which implies that $i + l \cdot s^j = l + i \cdot s^k$. This means that

$$l = \frac{s^k - 1}{s^j - 1} i.$$

We can write this as

$$l = \frac{\sum_{m=0}^{k-1} s^m}{\sum_{m=0}^{j-1} s^m} i.$$

For every choice of i, j, s, k this determines l. However the formula might give non-integer values for l, which does not correspond to an element of S. Let k_0 be the least non-negative integer for which the RHS is an integer when we substitute k_0 for k.

Let k_1 be the next least non-negative integer such that the RHS of the formula is an integer. We compute

$$\frac{\sum_{m=0}^{k_1-1} s^m}{\sum_{m=0}^{j-1} s^m} i = \frac{\sum_{m=0}^{k_0-1} s^m + s^{k_0} \sum_{m=0}^{k_1-k_0-1} s^m}{\sum_{m=0}^{j-1} s^m} i.$$

We see that (by the definition of k_0 and since $\gcd(s^{k_0}, \sum_{m=0}^{j-1} s^m) = 1$) that

$$\frac{\sum_{m=0}^{k_1-k_0-1} s^m}{\sum_{m=0}^{j-1} s^m} i$$

is an integer. This implies that $k_1 = 2k_0$, by the definition of k_0. Similarly, all k that give an integer value for l must be multiples of k_0. The result is now clear. □

Note that the proof of Proposition F.5.10 establishes that the generator of $C_S(P)$ is $y^l x^k$ where k is the least non-negative integer such that

$$\frac{\sum_{m=0}^{k-1} s^m}{\sum_{m=0}^{j-1} s^m} i$$

is an integer and l is the value of that integer.

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