Min-Mix Control in Discrete Time by Completion of Squares

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Abstract: This paper discusses the so called min-mix problem for discrete time systems. The min-mix problem is one of the mixed H_2/H_∞-control problems that capture the idea that disturbances may be both deterministic and stochastic. The deterministic behavior models worst case situations and stochastic disturbances model average properties. The problem is very rich and has connections both to recent results in H_∞ and also to classical results in stochastic control and differential games. Using the methods of [Bernhardsson, 1992] it is possible to show that the discrete time case also follows directly by *completion of squares.* The continuous time and discrete time cases are therefore unified. The solution is given by three coupled Riccati equations. The time-varying, finite time horizon, output feedback version of the problem is treated, and an explicit formula for the value of the game is also obtain. A simple example demonstrates how this information is useful for the infinite time horizon case.

Keywords: Mixed H_2/H_∞ Control; Stochastic Differential Games; Stochastic Optimal Control; H_∞-optimization; Linear-quadratic-Gaussian Control; Riccati Equations; Discrete-time Systems; Linear Systems.

1. Introduction

Standard H_∞-control can be applied to disturbance attenuation design, but it has then fundamental practical limitations. A description of the process disturbance as a worst case bounded signal has to include the effect of measurement noise. In H_∞-control the system is

$$\begin{cases} qx = Ax + B_1 d + B_2 u \\ y = Cx + De \end{cases}$$

Here d and e denote process and measurement disturbances, u control signals, y measurements and q the forward shift operator, i.e. $qx(k) := x(k+1)$. The H_∞-control problem optimizes system performance against worst disturbances d and e:

$$\min_K \max_{d,e} \|z\|_2^2, \qquad \text{when } \|d\|_2^2 + \|e\|_2^2 \leq 1$$

where z is a vector of signals to be minimized. The constraint means that process disturbances and measurement noise can not both be large. It is unnatural to couple process disturbance and measurement noise in this way. An alternativ is to assume that e.g. the measurement noise is stochastic. Several other motivations exist for considering a mixture of H_2 and H_∞ problems. It is important to separate between the many different so called "mixed H_2/H_∞ -problems", although some relations exist between the different setups, see e.g. [Doyle *et al.*, 1989], [Mustafa, 1989], [Rotea and Khargonekar, 1990], [Khargonekar and Rotea, 1991], [Limebeer *et al.*, 1991], [Peters and Stoorvogel, 1992], [Haddad *et al.*, 1991], [Kaminer *et al.*, 1992], [Zhou *et al.*, 1990],[Zhou *et al.*, 1992].

Much research is now concentrated on finding tools that make optimal controller design in the case of a combination of stochastic and deterministic disturbances possible. There have been many slightly different attempts. The most promising result seems to be the three coupled Riccati equations developed by [Doyle *et al.*, 1992]. See also [Nikoukhah and Delebecque, 1992]. In [Bernhardsson, 1992], [Bernhardsson and Hagander, 1993] a classical stochastic games approach is used to determine the finite time horizon, continuous time, version of these equations. The operations of minimizing, maximizing and taking expected values appear in the problem. The name min-mix is coined to emphasize the game theory background, see e.g. [Ho, 1970, Rhodes and Luenberger, 1969].

The cornerstone for the solution method presented here is a (conjectured) generalization of a recent dynamic programming separation principle [Bernhard, 1992]. The approach makes it possible to obtain the solution by completion of squares, to unify several results and to gain insight. An explicit formula for the value of the game is also determined. A simple example is used to investigate feasible stationary solutions to the three coupled Riccati equations.

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2. Discrete Time Min-Mix Controllers

The calculations in discrete time are analogous to continuous time, although there are some differences. The algebra is more complicated in discrete time. The calculations have hopefully been reduced to a near minimum in what follows. Let the, possibly time-varying, system be given by

$$\left\{\begin{aligned} qx &= Ax + B_0w_0 + B_1w_1 + B_2u \\ x(0) &= x_0 \\ y &= C_2x + D_{20}w_0 + D_{21}w_1 \end{aligned}\right. \tag{1}$$

Here w_0 denotes a stochastic disturbance modeled as Gaussian unit-variance white noise. Let the criterion be to find saddle equilibria for

$$\underset{w_0}{E}\left\{\sum_{k=0}^{t_f-1} x^T(k)Qx(k) + u^T(k)u(k) - \gamma^2 w_1^T(k)w_1(k)\right\} + \underset{w_0}{E}\, x^T(t_f)Q_f x(t_f) \tag{2}$$

where w_1 is a worst-case disturbance. The controller is assumed to be of the form $u(k) = f(y^{k-1})$, i. e. to contain a computational delay of one step. The disturbance $w_1(k)$ is allowed to be a function of x^k, y^{k-1} but not of future information.

The idea is now to use dynamic programming in the spirit of Isaacs' equation, see [Isaacs, 1965], applying a generalization of a recent separation principle, see [Bernhard, 1992], [Bernhardsson, 1992]. Inspired by the continuous time case, see also [Bernhardsson and Hagander, 1993], we define $\check{x}$ by

$$\left\{\begin{aligned} q\check{x} &= A\check{x} + B_2u + B_1\widehat{F}_1\check{x} - L(y - C_2\check{x} - D_{21}\widehat{F}_1\check{x}) \\ \check{x}(0) &= x_0 \end{aligned}\right. \tag{3}$$

where L and $\widehat{F}_1$ will be determined later. The interpretation of $\check{x}$ will also be clarified later; Consider it for the moment just to be a suitably defined signal. Similarly we investigate the following quadratic expression as a possible candidate for the future loss from time k and onwards:

$$\left\{\begin{aligned} \mathcal{V}(k, x, \tilde{x}) &= x^T X(k)x + \tilde{x}^T Y(k)\tilde{x} + \nu(k) \\ \nu(k) &= \sum_{j=k+1}^{t_f} \mathrm{Tr}\,\left(B_0^T X(j)B_0 + B_{0L}^T Y(j)B_{0L}\right) \end{aligned}\right. \tag{4}$$

Recursions for X and Y will be determined later. Here we introduce the notation $\tilde{x} = x - \check{x}$, $B_{0L} = B_0 + LD_{20}$ and $B_{1L} = B_1 + LD_{21}$. We have also

$$q\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} = \overline{A}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} + \overline{B}_0w_0 + \overline{B}\begin{pmatrix} w_1 \\ u \end{pmatrix}$$

$$\overline{A} = \begin{pmatrix} A & 0 \\ -B_{1L}\widehat{F}_1 & A + LC_2 + B_{1L}\widehat{F}_1 \end{pmatrix}$$

$$\overline{B}_0 = \begin{pmatrix} B_0 \\ B_{0L} \end{pmatrix}, \qquad \overline{B} = \begin{pmatrix} B_1 & B_2 \\ B_{1L} & 0 \end{pmatrix}$$

The loss per step in (2) can be rewritten as

$$x^TQx + u^Tu - \gamma^2 w_1^T w_1 = x^TQx + \begin{pmatrix} w_1 \\ u \end{pmatrix}^T J \begin{pmatrix} w_1 \\ u \end{pmatrix}$$

with $J = \begin{pmatrix} -\gamma^2 & 0 \\ 0 & I \end{pmatrix}$. For the dynamic programming we introduce

$$\overline{X} = \begin{pmatrix} X(k+1) & 0 \\ 0 & Y(k+1) \end{pmatrix}, \qquad \overline{Q} = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}$$

and study the future loss plus the loss at step k

$$\begin{aligned} f &= \underset{w_0}{E}\begin{pmatrix} qx \\ q\tilde{x} \end{pmatrix}^T \overline{X}\begin{pmatrix} qx \\ q\tilde{x} \end{pmatrix} + \\ &\quad + \underset{w_0}{E}\left\{\begin{pmatrix} x \\ \tilde{x} \end{pmatrix}^T \overline{Q}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} + \begin{pmatrix} w_1 \\ u \end{pmatrix}^T J\begin{pmatrix} w_1 \\ u \end{pmatrix}\right\} \\ &= \mathrm{Tr}\,\overline{B}_0^T\overline{X}\,\overline{B}_0 + \begin{pmatrix} x \\ \tilde{x} \end{pmatrix}^T\left\{\overline{A}^T\overline{X}\,\overline{A} + \overline{Q}\right\}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} + \\ &\quad + 2\begin{pmatrix} x \\ \tilde{x} \end{pmatrix}^T \overline{A}^T\overline{X}\,\overline{B}\begin{pmatrix} w_1 \\ u \end{pmatrix} + \\ &\quad + \begin{pmatrix} w_1 \\ u \end{pmatrix}^T\left\{\overline{B}^T\overline{X}\,\overline{B} + J\right\}\begin{pmatrix} w_1 \\ u \end{pmatrix} \end{aligned}$$

To complete the squares we determine F from

$$\left\{\begin{aligned} S &= \begin{pmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{pmatrix} = \overline{B}^T\overline{X}\,\overline{B} + J \\ SF &= -\overline{B}^T\overline{X}\,\overline{A} \\ F &= \begin{pmatrix} F_1 & F_{12} \\ F_2 & F_{22} \end{pmatrix} \end{aligned}\right. \tag{5}$$

giving

$$\begin{aligned} f &= \mathrm{Tr}\,\overline{B}_0^T\overline{X}\overline{B}_0 \\ &\quad + \begin{pmatrix} x \\ \tilde{x} \end{pmatrix}^T\left\{\overline{A}^T\overline{X}\,\overline{A} + \overline{Q} - F^TSF\right\}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} \\ &\quad + \left[\begin{pmatrix} w_1 \\ u \end{pmatrix} - F\begin{pmatrix} x \\ \tilde{x} \end{pmatrix}\right]^T S\left[\begin{pmatrix} w_1 \\ u \end{pmatrix} - F\begin{pmatrix} x \\ \tilde{x} \end{pmatrix}\right] \end{aligned} \tag{6}$$

We first evaluate the off-diagonal block of the second term in (6)

$$\begin{aligned} &\begin{pmatrix} I & 0 \end{pmatrix}\left\{\overline{A}^T\overline{X}\,\overline{A} + \overline{Q} - F^TSF\right\}\begin{pmatrix} 0 \\ I \end{pmatrix} = \\ &\begin{pmatrix} I & 0 \end{pmatrix}\left\{\overline{A}^T + F^T\overline{B}^T\right\}\overline{X}\,\overline{A}\begin{pmatrix} 0 \\ I \end{pmatrix} \\ &= -\left(\widehat{F}_1^T - F_1^T\right)B_{1L}^TY\left(A + LC_2 + B_{1L}\widehat{F}_1\right) \end{aligned} \tag{7}$$

and choose to make it zero by $\widehat{F}_1 = F_1$. Then rewrite the feedback-gain equation

$$S\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = -\overline{B}^T\overline{X}\,\overline{A}\begin{pmatrix} I \\ 0 \end{pmatrix}$$

$$\left\{[B_1B_2]^TX[B_1B_2] + [B_{1L}0]^TY[B_{1L}0] + J\right\}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

$$= -[B_1B_2]^TXA + [B_{1L}0]^TYB_{1L}F_1$$

which simplifies to

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = -J^{-1} \begin{pmatrix} B_1 & B_2 \end{pmatrix}^T VA \tag{8}$$

$$V^{-1} = X^{-1} + \begin{pmatrix} B_1 & B_2 \end{pmatrix} J^{-1} \begin{pmatrix} B_1 & B_2 \end{pmatrix}^T \tag{9}$$

This we recognize from standard discrete time H_∞-theory, see e.g. [Başar and Bernhard, 1991]. The following convexity-concavity conditions on the S-blocks in (5) are now required for the min-max problem

$$S_{11} = -\gamma^2 I + B_1^T X B_1 + B_{1L}^T Y B_{1L} < 0 \tag{10}$$

$$S_{12} = B_1^T X B_2 \tag{11}$$

$$S_{22} = I + B_2^T X B_2 > 0 \tag{12}$$

Using the completion of squares (6) and a lot of similar matrix algebra, we obtain that f has a maximum with respect to w_1 in case $u = u^* = F_2\check{x}$,

$$\max_{w_1} f = x^T X_k x + \tilde{x}^T Y_k \tilde{x} + \text{Tr}\,\left(B_0^T X_j B_0 + B_{0L}^T Y_j B_{0L}\right)$$

for $w_1 = w_1^*$ given by

$$w_1^* = F_1\check{x} + N\tilde{x} \tag{13}$$

$$-S_{11}N = B_1^T XA + B_{1L}^T Y(A + LC_2) \tag{14}$$

with S_{11} from (10), and $X(k)$ and $Y(k)$ given by the Riccati recursions

$$q^{-1}X = Q + A^T VA \tag{15}$$

$$\begin{aligned} q^{-1}Y &= A^T XA - A^T VA + \\ &\quad + (A + LC_2)^T Y(A + LC_2) - N^T S_{11} N \end{aligned} \tag{16}$$

where V is obtained by (9), and where $X(t_f) = Q_f$ and $Y(t_f) = 0$. To discuss the minimization with respect to u we rewrite the system using u^* and w_1^*:

$$\begin{cases} qx = A_c x + [B_1(N - F_1) - B_2F_2]\tilde{x} + B_0 w_0 + \\ \qquad + B_1(w_1 - w_1^*) + B_2(u - u^*) \\ q\tilde{x} = A_e\tilde{x} + B_{0L}w_0 + B_{1L}(w_1 - w_1^*) \end{cases} \tag{17}$$

$$\begin{cases} A_c = A + B_1F_1 + B_2F_2 \\ A_e = A + LC_2 + B_{1L}N \end{cases} \tag{18}$$

If we now assume $w_1 = w_1^*$, we thus have that the covariance matrix $P = E[\tilde{x}\tilde{x}^T \mid y^{k-1}]$, is given by

$$\begin{cases} qP = A_e P A_e^T + B_{0L}B_{0L}^T \\ P(0) = 0 \end{cases} \tag{19}$$

and the system equations could be written as

$$qx = (A + B_1F_1)x + B_2u + B_0w_0 + B_1(N - F_1)\tilde{x}$$

$$y = (C_2 + D_{21}F_1)x + D_{20}w_0 + D_{21}(N - F_1)\tilde{x}$$

Now define

$$\begin{pmatrix} V_a & V_{ab} \\ V_{ab}^T & V_b \end{pmatrix} = \text{Cov}\begin{pmatrix} qx - (A + B_1F_1)\check{x} - B_2u \\ y - (C_2 + D_{21}F_1)\check{x} \end{pmatrix}$$

$$= \text{Cov}\begin{pmatrix} (A + B_1N)\tilde{x} + B_0w_0 \\ (C_2 + D_{21}N)\tilde{x} + D_{20}w_0 \end{pmatrix} = \text{Cov}\begin{pmatrix} a \\ b \end{pmatrix}$$

so that

$$V_a = (A + B_1N)P(A + B_1N)^T + B_0B_0^T$$

$$V_{ab} = (A + B_1N)P(C_2 + D_{21}N)^T + B_0D_{20}^T$$

$$V_b = (C_2 + D_{21}N)P(C_2 + D_{21}N)^T + D_{20}D_{20}^T$$

Standard estimation theory now says that if L satisfies $LV_b + V_{ab} = 0$ simplifying to

$$-LD_{20}D_{20}^T = A_eP(C_2 + D_{21}N)^T + B_0D_{20}^T \tag{20}$$

then $\tilde{x}(k+1) \perp \mathcal{Y}_k$ where $\mathcal{Y}_k$ denotes all past information, since we in (3) made the choice

$$\begin{aligned} &\check{x}(k+1) - (A + B_1F_1(k))\check{x}(k) - B_2u(k) = \\ &\quad = -L(k)[y(k) - C_2\check{x}(k) - D_{21}F_1(k)\check{x}(k)] \end{aligned}$$

i.e. $a = -Lb$. This is the classical orthogonality principle. In summary the discrete time min-max problem boils down to the following:

THEOREM 1
Let the problem be given by (1)-(2). The discrete time optimal min-mix controllers satisfy

$$\begin{cases} q\check{x} = A\check{x} + B_2u + B_1F_1\check{x} - L(y - C_2\check{x} - D_{21}F_1\check{x}) \\ u = u^* = F_2\check{x}, \qquad w_1 = w_1^* = F_1x + (N - F_1)\tilde{x} \end{cases}$$

$$q^{-1}X = Q + A^T VA$$

$$\begin{aligned} q^{-1}Y &= A^T XA - A^T VA + \\ &\quad + (A + LC_2)^T Y(A + LC_2) - N^T S_{11} N \end{aligned}$$

$$qP = A_e P A_e^T + B_{0L}B_{0L}^T$$

$$-LD_{20}D_{20}^T = A_eP(C_2 + D_{21}N)^T + B_0D_{20}^T$$

where $\{F_1, F_2\}$, V, and N are given by (8), (9), and (14), and where A_c and A_e are defined in (18). The concavity-convexity conditions are given by (10), and (12).

Proof: The proof follows from the previous discussion in the following way: Complete the squares as in (6),and use the dynamic programming principle described in [Bernhard, 1992]. First prove that given $u = u^*$ the maximization over w_1 leads recursively to the value of the game given in (4). Here the Riccati equations for X and Y are used. Then put $w_1 = w_1^*$ instead and prove, using the orthogonality described above, that the minimization

$$\min_u E\,[f \mid y^{k-1}]$$

using (6) gives $u = u^*$ and thus also recursively the same value of the game in (4). The convexity-concavity conditions apply at each time step. □

Conjecture: The stationary controller now follows, if in addition A_c and A_e, in (18), are stable.

3. A Simple Example

Consider the following first order SISO system

$$qx = ax + u + w_1, \qquad x(0) = 0$$

$$y = x + w_0$$

Let the criteria function be given by

$$\sum \left(Qx^2 + u^2 - g^{-1}w_1^2\right)$$

where we have introduced $g := \gamma^{-2}$ for convenience. The stationary equations for the controller are the following

$$
\begin{aligned}
X &= Q + a^2 V, \qquad V = 1/(1/X + 1 - g) \\
A_c &= a - aV(1-g), \quad A_e = a + L + N \\
Y &= a^2 X - a^2 V + (a+L)^2 Y - N^2(X + Y - 1/g) \\
N &= (aX + (a+L)Y)/(1/g - X - Y) \\
P &= A_e^2 P + L^2 \sigma^2 \\
L\sigma^2 &= -A_e P
\end{aligned}
$$

These equations have two sets of solutions, the open loop solution with $L = 0, P = 0, A_e = a + N$, and the solution $L = A_e - 1/A_e$, $P/\sigma^2 = 1/A_e^2 - 1$ and $A_e = 1/(a+N)$. An interesting observation is now that different solutions should be chosen for different levels of γ. The open loop solution with $L = 0$ is for a stable system optimal if $\gamma^2 > \gamma_o^2 = Q/(1-a)^2$. For smaller γ there is a region where the other solution should be chosen. For $\gamma < \gamma_{mix}$ there are no solutions to the equations satisfying both the stability and convexity-concavity conditions. There are several open questions for future research here.

4. Conclusions

The paper has presented a solution of the min-mix problem for the discrete time, output feedback, finite horizon case. The solution was obtained by completion of squares. It is rewarding for the understanding to study how LQG and H_∞ follows as special cases from the min-mix formulas above. From a practical point of view the main drawback is that the obtained three Riccati equations are coupled and constitute a two point boundary value problem in the finite horizon case. It is nontrivial to find numerical solutions. There might be a possibility to triangularize these equations by clever change of variables.

The discrete time parallels the continuous time, e.g. in the use of a dynamic programming separation principle. The main difference is that the algebra is more complicated. There are also some other differences. The stochastic theory needed is simpler in discrete time since Ito-calculus does not have to be used. The conjugate-point theory connected to the Riccati equations is also simpler. It is replaced by the concavity-convexity conditions (10) and (12).

We have in this paper restricted ourselves to the case with a time delay of one step in the controller. The case with a direct term in the controller involves even more tedious algebra. We have also presented a simple example illustrating the equations and the solutions. More work has to be done to judge the practical merits of this controller design method.

The problem is quite rich and we have found many possibilities for future work. One goal could be to tie the results closer to the elegant operator factorization approach for the risk-sensitive problem treated by Whittle, see eg. [Whittle, 1990]. It might then be possible to describe the solution using operator factorizations, see [Hagander, 1973] for the LQG case.

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