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Georgios Kotsalis and Anders Rantzer

***Abstract*—This technical note investigates the model reduction problem for mean square stable discrete time Markov jump linear systems. For this class of systems a balanced truncation algorithm is developed. The reduced order model is suboptimal, however the approximation error, which is captured by means of the stochastic L_2 gain, is bounded from above by twice the sum of singular numbers associated to the truncated states of each mode. Such a result allows rigorous simplification of the dynamics of each mode in an independent manner with respect to a metric which is relevant from a robust control point of view.**

***Index Terms*—Jump linear systems (JLS's), linear time invariant (LTI) systems, Markov jump linear systems (MJLS's).**

I. Introduction

Jump linear systems (JLS's) form an important class of hybrid systems that combine continuous and discrete dynamics. They present an extension of linear time invariant (LTI) systems, in the sense that they use state update laws that are linear with respect to the analog state, with matrix coefficients depending on a quantized auxiliary input, frequently referred to as the switching signal. The transition between the different modes of operation is controlled by this exogenous parametric input. In this work it is assumed that the switching signal takes values in a finite set and that it follows an unconstrained evolution, modeled by a finite memory stochastic process.

There is a large body of literature in the fields of econometrics and system theory pertaining to the class of JLS's with randomly varying parameters. Various analysis and synthesis results applicable to LTI systems have been extended to the class of Markov jump linear systems (MJLS's). A comprehensive review of this material, and in particular robust control design algorithms using the stochastic L_2 gain as a sensitivity measure, can be found in [1] and the references therein.

A major question associated with MJLS's is that of complexity reduction. The work in [2] investigates the problem of obtaining an optimal in terms of the stochastic L_2 gain reduced model of fixed order. The formulation in [2] leads to a non convex optimization problem and the proposed algorithms do not guarantee convergence to the global optimum. In contrast to [2] the search of a reduced model in the current paper is based on a convex programming formulation, the obtained reduced model is suboptimal in terms of the stochastic L_2 gain, however it is accompanied by an a priori computable upper bound to the approximation error. The reduction algorithm in this work can be interpreted as an extension of the well known balanced truncation algorithm for LTI systems to the wider class of MJLS's. Balanced realizations were originally proposed in the controls literature in [3]. Their utilization for model reduction purposes of LTI systems and associated error bounds

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in continuous and discrete-time settings can be found in [4]–[6]. Balanced truncation has been investigated also outside the realm of LTI systems. In [7] a generalization to multidimensional and uncertain systems in the linear-fractional framework is presented. The case of linear parameter-varying systems is the subject of [8] and linear time-varying systems are handled in [9] and [10]. Balanced truncation of JLS's with independent identically distributed parameters is investigated in [11]. Approximation algorithms for various classes of stochastic hybrid systems based on the concept of approximate bisimulation are developed in [12].

A. Notation

The set of nonnegative integers is $\mathbf{N}$, the set of positive integers is $\mathbf{Z}_+$ and the set of real numbers is $\mathbf{R}$. For $n \in \mathbf{Z}_+$, $\mathbf{R}^n$ denotes the Euclidean n-space. The transpose of a column vector $x \in \mathbf{R}^n$ is x'. For $x \in \mathbf{R}^n$ let $|x|^2 = x'x$. For $P \in \mathbf{R}^{n\times n}$ let $P > 0$ indicate that it is a positive definite matrix and $|x|_P^2 = x'Px$. The positive definite square root of P is denoted by $P^{1/2}$. The identity matrix in $\mathbf{R}^{n\times n}$ is denoted by I_n. For $A \in \mathbf{R}^{n\times n}$, $r_\sigma[A]$ denotes the spectral radius of A. For $P, Q \in \mathbf{R}^{n\times n}$, the inner product of these two matrices is defined as $\langle P, Q\rangle = \mathrm{Tr}[P'Q]$. For $f : \mathbf{N} \to \mathbf{R}^n$ let $l_2^n = \{f : \sum_{k=0}^{\infty} |f(k)|^2 < \infty\}$. For $f \in l_2^n$ let $\|f\|_2^2 = \sum_{k=0}^{\infty} |f(k)|^2$, $S_2^n = \{f \in l_2^n : \|f\|_2 = 1\}$. The expected value of the random variable x is denoted by $\mathbf{E}[x]$. For $N \in \mathbf{Z}_+, \boldsymbol{\Theta} = \{1,\ldots,N\}$, $n : \boldsymbol{\Theta} \to \mathbf{Z}_+$, $r : \boldsymbol{\Theta} \to \mathbf{N}$ with $r[i] < n[i], \forall i \in \boldsymbol{\Theta}$, define $\mathbf{H}^n = \mathbf{R}^{n[1]\times n[1]} \times \ldots \times \mathbf{R}^{n[N]\times n[N]}$, its subset $\mathbf{H}_+^n = \{U \in \mathbf{H}^n : U[i] > 0, i \in \boldsymbol{\Theta}\}$. and $VC_{n[i]-r[i]} = \{x \in \mathbf{R}^{n[i]} : x[j] = 0, j > n[i] - r[i]\}$. Let $A[i] \in \mathbf{R}^{n[i]\times n[i]}, i \in \boldsymbol{\Theta}$, $\mathrm{diag}\{A[1],\ldots,A[N]\}$ denotes the block diagonal matrix

$$\begin{bmatrix} A[1] & & 0 \\ & \ddots & \\ 0 & & A[N] \end{bmatrix}.$$

II. Preliminaries

A. System Model

Let $m, q, N \in \mathbf{Z}_+, \boldsymbol{\Theta} = \{1,\ldots,N\}$ and $n : \boldsymbol{\Theta} \to \mathbf{Z}_+$. Define a MJLS $\mathbf{L}$ as a dynamical system with input, $f(k) \in \mathbf{R}^m$, discrete valued state variable $\theta(k) \in \boldsymbol{\Theta}$, also referred to as the system mode, mode transition signal $\phi(k) = (\theta(k-1), \theta(k))$, continuous valued state variable $x(k) \in \mathbf{R}^{n[\theta(k)]}$, and output $y(k) \in \mathbf{R}^q$, related by the state space equations

$$\begin{aligned} x(k+1) &= A[\phi(k+1)]\, x(k) + B[\phi(k+1)]\, f(k), \\ y(k) &= C[\theta(k)]\, x(k), \quad k \in \mathbf{N}. \end{aligned} \tag{1}$$

The system mode θ follows an unconstrained stochastic evolution, modeled as a Markov process on $\boldsymbol{\Theta}$. The transition probability matrix of the Markov chain is denoted by $P = [p_{ij}]$, $i, j \in \boldsymbol{\Theta}$, $p_{ij} = \mathbf{P}[\theta(k+1) = j|\theta(k) = i]$. The input f is assumed to be deterministic. The state space matrices have compatible dimensions, in particular $A[\phi(k+1)] \in \mathbf{R}^{n[\theta(k+1)]\times n[\theta(k)]}$, $B[\phi(k+1)] \in \mathbf{R}^{n[\theta(k+1)]\times m}$, $C[\theta(k)] \in \mathbf{R}^{q\times n[\theta(k)]}$, $\theta(k), \theta(k+1) \in \boldsymbol{\Theta}$. The matrices in the state space recursion depend on the mode transition allowing the dimension of the continuous valued part of the state variable to vary depending on which discrete mode the system resides in. Similar type of MJLS's as in (1) were considered in [13], [14] and have been also used in the study of networked control systems in a probabilistic framework, a review paper in that area is [15].

B. Stability

Definition 2.1: The MJLS $\mathbf{L}$ with $f(k) = 0, \forall k \in \mathbf{N}$ is mean square stable, if for every initial condition $\theta(0) \in \boldsymbol{\Theta}$, $x(0) \in \mathbf{R}^{n[\theta(0)]}$, $\mathbf{E}[|x(k)|^2] \to 0$ as $k \to \infty$.

Definition 2.2: Assume that $x(0) = 0$. The stochastic L_2 gain for the MJLS $\mathbf{L}$ is denoted by $\gamma_{\mathbf{L}}$ and is defined by

$$\gamma_{\mathbf{L}}^2 = \sup_{\theta(0)\in\boldsymbol{\Theta}} \sup_{f\in S_2^m} \sum_{k=0}^{\infty} \mathbf{E}\left[|y(k)|^2\right].$$

Theorem 2.1: Consider a MJLS $\mathbf{L}$. Let $F \in \mathbf{H}_+^n$. System $\mathbf{L}$ is mean square stable if and only if there exists a unique $G \in \mathbf{H}_+^n$ such that:

$$G[i] - \sum_{j\in\boldsymbol{\Theta}} p_{ij} A[i,j]' G[j] A[i,j] = F[i], \quad \forall i \in \boldsymbol{\Theta}. \tag{2}$$

Define the linear operator $\mathcal{L} : \mathbf{H}^n \to \mathbf{H}^n$

$$\mathcal{L}[V] = W,\ W[i] = \sum_{j\in\boldsymbol{\Theta}} p_{ij} A[i,j]' V[j] A[i,j],\ i \in \boldsymbol{\Theta}. \tag{3}$$

The equations in (2) are equivalent to $\mathcal{L}[G] - G = -F$.

Lemma 2.1: Consider a MJLS $\mathbf{L}$, and let $\gamma > 0$. Consider a nonnegative, real valued, measurable function $V[x,\theta]$, $x \in \mathbf{R}^{n[\theta]}$, $\theta \in \boldsymbol{\Theta}$, with $V[0,\theta] = 0$ and $\mathbf{E}[V[x(k),\theta(k)]] < \infty$ for all trajectories of $\mathbf{L}$. Suppose that $\forall f(k) \in \mathbf{R}^m$, $\forall x(k) \in \mathbf{R}^{n[\theta(k)]}$, $\forall \theta(k) \in \boldsymbol{\Theta}$

$$\begin{aligned} |y(k)|^2 + \mathbf{E}\left[V\left[x(k+1), \theta(k+1)\right] | x(k), \theta(k)\right] \\ \le V\left[x(k), \theta(k)\right] + \gamma^2 |f(k)|^2 \end{aligned} \tag{4}$$

then the stochastic L_2 gain of $\mathbf{L}$ does not exceed γ.

Lemma 2.2: If the MJLS $\mathbf{L}$ is mean square stable, then its stochastic L_2 gain is finite.

Proofs for the above three statements can be found in [16].

C. Reduced Order Model and State Truncation

Let $\hat{n} : \boldsymbol{\Theta} \to \mathbf{Z}_+$ and $\hat{n}[i] \le n[i], \forall i \in \boldsymbol{\Theta}$ with the inequality being strict for at least one of the modes. A reduced order MJLS is denoted by $\hat{\mathbf{L}}$ and has state space representation

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}[\phi(k+1)]\, \hat{x}(k) + \hat{B}[\phi(k+1)]\, f(k), \\ \hat{y}(k) &= \hat{C}[\theta(k)]\, \hat{x}(k), \quad k \in \mathbf{N} \end{aligned} \tag{5}$$

where $\hat{y}(k) \in \mathbf{R}^q$, $f(k) \in \mathbf{R}^m$, $\theta(k) \in \boldsymbol{\Theta}$ and $\hat{x}(k) \in \mathbf{R}^{\hat{n}[\theta(k)]}$. In order to quantify the fidelity of $\hat{\mathbf{L}}$, an error system $\mathbf{E}_{\{\mathbf{L},\hat{\mathbf{L}}\}}$ is introduced, whose inputs are the common inputs $f(k), \theta(k)$ of $\mathbf{L}$ and $\hat{\mathbf{L}}$ and whose output is $e(k) = y(k) - \hat{y}(k)$. The state space representation of $\mathbf{E}_{\{\mathbf{L},\hat{\mathbf{L}}\}}$ is

$$\begin{aligned} \begin{bmatrix} x(k+1) \\ \hat{x}(k+1) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} [\phi(k+1)] \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix} \\ &\quad + \begin{bmatrix} B \\ \hat{B} \end{bmatrix} [\phi(k+1)]\, f(k), \quad k \in \mathbf{N}, \\ e(k) &= [C \quad -\hat{C}]\, [\theta(k)] \begin{bmatrix} x(k) \\ \hat{x}(k) \end{bmatrix}. \end{aligned} \tag{6}$$

The objective of model reduction is to find a reduced order model such that the stochastic L_2 gain of the error system $\mathbf{E}_{\{\mathbf{L},\hat{\mathbf{L}}\}}$ is small. Reduced order models are obtained by means of truncation. The number of truncated states at a particular mode is given by $r[\theta(k)] = n[\theta(k)] - \hat{n}[\theta(k)], \theta(k) \in \mathbf{\Theta}$. The following partitions are used:

$$\begin{aligned} A[\phi(k+1)] &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} [\phi(k+1)], \\ &\quad A_{11}[\phi(k+1)] \in \mathbf{R}^{\hat{n}[\theta(k+1)] \times \hat{n}[\theta(k)]}, \\ B[\phi(k+1)] &= \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} [\phi(k+1)], \\ &\quad B_1[\phi(k+1)] \in \mathbf{R}^{\hat{n}[\theta(k)] \times m}, \\ C[\theta(k)] &= [C_1 \quad C_2][\theta(k)], \\ &\quad C_1[\theta(k)] \in \mathbf{R}^{q \times \hat{n}[\theta(k)]}, \\ x(k)' &= [x_1(k)' \quad x_2(k)'], \quad x_1(k) \in \mathbf{R}^{\hat{n}[\theta(k)]}. \end{aligned}$$

The state space matrices of the reduced order model are given by $\{A_{11}[\phi(k+1)], B_1[\phi(k+1)], C_1[\theta(k)]\}$. It will be convenient to think of the continuous part of the state variable of the reduced system submerged in the original state space. Let $\bar{x}(k)' = (x_1(k)', 0') \in \mathbf{R}^{n[\theta(k)]}$. Consider the system $\bar{\mathbf{L}}$

$$\begin{aligned} \bar{x}(k+1) &= \left(I_{n[\theta(k+1)]} - E_{r[\theta(k+1)]}\right) \\ &\quad \times \left(A[\phi(k+1)]\bar{x}(k) + B[\phi(k+1)] f(k)\right), \\ \bar{y}(k) &= C[\theta(k)]\bar{x}(k), \quad k \in \mathbf{N}, \\ E_{r[\theta(k)]} &= \operatorname{diag}\left\{0, I_{r[\theta(k)]}\right\} \in \mathbf{R}^{n[\theta(k)] \times n[\theta(k)].} \end{aligned} \tag{7}$$

Evidently one can identify $\bar{\mathbf{L}}$ with $\hat{\mathbf{L}}$, since for the same input signal $\{f(k)\}_{k\in\mathbf{N}}$ in (5) and (7) and if $\bar{x}(0)' = (\hat{x}(0)', 0')$, one has $\bar{x}(k)' = (\hat{x}(k)', 0')$ and $\hat{y}(k) = \bar{y}(k), \forall k \in \mathbf{N}$. On these grounds, $\hat{\mathbf{L}}$ will be used for both state space representations (5), (7), which one is meant will be clear from the context. The idea of truncation presupposes that the states $x_2(k)$ are small in some appropriate sense. Mode dependent transformation matrices will be utilized to achieve this objective. Let $T[i] \in \mathbf{R}^{n[i] \times n[i]}, i \in \mathbf{\Theta}$ be invertible matrices. Consider the change in coordinate system $x(k) = T[\theta(k)]\tilde{x}(k)$, one has

$$\begin{aligned} &\left\{\tilde{A}[\phi(k+1)], \tilde{B}[\phi(k+1)], \tilde{C}[\theta(k)]\right\} \\ &\quad = \left\{T[\theta(k+1)]^{-1} A[\phi(k+1)] T[\theta(k)],\right. \\ &\qquad \left. T[\theta(k+1)]^{-1} B[\phi(k+1)], C[\theta(k)] T[\theta(k)]\right\}. \end{aligned} \tag{8}$$

III. Balanced Truncation for Markov Jump Linear Systems

A. Dissipation Inequalities

A balanced truncation procedure for mean square stable MJLS's will be developed. Central to the reduction algorithm are two sets of dissipation inequalities, expressed in the form of linear matrix inequalities (LMI's) that will be referred to as input and output dissipation inequalities respectively. The mean square stability assumption guarantees solutions to these LMI's of a particular diagonal structure.

1) Output Dissipation Inequalities: Let $U \in \mathbf{H}_+^n$, the output dissipation inequalities are $\forall x \in \mathbf{R}^{n[i]}, \forall i \in \mathbf{\Theta}$

$$|x|_{U[i]}^2 \geq \sum_{j\in\mathbf{\Theta}} p_{ij} \left(|A[i,j]x|_{U[j]}^2\right) + |C[i]x|^2. \tag{9}$$

The above relations are LMI's and using the operator $\mathcal{L}$ introduced in (3) they can be written more compactly as

$$\mathcal{L}[U] - U \leq -Q \tag{10}$$

where $Q = [Q[1], \ldots, Q[N]], Q[i] = C[i]'C[i] \geq 0, i \in \mathbf{\Theta}$.

Lemma 3.1: Given a mean square stable MJLS $\mathbf{L}$, there exists $U \in \mathbf{H}_+^n$, such that (9) is satisfied.

Proof: The proof follows directly from theorem 2.1. Let $\tilde{Q}[i] = Q[i] + \epsilon I_{n[i]}, i \in \mathbf{\Theta}$, where $\epsilon > 0$ is chosen so that $\tilde{Q}[i] > 0$. Mean square stability is equivalent to the existence of a unique solution $U \in \mathbf{H}_+^n$ to $\mathcal{L}[U] - U = -\tilde{Q} \leq -Q$. Thus the N-tuple U satisfies (10) and therefore (9). ■

2) Input Dissipation Inequalities: Let $R \in \mathbf{H}_+^n$, the input dissipation inequalities are $\forall x \in \mathbf{R}^{n[i]}, \forall f \in \mathbf{R}^m, \forall i \in \mathbf{\Theta}$

$$|x|_{R[i]}^2 + |f|^2 \geq \sum_{j\in\mathbf{\Theta}} p_{ij} \left(|A[i,j]x + B[i,j]f|_{R[j]}^2\right). \tag{11}$$

The above set of LMI's can be written equivalently as $\forall i \in \mathbf{\Theta}$

$$\begin{aligned} K[i] &= \begin{bmatrix} K_{11}[i] & K_{12}[i] \\ K_{12}'[i] & K_{22}[i] \end{bmatrix} \leq \operatorname{diag}\left\{R[i], I_{n[i]}\right\}, \\ K[i] &= \sum_{j\in\mathbf{\Theta}} p_{ij} \begin{bmatrix} A[i,j]' \\ B[i,j]' \end{bmatrix} R[j] \, [A[i,j] \quad B[i,j]]. \end{aligned} \tag{12}$$

Lemma 3.2: Given a mean square stable MJLS $\mathbf{L}$, there exists $R \in \mathbf{H}_+^n$ such that (11) is satisfied.

Proof: By application of the Schur lemma to (12)it suffices to find $R \in \mathbf{H}_+^n$ such that $\forall i \in \mathbf{\Theta}$

$$K_{11}[i] < R[i] \tag{13}$$

$$K_{22}[i] - K_{12}'[i] \left(K_{11}[i] - R[i]\right)^{-1} K_{12}[i] < I_{n[i].} \tag{14}$$

Mean square stability is equivalent to the existence of $\tilde{R} \in \mathbf{H}_+^n$ such that $\forall i \in \mathbf{\Theta}, -\tilde{R}[i] + \sum_{j\in\mathbf{\Theta}} p_{ij} A[i,j]'\tilde{R}[j]A[i,j] < 0$. Set $R[i] = \alpha\tilde{R}[i], i \in \mathbf{\Theta}, \alpha > 0$. Condition (13) is automatically satisfied. Both terms in the left hand side of (14) scale linearly with α, thus (14) is satisfied by taking α small enough. ■

The relations in (12) can be expressed in an equivalent form where the search variables are $Z[i] = R[i]^{-1}, i \in \mathbf{\Theta}$. In particular by using the Schur lemma and accounting for the fact that $Z[i] > 0, i \in \mathbf{\Theta}$ relations (12) are equivalent to

$$\begin{aligned} V[i] &= \begin{bmatrix} V_{11}[i] & V_{12}[i] \\ V_{12}'[i] & V_{22}[i] \end{bmatrix} \geq 0, \\ V_{11}[i] &= \operatorname{diag}\left\{R[i], I_{n[i]}\right\}, \\ V_{12}[i] &= \begin{bmatrix} \sqrt{p_{i1}}A[i,1]' & \ldots & \sqrt{p_{iN}}A[i,N]' \\ \sqrt{p_{i1}}B[i,1]' & \ldots & \sqrt{p_{iN}}B[i,N]' \end{bmatrix}, \\ V_{22}[i] &= \operatorname{diag}\{Z[1], \ldots, Z[N]\}, \quad i \in \Theta \end{aligned}$$

and the latter set of LMI's are equivalent to

$$\begin{aligned} &\operatorname{diag}\{Z[1], \ldots, Z[N]\} \geq \bar{A}[i]Z[i]\bar{A}[i]' + \bar{B}[i]\bar{B}[i]', \\ &\bar{A}[i]' = \left[\sqrt{p_{i1}}A[i,1]' \ \ldots \ \sqrt{p_{iN}}A[i,N]'\right], \\ &\bar{B}[i]' = \left[\sqrt{p_{i1}}B[i,1]' \ \ldots \ \sqrt{p_{iN}}B[i,N]'\right], \quad i \in \mathbf{\Theta}. \end{aligned} \tag{15}$$

3) Obtaining Diagonal Solutions to the Dissipation Inequalities: Certain proofs become more transparent if the solutions to the dissipation inequalities are simultaneously transformed to diagonal matrices.

Lemma 3.3: Let $U \in \mathbf{H}_+^n, R \in \mathbf{H}_+^n$ satisfy the dissipation inequalities (9) and (11) respectively. Consider the mode dependent coordinate transformation $x = T[i]\tilde{x}$, $i \in \mathbf{\Theta}$, where $T \in \mathbf{H}^n$, $T[i]$ invertible, $i \in \mathbf{\Theta}$. In the new coordinates, one has $\forall \tilde{x} \in \mathbf{R}^{n[i]}, \forall f \in \mathbf{R}^m, \forall i \in \mathbf{\Theta}$

$$|\tilde{x}|^2_{\tilde{U}[i]} \geq \sum_{j\in\mathbf{\Theta}} p_{ij}\left(\left|\tilde{A}[i,j]\tilde{x}\right|^2_{\tilde{U}[j]}\right) + \left|\tilde{C}[i]\tilde{x}\right|^2,$$
$$|\tilde{x}|^2_{\tilde{R}[i]} + |f|^2 \geq \sum_{j\in\mathbf{\Theta}} p_{ij}\left(\left|\tilde{A}[i,j]\tilde{x} + \tilde{B}[i,j]f\right|^2_{\tilde{R}[j]}\right),$$
$$\tilde{U}[i] = T'[i]U[i]T[i],\ \ \tilde{R}[i] = T'[i]R[i]T[i]. \quad (16)$$

From (16) one can conclude that the eigenvalues of $Z[i]U[i]$, $i \in \mathbf{\Theta}$ remain invariant under mode dependent coordinate transformations since

$$\tilde{Z}[i]\tilde{U}[i] = T[i]^{-1}Z[i]U[i]T[i], \quad i \in \mathbf{\Theta}. \quad (17)$$

Lemma 3.4: Let $U \in \mathbf{H}^n_+$, $R \in \mathbf{H}^n_+$ satisfy the dissipation inequalities (9) and (11) respectively. There exists a mode dependent coordinate transformation $x = T[i]\tilde{x}$, $i \in \mathbf{\Theta}$, where $T \in \mathbf{H}^n$, $T[i]$ invertible, $i \in \mathbf{\Theta}$, such that $\tilde{U}[i] = \tilde{Z}[i] = \mathrm{diag}\{\beta_{1i}, \ldots, \beta_{n[i]i}\}$, $i \in \mathbf{\Theta}$.

Proofs of the above two statements can be found in [16].

B. Upper Bound on the Approximation Error

This section is devoted in proving an upper bound to the approximation error with respect to the stochastic L_2 gain when the dimension of the continuous valued part of the states is reduced by means of truncation.

Theorem 3.1: Consider a mean square stable system $\mathbf{L}$. Suppose that $U \in \mathbf{H}^n_+$, $R \in \mathbf{H}^n_+$ satisfy the dissipation inequalities (9), (11) respectively. Assume that for a particular mode $i^* \in \mathbf{\Theta}$, $U[i^*] = \mathrm{diag}\{\Sigma_{U_{i^*}}, \beta I_{r[i^*]}\}$ and $R[i^*] = \mathrm{diag}\{\Sigma_{R_{i^*}}, (1/\beta)I_{r[i^*]}\}$. Let $\hat{\mathbf{L}}$ be the reduced order model obtained by truncating the last $r[i^*]$ continuous states corresponding to the mode i^* of $\mathbf{L}$. The stochastic L_2 gain of the error system $\mathbf{E}_{\mathbf{L},\hat{\mathbf{L}}}$ is bounded from above by

$$\gamma_{\mathbf{E}_{\mathbf{L},\hat{\mathbf{L}}}} \leq 2\beta. \quad (18)$$

Proof: Introduce the matrix $E_{r[i]} = \mathrm{diag}\{0, I_{r[i]}\} \in \mathbf{R}^{n[i]\times n[i]}$ and note that $E_{r[i]} = 0$ unless $i = i^*$. Let $\hat{x}(k)' = (x_1(k)', 0')$ be the continuous part of the state variable of the reduced order model submerged in the original state space. The dynamics of the reduced order system are given by (7). The following variables are introduced to shorten subsequent notation, $z(k) = x(k)+\hat{x}(k)$, $\delta(k) = x(k)-\hat{x}(k)$, $e(k) = y(k) - \hat{y}(k)$

$$h[\phi(k+1)] = A[\phi(k+1)]\hat{x}(k) + B[\phi(k+1)]f(k),$$
$$z(k+1) = A[\phi(k+1)]z(k) + 2B[\phi(k+1)]f(k) - E_{r[\theta(k+1)]}h[\phi(k+1)],$$
$$\delta(k+1) = A[\phi(k+1)]\delta(k) + E_{r[\theta(k+1)]}h[\phi(k+1)],$$
$$e(k) = C[\theta(k)]\delta(k), \quad k \in \mathbf{N}.$$

According to Lemma 2.1 it is sufficient to find a storage function such that $\forall x \in \mathbf{R}^{n[i]}, \forall \hat{x} \in VC_{n[i]-r[i]}, \forall f \in \mathbf{R}^m, \forall i \in \mathbf{\Theta}$

$$|C[i]\delta|^2 + \Delta V_i \leq 4\beta^2|f|^2,$$
$$\Delta V_i = \sum_{j\in\mathbf{\Theta}} p_{ij}V[x(+),\hat{x}(+),j] - V[x,\hat{x},i]$$
$$x(+) = A[i,j]x + B[i,j]f$$
$$\hat{x}(+) = \left(I_{n[j]} - E_{r[j]}\right)(A[i,j]\hat{x} + B[i,j]f). \quad (19)$$

A quadratic storage function candidate is given by

$$V[x,\hat{x},i] = \beta^2|x+\hat{x}|^2_{R[i]} + |x-\hat{x}|^2_{U[i]} = \beta^2|z|^2_{R[i]} + |\delta|^2_{U[i]}.$$

One needs to verify (19). Let $x \in \mathbf{R}^{n[i]}$, $\hat{x} \in VC_{n[i]-r[i]}$, $f \in \mathbf{R}^m$, $i \in \mathbf{\Theta}$, one has

$$\Delta V_i = \sum_{j\in\mathbf{\Theta}} p_{ij}\left|A[i,j]\delta + E_{r[j]}h[i,j]\right|^2_{U[j]} + \beta^2\sum_{j\in\mathbf{\Theta}} p_{ij}\left|A[i,j]z + 2B[i,j]f - E_{r[j]}h[i,j]\right|^2_{R[j]} + - \beta^2|z|^2_{R[i]} - |\delta|^2_{U[i]}.$$

Expanding the individual terms in the above expressions, gives

$$\begin{aligned}\Delta V_i = &\sum_{j\in\mathbf{\Theta}} p_{ij}|A[i,j]\delta|^2_{U[j]} - |\delta|^2_{U[i]} \\ &+ \beta^2\sum_{j\in\mathbf{\Theta}} p_{ij}|A[i,j]z + 2B[i,j]f|^2_{R[j]} - \beta^2|z|^2_{R[i]} \\ &+ 2\beta\sum_{j\in\mathbf{\Theta}} p_{ij}\left|E_{r[j]}h(i,j)\right|^2 \\ &- 2\beta\sum_{j\in\mathbf{\Theta}} p_{ij}\left(E_{r[j]}h(i,j)\right)' \\ &\times (A[i,j]z + 2B[i,j]f - A[i,j]\delta).\end{aligned} \quad (20)$$

Applying the dissipation inequality (9) to the first two terms of (20) gives $\sum_{j\in\mathbf{\Theta}} p_{ij}|A[i,j]\delta|^2_{U[j]} - |\delta|^2_{U[i]} \leq -|C[i]\delta|^2$. Applying the dissipation inequality (11) to the second line in (20) gives

$$\beta^2\sum_{j\in\mathbf{\Theta}} p_{ij}|A[i,j]z + 2B[i,j]f|^2_{R[j]} - \beta^2|z|^2_{R[i]} \leq 4\beta^2|f|^2.$$

For the last term of (20) note that $A[i,j]z + 2B[i,j]f - A[i,j]\delta = 2h[i,j]$, and that $E^2_{r[j]} = E_{r[j]}$. Using the above relations we obtain

$$\Delta V_i + |C[i]\delta|^2 \leq 4\beta^2|f|^2 - 2\beta\sum_{j\in\mathbf{\Theta}} p_{ij}\left|E_{r[j]}h[i,j]\right|^2.$$

Since $2\beta\sum_{j\in\mathbf{\Theta}} p_{ij}|E_{r[j]}h[i,j]|^2 \geq 0$ (19) is satisfied. ∎

The above result can be generalized to the case where truncation is applied recursively in order to achieve further reduction. The recursive truncation is enabled by the following lemma whose proof is in [16].

Lemma 3.5: Consider the same setting as in theorem 3.1. Let $\hat{U}[i] = U[i]$, $\hat{R}[i] = R[i]$, when $i \in \mathbf{\Theta}$, $i \neq i^*$, $\hat{U}[i^*] = \Sigma_{U_{i^*}}$, $\hat{R}[i^*] = \Sigma_{R_{i^*}}$. For the reduced order model $\hat{\mathbf{L}}$ one has $\forall \hat{x} \in \mathbf{R}^{\hat{n}[i]}, \forall f \in \mathbf{R}^m, \forall i \in \mathbf{\Theta}$

$$|\hat{x}|^2_{\hat{U}[i]} \geq \sum_{j\in\mathbf{\Theta}} p_{ij}\left(\left|\hat{A}[i,j]\hat{x}\right|^2_{\hat{U}[j]}\right) + \left|\hat{C}[i]\hat{x}\right|^2,$$
$$|\hat{x}|^2_{\hat{R}[i]} + |f|^2 \geq \sum_{j\in\mathbf{\Theta}} p_{ij}\left(\left|\hat{A}[i,j]\hat{x} + \hat{B}[i,j]f\right|^2_{\hat{R}[j]}\right).$$

The next theorem is now obtained readily.

Theorem 3.2: Given a mean square stable system $\mathbf{L}$ and matrices $U \in \mathbf{H}^n_+$, $R \in \mathbf{H}^n_+$ such that the dissipation inequalities (9), (11) are satisfied, and suppose for mode $i^* \in \mathbf{\Theta}$, $U[i^*] = \mathrm{diag}\{\Sigma_{1i^*}, \beta_1 I_{r_1[i^*]}, \ldots, \beta_s I_{r_s[i^*]}\}$ and $R[i^*] = \mathrm{diag}\{\bar{\Sigma}_{1i^*}, (1/\beta_1)I_{r_1[i^*]}, \ldots, (1/\beta_s)I_{r_s[i^*]}\}$. Let $\hat{\mathbf{L}}$ be the reduced order model obtained by truncating the last $r_1[i^*] + \ldots + r_s[i^*]$ continuous states corresponding to the mode i^* of $\mathbf{L}$. Then, the stochastic L_2 gain of the error system $\mathbf{E}_{\mathbf{L},\hat{\mathbf{L}}}$ is bounded from above by

$$\gamma_{\mathbf{E}_{\mathbf{L},\hat{\mathbf{L}}}} \leq 2(\beta_1 + \ldots + \beta_s). \quad (21)$$

Proof: First remove the last $r_s[i^*]$ and call the truncated system $\mathbf{L}_s$. By theorem 3.1 one has $\gamma_{\mathbf{E}_{\mathbf{L},\mathbf{L}_s}} \leq 2\beta_s$. Due to lemma 3.5 the truncated system $\mathbf{L}_s$ still satisfies the dissipation inequalities (9), (11), thus one can proceed iteratively and repeat the truncation process until $\mathbf{L}_1 =$

$\hat{\mathbf{L}}$ is reached. By invoking the triangle inequality one has $\gamma_{\mathbf{E}_{\mathbf{L},\hat{\mathbf{L}}}} \leq \gamma_{\mathbf{E}_{\mathbf{L},\mathbf{L_s}}} + \ldots + \gamma_{\mathbf{E}_{\mathbf{L}_2,\mathbf{L}_1}} = 2(\beta_1 + \ldots + \beta_s)$. ■

The derived error bound readily generalizes to the case where continuous states associated with different modes are truncated. Each mode can be treated successively by virtue of lemma 3.5.

C. Computational Considerations

In this section it will be discussed how to obtain solutions to the dissipation inequalities that are suitable for truncating the continuous valued part of the state of a particular discrete mode, call it i. Suppose that $U \in \mathbf{H}_+^n$, $R \in \mathbf{H}_+^n$ satisfy the dissipation inequalities (9), (11). In lemma 3.4 it was established that there exists a mode dependent coordinate transformation $x = T[i]\tilde{x}$, $i \in \mathbf{\Theta}$, where $T \in \mathbf{H}^n$, $T[i]$ invertible, $i \in \mathbf{\Theta}$, such that

$$\tilde{U}[i] = \tilde{Z}[i] = W[i] = \operatorname{diag}\left\{\beta_{1i}, \ldots, \beta_{n[i]i}\right\}. \tag{22}$$

Furthermore (17) implies that $\operatorname{Tr}[U[i]Z[i]] = \operatorname{Tr}[\tilde{U}[i]\tilde{Z}[i]] = \sum_{j=1}^{n[i]} \beta_{ji}^2, \forall i \in \Theta$. Denote the subset of $\mathbf{H}_+^n$, whose elements satisfy (10) with $\mathbf{H}_U^n$. Similarly let $\mathbf{H}_Z^n$ denote the subset of $\mathbf{H}_+^n$, whose elements satisfy (15). Given that the error bound (21) is controlled by the sum of the nonrepeated eigenvalues corresponding to the truncated states a reasonable objective is

$$\min_{U \in \mathbf{H}_U^n, Z \in \mathbf{H}_Z^n} \operatorname{Tr}\left[U[i]Z[i]\right]. \tag{23}$$

This is a nonconvex optimization problem, which needs to be relaxed for the sake of computation tractability. Note for fixed $Z[i]$, the objective function in (23) is monotonic in $U[i]$. Thus from an error bound point of view it is desirable to find a minimal solution $U_- \in \mathbf{H}_U^n$, in the sense that $U_-[j] \leq U[j], \forall j \in \mathbf{\Theta}, \forall U \in \mathbf{H}_U^n$.

Lemma 3.6: The output dissipation inequalities possess a minimal solution.

Proof: Let $Q[i] = C[i]'C[i] \geq 0$, $i \in \mathbf{\Theta}$. Consider (10) and the corresponding Lyapunov like equation

$$\mathcal{L}[U_-] - U_- = -Q. \tag{24}$$

Subtracting (24) from (10) and by letting $\Delta = U - U_-$, one gets $\mathcal{L}[\Delta] - \Delta = -Q_\Delta \leq 0$. Mean square stability implies $r_\sigma[\mathcal{L}] < 1$ and $\Delta = \sum_{i=0}^{\infty} \mathcal{L}^i[Q_\Delta]$ solves the above Lyapunov like equation. By construction $\Delta \geq 0$ proving the minimality of U_- among all solutions of (9). ■

The N-tuple of matrices U_- can be computed as the limit of the nondecreasing sequence $\{U(k)\}$, where $U(k+1) = Q + \mathcal{L}[U(k)]$, $U(0) = Q$, $k \in \mathbf{N}$. The convergence to the fixed point U_- is exponential. The situation concerning the computation of U_- is completely analogous to the balanced truncation algorithm for the LTI case. For $N = 2$ one can compute U_- for systems up to about 1000 states per discrete mode on a standard PC. Having obtained U_- and in particular $U_-[i]$ one can revisit the objective function in (23). The matrix $Z[i]$ can now be obtained as the result of the optimization problem

$$\min_{Z \in \mathbf{H}_Z^n} \operatorname{Tr}\left[U_-[i]Z[i]\right]. \tag{25}$$

The optimization problem in (25) is a semidefinite program that is convex and can be solved efficiently using interior point methods [17]. This step of the reduction algorithm is the limiting factor since the computational cost for obtaining Z is higher than the the matrix product iterations required for computing U. On a standard PC using SeDuMi [18] together with YALMIP [19] one can compute solutions to (25), when $N = 2$, for systems up to about 100 states per discrete mode.

D. Remarks

Markov jump linear systems contain as special cases LTI systems as well linear time varying periodic systems. For the latter two classes of systems balanced truncation algorithms have already been developed in the literature. The two sets of output and input dissipation inequalities proposed in this work reduce for these special cases to the observability and reachability Lyapunov inequality, respectively, see for instance [10] for the case of periodic systems.

In this technical note the matrices in the state space recursion are allowed to depend on the mode transition rather than the mode alone as is the case with standard MJLS's [1]. This was done to accommodate mode varying dimension of the continuous valued part of the state. Applying the balanced truncation algorithm to a standard MJLS with state space representation

$$\begin{aligned} x(k+1) &= A\left[\theta(k)\right]x(k) + B\left[\theta(k)\right]f(k), \\ y(k) &= C\left[\theta(k)\right]x(k), \quad k \in \mathbf{N} \end{aligned}$$

will lead to a reduced order model where again the matrices in the state space recursion will depend on the mode transition. The only way of getting a reduced order model in the standard form is by computing mode independent solutions to the corresponding dissipation inequalities.

IV. A Numerical Example

To illustrate the model reduction algorithm developed in this technical note, consider a network control example based on [20], [21]. A one dimensional platoon consists of $m+1$ vehicles. Let x_0 denote the position of the lead car and x_i, $i \in \{1, \ldots, m\}$ denote the position of the i'th follower in the platoon. The spacing error is given by $e_i(t) = x_{i-1}(t) - x_i(t) - \delta$, $i \in \{1, \ldots, m\}$, where δ is the desired vehicle spacing, which is constant. It is assumed that $x_0(0) = 0$ and that there is no initial spacing error, $e_i(0) = 0$, $i \in \{1, \ldots, m\}$. Two control schemes have been designed, whose goal is to achieve disturbance attenuation between the leader motion, which is considered as a reference signal and the spacing error among any two successive followers in the platoon.

The first scheme is decentralized, based on local measurements from on-board sensors. Its performance cannot be satisfactory due to fundamental limitations, which have been elaborated in [20]. The second control scheme utilizes information about the lead car and exhibits better performance. However it requires communication between the lead car and the followers $\{2, \ldots, m\}$, which occurs through a wireless network idealized as a two state Markov chain. State one corresponds to low load and state two to high load in the network. If there is a transition from high load to high load the leader motion is not transmitted to the followers $\{2, \ldots, m\}$ and the first control scheme is implemented for these vehicles in that particular sample. If there is a transition from low load to low load the leader motion is transmitted to the followers $\{2, \ldots, m\}$ and the second control scheme is utilized for these vehicles. If there is a transition from high load to low load or vice versa then only the followers 2 and 3 get information about the leader motion, they implement the second control scheme, whereas followers $\{4, \ldots, m\}$ receive no information about the leader motion and utilize the first control scheme. The transition probability matrix of the two state Markov chain is denoted by P.

All the necessary information on the vehicle dynamics and the actual parameters of the control algorithms can be found in [16]. For the exposition of this technical note, what is important is that the closed loop system is a MJLS that can serve the purpose of demonstrating the reduction algorithm. An example where $m = 8$ is considered. The input to the system is the position of the lead car and the output is taken to be

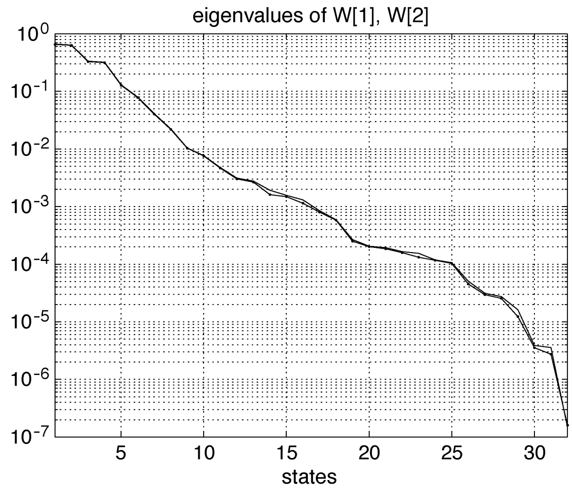

Fig. 1. Entries in the diagonal of $\boldsymbol{W}[\mathbf{1}], \boldsymbol{W}[\mathbf{2}]$ in logarithmic scale.

TABLE I
ACTUAL APPROXIMATION $\boldsymbol{\delta}_{\mathbf{L},\hat{\mathbf{L}}}$ AND BOUND $\boldsymbol{\gamma}_{\mathbf{L},\hat{\mathbf{L}}}$

$\hat{n}[1] = \hat{n}[2]$	$\delta_{\mathbf{L},\hat{\mathbf{L}}}$	$\gamma_{\mathbf{L},\hat{\mathbf{L}}}$
7	0.0257	0.2319
6	0.0497	0.3951
5	0.1809	0.7122
4	0.1889	1.2274
3	0.5616	2.5027

the spacing error between the last two followers. The transition probability matrix is chosen to be

$$P = \begin{bmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{bmatrix}.$$

The original model has 32 states per discrete mode. We compute diagonal matrices $W[1], W[2]$ with positive entries as in (22). The diagonal entries of $W[1], W[2]$ control the error bound in terms of the stochastic L_2 gain and are depicted in Fig. 1. The approximation error and the upper bounds to the approximation error are depicted for various truncation levels, showing that for this particular example the bound is rather conservative (see Table I). Future research should focus in obtaining sharper upper bounds as well as a lower bound similar to the $n'th$ Hankel singular value for LTI systems.

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