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The Linear Sampling Method and the MUSIC Algorithm

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Abstract

This article gives a short tutorial on the MUSIC algorithm [2, 10] and the linear sampling method of [3], and explains how the latter is an extension of the former. In particular, for the case of scattering from a finite number of weakly scattering targets, the two algorithms are identical.

Section 1 outlines the MUSIC algorithm and its use in signal processing and imaging. Section 2 outlines the linear sampling method and discusses its similarity with the MUSIC algorithm. The paper ends with Section 3, a discussion and listing of open questions.

1 MUSIC

MUSIC is an abbreviation for MUltiple SIgnal Classification [10]. Section 1.1 outlines the general MUSIC algorithm; section 1.2 explains how it is applied in signal processing (in order to explain the name); section 1.3 explains how it applies to imaging [2].

1.1 The basics of MUSIC

MUSIC is essentially a method of characterizing the range of a self-adjoint operator. Suppose A is a self-adjoint operator with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots$, and corresponding eigenvectors $v_1, v_2, \ldots$. Suppose the eigenvalues $\lambda_{M+1}, \lambda_{M+2}, \ldots$ are all zero, so that the vectors $v_{M+1}, v_{M+2}, \ldots$ span the null space of A. Alternatively, $\lambda_{M+1}, \lambda_{M+2}, \ldots$ could merely be very small, below the noise level of the system represented by A; in this case we say that the vectors $v_{M+1}, v_{M+2}, \ldots$ span the *noise subspace* of A. We can form the projection onto the noise subspace; this projection is given explicitly by

$$P_{\text{noise}} = \sum_{j>M} v_j \overline{v_j}^T \tag{1.1}$$

where the superscript T denotes transpose, the bar denotes complex conjugate, and $\overline{v_j}^T$ is the linear functional that maps a vector f to the inner product $\langle v_j, f \rangle$.

The (essential) range of A, meanwhile, is spanned by the vectors $v_1, v_2, \ldots, v_M$.

The key idea of MUSIC is this: because A is self-adjoint, we know that the noise subspace is orthogonal to the (essential) range. Therefore, a vector f is in the range if and only if its projection onto the noise subspace is zero, i.e., if $\|P_{\text{noise}} f\| = 0$. And this, in turn, happens only if

$$\frac{1}{\|P_{\text{noise}} f\|} = \infty \tag{1.2}$$

Equation (1.2) is the MUSIC characterization of the range of A.

We note that for an operator that is not self-adjoint, MUSIC can be used with the singular value decomposition instead of the eigenvalue decomposition.

1.2 The use of MUSIC in signal processing

MUSIC is generally used in signal processing problems. In this case, we make measurements of some signal $x(t)$ at discrete times $t_n = n$. The resulting samples $x_n = x(t_n)$ are considered random variables. We form the correlation matrix $A_{n,m} = E(x_n \overline{x_m})$, where E denotes the expected value.

We consider the special case when the signal is composed of two time-harmonic signals of different frequencies, plus noise. Thus $x_n = a_1 e^{i\omega_1 n} + a_2 e^{i\omega_2 n} + w_n$. We assume that the random variables w_n are identically distributed. The goal is to estimate the frequencies of the signals.

Because the different terms of x_n are mutually independent, the self-adjoint matrix A can be written [10]

$$A = E(|a_1|^2) s^1 \overline{s^1}^T + E(|a_2|^2) s^2 \overline{s^2}^T + \sigma_0^2 I \tag{1.3}$$

where the nth component of the vector s^j is given by $s_n^j = e^{i\omega_j n}$, I denotes the identity operator, and $\sigma_0^2 = E(|w_n|^2)$. Thus we see that the the range is spanned by s^1 and s^2; The orthogonal complement is the noise subspace.

The MUSIC algorithm for estimating the frequencies ω_1 and ω_2 is to plot, as a function of ω, the quotient

$$\frac{1}{\|P_{\text{noise}} s^\omega\|} \tag{1.4}$$

were s^ω is the vector whose nth component is $e^{i\omega n}$. The resulting plot, which has large peaks at the frequencies ω_1 and ω_2, is called the *MUSIC pseudospectrum.*

We note that the MUSIC algorithm involves applying a test to a large number of trial signals s^ω.

The appropriateness of the name MUSIC is now clear: MUSIC is a method for estimating the individual frequencies of multiple time-harmonic signals.

1.3 The use of MUSIC in imaging

Devaney [2] has recently applied the MUSIC algorithm to the problem of estimating the locations of a number of point-like scatterers. The following is an outline of his approach.

We consider the mathematical model in which wave propagation is governed in free space by the Helmholtz equation

$$(\nabla^2 + k^2)\psi = 0 \tag{1.5}$$

where k corresponds to the frequency of the propagating wave. We imagine that we have N antennas or transducers, positioned at the points $R_1, R_2, \ldots, R_N$, that transmit spherically spreading waves. If the jth antenna is excited by an input voltage e_j, the field produced at the point x by the jth antenna is $\psi_j^{\text{in}}(x) = G(x, R_j) e_j$.

We assume that the scatterers, positioned at the points $X_1, X_2, \ldots, X_M$, are small, weak, and well-separated, so that they scatter according to the Born approximation. Thus if the field ψ^{in} is incident on the mth scatterer, it produces at x the

scattered field $G(x, X_m)\tau_m\psi^{\text{in}}(X_m)$, where τ_m is the strength of the mth scatterer and $G(x, y)$ denotes the outgoing Green's function. The scattered field from the whole cloud of scatterers is $\sum_m G(x, X_m)\tau_m\psi^{\text{in}}(X_m)$. Thus the total scattered field due to the field emanating from the jth antenna is

$$\psi_j^{\text{sc}}(x) = \sum_m G(x, X_m)\tau_m G(X_m, R_j)e_j$$

If this field is measured at the lth antenna, the result is

$$\psi_j^{\text{sc}}(R_l) = \sum_m G(R_l, X_m)\tau_m G(X_m, R_j)e_j$$

This expression gives rise to the *multi-static response* matrix K, whose (l, j)th element is

$$K_{l,j} = \sum_m G(R_l, X_m)\tau_m G(X_m, R_j) \tag{1.6}$$

The multi-static response matrix maps the vector of input amplitudes $(e_1, e_2, \ldots, e_N)^T$ to the vector of measured amplitudes on the N antennas. This matrix can be written

$$K = \sum_m \tau_m g_m g_m^T \tag{1.7}$$

where we have used the notation

$$g_m = (G(R_1, X_m), G(R_2, X_m), \ldots, G(R_N, X_m)^T. \tag{1.8}$$

For simplicity we consider only the case $N > M$, i.e., more antennas than scatterers.

Because the Green's function is symmetric, K is symmetric, but it is not self-adjoint. We form a self-adjoint matrix $A = K^*K = \overline{K}K$, where the star denotes the adjoint and the bar denotes the complex conjugate (which is the same as the adjoint, since K is symmetric). We note that $\overline{K}$ is the frequency-domain version of a time-reversed multi-static response matrix; thus $\overline{K}K$ corresponds to performing a scattering experiment, time-reversing the received signals, and using them as input for a second scattering experiment [8], [5], [1].

The matrix A can be written

$$A = \sum_m \overline{\tau_m}\ \overline{g_m}\ \overline{g_m}^T \sum_l \tau_l g_l g_l^T \tag{1.9}$$

from which we see immediately that the eigenvectors of A are the $\overline{g_m}$. This means that the range of A is spanned by the M vectors $\overline{g_m}$.

Devaney's insight is that the MUSIC algorithm can now be used as follows to determine the location of the scatterers. Given any point p, form the vector $g^p = (G(R_1, p), G(R_2, p), \ldots, G(R_N, p))^T$. The point p coincides with the location of a scatterer if and only if

$$P_{\text{noise}} g^p = 0. \tag{1.10}$$

Thus we can form an image of the scatterers by plotting, at each point p, the quantity $1/\|P_{\text{noise}} g^p\|$. The resulting plot will have large peaks at the locations of the scatterers. We note that the condition (1.10) depends only on the operator P_{noise} and not on the particular basis $\{\overline{g_m}\}$.

2 The linear sampling method

The linear sampling method is a linear method for finding the boundary of one or more impenetrable objects from scattering data.

2.1 The basics of the linear sampling method

Kirsch [3] considers the scattering problem in which incident plane waves scatter off one or more impenetrable objects. He considers the far-field operator F, which is an integral operator whose kernel is the far-field scattering amplitude. The operator F satisfies a reciprocity condition but is not self-adjoint. Kirsch forms the self-adjoint operator $A = F^*F = \overline{F}F$, and considers the eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots$ and corresponding eigenfunctions $v_1, v_2, \ldots$.

The linear sampling method is based on the theorem [3] that the range of $A^{1/4}$ coincides with the range of the operator H, which is defined as follows. Suppose ψ is equal to h on the boundary of the object, satisfies (1.5) in the region exterior to the object, and satisfies an outgoing radiation condition. Then H maps the Dirichlet data h to the far-field pattern of ψ.

In the linear sampling method, one determines the boundary of the object by testing points p as follows. We denote by g^p the far-field amplitude corresponding to the Green's function $G(x,p)$. If p is inside one of the objects, then in the region exterior to the object, $G(x,p)$ satisfies (1.5), so g^p is in the range of H. But if p is exterior to the object, then because $G(x,p)$ has a singularity at p, it cannot satisfy (1.5) there, so g^p cannot be in the range of H.

The range of H, which Kirsch showed is identical to the range of $A^{1/4}$, can be determined from the eigenvalues and eigenfunctions of A. In particular, the range of $A^{1/4}$ is given by

$$\mathrm{Ran} A^{1/4} = \{f : \sum_j \frac{|\langle v_j, f\rangle|^2}{|\lambda_j|^{1/2}} < \infty\} \tag{2.1}$$

The algorithm of the linear sampling method is to plot, at each point p, the quantity $1/\left(\sum_j |\lambda_j|^{-1/2} |\langle v_j, g^p\rangle|^2\right)$. The plot will be identically zero whenever p is outside all the scattering objects, and nonzero whenever p is inside one of the scatterers.

2.2 Linear sampling for M point scatterers

To see the connection between MUSIC and linear sampling, let us consider the linear sampling algorithm for the same case considered in section 1.3, namely when the scattering object is composed of M weakly scattering point-like scatterers. Then, from the arguments of section 1.3, we find that the operator A has a finite-dimensional range, so that the eigenvalues $\lambda_{M+1}, \lambda_{M+2}, \ldots$ are all zero. In that case, the condition (2.1) for g^p to be in the range of H becomes

$$\langle v_j, g^p\rangle = 0, \qquad j = M+1, M+2, \ldots \tag{2.2}$$

which can also be written $P_{\text{noise}} g^p = 0$. This is precisely the MUSIC condition (1.10). Plotting $1/\|P_{\text{noise}} g^p\|$ will give an image with very large values at the locations of the scatterers.

3 Discussion and open questions

It appears that the linear sampling method is an extension of the MUSIC imaging algorithm of [2] to the case of extended objects and infinite-dimensional scattering operators.

Many questions arise in connection with these algorithms. First, the MUSIC algorithm uses only the null space of the operator K. But we know that the eigenvectors of K contain information about the scatterers. In particular, the eigenvector corresponding to the largest eigenvalue corresponds to a wave focusing on the strongest scatterer, and the eigenvalue contains information about the strength of the scatterer [8], [1]. How can MUSIC be modified to make use of this information?

The linear sampling method, on the other hand, uses all the eigenvalues and eigenfunctions but produces only the location of the boundary of the scattering object. Yet we know that the eigenvalues and eigenfunctions contain all the information about the scatterer [9], [4]. How can the eigenvalues and eigenfunctions be used to recover more information?

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