Controller Reduction with Closed Loop Performance Guarantee

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2010

49th IEEE Conference on Decision and Control
December 15-17, 2010
Hilton Atlanta Hotel, Atlanta, GA, USA

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Abstract— **This paper describes a modification of a singular value decomposition (SVD) based controller reduction method recently proposed in [13]. Instead of formulating a $\mathcal{H}_2$ norm characterizing generalized controllability Gramian inequality as in the previous case, the current method applies the bounded-real lemma to certify the closed loop performances in $\mathcal{H}_\infty$ norm. In addition, unlike the previous method, the current one does not suffer from the lack of symmetry that all the data from the plant is not utilized. Yet, the current method inherits the same merit as the previous one that the formulated problem can be solved as a generalized minimum rank matrix approximation problem which can be solved efficiently using SVD. Extensions and numerical examples are shown in the end.**

I. Introduction

In this paper the controller reduction problem in Fig. 1 is considered. In this reduction problem, the feedback structure should remain and only the controller can be simplified. This can be viewed as a special case of the more general problem of structured model reduction (e.g. [1], [2], [3]). However, the controller reduction problem, by itself, has also been extensively studied because of its potential values. For example, the optimal $\mathcal{H}_\infty$ or $\mathcal{H}_2$ controllers can sometimes be very complicated, and reduced order controllers are oftentimes desirable for implementation purposes. Methods for controller reduction mostly fall into two categories. One category is based on balanced truncation/Hankel norm reduction and their variants (e.g. [4], [5], [6], [7], [8]), and the other category solves some optimization problems derived from some forms of the KYP lemma (e.g. [9], [10], [11], [12]). The two groups of controller reduction methods are based on different principles and they have their own merits. Balanced truncation methods find very good reduced order approximations to the full order controller and/or closed loop system using relatively inexpensive singular value decomposition (SVD) computations. On the other hand, the KYP lemma based approaches bypass the full controller, and instead directly search for a globally optimal reduced order solution which satisfies some pre-specified closed loop performance constraints. However, the global optimization problems from the KYP approaches are typically difficult to solve. In an attempt to take advantage of the benefits of the two groups of controller reduction methods, [13] was recently proposed to investigate the use of the information of the full controller to conduct a local search for an optimization problem reminiscent of the KYP approach. In the end, [13] sets up a rank minimization problem with a generalized controllability Gramian inequality constraint, which happens to be solvable using SVD. However, in terms of closed loop performance the method in [13] can only establish an inequality between the full and reduced Gramians. In order to translate the Gramian relationship into a $\mathcal{H}_2$ norm one, the C and D matrices of the reduced controller are required to be the same as those of the full order controller. In addition, the method in [13] cannot enforce both the controllability and observability inequalities together because the resulted optimization problem would become computationally intractable. This causes a lack of symmetry in terms of plant data use, which in turn motivates the current research. In this paper, the proposed approach is based on another KYP type lemma which is commonly known as the bounded-real lemma. The bounded-real lemma is a direct characterization of the system input/output behavior, and it explicitly involves the use of all the system matrices of the plant and the controller. As a result, the symmetry issue is avoided and none of the system matrices of the reduced controller need to be fixed a priori. The rest of the paper is organized as follows. In Sec. II the notations and technical background materials are introduced. In Sec. III the problem which forms the core of the proposed controller reduction scheme is described. The properties of the proposed problem are also discussed in detail. Then in Sec. IV the SVD based solution procedure for the proposed problem is discussed. Sec. V describes an extension of the proposed method pertaining further order reduction. In Sec. VI some numerical examples are shown. Finally in Sec. VII the differences between the proposed method in this paper and the one in [13] are summarized.

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II. Background and Notations

A. Controller order reduction problem

In this paper, all the systems involved are discrete-time linear time-invariant systems. Consider the standard feedback setup of a plant and a controller in Fig. 1. In this figure, the

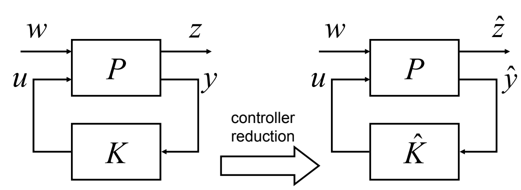

Fig. 1. The block diagram in the left is the feedback interconnection of a plant P and a controller K. The problem of controller reduction is to find a lower order controller $\hat{K}$ to replace K, so that the performances of the reduced closed loop system (in the right) do not degrade too much.

plant is denoted as P and the controller is denoted as K. The feedback closed loop system is denoted as $P \star K$, and it is assumed to be stable. The $\mathcal{H}_\infty$ norm of $P \star K$ is denoted as $\|P \star K\|_\infty$. The controller reduction problem in this paper seeks a minimum order reduced controller $\hat{K}$ to replace K, so that the corresponding reduced closed loop system $P \star \hat{K}$ is stable and the $\mathcal{H}_\infty$ norm, $\|P \star \hat{K}\|_\infty$, is less than some user-specified value.

B. State space characterization of the closed loop systems

In Fig. 1 the dimensions of the signals w, u, z, y are n_w, n_u, n_z, n_y respectively. The plant P is described in state space matrices as follows.

$$P = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right) \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$, $D_{12} \in \mathbb{R}^{n_z \times n_u}$, $D_{21} \in \mathbb{R}^{n_y \times n_w}$. In this paper, a technical assumption is made that $D_{22} = 0$. If this is not the case in the original plant/controller setup, an equivalent setup can be obtained by removing the D_{22} term from the plant and incorporating it into the controller as a positive feedback.

The given full order controller and the to-be-found reduced controller are described by their respective state space matrices as the following transfer matrices.

$$K = \left(\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right) \tag{2a}$$

$$\hat{K} = \left(\begin{array}{c|c} \hat{A}_k & \hat{B}_k \\ \hline \hat{C}_k & \hat{D}_k \end{array}\right) \tag{2b}$$

where $A_k \in \mathbb{R}^{n_k \times n_k}$, $D_k \in \mathbb{R}^{n_u \times n_y}$ and $\hat{A}_k \in \mathbb{R}^{\hat{n}_k \times \hat{n}_k}$, $\hat{D}_k \in \mathbb{R}^{n_u \times n_y}$. The full and reduced controllers are also characterized by the following two matrices respectively.

$$L \triangleq \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \in \mathbb{R}^{(n_k+n_u)\times(n_k+n_y)} \tag{3a}$$

$$\hat{L} \triangleq \begin{bmatrix} \hat{A}_k & \hat{B}_k \\ \hat{C}_k & \hat{D}_k \end{bmatrix} \in \mathbb{R}^{(\hat{n}_k+n_u)\times(\hat{n}_k+n_y)} \tag{3b}$$

Subsequently, K and L (also $\hat{K}$ and $\hat{L}$) will be used interchangeably to describe the controllers.

Define the data matrices

$$\begin{aligned} &\bar{A} \triangleq \begin{bmatrix} A & 0 \\ 0 & 0_{\hat{n}_k} \end{bmatrix}, \quad \bar{B}_1 \triangleq \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad \bar{B}_2 \triangleq \begin{bmatrix} 0 & B_2 \\ I_{\hat{n}_k} & 0 \end{bmatrix} \\ &\bar{C}_1 \triangleq \begin{bmatrix} C_1 & 0 \end{bmatrix}, \quad \bar{C}_2 \triangleq \begin{bmatrix} 0 & I_{\hat{n}_k} \\ C_2 & 0 \end{bmatrix} \\ &\bar{D}_{12} \triangleq \begin{bmatrix} 0 & D_{12} \end{bmatrix}, \quad \bar{D}_{21} \triangleq \begin{bmatrix} 0 \\ D_{21} \end{bmatrix} \end{aligned} \tag{4}$$

With the matrices defined in (4), the state space matrices of the reduced closed loop system $P \star \hat{K}$ can be described as affine functions of the reduced controller $\hat{L}$ (cf. (3b)) as

$$\begin{aligned} A_{cl}(\hat{L}) &\triangleq \bar{A} + \bar{B}_2 \hat{L} \bar{C}_2 \\ B_{cl}(\hat{L}) &\triangleq \bar{B}_1 + \bar{B}_2 \hat{L} \bar{D}_{21} \\ C_{cl}(\hat{L}) &\triangleq \bar{C}_1 + \bar{D}_{12} \hat{L} \bar{C}_2 \\ D_{cl}(\hat{L}) &\triangleq D_{11} + \bar{D}_{12} \hat{L} \bar{D}_{21} \end{aligned} \tag{5}$$

Notice that the dimensions of the closed loop state space matrices above depend on the dimension of $\hat{L}$ (i.e. $\hat{n}_k$). In the case where $n_k = \hat{n}_k$ (which is the focus of this paper), the full order closed loop state space matrices of $P \star K$ can also be described in terms of L (cf. (3a)) as

$$\begin{aligned} A_{cl}(L) &\triangleq \bar{A} + \bar{B}_2 L \bar{C}_2 &&\triangleq A_{cl} \\ B_{cl}(L) &\triangleq \bar{B}_1 + \bar{B}_2 L \bar{D}_{21} &&\triangleq B_{cl} \\ C_{cl}(L) &\triangleq \bar{C}_1 + \bar{D}_{12} L \bar{C}_2 &&\triangleq C_{cl} \\ D_{cl}(L) &\triangleq D_{11} + \bar{D}_{12} L \bar{D}_{21} &&\triangleq D_{cl} \end{aligned} \tag{6}$$

If no confusion is expected, the data matrices $A_{cl}(L)$, $B_{cl}(L)$, $C_{cl}(L)$ and $D_{cl}(L)$ will be shortened to A_{cl}, B_{cl}, C_{cl} and D_{cl} respectively.

C. Bounded-real lemma

The bounded-real lemma (e.g. [14]) is the result used in this paper to characterize the performance of the reduced controller. A statement can be given as follows.

Theorem 1 ([14]): Consider a discrete-time transfer matrix described by $G(z) = C\left(zI - A\right)^{-1} B + D$ and a positive scalar γ. Then the following statements are equivalent.

- G is stable and the $\mathcal{H}_\infty$ norm of G is less than γ (7a)
- there exists $X \succ 0$ such that

$$\begin{bmatrix} X & 0 \\ 0 & \gamma I \end{bmatrix} \succ \begin{bmatrix} A & B \\ C & D \end{bmatrix}^T \begin{bmatrix} X & 0 \\ 0 & \frac{1}{\gamma} I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{7b}$$

■

The bounded-real lemma provides a sufficient and necessary condition on the state space matrices, in the form of a matrix inequality in (7b), to characterize the system properties in (7a). To illustrate the use of the bounded-real lemma and to motivate the proposed problem in the next section, the problem in [9], [10] is now described first. To design an optimal $\mathcal{H}_\infty$ norm controller of order $\hat{n}_k < n_k$, it suffices to apply (7b) in which the matrices A, B, C, D are replaced by the reduced closed loop state space matrices $A_{cl}(\hat{L})$, $B_{cl}(\hat{L})$, $C_{cl}(\hat{L})$, $D_{cl}(\hat{L})$ defined in (5). Then an optimization problem can be set up minimizing the $\mathcal{H}_\infty$ norm upper bound γ with the decision variables being γ, X and $\hat{L}$. The minimization regarding γ can be achieved through bisection. However, the optimization with respect to $\hat{L}$ and X turns out to be computationally intractable because of their cross terms. This is where the proposed method and [9], [10] diverge. The main difference is in terms of the determination of the matrix X. The details will be given in the next section.

III. Problem Formulation

A. Description of the proposed problem

The data of the proposed problem are the plant matrices in (4) and L in (3a) describing the full order controller. There is a scalar parameter γ which is required to satisfy $\gamma > \|P \star K\|_\infty$. The parameter γ is a user-specified upper bound of $\left\|P \star \hat{K}\right\|_\infty$ controlling the tradeoff between performance loss and controller order reduction. The requirement that $\gamma > \|P \star K\|_\infty$ is needed to ensure that the full order controller L is a feasible solution of the proposed problem. The decision variable of the proposed problem is $\hat{L}$ characterizing the reduced controller. It is required that $\hat{L}$ and L have the ***same dimensions*** (i.e. $\hat{n}_k = n_k$, reason to be given shortly). The proposed problem is

$$\underset{\hat{L}}{\text{minimize}} \quad \operatorname{rank}(\hat{L}) \qquad \text{subject to} \tag{8a}$$

$$\begin{bmatrix} X & 0 \\ 0 & \gamma I \end{bmatrix} \succ \begin{bmatrix} A_{cl}(\hat{L}) & B_{cl}(\hat{L}) \\ C_{cl}(\hat{L}) & D_{cl}(\hat{L}) \end{bmatrix}^T \begin{bmatrix} X & 0 \\ 0 & \frac{1}{\gamma} I \end{bmatrix} \begin{bmatrix} A_{cl}(\hat{L}) & B_{cl}(\hat{L}) \\ C_{cl}(\hat{L}) & D_{cl}(\hat{L}) \end{bmatrix} \tag{8b}$$

where $X \succ 0$ can be computed from the data by solving

$$\begin{bmatrix} X & 0 \\ 0 & \gamma I \end{bmatrix} \succ \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix}^T \begin{bmatrix} X & 0 \\ 0 & \frac{1}{\gamma} I \end{bmatrix} \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} \tag{8c}$$

The definitions of the closed loop system matrices in (8b) and (8c) above can be found in (5) and (6). Note that X in (8c) is not part of the decision variable because it can be computed beforehand. Also note that the reason for imposing the dimensions of $\hat{L}$ and L to be the same (i.e. $\hat{n}_k = n_k$) is that X satisfies both (8b) and (8c). Once an optimal solution of $\hat{L}$ is found solving (8a) and (8b), controller $\hat{K}$ in (2b) can be defined, and a minimal realization can be obtained as the proposed reduced controller for the problem in Fig. 1. Finally, it is emphasized that while the rank minimization in (8a) might seem to be intractable, the proposed problem can in fact be solved efficiently using SVD. This is the same situation as in [13]. The details will be discussed in Sec. IV.

B. Discussions on the proposed problem

The proposed problem in (8a), (8b), (8c) is formulated roughly according to the problem description in Sec. II-A. The detailed motivations for (8a), (8b), (8c) are now discussed separately.

Since $P \star K$ is assumed to be stable and γ is chosen such that $\gamma > \|P \star K\|_\infty$, $P \star K$ satisfies the bounded-real lemma statement in (7a). Hence, the corresponding statement in (7b) is true, and a solution $X \succ 0$ satisfying (8c) exists. In addition, since $\hat{L}$ has the same dimension as L (i.e. $\hat{n}_k = n_k$), L is always a feasible solution of (8b) because (8b) and (8c) are the same when $\hat{L} = L$. This concludes that the proposed problem in (8a), (8b), (8c) is always feasible.

The dimension constraint on $\hat{L}$ makes it impossible to specify the order of the reduced controller via the dimension of the matrix $\hat{A}_k$, which is fixed to be $\hat{n}_k = n_k$. Instead, the order reduction is achieved through the minimization of the rank of $\hat{L}$ as in (8a). This is the same technique as in [13], which shows that $\operatorname{order}(\hat{K}) \leq \operatorname{rank}(\hat{L})$. Therefore, while $\hat{L}$ has the same dimension as L, the rank minimization in (8a) can still lead to a low order minimal reduced controller. However, note that the inequality $\operatorname{order}(\hat{K}) \leq \operatorname{rank}(\hat{L})$ can be strict, and this can result in a restriction.

The closed loop performance of the reduced controller is guaranteed by (8b). Since $X \succ 0$, condition (8b) can be interpreted as the bounded-real lemma statement (7b) applied to the closed loop system $P \star \hat{K}$. Therefore, for any reduced controller $\hat{K}$ satisfying (8b), $P \star \hat{K}$ is guaranteed to be stable and $\left\| P \star \hat{K} \right\|_\infty < \gamma$.

While it is necessary to introduce (8a) for order reduction and (8b) for closed loop performances, the motivation for (8c) deserves more elaborations. In principle, a positive definite X does not need to satisfy (8c) to be "valid" for (8b) in the sense that there is at least one (any one) reduced controller satisfying (8b). However, as it turns out in [9], [10], the reduction problem with the entire set of valid X is intractable. Therefore, this paper only considers a subset of the valid X, which is defined by (8c). Any X satisfying (8c) is automatically valid for (8b) because L satisfies the (8b) with this X. The restriction on the candidate set of X has two implications. On one hand, the reduced controller candidates are now confined to a neighborhood of L. If $\hat{L}$ is a member of the neighborhood and X is a matrix that makes $\hat{L}$ satisfy (8b), then L has to satisfy the same inequality. This is in general not the case for the globally optimal reduced controller in [9], [10]. For this reason, the proposed method is only a local search for the optimal reduced order controller. An important question then arises as to whether the proposed local search is too restrictive. This question is positively answered by the numerical examples in Sec. VI. On the other hand, the restriction of (8c) results in a significant gain in terms of computation. (8c) is a linear matrix inequality (LMI) with respect to X when γ is fixed. For moderate size instances, X can be solved from (8c) as a semidefinite program to allows some flexibilities. This will be explained in Sec. V. Conversely, when the semidefinite program approach is deemed too expensive, the Schur's complement can be applied to (8c) to obtain an equivalence given by a Riccati equation in (9) and a LMI in (10).

$$V = X - A_{cl}^T X A_{cl} - \tfrac{1}{\gamma} C_{cl}^T C_{cl} + \left(A_{cl}^T X B_{cl} + \tfrac{1}{\gamma} C_{cl}^T D_{cl} \right) \left(B_{cl}^T X B_{cl} + \tfrac{1}{\gamma} D_{cl}^T D_{cl} - \tfrac{1}{\gamma} I \right)^{-1} \left(A_{cl}^T X B_{cl} + \tfrac{1}{\gamma} C_{cl}^T D_{cl} \right)^T \tag{9}$$

$$B_{cl}^T X B_{cl} + \tfrac{1}{\gamma} D_{cl}^T D_{cl} - \tfrac{1}{\gamma} I \prec 0 \tag{10}$$

In (9), V is a small positive definite matrix to ensure that (8c) is satisfied strictly. In practice, V can be chosen as $V = \epsilon I$ for some small $\epsilon > 0$. Equation (9), as a Riccati equation, can be solved much more efficiently than a LMI. If an solution $X \succ 0$ of (9) also satisfies (10), then this X satisfies the bounded-real inequality (8c). To recap, inequality (8c) results in a restriction on the choice of the bounded-real lemma related matrix X. It is introduced to enable the efficient use of the full controller information in the search of valid X.

IV. Solution Via Matrix Approximation

This section is concerned with the solving of the proposed problem in (8a), (8b). Similar to [13], the main result here is developed in two steps. First, condition (8b) is shown to be equivalent to a simplified matrix approximation inequality. Then using the simplified inequality, the proposed problem in (8a), (8b) can be shown to be equivalent to a classical minimum rank matrix approximation problem which can be solved efficiently using SVD.

A. Reformulation of (8b) to a simplified matrix approximation constraint in (17)

First note that, by using (5), the following in (8b) is true.

$$\begin{bmatrix} A_{cl}(\hat{L}) & B_{cl}(\hat{L}) \\ C_{cl}(\hat{L}) & D_{cl}(\hat{L}) \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B}_1 \\ \bar{C}_1 & D_{11} \end{bmatrix} + \begin{bmatrix} \bar{B}_2 \\ \bar{D}_{12} \end{bmatrix} \hat{L} \begin{bmatrix} \bar{C}_2 & \bar{D}_{21} \end{bmatrix}$$

Insert the above expression into (8b) and multiply both sides of (8b) with $\text{blkdiag}\left(X^{-\frac{1}{2}}, \gamma^{-\frac{1}{2}} I\right)$ (this matrix can be defined since $X \succ 0$ and $\gamma > 0$). Then (8b) becomes

$$\bar{\sigma}\left(F + G\hat{L}H\right) < 1 \tag{11}$$

where $\bar{\sigma}(\cdot)$ denotes the maximum singular value of a matrix, and F, G and H can be defined from the problem data as

$$\begin{aligned} F &\triangleq \begin{bmatrix} X^{\frac{1}{2}} & 0 \\ 0 & \gamma^{-\frac{1}{2}} I \end{bmatrix} \begin{bmatrix} \bar{A} & \bar{B}_1 \\ \bar{C}_1 & D_{11} \end{bmatrix} \begin{bmatrix} X^{-\frac{1}{2}} & 0 \\ 0 & \gamma^{-\frac{1}{2}} I \end{bmatrix} \\ G &\triangleq \begin{bmatrix} X^{\frac{1}{2}} & 0 \\ 0 & \gamma^{-\frac{1}{2}} I \end{bmatrix} \begin{bmatrix} \bar{B}_2 \\ \bar{D}_{12} \end{bmatrix} \\ H &\triangleq \begin{bmatrix} \bar{C}_2 & \bar{D}_{21} \end{bmatrix} \begin{bmatrix} X^{-\frac{1}{2}} & 0 \\ 0 & \gamma^{-\frac{1}{2}} I \end{bmatrix} \end{aligned} \tag{12}$$

For notation convenience, it is denoted that $F \in \mathbb{R}^{m_F \times n_F}$ and $\hat{L} \in \mathbb{R}^{m_L \times n_L}$. Whether (11), an equivalence of (8b), is easy to handle depends on the dimensions of G and H. In this paper the most difficult (and typical) case will be considered. That is, the dimensions of data matrices such as B_2 and D_{12} are assumed so that G is a full rank "thin" matrix with more rows than columns. Similarly, H is assumed to be a full rank "fat" matrix with more columns than rows. The rest of this subsection presents the result from [15] to reduce (11) into a simplified matrix approximation constraint to be defined in (17). To begin with the discussion, denote the (rectangular, or "economy size") SVD of G and H as

$$G = U_G S_G V_G^T \quad \text{s.t.} \quad \begin{aligned} & U_G \in \mathbb{R}^{m_F \times m_L},\ {U_G}^T U_G = I_{m_L} \\ & S_G \in \mathbb{R}^{m_L \times m_L},\ \text{positive diagonal} \\ & V_G \in \mathbb{R}^{m_L \times m_L},\ {V_G}^T V_G = I_{m_L} \end{aligned} \tag{13}$$

$$H = U_H S_H V_H^T \quad \text{s.t.} \quad \begin{aligned} & U_H \in \mathbb{R}^{n_L \times n_L},\ {U_H}^T U_H = I_{n_L} \\ & S_H \in \mathbb{R}^{n_L \times n_L},\ \text{positive diagonal} \\ & V_H \in \mathbb{R}^{n_F \times n_L},\ {V_H}^T V_H = I_{n_L} \end{aligned} \tag{14}$$

In addition, define the basis matrices N_G and N_H for the kernels of U_G in (13) and V_H in (14) as, respectively

$$\begin{aligned} & N_G \in \mathbb{R}^{m_F \times (m_F - m_L)},\ {N_G}^T N_G = I,\ {U_G}^T N_G = 0 \\ & N_H \in \mathbb{R}^{n_F \times (n_F - n_L)},\ {N_H}^T N_H = I,\ {V_H}^T N_H = 0 \end{aligned} \tag{15}$$

Then [15] states that if (11) is feasible, which is the case in this paper, then the following two matrices can be defined.

$$\begin{aligned} \Delta_G &\triangleq \left(I_{n_F} - F^T N_G {N_G}^T F\right)^{-1} \succ 0 \\ \Delta_H &\triangleq \left(I_{m_F} - F N_H {N_H}^T F^T\right)^{-1} \succ 0 \end{aligned} \tag{16}$$

In addition, (11) is equivalent to the following inequality.

$$\bar{\sigma}\left(\check{F} + \check{L}\right) < 1 \tag{17}$$

where

$$\begin{aligned} \check{F} &\triangleq \left({U_G}^T \Delta_H U_G\right)^{-\frac{1}{2}} {U_G}^T \Delta_H F V_H \left({V_H}^T \Delta_G V_H\right)^{\frac{1}{2}} \\ \check{L} &\triangleq (P_1)^{-1} \hat{L} (P_2)^{-1} \\ P_1 &\triangleq V_G (S_G)^{-1} \left({U_G}^T \Delta_H U_G\right)^{-\frac{1}{2}} \\ P_2 &\triangleq \left({V_H}^T \Delta_G V_H\right)^{-\frac{1}{2}} (S_H)^{-1} {U_H}^T \end{aligned} \tag{18}$$

with data matrices computed in (13), (14), (15) and (16). To summarize, constraint (8b) is equivalent to the constraint in (11), which in turn is equivalent to (17) with a new decision variable $\check{L}$ defined in (18). Additionally, compared with (11) the "$\check{G}$" and "$\check{H}$" matrices in (17) are now identities.

B. Reformulation of (8a), (8b) into a classical minimum rank matrix approximation problem in (20)

In the previous subsection the proposed problem in (8a), (8b) was shown to be the same as

$$\begin{aligned} \underset{\hat{L}}{\text{minimize}} \quad & \text{rank}\left(\hat{L}\right) \\ \text{subject to} \quad & \bar{\sigma}\left(F + G\hat{L}H\right) < 1 \end{aligned} \tag{19}$$

with F, G and H defined in (12). In addition, the constraint $\bar{\sigma}\left(F + G\hat{L}H\right) < 1$ above (i.e. (11)) was shown to be equivalent to (17) with a new variable $\check{L}$ defined in (18). Since in (18) $\check{L} = (P_1)^{-1} \hat{L} (P_2)^{-1}$, the ranks of $\check{L}$ and $\hat{L}$ are the same. Hence, the optimization problem in (19), as well as the proposed problem in (8a), (8b) is equivalent to

$$\begin{aligned} \underset{\check{L}}{\text{minimize}} \quad & \text{rank}\left(\check{L}\right) \\ \text{subject to} \quad & \bar{\sigma}\left(\check{F} + \check{L}\right) < 1 \end{aligned} \tag{20}$$

where $\check{F}$ is defined in (18). For computation purposes, the problem in (20) can be put in an equivalent form as follows.

$$\begin{aligned} \underset{r}{\text{minimize}} \quad & r \\ \text{subject to} \quad & \left(\begin{array}{ll} \underset{\check{L}}{\min} & \bar{\sigma}(\check{F} + \check{L}) \\ \text{s.t.} & \text{rank}(\check{L}) \le r \end{array} \right) < 1 \end{aligned}$$

This can be solved using SVD via the well-known theorem by Eckart-Young-Mirsky (e.g. [16]). From the description above, it can be seen that the computation cost of forming and solving (20) is dominated by the SVD's of matrices of dimension $n + n_k$. Therefore, the proposed problem can be solved by a $O((n + n_k)^3)$ algorithm.

V. Solving (8c) as a Semidefinite Program

When $n + n_k$ is relatively small (e.g. less than 40), it might be realistic to set up a semidefinite program to solve for the bounded-real matrix X in (8c). To set up the optimization problem, the following result from [15] is needed.

Theorem 2 ([15]): If the problem in (19) is feasible (which is the case in this paper), then the rank of the optimal $\hat{L}$ is the number of singular values of F which are greater than or equal to one. ■

Denote the number of singular values of F which are greater than or equal to one as $\text{sve}(F)$ (shorthand for singular value excess), then the above theorem states that the minimum rank of $\hat{L}$ in (19) is $\text{sve}(F)$. Since F is a function of X (cf. (12)), it is possible to pose an optimization problem to solve for X satisfying (8c) and minimizing $\text{sve}(F)$. More specifically, notice that $\text{sve}(F)$ is the same as the number of nonnegative eigenvalues of $F^T F - I$. By the Sylvester's law of inertia (e.g. [16], pp. 223), multiplying both sides of $F^T F - I$ with a symmetric matrix $\text{blkdiag}\left(X^{\frac{1}{2}}, \gamma^{\frac{1}{2}} I\right)$ does not change its number of nonnegative eigenvalues. Therefore, for any given X, $\text{sve}(F)$ is the rank of the minimum rank matrix in the following set.

$$\mathcal{Y} \triangleq \left\{ Y \succeq 0 \quad \text{s.t.} \quad Y \succ \begin{bmatrix} \bar{A} & \bar{B}_1 \\ \bar{C}_1 & D_{11} \end{bmatrix}^T \begin{bmatrix} X & 0 \\ 0 & \frac{1}{\gamma} I \end{bmatrix} \cdot \begin{bmatrix} \bar{A} & \bar{B}_1 \\ \bar{C}_1 & D_{11} \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & \gamma I \end{bmatrix} \right\} \tag{21}$$

Note that in above if X, in addition to Y, is treated as a decision variable, then the minimization problem of $\text{sve}(F)$ with respect to X can be obtained. The resulted optimization problem is a rank minimization problem subject to a joint LMI constraint with X and Y. Unfortunately this problem turns out to be computationally intractable. Nevertheless, the trace heuristics (e.g. [17]) can be applied to replace the rank in the objective as the trace of Y. The heuristic problem can be formulated as follows.

$$\begin{array}{ll} \underset{X,Y}{\text{minimize}} & \text{Tr}(Y) \\ \text{subject to} & X \succ 0 \text{ and } X \text{ satisfies (8c)} \\ & Y \in \mathcal{Y} \text{ in (21)} \end{array} \tag{22}$$

The trace heuristics turns the rank minimization problem into a semidefinite program, and in practice it is observed that it oftentimes yields reasonably low rank solutions. Finally, notice the similarity between (8c) and the second LMI in (21). For example, while A_{cl} is $A_{cl}(L)$, $\bar{A}$ is in fact $A_{cl}(0)$. Analogous relationships with the B, C and D matrices are also true. Therefore, on one hand X is chosen to satisfy (8c) with the full controller L. On the other hand, X is also chosen to satisfy as much as possible the open loop bounded-real inequality to obtain a good rank reduction of $\hat{L}$.

VI. Numerical Examples

A. Reduction of a Youla optimized controller

This example is the same as the one in [13] and data are taken from [19]. In this problem, a reduced controller $\hat{K}$ is sought so that the $\mathcal{H}_\infty$ norm difference $\left\| P \star K - P \star \hat{K} \right\|_\infty$, instead of $\left\| P \star \hat{K} \right\|_\infty$, should be small. To apply the proposed method for controller reduction, a modified plant, denoted as $\tilde{P}$, is defined so that $\tilde{P} \star \hat{K} = P \star K - P \star \hat{K}$. Then the feedback setup of $\tilde{P}$ and K is considered for controller reduction purposes. The full controller K has 152 states and also an integrator. The same approach that was in [13] is applied in here to make sure that the reduced controller has an integrator. See [13] for detail. Because the order of K is too large for a semidefinite program to be solved in reasonable amount of time, the Riccati equation in (9) is solved to obtain an X, and then the proposed problem in (8a) and (8b) is solved using SVD. In the end, a reduced controller with 18 states is obtained. As a comparison, two other reduced controllers of order 18 are obtained. One is by directly applying balanced truncation (BT) (see, e.g. [8] pp.7-8) to the full controller K, and the other is by frequency-weighted balanced truncation (FWBT) [8] pp.106-107 with the input and output weights being $(I - P_0(2,2)K)^{-1} P_0(2,1)$ and $(I - P_0(2,2)K)^{-1} P_0(1,2)$ respectively (to minimize first order closed loop difference). Fig. 2 shows the magnitude Bode diagram of the error of closed loop systems due to different reduced controllers, as well as the magnitude Bode diagram of the full order closed loop system. It is observed that the three different methods have similar approximation qualities in this example.

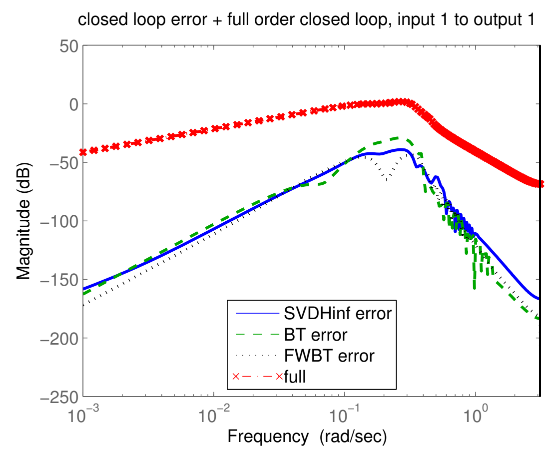

Fig. 2. Magnitude Bode diagrams of the closed loop system error. Solid: error with SVD reduced controller. Dash: error with FWBT reduced controller. Dot: error with BT reduced controller. Dash-dot: full order closed loop for reference. The closed loop approximation qualities of the three different controller reduction methods are similar.

B. Controller reduction of the HIMAT example

In this example the standard controller reduction problem in Fig. 1 is considered. The plant P models the HIMAT aircraft (data available from the MathWorks website), and the full controller K is the optimal $\mathcal{H}_\infty$ norm controller obtained using MATLAB's routine `hinfsyn`. The full controller has 10 states. In this example, the proposed controller reduction method is applied where the bounded-real matrix X is obtained by solving (22). In the end an 8th order reduced controller is obtained. As a comparison, two other reduced controllers of the same order are obtained using HIFOO [18] and balanced truncation. The reduction results are shown in Table I. In this example, balanced truncation fails to return a stabilizing controller. HIFOO and the proposed method have similar closed loop performances, but the proposed method is much more efficient than HIFOO (generally true when the order of $\hat{K}$ is relatively large).

VII. Comparison with the Method in [13]

The discussion in this section is not meant to be self-contained. Some properties of the method in [13] are listed here without any explanation. The purpose of the discussion

TABLE I

CONTROLLER REDUCTION RESULTS FOR THE HIMAT EXAMPLE

method	order	$\|CL\|_\infty$	CPU time (sec)
full	10	0.7885	≈ 0
proposed	8	0.8242	38
HIFOO	8	0.8487	5602
BT	8	unstable	≈ 0

in this section is to collect the differences between the two similar SVD based methods in this paper and in [13].

1) In [13], the proposed reduction problem is based on a generalized controllability Gramian inequality, hence the method therein characterizes closed loop performance in $\mathcal{H}_2$ norm. On the other hand, the proposed problem in this paper is based on the bounded-real lemma in Sec. II-C. The corresponding reduced controller has instead $\mathcal{H}_\infty$ norm performance guarantee.
2) The generalized controllability Gramian in [13] can be computed by solving a Lyapunov equation. On the other hand, the bounded-real matrix X is obtained either by solving a Riccati equation in (9) or a semidefinite program in (22). Regarding computation the method in [13] has an edge because it is less expensive to solve a Lyapunov equation than a Riccati equation of the same size.
3) The method in [13] can only enforce one of the two Gramian inequalities (either controllability or observability but not both). Hence, some of the plant data never appears in the search of the reduced controller in [13]. On the other hand, the bounded-real lemma statement in (8b) involves the use of all plant data.
4) In [13] the C and D matrices of the reduced controller have to be the same as those of the full controller. This is necessary for the generalized controllability Gramian inequality to be meaningful in terms of $\mathcal{H}_2$ norm performance. On the other hand, the application of the bounded-real lemma in this paper poses no a prior restriction on the reduced controller matrices.
5) In [13] the controller order/accuracy tradeoff is mostly determined by a *matrix*. This is not very intuitive. On the other hand, in this paper the tradeoff comes very naturally as the $\mathcal{H}_\infty$ norm upper bound γ.

VIII. CONCLUSION

In this paper a modification of the SVD based controller reduction method in [13] is described. Compared with the method in [13], the current method fixes some lack of symmetry issues listed in Sec. VII (i.e. 3) and 4)) and allows a more natural order/accuracy tradeoff. With the introduction of the proposed method in this paper, the connection between the SVD based methods and the ones in [9], [10] can now be better understood. This is because the proposed method in this paper can be interpreted as a local search step of the global search of the optimal reduced order controller in [9], [10]. Furthermore, the result presented in this paper suggests that the local search can be performed efficiently by solving a generalized matrix approximation problem in the form of (19). Whether (19) comes from a $\mathcal{H}_\infty$ norm description as in (8a), (8b) (8c) or a $\mathcal{H}_2$ norm description as in [13] depends on the application. Finally, numerical examples demonstrate the practical value of the proposed reduction method.

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