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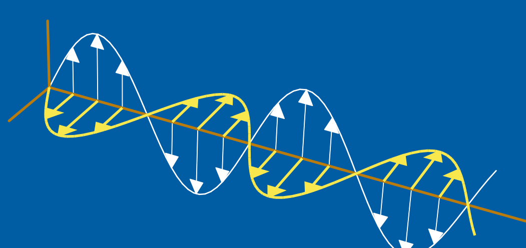

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Abstract

An inverse source problem associated with a semi-linear transport or one-way wave equation in one spatial dimension is considered. It is shown an analytic solution to the inverse problem can be given and furthermore, that this inverse problem of determination of a source function is ill-posed, and must be regularised. A novel regularisation scheme which combines least squares monotone approximation and mollification of the noisy data is used to provide this regularisation. Proof of convergence of this regularisation scheme of *monotone smoothing* is given. Numerical solutions from the inverse problems are presented showing that the method is robust to noisy signals.

The solution of this inverse problem is also shown to illustrate the behaviour of more complex problems from electromagnetism and nonlinear optics. The mathematical techniques that are developed are therefore applicable to other sets of nonlinear first order equations. The method is therefore model independent.

1 Introduction

A form of the one-way wave equation or transport equation occurs in many areas of applied mathematics describing wave motion. Many wave phenomema are modelled by nonlinear second-order partial differential equations. For such difficult problems any simplification possible will aid the advancement of the problem. For many such nonlinear problems reduction to a lower order partial differential equation is often employed. Scalar one-way wave equations are obtained through multiple scale perturbation analysis (see [25, 28] and [6]), for some phenomena described by nonlinear second-order equations. Equations such as the second order wave equation, $u_{tt} - u_{xx} - \sigma(u_t, u_x) = 0$ which is of Klein-Gordon type ([17], pp. 254 *et. seq.*), ([18], pp. 562 *et. seq.*), and is often used in nonlinear optics, can be reduced into a system of two one-way scalar wave equations. Under some conditions this system can be further reduced to a single one-way equation [9]. The equation we study in this paper is also similar to one used to study hyperbolic conservation laws for reacting flows [12], namely it is a semi-linear one-way wave equation.

The linear second-order partial differential equations modelling physical wave phenomena, can always be reduced to a first-order system, and by the technique of wave splitting [1, 8, 36] these can be considered as a system of one-way wave equations; albeit that they are frequently coupled. Such systems will generally involve two-way wave propagation, and so cannot be treated as a scalar one-way equation. Even for such problems, however, the results obtained from the scalar one-way wave equation will often apply in the limit that the coupling reduces to zero or is small. The source reconstruction problem for linear second-order partial differential equations has been examined in several publications [34], [35], and [36]. Inverse source problems for a linear second order equation from electromagnetics involving lightning strikes has also been examined by [21].

There has been little treatment of inverse problems for which the wave equation is nonlinear as compared to the literature for linear wave equations. For other nonlinear wave equations for which inverse problems have been tackled see [3, 6, 13, 15]. Inverse problems for nonlinear reaction diffusion and reaction advection diffusion equations have been considered by [4, 14], for example. Another example for which the inverse problem reduces from one of consideration of a nonlinear second order equation to one of involving a scalar one-way wave equation occurs in an electromagnetic problem for an optical Kerr medium [19, 33, 38].

The inverse problem analysed in this paper also has applications to an equation utilised in age dependent population dynamics, namely

$$u_t + u_a = \sigma(u),$$

where here $u = u(a,t)$ is the population density, a represents the age of the population, and t is the time. The left-hand-side of this equation is the advective derivative of the population when the advection velocity of the population $c(a) = 1$, *i.e.* the population ages at the rate t, and $\sigma(u)$ describes a death (birth) rate of the population nonlinearly dependent on the population size. Side conditions appropriate to this equation are the initial condition $u(x,0) = f(x)$ and either of the boundary conditions

$$u(0,t) = h(t), \quad \text{or} \tag{1.1}$$

$$u(0,t) = \int_0^L w(x)u(x,t)\,dx. \tag{1.2}$$

Here (1.1) represents the simplest prescribed birth condition, and where (1.2) represents a more realistic renewal condition for the population birth as given at time t by the weighted integral over the total population to the maximum age L, with weighting function w. Inverse problems in this area have previously been considered by [10, 23, 24, 26, 27], and the inverse problem we analyse here has application to this type of problem.

The solution method we utilise is based on a novel form of the solution of the inverse problem to the semi-linear one-way equation that was first presented in [6]. This solution has not been used previously except in a simple application without the addition of measurement noise and the consequent requirement of regularisation [38]. This paper considerably extends the analysis of the previous work, and uses a new method of regularisation of this ill-posed inverse problem. We call this approach *monotone smoothing* and it involves the sequential application to the measured data of monotone approximation followed by a mollification scheme. It is shown in §4 that our mollification scheme preserves the monotone property of the monotone approximation. Furthermore it is shown that this regularisation leads to a well posed problem.

In §2 we consider the prototype semi-linear one-way wave equation. In that section we define the direct transmission problem and provide an implicit solution. The well-posedness of this forward problem is considered through study of the monotonicity and regularity properties of the map from the inital/boundary data to the transmission function. The inverse problem posed in §2, for a one-way wave equation, is one of determination of a source function from measurement of the transmitted wave. Similar inverse problems which measure transmitted waves to reconstruct some medium properties have been posed to determine *Kerr* medium properties [19], and for mechanical medium properties [13], from nonlinear second order equations. Inverse problems for source determination for the linear one-way wave equation have been analysed in [39, 40].

In §3 we consider the inverse problem of source function determination. We show that the inverse problem, by linearisation, is Lipschitz continuous in appropriate function spaces. In §4 it is shown that the inverse problem although ill-posed in typical measurement function spaces can be made to be well-posed by monotone smoothing of the data. The inverse problem is nonlinear for this transmission problem, which is in contrast with those we have studied in [29, 39] and in the second order advection diffusion inverse problem we consider in [31, 32].

Numerical simulation illustrating the effectiveness of our method of solving the nonlinear inverse source problem and of the regularisation technique of monotone smoothing is presented in §5.

In Appendix A we show that under suitable conditions the semi-linear one-way wave equation studied here can be used to approximate semi-linear second order equations. In Appendix B we find the Lipschitz continuity constant of the inverse function operation — this is fundamental to our inversion technique.

2 Semi-linear Source Problem

Consider the semi-linear one-way wave equation, with a source term σ,

$$u_t + c(z)u_z = \sigma(u), \quad 0 < z < \infty, \quad t > 0, \tag{2.1}$$

and with the dependent variable $u(z,t)$, as an initial-boundary value problem with the side conditions

$$u(0,t) = h(t), \quad 0 \le t \le \infty, \text{ and } \quad u(z,0) = \overline{f}(z), \quad 0 \le z \le \ell. \tag{2.2}$$

Here we assume the wave speed parameter c, is only a function of the spatial variable z. This will mean that the linear part of the equation is inhomogeneous but will produce only non-dispersive effects, however the nonlinear dependence of the source term on u will cause dispersion. In this equation it is to be noted that the source term maybe considered as a reaction term, which is a nonlinear function of only u. As equation (2.1) only admits unilateral waves we need not consider wave motion for $z < 0$; also as the equation is semi-linear shocks do *not* appear. For this problem the characteristic traces can easily be straightened, via a transformation of the form

$$x = \zeta(z) = \int_0^z \varsigma(x')\,dx', \tag{2.3}$$

with the slowness $\varsigma = 1/c$. This transformation converts (2.1) into the form

$$u_t + u_x = \sigma(u), \quad 0 < x < \infty, \quad t > 0, \tag{2.4}$$

with the modified side conditions

$$u(0,t) = h(t),\ 0 \le t \le \infty,\ \text{and} \quad u(x,0) = f(x) \equiv \left(\overline{f} \circ \zeta^{-1}\right)(x),\ 0 \le x \le L = \zeta(\ell), \tag{2.5}$$

and the initial condition in the x-coordinates has been redefined by the f function[1]. This reformulation is more convenient for our subsequent development.

An essential feature in the sequel is that $\sigma \neq 0$ for all u under consideration, then an implicit solution to equation (2.4) can be found to be

$$\begin{cases} Tu(x,t) - Tf(x-t) = t, & 0 < t < x, \\ \\ Tu(x,t) - Th(t-x) = x, & 0 < x < t, \end{cases} \tag{2.6}$$

with the definition of the linear functional

$$T\eta = \int_0^\eta \frac{d\xi}{\sigma(\xi)}. \tag{2.7}$$

It can be easily verified that (2.6), is a classical solution to (2.4), with the given side conditions if $h \in C^1[0,\infty)$, $f \in C^1[0,\infty]$, and σ integrable with $\sigma \neq 0$. Observe from (2.6) (see also (2.10)) unless $h(0) = f(0)$ there is a presence of a discontinuity in the solution $u(x,t)$ at the time $t = x = \int_0^z \zeta(x')\,dx'$.

In the sequel for convenience we assume that σ is continuous. Furthermore with $c > 0$ and $c \in C$ then (2.6) transformed via (2.3) is also a solution to (2.1).

The transmission problem under consideration in this paper is the direct problem that is described by (2.1), equivalently (2.4), and given σ and side data (2.2), equivalently (2.5), determine the transmitted signal $\overline{u}(t) = u(\ell,t) = u(L,t)$.

For convenience in the sequel we define define the map, from the source term σ, and also the side conditions (2.2), to the measurement $\overline{u}$, by the operator $\mathbb{T}$, and assume that the measurement function, the source function and the side conditions are in appropriate function spaces represented by Σ, $\mathbb{S}$, and $\mathbb{F}$, respectively. Then

$$\mathbb{T} : (\mathbb{S};\mathbb{F}) \to \Sigma, \tag{2.8}$$

is the forward problem map. The solution (2.6) provides an implicit representation of this map.

It is now possible to pose an inverse problem for reconstruction of the function $\sigma(u)$ from (2.1), when the measured data is the time history of u down-stream, *i.e.* $u(\ell,t)$. In order to do this $\overline{f}$, c, and h, must be known functions. Here the given data are at location $z = \ell$, or equivalently at $L = \zeta(\ell)$. If $\overline{u}(t) = u(\ell,t)$ is the measured function then an inverse reconstruction map is determined by the inverse of the map $\mathbb{T}$ previously defined in (2.8). This inverse map can be written

$$\mathbb{T}^{-1} : \Sigma \to \mathbb{S}, \tag{2.9}$$

where we wish to have a one-to-one map, so we have only written the inverse operator into the space $\mathbb{S}$, assuming that $\overline{f}$, c, and h are implicit and known. The main objective of this paper is to provide computational algorithms that can be used to implement this inverse map.

[1]Note ζ^{-1} exists as $\varsigma > 0$ because $c > 0$.

2.1 Monotonicity of measured field $\overline{u}$

Prior to looking at the regularity of the solution we show the conditions required to ensure that $\overline{u}$ is a monotone function. This is an essential feature for the numerical algorithms which we develop.

For $\overline{u}$ we have two cases to consider:

1. For $t < L$, by differentiation of (2.6)(a), with respect to t, it follows that
$$\overline{u}'(t) = \sigma\big(\overline{u}(t)\big)\big(1 - \frac{f'(L-t)}{\sigma\big(f(L-t)\big)}\big).$$
where we have used the prime to denote differentiation with respect to the function argument. So that a sufficiency condition for monotonicity in this case is f be strictly monotone and also σ and f' have opposite sign. It can then can be seen that $\overline{u}'(t) > 0$, or $\overline{u}'(t) < 0$, as required.

2. For $t > L$, from differentiation of (2.6)(b) with respect to t, it follows that
$$\overline{u}'(t) = \frac{\sigma\big(\overline{u}(t)\big)h'(t-L)}{\sigma\big(h(t-L)\big)},$$
which is seen to be of one sign if h is strictly monotone and σ is of one sign.

Lemma 2.1. *A necessary and sufficient condition for the measurement function $\overline{u}$ to be strictly monotone function, for $t > L$, is that σ be of one sign and h to be strictly increasing or decreasing.*

A sufficient condition for the function $\overline{u}$ to be a strictly monotone function, for $t < L$, is that σ be of one sign and h to be strictly increasing or decreasing and that both σ and f' be of one sign, but of opposite signs to each other.

Corollary 2.1. *When the initial condition $f \equiv 0$ a necessary and sufficient condition for the measurement function $\overline{u}$ to be strictly monotone is that σ be of one sign and h to be strictly increasing or decreasing.*

We note that our results are slightly more general than those quoted as Lemma 3.1 and Corollary 3.2 in [6].

2.2 Regularity of the Direct problem

In this section we examine the regularity of the semi-linear direct problem defined in the last section via the implicit solution (2.6). We obtain results for the continuous dependence of the measured function upon the side conditions, *i.e.* the map $\mathbb{T}_1 : \mathbb{F} \to \Sigma$, with arbitrary source, as given by implicitly by (2.6); to distinguish this map from $\mathbb{T}$ mapping the source to the measurement data, we add the subscript 1.

We assume that besides the conditions $f, h \in C^1$ and $\sigma \neq 0$, that $\sigma \in L^\infty$, and the source function has a two-sided bound such as $0 < m \leq |\sigma| < M$, and that it also does not change sign. We note the upper bound on σ is not too severe as most physically realistic nonlinear problems have such a bound [19]. The choice of the Banach space L^∞ for σ would enable us to allow for the reconstruction of discontinuous source functions, unlike classical theory which would require σ to be Lipschitz continuous. However we will restrict our numerical results in §5 to continuous σ. We there assume the function spaces $\mathbb{F}$, Σ are in C and

Suitable norms in the spaces we use are:

$$\begin{aligned}
\text{in the space } C: \quad & \|f\|_\infty = \sup |f|, \\
\text{in } L^\infty: \quad & \|f\|_\infty = \text{ess. sup}|f|, \\
\text{in } L^p(I): \quad & \|f\|_p = \left\{ \int_I |f|^p \right\}^{1/p}, \\
\text{and in the Sobolev space } W^{p,q}: \quad & \|f\|_{p,q} = \left\{ \sum_{i=0}^{p} \|f^{(i)}\|_q \right\}^{1/p},
\end{aligned}$$

note that $f^{(i)}$ denotes the i-th derivative of f, for further details see *e.g.* [2, 16]. It should also be noted that the standard Sobolev space $H^2 \equiv W^{2,2}$.

Returning now to determining the continuity of the direct map $\mathbb{T}_1$. An explicit form of the map can be found from (2.6) by use of the inverse of the functional T. The inverse will exist as T is strictly monotone as shown in Lemma 2.1. By applying T^{-1} to (2.6) it is found

$$\overline{u}(t) = \begin{cases} T^{-1}\Big(Tf(L-t)+t\Big), & 0 < t < L; \\ \\ T^{-1}\Big(Th(t-L)+L\Big), & t > L. \end{cases} \tag{2.10}$$

Now as T is given explicitly by (2.7), and as we have assumed that $m \le |\sigma(t)| \le M$, it is clear that

$$\begin{aligned} \frac{\eta}{M} \le |T\eta| \le \frac{\eta}{m}&, \\ \text{and} \quad m\xi \le |T^{-1}\xi| \le M\xi&, \end{aligned}$$

where the second inequality follows from the first, on the definition $T\eta = \xi$. It therefore follows that

$$\|\overline{u}\|_\infty \le M\Bigg(\frac{\max[\|f\|_\infty, \|h\|_\infty]}{m} + L\Bigg). \tag{2.11}$$

This result therefore demonstrates the continuous dependence of the solution $\overline{u}(t)$ on the initial/boundary data. By utilising the source term $\sigma = M$ it is readily seen that the bounds we have obtained are strict. Continuous dependence on the source function may be shown in a similar manner to Fréchet differentiability of the forward map, *i.e.* via the implicit function theorem [7]. We do not pursue this here.

We observe from (2.10) unless $f(0) = h(0)$ then $\overline{u}(t)$ will have a discontinuity at $t = L$. Looking ahead at (3.1) and after some manipulation it can be seen that $\overline{u}\,'(t)$ will also have a discontinuity at $t = L$, unless $h'(0)/\sigma(h(0)) = 1$.

We have chosen to show this continuity for $\overline{u}$, it is straightforward to extend this to $u(x,t)$, the solution of (2.1).

To show C^1 regularity of the forward map from the data, again consider first the problem for $t \le x$, so that

$$T(u) = T\big(f(x-t)\big) + t, \quad 0 \le t \le x, \tag{2.12}$$

and then by differentiating this equation with respect to t we obtain

$$u_t = \left(1 - \frac{f'(x-t)}{\sigma\big(f(x-t)\big)}\right)\sigma(u).$$

Similarly by differentiation of (2.12) with respect to x it is seen that

$$u_x = \left(\frac{f'(x-t)}{\sigma\big(f(x-t)\big)}\right)\sigma\big(u\big).$$

By use of (2.6)(b) and by similar differentiation to the above the u_t and u_x dependence upon h, for $t > x$, can be found. It then follows from all these, and similar procedures to those used to get (2.11), that

$$\begin{aligned} &t < x, \quad \|u_t\|_\infty \le \left(1 + \frac{\|f'\|_\infty}{m}\right)M, \quad \text{and} \quad && t > x, \quad \|u_t\|_\infty \le \left(\frac{M}{m}\right)\|h'\|_\infty, \\ &t < x, \quad \|u_x\|_\infty \le \left(\frac{M}{m}\right)\|f'\|_\infty, \quad \text{and} \quad && t > x, \quad \|u_x\|_\infty \le \left(1 + \frac{\|h'\|_\infty}{m}\right)M. \end{aligned}$$

We therefore have the result:

Theorem 2.2. *With $\sigma \in L^\infty$, and σ of one sign together with the bounds $0 < m \le |\sigma| < M$, and with $(h, h') \in L^\infty[0,\infty)$, $(f, f') \in L^\infty[0, L]$, and the conditions required in Lemma 2.1, the solution of (2.1) and its first derivative, with respect to x or t, depends continuously upon the inital/boundary conditions, except possibly across the characteristic trace $x = t$. Across this line the solution and/or its derivatives may have a jump discontinuity.*

We note that this continuous dependence can be straightforwardly moved to the case when $\sigma \in C$ and this theorem shows the map $\mathbb{T}_1$ is bounded.

3 Inverse Source Problem with measured transmission data

It is now possible to pose an inverse problem for reconstruction of the functional $\sigma(u)$ from (2.1), when the measured data is the time history of u down-stream, *i.e.*

$\overline{u}(t) = u(\ell, t)$. In order to do this $\overline{f}$, c, and h, must be known functions. What is required for the resolution of the inverse problem is an inverse operator map $\mathbb{T}^{-1}$.

Equations (2.6) are functional equations, the solution of which defines $\mathbb{T}^{-1}$ and yields the σ function.

3.1 Reconstruction of the source functional with an applied boundary condition

We consider the problem of reconstruction of $\sigma(u)$ from measurement of $u(\ell, t) = \overline{u}(t)$ when the initial value $f \equiv 0$. The operator mapping σ to $\overline{u}$ is implicitly defined by (2.6). By differentiation of this equation with respect to t, and after some manipulation an explicit formula maybe found for the source functional as

$$\begin{cases} \sigma(\overline{u}(t)) = \overline{u}\,'(t), & 0 < t < L, \\ \\ \sigma(\overline{u}(t)) = \dfrac{\overline{u}\,'(t)\sigma(h(t-L))}{h'(t-L)}, & t > L, \end{cases} \tag{3.1}$$

where use has been made of the fact that $T'(\xi) = 1/\sigma(\xi)$. Now with $\xi = \overline{u}(t)$ we may define $t = \overline{u}^{\,-1}(\xi)$ when $\overline{u}$ is strictly monotone and the conditions on f, h, and σ to achieve this are prescribed in Lemma 2.1. It can be further shown that $\overline{u}$ is a homeomorphism; hence $\overline{u}^{\,-1}$ exists. It now follows (3.1) may be rewritten as

$$\begin{cases} \sigma(\xi) = \overline{u}\,'(\overline{u}^{\,-1}(\xi)), & \overline{u}(0) < \xi < \overline{u}(L), \\ \\ \sigma(\xi) = \dfrac{\overline{u}\,'(\overline{u}^{\,-1}(\xi))\sigma(h(\overline{u}^{\,-1}(\xi) - L))}{h'(\overline{u}^{\,-1}(\xi) - L)}, & \overline{u}(L) < \xi. \end{cases} \tag{3.2}$$

First restricting our considerations to the first of these formula, we note with $\xi = \overline{u}(t)$ it may also be written as

$$\sigma(\xi) = \frac{1}{[\overline{u}^{\,-1}(\xi)]'}, \quad \overline{u}(0) < \xi < \overline{u}(L). \tag{3.3}$$

This formula can be readily found from the use of the chain rule on the composite function. Equation (3.2) along with (3.3) now provides the basis of an inverse source reconstruction algorithm, when $h(t)$ is known and is used in §5 for the reconstruction.

It is readily seen that from equation (2.6) that when $t = 0$

$$\int_0^{\overline{u}(0)} \frac{d\xi}{\sigma(\xi)} = 0,$$

and this implies that $\overline{u}(0) = 0$, as $\sigma \neq 0$. We shall now for illustrative purposes assume that $h(0) = 0$ and $h(t) \to \infty$ as $t \to \infty$. Hence provided $\overline{u} \in C^1$, the first part of (3.2) provides a functional solution for $\sigma(\xi)$, with $\xi \in [0, \overline{u}(L)]$. Now with $\xi \in [\overline{u}(L), \overline{u}\big(L + h^{-1}(\overline{u}(L))\big)]$ it is seen that the argument of the term $\sigma(h(\overline{u}^{\,-1}(\xi) - L))$, on the right-hand-side of the second part of (3.2), is in the range $[0, \overline{u}(L)]$. Hence it is now possible to reconstruct $\sigma(\xi)$, with $\xi \in [\overline{u}(L), \overline{u}\big(L + h^{-1}(\overline{u}(L))\big)]$. Then for $\xi > \overline{u}\big(L + h^{-1}(\overline{u}(L))\big)$ we may continue to reconstruct $\sigma(\xi)$ in a similar manner. It is therefore seen that σ can therefore be completely reconstructed from the inverse map equations (3.2). It follows that the source functional $\sigma(u)$ can be uniquely reconstructed, from transmission data $\overline{u}$, with an appropriately chosen boundary function h.

We see that the resolution of this inverse problem is ill-posed, and this follows as it requires a differentiation operation on the measurement function. If $\overline{u}$ is mollified (see [22, 37]), it will follow that the resultant regularised differentiation operation will depend continuously on the measurement data, and so be well-posed. However the constraint that $\overline{u}$ be strictly monotone is *not* conserved by mollification and the inverse function $\overline{u}^{\,-1}$ may therefore not exist after mollification . We therefore smooth the noisy data $\overline{u}_m$ with a monotone approximant and subsequently apply mollification; we call this regularisation technique *monotone smoothing*. It is shown in §4 that the posterior application of mollification does not destroy the monotonicity of $\overline{u}$. The differentiation nature of the ill-conditioning is also in accordance with the results obtained for the source reconstruction problem associated with a linear one-way equation [39].

We next wish to show the continuous dependence of *source* on the measurement function and the data f and h, *i.e.* to prove the continuity of the inverse map $\mathbb{T}^{-1}$.

3.2 Lipschitz continuity of the inverse problem

We prove this result in two norms for full generality, although only the L^2 norm is used in our numerical procedures discussed in §5. We consider the L^∞ norm first as this is the norm we used for obtaining regularity of the forward problem.

3.2.1 Lipschitz continuity of the inverse problem in the L^∞ norm

To obtain a Lipschitz constant for continuity of the inverse map it is necessary to find the norm of the Fréchet derivative of the map $\mathbb{T}^{-1}(\overline{u}^{\,-1}, \overline{u}\,') \to \sigma$. Now the the explicit form of this map is seen in (3.2), and $\mathbb{T}^{-1\prime}$ is the Fréchet derivative, with respect to $\overline{u}$ of this map.

To find the Fréchet differential examine the change in σ namely $\delta\sigma$, to a perturbation in the data $\overline{u}$, namely $\delta\overline{u}$. To first order terms, when $t \leq L$, this can be shown to be

$$\begin{aligned} \delta\sigma(\xi) + \sigma'(\xi)\delta\xi &= \delta\overline{u}'(t), \\ \text{or} \quad \delta\sigma(\xi) &= -\sigma'(\xi)\delta\xi + \delta\overline{u}\,'(t). \end{aligned}$$

Alternatively the Fréchet differential for the map $\mathbb{T}^{-1}$ may be written

$$\begin{aligned} \delta\sigma(\overline{u}(t)) &= -\sigma'(\overline{u}(t))\delta\overline{u}(t) + \delta\overline{u}\,'(t), \\ \delta\sigma(\xi) &= -\sigma'(\xi)\delta\overline{u}(\overline{u}^{\,-1}(\xi)) + \delta\overline{u}\,'(\overline{u}^{\,-1}(\xi)). \end{aligned} \tag{3.4}$$

So by taking the L^∞ norm of (3.4)

$$\begin{aligned}\|\delta\sigma\|_\infty \leq & \|\sigma'\|_\infty \|\delta\overline{u}\|_\infty + \|\delta\overline{u}\,'\|_\infty \\ \leq & \left\|\frac{\overline{u}''}{\overline{u}\,'}\right\|_\infty \|\delta\overline{u}\|_\infty + \|\delta\overline{u}\,'\|_\infty .\end{aligned}$$

The first term on the right-hand-side of the second form of this equation has been found from differentiation of the first of (3.1), to get $\sigma'(\overline{u}(t))\overline{u}\,'(t) = \overline{u}''(t)$ when $t < L$. As expected, from (3.2), we have shown that the solution depends continuously on the derivative of the data, but the Lipschitz constant is also dependent upon the second derivative of the data and $1/\inf|\overline{u}\,'|$.

The inverse problem solution, for $t < L$, can then be seen to be well-posed, if $\overline{u} \in C^2$ *a.e.*, $\sup|\overline{u}''| \leq C$ and $\inf|\overline{u}\,'| \geq c > 0$, and it then depends Lipschitz continuously on the data $\overline{u}$, with

$$\begin{aligned}\|\delta\sigma\|_\infty \leq & \frac{C}{c}\|\delta\overline{u}\|_\infty + \|\delta\overline{u}\,'\|_\infty \\ \leq & K\|\delta\overline{u}\|_{1,\infty} .\end{aligned} \tag{3.5}$$

As discussed above this would require the measurement data $\overline{u}$ to be in the Sobolev function space $W^{2,\infty}$, *i.e.* $\Sigma \in W^{2,\infty}$.

We now consider the recursive part of the solution in order to get the Lipschitz constant for $t > L$. Observe that the formula (3.1), or equivalently (3.2), provides a recursion over the interval $0 < t < \infty$; *i.e.* for $0 < t < L$ use (3.2)(a), then use (3.2)(b) over the interval $L < t < L + h^{-1}(\overline{u}(L))$. Continue to use (3.2)(b) in increasing t intervals in recursion; *i.e.* recursion over the interval $L < t < \infty$ is provided by:

$$\sigma\big(\overline{u}(t)\big) = \frac{\overline{u}\,'(\overline{u}^{\,-1}(\xi))\sigma(h(\overline{u}^{\,-1}(\xi) - L))}{h'(\overline{u}^{\,-1}(\xi) - L)}.$$

Now, again examining the change in σ namely $\delta\sigma$, to a perturbation in the data $\overline{u}$, namely $\delta\overline{u}$. To first order terms this can be shown to be

$$\delta\sigma(\xi) + \sigma'(\xi)\delta\xi = \frac{\sigma(h(t-L))\delta\overline{u}'(t) + \overline{u}'(t)\delta\sigma(h(t-L))}{h'(t-L)}.$$

Hence the Fréchet differential for the map $\mathbb{T}^{-1}$ is

$$\delta\sigma(\xi) = -\sigma'(\xi)\delta\xi + \frac{\sigma(h(t-L))}{h'(t-L)}\delta\overline{u}'(t) + \frac{\overline{u}'(t)}{h'(t-L)}\delta\sigma(h(t-L)),$$

where the first and second terms on the right-hand-side of this equation depend upon $\delta\overline{u}$ and its derivatives, but the third term depends on the perturbation of $\delta\sigma$ in the previous equation in the recursion scheme. On taking norms of this formula we obtain

$$\|\delta\sigma^k(\xi)\|_\infty \leq \|\sigma^{k\prime}\|_\infty\|\delta\overline{u}\|_\infty + \left\|\frac{\sigma^k}{h'}\right\|_\infty \|\delta\overline{u}'\|_\infty + \left\|\frac{\overline{u}\,'}{h'}\right\|_\infty \|\delta\sigma^{k-1}\|_\infty, \tag{3.6}$$

$$\leq K_1^k\|\delta\overline{u}\|_{1,\infty} + K_2^k\|\delta\sigma^{k-1}\|_\infty, \tag{3.7}$$

where the superscript k on the source function, as σ^k, denotes the source on the k-th interval in the recursion and the superscript on K denotes the k-th constant. To obtain this formula we have assumed that $\sigma^{k\prime}$ is bounded. To prove Lipschitz continuity we need now to relate the terms σ^k and $\sigma^{k\prime}$ in terms of $\overline{u}$ and its derivatives. For the next interval differentiate (3.2)(b) with respect to t to obtain, after some rearrangement

$$\sigma'(\xi) = \frac{[\overline{u}''(t)\sigma\big(h(t-L)\big) - h''(t-L)\sigma(\xi) + \overline{u}\,'(t)\sigma'\big(h(t-L)\big)h'(t-L)]}{h'(t-L)\overline{u}\,'(t)}, \quad (3.8)$$

which is again recursive. We must bound $\|\sigma\|_\infty$ as we observe from (3.8) that σ' depends on σ in earlier intervals. From (3.2)(a)

$$\|\sigma^1\|_\infty \leq \|\overline{u}\,'\|_\infty, \quad t < L,$$

and from (3.2)(b), *i.e.* in the next interval,

$$\|\sigma^2\|_\infty \leq \frac{\|\overline{u}\|_\infty \|\sigma^1\|_\infty}{\inf |h'|},$$

which is bounded, from a recursive argument, if $\overline{u} \in C^1$, $\inf |h'| > 0$. We may continue to bound σ from a recursive argument, in the subsequent intervals.

It can be seen that the terms on the right-hand-side of equation (3.8) will be bounded provided $h, \overline{u} \in C^2$ *a.e.*, $\inf |h'| > 0$ and $\inf |\overline{u}\,'| > C$. We may continue to bound σ' for subsequent intervals in a similar manner.

To recap we have shown on the first interval that

$$\|\delta\sigma^{(1)}(\xi)\|_\infty \leq K^1 \|\delta\overline{u}\|_{1,\infty},$$

and on the k-th interval that

$$\|\delta\sigma^k(\xi)\|_\infty \leq K_1^k \|\delta\overline{u}\|_{1,\infty} + K_2^k \|\delta\sigma^{k-1}\|_\infty,$$

it follows we have proven that

$$\|\delta\sigma^k(\xi)\|_\infty \leq \overline{K}^k \|\delta\overline{u}\|_{1,\infty},$$

where $\overline{K}^k$ is dependent upon $\|\overline{u}\|_{2,\infty}$, $\|h'\|_\infty$, $\|h''\|_\infty$ together with $\inf |\overline{u}\,'|$, and $\inf |h'|$.

We can now state the well-posedness result for the inverse problem.

Theorem 3.1. *With the conditions of Lemma 2.1, $\overline{u} \in W^{2,\infty}$* a.e., *and $\sigma \in L^\infty$ the solution of the inverse problem of determination of σ from the measured data $\overline{u}$ is well conditioned and is Lipschitz continuous in $\overline{u}$; except possibly across the characteristic trace $t = L$.*

Note that the Lipschitz constant is larger when c, where $\inf |\overline{u}'| \geq c$, is small, this is as expected by knowledge of the behaviour of an inverse function *c.f.* Appendix B. Hence we can expect the conditioning of the numerical solution to be poor when $|\overline{u}'|$ is very small.

3.2.2 Lipschitz continuity of the inverse problem in the L^2 norm

We only explicitly show the result for the first part of the recursion as the proof then proceeds similar to that in the L^∞ norm. The corresponding result for the inverse function operation alone is considered in Appendix B.

With the variation in $\delta\sigma$ given by (3.4) take the 2-norm and use the triangle inequality to yield

$$\begin{aligned}\|\delta\sigma\|_2^2 \leq& \|\overline{u}'\|_\infty \|\delta\overline{u}'\|_2^2 + \frac{\|\overline{u}''\|_\infty^2}{\inf|\overline{u}'|}\|\delta\overline{u}\|_2^2 + 2\|\delta\overline{u}\|_2\|\delta\overline{u}'\|_2\|\overline{u}''\|_\infty \\ \leq& \frac{M}{c}\|\delta\overline{u}\|_{2,2}^2,\end{aligned}$$

for some constant M. Observe as noted in §2.2 the equivalence of the Sobolev spaces $H^2 \equiv W^{2,2}$. The result for the recursive part of the formula (3.2)(b) follows in a similar manner to that used in the last section.

Theorem 3.2. *With the conditions of Lemma 2.1, $\overline{u} \in H^2$, and $\sigma \in L^2$, the solution of the inverse problem of determination of σ from the measured data $\overline{u}$ is well posed, and is Lipschitz continuous on $\overline{u}$ except possibly across the line $t = L$.*

This result is the basis of our numerical procedures, and in §4 it is used in the regularisation result. Note that the proof requires the existence of the inverse function $\overline{u}^{-1}$; which is assured by Lemma 2.1.

3.3 Reconstruction of the source function with only initial conditions

If $h \equiv 0$ and $f \neq 0$ then the operator $\mathbb{T}$ is defined implicitly by

$$\begin{cases} T\overline{u}(t) - Tf(L-t) = t, & 0 < t < L, \\ T\overline{u}(t) = L, & t > L, \end{cases} \tag{3.9}$$

By differentiation of the formulae in (3.9) with respect to t, and after some manipulation, the source functional may be written as

$$\sigma(\overline{u}(t)) = \frac{\overline{u}'(t)\sigma(f(L-t))}{\big(\sigma(f(L-t)) - f'(t-L)\big)}, \quad 0 < t < L. \tag{3.10}$$

For this case $\overline{u}'(t) \equiv 0$, when $t > L$, as the initial condition function has completely propagated past the point $x = L$ after a time $t = L$. Therefore the only information on σ is obtained from the measurement for $0 < t < L$. We see that the resolution of this inverse problem is ill-posed, and this follows as it requires a differentiation operation on the measurement function. The functional equation (3.10) does not admit an explicit solution.

The monotonicity of $\overline{u}$ proven in Lemma 2.1 would however enable a numerical algorithm for solution of this inverse problem to be formulated; through a basis function expansion and collocation of (3.10). This would result in a system of nonlinear algebraic equations to be solved numerically to reconstruct σ. This inverse problem is the relevant problem for solution of the population model discussed in §1. However, this inverse problem is not considered further here.

4 Problem regularization

In this section we use a new method of regularisation called monotone smoothing. This method utilises the techniques of monotone approximation in addition to mollification [22, 37].

As we have seen in section §3.2, the ill-posed nature of reconstructing the source function from noisy transmission measurements is in part equivalent to differentiation. Because the measured data lie in function spaces where differentiation operators are unbounded it is therefore necessary to restore continuity with respect to the data to solve the problem. The regularization procedure for solution of (3.2) is to compute $\overline{u}'$ in a stable manner. It is well known that numerical differentiation can be made a well-posed problem, and there are a number of regularization methods available. Any of these regularization methods can be used to yield stable solutions to the numerical differentiation required in (3.2). However, more is required of the solution in order to stably reconstruct σ. The inverse problem also requires a stable resolution of the inverse function operation. This is well known to be ill-conditioned if the derivative of the data function approaches zero. It is shown in Appendix B that the inverse function will only be well conditioned if the approximation to $\overline{u}$ is strictly monotone. The algorithm must therefore maintain the strict monotone property[2] of $\overline{u}$ else the inverse map for $\overline{u}$ is not defined. We do this through the use of a least squares monotone approximation scheme to $\overline{u}_m$, the noisy measured data and then we mollify the resultant approximation to form $\overline{u}'_m$ in a stable manner. It is shown in Theorem 4.1 that the final approximation to $\overline{u}$ is both monotone and depends continuously upon the data. We call the resultant approximation *monotone smoothing* and it is the result of performing *monotone approximation* in addition to *mollification*. We do this in a sequential manner.

We only illustrate our results in L^2 however it is possible to proceed in the space L^∞.

4.1 Monotone approximation by B-splines

The regularized solution to the inverse problem consists of first fitting a monotone B-spline through the noisy data $\overline{u}_m$.

Define the natural numbers $i, j, M, N \in \mathbb{N}$, and then a mesh $\{\bar{t}_i\}_{i=0}^N$, is established on the time axis. This is a mesh with possibly a non-uniform mesh interval, $\bar{t}_0 = 0$, and $\bar{t}_N = T$ is the maximum time used to collect data. The time intervals correspond to the $N+1$ time measurements of $\overline{u}$; namely $\overline{u}_m$. Define another mesh $\{t_i\}_{i=0}^M$, with

[2] As shown in Lemma 2.1.

a uniform mesh interval $h = T/M$, and $t_0 = 0$, $t_i = t_{i-1} + h$, $1 \le i \le M$, which is also established on the time axis.

The measured signal function $\overline{u}_m(t)$ is to be approximated in a least squares sense by a B-spline of degree n, denoted by S_n, such that

$$S_n(t) = \sum_{i=0}^{M} \alpha_i b_i(t), \tag{4.1}$$

where $M+1$ is the cardinality of the B-spline basis, $\{b_i(t)\}_{i=0}^{M} \in \mathcal{S}_n[0,T]$ and the knots are at the uniform t_i. Here $\mathcal{S}_n$ is the space of polynomial splines of degree n on the interval $[0,T]$.

We now require a least squares fit to the $\overline{u}_m$ at the $\overline{t}_i$; where it is assumed that $N \gg M$. To ensure that the least squares B-spline is a monotone approximate to $\overline{u}$ we are required to include the constraint that the derivative of S is bounded from zero. In general if we make this restriction over $0 < t < T$ this will lead to a semi-infinite programming problem. These problems are computationally expense and more difficult to solve, and this technique applied here would be excessive. So we use the constraint to be that the derivative has one sign at the $\{\overline{t}_i\}_{i=0}^{M}$. Therefore on restricting consideration, without loss of generality, to the case where the monotonicity of $\overline{u}$ is strictly nondecreasing, this leads to the constrained least squares problem:

$$\min_{\boldsymbol{\alpha}} \|\overline{u}_m(\overline{t}_i) - S_n(\overline{t}_i)\|_2, \quad \text{subject to the constraint } S_n'(\overline{t}_i) \ge \epsilon, \quad \epsilon > 0, \quad 0 \le i \le N, \tag{4.2}$$

with the discrete 2-norm being used. The parameter ϵ is used to ensure that the spline is *strictly* monotone. The existence of the inverse function map provided by the approximating B-spline S_n is then assured.

The convergence of the monotone approximation scheme is now discussed. When we wish to denote the solution of (4.2) to an appropriate function $\overline{u}_m(t) = f(t)$ we use $S_n(f)$. For treatment of unconstrained least squares approximation by splines see [11]. What we require is in the form of a Jackson type estimate for the consistency result [5]

$$\begin{aligned} \|\overline{u} - S_n(\overline{u})\|_2 &= \left\{ \min_{\boldsymbol{\alpha} \text{ s.t. } S_n' \ge \epsilon} \int_0^T |\overline{u}(t) - S_n(t)|^2 \, dt \right\}^{\frac{1}{2}}, \\ &\le C(\epsilon)\omega(h), \end{aligned}$$

provided $\overline{u}' \ge \epsilon$, and ω is the modulus of continuity of $\overline{u}$, *i.e.* if $\overline{u} \in C^1$ then $\omega(h) = \max_{0<t<T} |\overline{u}'(t)| h$. Given a higher regularity on $\overline{u}$ the power of h and the order of the derivative can be raised to $n+1$.

A continuity result to the monotone approximation is of the form

$$\|S_n(\overline{u}) - S_n(\overline{u}_m)\|_2 \le C_1 \|\overline{u} - \overline{u}_m\|_2,$$

and this states that approximation depends continuously on the function being approximated. It is to be observed that only the data is being changed on the left-hand-side, and that C_1 depends on the degree of the spline, the knots and the regularity of $\overline{u}, \overline{u}_m$. Now if the noise level between the measured $\overline{u}_m$ and the exact $\overline{u}$ is κ, *i.e.*

$$\kappa = \|n\|_2 = \|\overline{u} - \overline{u}_m\|_2,$$

then the convergence result

$$\begin{aligned} \|\overline{u} - S_n(\overline{u}_m)\|_2 \leq & \|\overline{u} - S_n(\overline{u})\|_2 + \|S_n(\overline{u}) - S_n(\overline{u}_m)\|_2, && (4.3)\\ \leq & \, C_1\kappa + C(\epsilon)\omega(h), && (4.4)\end{aligned}$$

as $\kappa \to 0$ and $h \to 0$, follows by the triangle inequality.

4.2 Mollification in L^2

We secondly discuss the technique of mollification to provide a numerical stable algorithm to approximate $\overline{u}'_m$.

Suppose that due to measurement difficulties an ideal continuous data function g has been corrupted by noise n and is measured as g_m. That is $g_m(t) = g(t) + n(t)$, $t \in I$, where the functions are defined on some interval $I = [0, T]$, for some $T > 0$. Let the extension of the data function g_m to the interval $I_\delta = [-3\delta, T+3\delta]$ be defined from an appropriate extrapolation extension of g_m to I_δ. Consider the mollification, or Gaussian function

$$\rho_\delta = \frac{1}{\delta\sqrt{\pi}} e^{-x^2/\delta^2}, \quad x \in \mathcal{R}, \tag{4.5}$$

and define the convolution, or mollification of g by

$$\begin{aligned} J_\delta g(t) = \big(\rho_\delta * g\big)(t) = & \int_{-\infty}^{\infty} \rho_\delta(t-s) g(s)\, ds, \\ \approxeq & \int_{t-3\delta}^{t+3\delta} \rho_\delta(t-s) g(s)\, ds, \end{aligned} \tag{4.6}$$

where δ is the radius of mollification. Then the following results can be obtained from Fourier transform theory and Parseval's equality; see ([22], pp. 62 *et. seq.*)

Lemma 4.1. *If* $\|g'\|_2 \leq M_1$ *then*

$$\|\big(J_\delta g\big) - g\|_2 \leq \delta M_1/2.$$

This lemma shows that $\big(J_\delta g\big) \to g$ as $\delta \to 0$, we also need this behaviour for the derivative.

Lemma 4.2. *If* $\|g''\|_2 \leq M_2$ *then*

$$\|\big(J_\delta g\big)' - g'\|_2 \leq \delta M_2/2,$$

This consistency result shows that $\big(J_\delta g\big)' \to g'$ as $\delta \to 0$. The following continuity result also follows in a straight-forward manner by Fourier transform theory ([22], pp. 62 *et. seq.*)

Lemma 4.3. *With the extended noisy measurement function $g_m \in L^2(I_\delta)$*

$$\|\big(J_\delta g\big)' - \big(J_\delta g_m\big)'\|_2 \le \frac{e^{-1/2}}{\delta\sqrt{\pi}}\|g_m - g\|_2.$$

Therefore the mollification method provides the differentiation operator with a Lipschitz continuity result provided that the data $g_m \in L^2$, and the mollification radius $\delta > 0$ is fixed. Furthermore as $\|g_m - g\| \to 0$, δ can be reduced, and the consistency error then decreases provided that $g'' \in L^2$.

4.3 Monotone Smoothing

By combining the monotone approximation and the mollification in a sequential manner we will have the continuous dependence of the solution of the inverse problem on the measured data.

Theorem 4.1. *The source function reconstruction problem, as stated in §3.1, with monotone smoothed measurement data $J_\delta S_n(\overline{u}_m)$, has a well-posed solution and the solution converges to the true σ as the noise and mesh interval is reduced to zero.*

Proof. This result follows directly from Theorem 3.2, the monotone spline approximation property (4.4), Lemma 4.2 –4.3 and the existence of the inverse map of the monotone B-spline approximating $\overline{u}$.

As the monotone smoothing technique is sequential the inverse map existence will be true provided the mollification does not destroy the monotone behaviour of the B-spline. We show this is true with a suitable choice of the mollification kernel. Now as

$$J_\delta g(t) = \big(\rho_\delta * g\big)(t), \quad \text{it follows} \quad \big(J_\delta g\big)'(t) = \big(\rho_\delta * g'\big)(t),$$

and if $g' > 0$ and $\rho_\delta > 0$, as follows from our definition (4.5), it is determined that $\big(J_\delta g\big)'(t) > 0$, and we have that mollification preserves monotonicity. What we require for our algorithm is that the condition $g' > \epsilon$ is preserved; where ϵ corresponds to $\inf|\overline{u}'(t)|$. It is straightforward to show this by applying the above argument to $\widehat{g} = g - \epsilon t$, so $\widehat{g}' > 0 \implies g' > 0$, and also $\rho_\delta * \epsilon = \epsilon$; so we the result. We should note that if a non-positive mollification kernel is chosen, such as the Dirichlet kernel, that this result is not obtained.

If g_m is the measured function, then from Lemmata 4.2 and 4.3 and the triangle inequality, it follows

$$\begin{aligned}\|g' - \big(J_\delta g_m\big)'\|_2 \le& \|g' - \big(J_\delta g\big)'\|_2 + \|\big(J_\delta g\big)' - \big(J_\delta g_m\big)'\|_2, \\ \le& \frac{\delta M}{2} + \frac{e^{-1/2}}{\delta\sqrt{\pi}}\|g_m - g\|_2,\end{aligned}$$

which provides an error estimate for the mollified derivative.

First consider the reconstruction for $0 < t < L$, in this range $\mathbb{T}^{-1} : \overline{u} \to \sigma(\xi)$ has functional form $1/(\overline{u}^{-1}(\xi))'$ and is nonlinear, but maybe found from solving the linear system $\sigma(\overline{u}(t)) = \overline{u}'(t)$, and then the convergence result we require is straightforward. Denote by σ_m the value of σ that has been reconstructed from $\overline{u}_m$, where here $\sigma_m = \big(J_\delta S_n(\overline{u}_m)\big)'$, and it then follows that

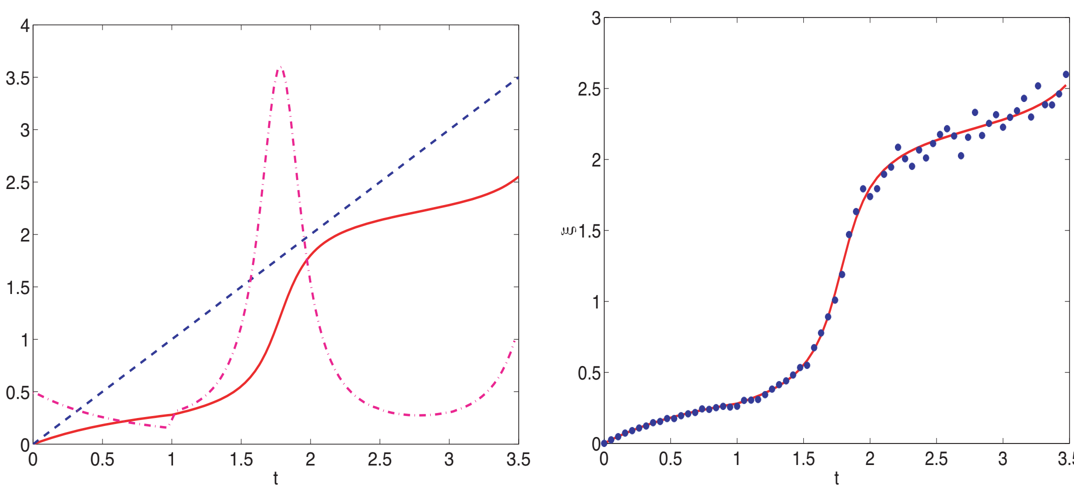
$$\begin{aligned}\|\sigma - \sigma_m\|_2 = &\|\overline{u}' - \big(J_\delta S_n(\overline{u}_m)\big)'\|_2,\\ \leq &\|\overline{u}\,' - \big(J_\delta \overline{u}_m\big)'\|_2 + \|\big(J_\delta S_n(\overline{u}_m)\big)' - \big(J_\delta \overline{u}_m\big)'\|_2,\\ \leq &K_1\|\overline{u} - \overline{u}_m\|_2 + K_2\|S_n(\overline{u}_m) - \overline{u}_m\|_2,\\ \leq &K_1\kappa + K_2 C(\epsilon)\omega(h),\end{aligned}$$

where $K_1 = \delta M_2/2$ and $K_2 = \frac{e^{-1/2}}{\delta\sqrt{\pi}}$.

Now for $t > L$, it is not as straightforward as the operator $\mathbb{T}^{-1}$ in this region is more complicated, but is recursive as it relies on previous values of σ; this is shown in the discussion following (3.5). However, it is seen that following the argument after (3.8) that the term $|\sigma(t-L)/h'(t-L)|$ is therefore bounded, by say K_3, so that we can write from the above

$$\|\sigma - \sigma_m\|_2 \leq K_3\mathcal{O}\big(\kappa\big) + K_3\mathcal{O}\big(h\big).$$

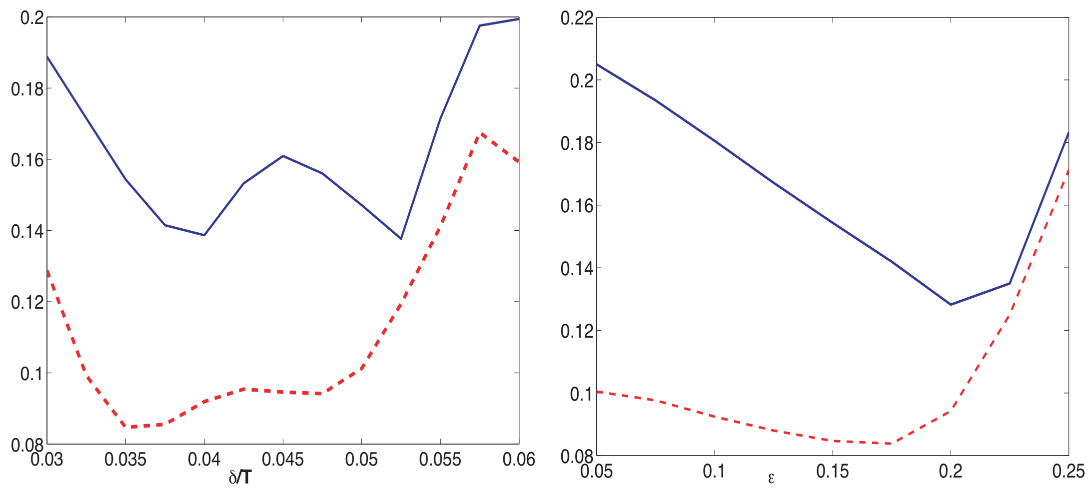
We again observe convergence can only occur if ϵ is not too big for a given σ. □



(a) The time signal, $\overline{u}(t)$ ——, and its derivative $\overline{u}\,'$ $- \cdot -$, also shown is the applied boundary condition $h(t)$ $- - -$.

(b) The time signal, $\overline{u}(t)$ ——, and the relative, noisy measured time signal $\overline{u}_m(t)$ $* * *$, with noise to signal ratio of 2.17% in the ℓ^2 norm; this corresponds to a decibel rating of -33dB.

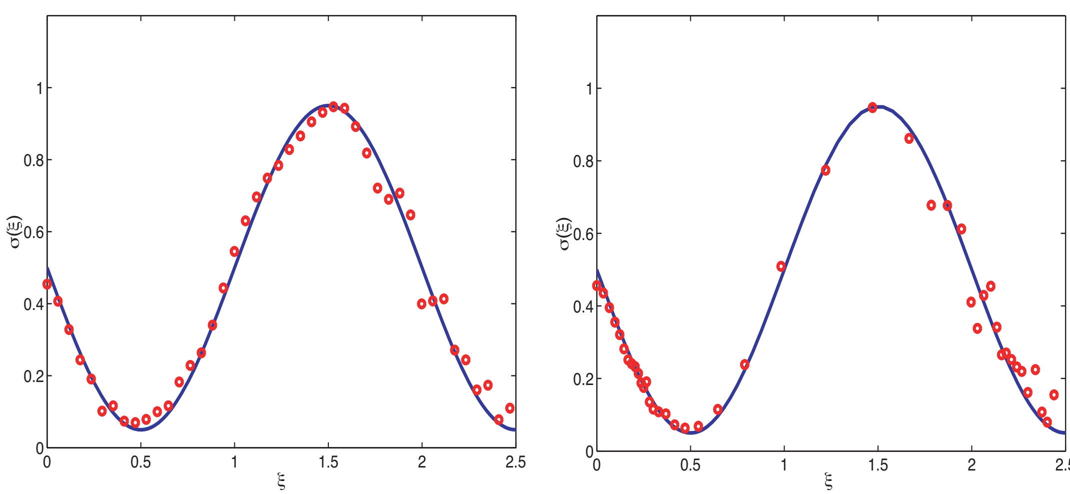
Figure 1: The exact simulated $\overline{u}$, its derivative $\overline{u}\,'(t)$ and the applied boundary condition $h(t)$. Shown in (b) is the measured noisy signal $\overline{u}_m$ with relative noise from which the reconstruction of σ is attempted.



(a) The error of the reconstruction, relative ℓ_2 error $- - -$ and uniform error ___ , plotted against δ/T, for $\epsilon = 0.15$.

(b) The error of the reconstruction, relative ℓ_2 error $- - -$ and uniform error ___ , plotted against ϵ, for $\delta/T = 3.5\%$.

Figure 2: The reconstruction error plotted against δ/T and ϵ; from the noisy signal of Fig 1 which is a relatively noisy signal with a noise to signal ratio of 2.17% in the ℓ^2 norm.



(a) The reconstructed source term $\sigma(\xi)$ $\circ\ \circ\ \circ$, and the exact source term ___ . Reconstruction using monotone smoothing for uniform ξ intervals.

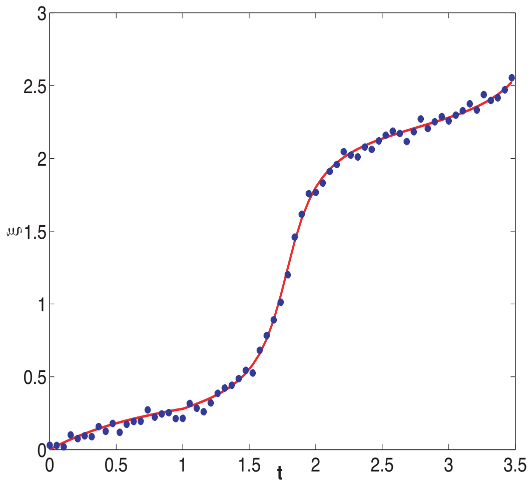
(b) The reconstructed source term $\sigma(\xi)$ $\circ\ \circ\ \circ$, and the exact source term ___ . Reconstruction using monotone smoothing for uniform t intervals.

Figure 3: Contrasting the reconstructed source term $\sigma(\xi)$ for uniform ξ, and uniform t from the noisy signal of Fig 1. Monotone smoothing used with mollification parameter $\delta/T = 3.5\%$, monotone constraint parameter $\epsilon = 0.15$. Also shown in (a), but not discernible from the true σ is the reconstruction from noise free data with *no* regularisation.

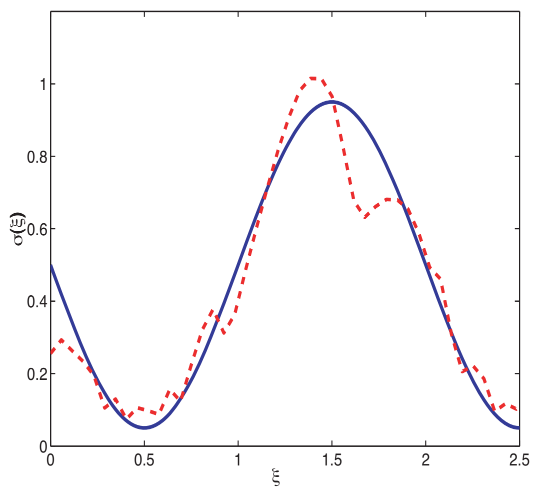
5 Numerical methods and Results

The basis of the uniform ξ algorithm is implemented by the formula

$$\begin{cases} \sigma(\xi) = \dfrac{1}{\overline{u}^{\,-1\,\prime}(\xi)}, & \overline{u}(0) < \xi < \overline{u}(L), \end{cases} \tag{5.1}$$



(a) The time signal, $\overline{u}(t)$ ___, and the uniform noisy measured time signal $\overline{u}_m(t)$ $**$ with noise to signal ratio of 2.17% in the ℓ^2 norm.



(b) The reconstructed source term $\sigma(\xi)$ _ _ and the exact source term ___. Using monotone smoothing with $\epsilon = 0.15$, $\delta/T = 4.7\%$.

Figure 4: The measured noisy signal $\overline{u}_m$ for uniform noise and the reconstruction of σ with monotone smoothing.

We now describe the numerical algorithm used to solve the inverse problem. We use the two mesh structures on $t \in [0, T]$ as described in §4. As explained in that section the signal function $\overline{u}(t)$ is to be approximated by a B-spline from the spline spaces $\mathcal{S}_1$ or $\mathcal{S}_3$, so that the spline is linear or cubic, respectively. This B-spline is to be constrained to be monotone, so that when the derivative operator is applied to (4.1) this results in an equation of constraints which is equivalent to $G\boldsymbol{\alpha} \geq \boldsymbol{\epsilon}$. Here G is a bi-diagonal lower triangular matrix when $n = 1$, and a quad-diagonal matrix when $n = 3$, and where $\boldsymbol{\alpha}$ is a column vector of the spline coefficients α_i, and $\boldsymbol{\epsilon}$ is a column vector of scalars ϵ. The equation (4.2) is thereby converted into a least squares problem with linear inequality constraints, namely

$$\min_{\boldsymbol{\alpha}} \|\overline{u}_m(\bar{t}_i) - S(\bar{t}_i)\|_2, \quad \text{subject to the constraint } G\boldsymbol{\alpha} \geq \boldsymbol{\epsilon}, \quad 0 \leq i \leq N, \tag{5.2}$$

where ϵ is a priori chosen for our algorithm. A higher value on ϵ will reduce the Lipschitz constant on $\mathbb{T}^{-1}$, but this must be chosen consistent with $\overline{u}'$. Our method of solution for this problem follows the techniques discussed by ([20], pp. 158 *et. seq.*), who call the (5.2), Problem LSI (least squares inequality constraints). Their method is to reduce the problem LSI into the nonnegative least squares problem (NNLS)

$$\min \|E\mathbf{x} - \mathbf{f}\|_2, \text{ subject to the constraint } \mathbf{x} \geq \mathbf{0},\ E \in (\mathcal{R}^m, \mathcal{R}^n),\ \mathbf{x} \in \mathcal{R}^n, \quad \mathbf{f} \in \mathcal{R}^m.$$

Our implementation of the NNLS algorithm is the one supplied in MATLAB. In fact our LSI problem is first reduced to the least distance programming problem (LDP) of

$$\min \|\mathbf{x}\|_2, \quad \text{subject to the constraint } G\mathbf{x} \geq \mathbf{h}, \quad \mathbf{x} \in \mathcal{R}^n, \mathbf{h} \in \mathcal{R}^m.$$

prior to being reduced to the NNLS problem. The requirement of the Theorem 4.1 has therefore been met.

After finding the monotone approximation to $\overline{u}_m$, which we denote by $S_n(\overline{u}_m)$, this approximate is then mollified by forming $J_\delta S_n(\overline{u}_m)$, and then solving the inverse problem (3.2) with this regularised data. For a concrete implementation of the algorithm, the mollified integral is approximated by the trapezoidal rule. Our use of mollification is in a similar manner to [29, 30]. The derivative required in (5.1) is formed by fitting another cubic spline to $J_\delta S_n(\overline{u}_m)$, which we call the reconstruction spline, and then taking its derivative.

For the numerical experiments illustrated in this section the boundary function h is a ramp function, this and the resultant transmission signal $\overline{u}$ are shown in Figure 1 where $T = 3.5$. Observe that σ can only be reconstructed depending on the range of $\overline{u}$ measured; namely $\sigma(\xi)$ for $0 < \xi < \overline{u}(T)$. The simulation data was formed from (2.6) and was also checked from a finite difference solution of (2.4)–(2.5). It is seen from Figure 1 that $\overline{u}(t)$ is strictly monotone, as is proven in §2.1. The derivative $\overline{u}'$ is also illustrated in this figure and it is important to note that $\overline{u}' \geq 0.15$ so that this number will be an upper limit for ϵ in the monotone algorithm as an any larger value will degrade the solution for σ; as is shown in Figure 2(b). Note that $\overline{u}$ is continuous and that $\overline{u}'$ has a jump at $t = 1$, as expected. The source function σ is an oscillatory function that is used throughout the simulation and is shown as the *true* source term in the reconstruction figures, *e.g.* Figures 3–6.

Throughout this section the number of measurement data points is $N = 67$ and the number of constraint spline nodes is $M = 25$; so ensuring the least-squares nature as required in §4. The degree of the constraint spline is 3, unless stated otherwise. We always use $L = 1$ and a number of reconstruction nodes of 180; this is for the reconstruction spline that we use for evaluation of the monotone smoothed derivative for the solution σ. A linear polynomial is fitted to the spline monotone approximation in $x \in [0, h]$ where h is the interval described in §4.1, and this polynomial is then used to extend the data to $[-3\delta, 0]$ as needed in the mollification procedure. A similar extension is used at the end of the data at $[T, T + 3\delta]$. This technique ensures that the derivatives estimated at each end of the data are superior to those obtained by simple extension of the data by zero.

To provide the noisy signal $\overline{u}_m$ the calculated data $\overline{u}$ was corrupted with white noise having a normal distribution and zero mean. The quoted value of the noise level, in our results, is a relative measure of $\|n\|_2/\|\overline{u}\|_2$[3], where n is the noise and $\|\cdot\|_2$ denotes the ℓ_2 sequence space norm. We utilise two types of noisy data in the experiments. The first we denote by the term uniform noise and the noise is simply added to the signal as $\overline{u}_m = \overline{u} + n$. Whereas the relative noisy signal the noise is relative to the signal amplitude and is formed by $\overline{u}_m = \overline{u}(1 + n)$; it is observed this will mean that the signal to noise ratio is constant with this second type of noisy signal. In the first type the noise can exceed the signal, in magnitude, for the first part of the signal; *e.g.* see Figure 6(a).

We start our results with a low noise example and the noisy $\overline{u}_m$ signal with

[3]Equivalently here, the the standard deviation of the the noise to the ℓ_2 norm of the signal.

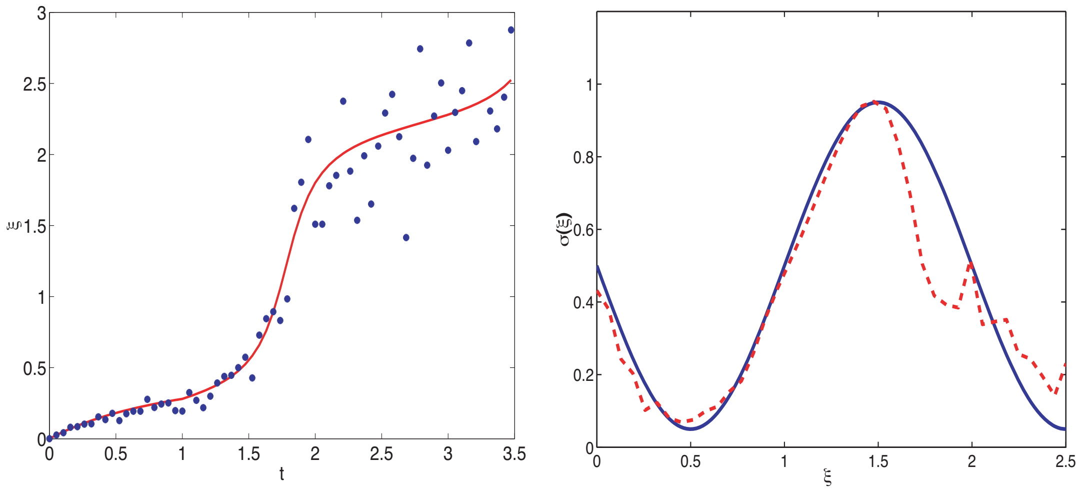
relative noise in the ℓ^2 norm of 2.1% is shown in Figure 1(b) and the reconstruction of the associated σ when monotone smoothing is used is shown in Figure 3. It is seen from comparing Figure 3(a) and (b) that the reconstruction with uniform ξ intervals is preferable to the uniform t intervals, although the reconstruction is similar in character. The ℓ_2 relative error is 0.08% and 0.1% for uniform ξ and t spacing, respectively. The uniform ξ interval algorithm is based on equation (5.1) and is one used for all remaining reconstructions in this section. The uniform t interval algorithm is based on equation (3.2) and is not as effective because of bunching of the data points. It should be pointed out that if reconstruction is attempted, even with this low noise problem, without regularisation, no approximation of the source is obtained.

We use cubic splines throughout these results because we found they provide a slightly smoother approximation to σ than lower order splines; this is only a very small effect on the reconstruction. More noticeable effects are obtained by changing the number of spline nodes. The optimal choice should allow for accurate approximation of the source function and is determined by the approximation error estimate as given by higher order forms of (4.3). This is achieved for our experiments here when $M = 25$. With $M = 25$ when the noise free $\overline{u}$ is used in the algorithm, with no regularisation, it produces a reconstruction as shown in Figure 3(a), but is not discernable from the exact σ in this figure; an almost exact replication as it has a relative ℓ_2 error of 0.02.

The amount of mollification utilised in our numerical reconstructions is stated as a relative percentage of δ/T. In Figure 2(a) we show the reconstruction error as δ/T varies for the noisy signal of Figure 1(b). It is seen that the reconstruction error is minimised when δ/T is about 3.5%. The minimum is fairly broad and it can be shown through experimentation that it is more so when the noise level increases (not shown). For the results shown in Figure 4 it is flatter and lies between 3%–6%, and similarly for Figure 5. Methods of optimising δ, such as those in [22], do not apply to the nonlinear and monotone smoothing technique used here. Our experience on this problem indicates the choice of δ is not *too* critical. The value of the mollification parameter δ/T, around 3.5% seems to produce reasonable results with the levels of noise used here. It appears that the effect of the mollification parameter δ is not as dominant as in pure mollification, because of the smoothing produced by the monotone approximation.

The choice of ϵ is less difficult to see, as it can have a range of values with similar reconstruction quality. Its upper value however must be less than the minimum value of the exact $\overline{u}'$. Smaller values of ϵ enable the monotone approximation to the noisy $\overline{u}_m$ to oscillate more. In Figure 2(b) the reconstruction error as ϵ is varied is shown. It can be readily seen that the upper limit of 0.15, which is the smallest exact value of $\overline{u}'$ in this simulation, is where the error starts increasing.

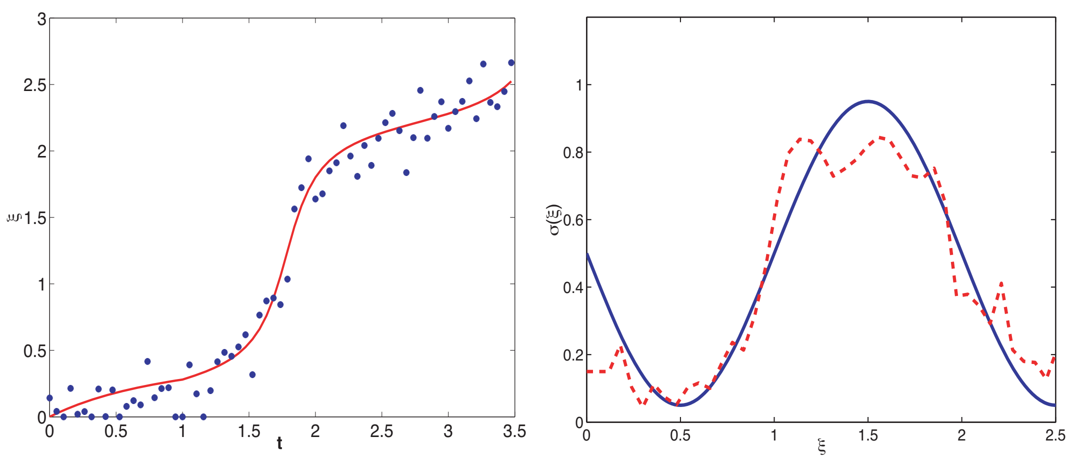
Figure 4(a) shows a noisy $\overline{u}_m$ signal with uniform noise of 2.17% in the ℓ_2 norm, and the reconstruction of the associated σ when monotone smoothing is used is shown in Figure 4(b). It is seen that the uniform noise signal provides a high level of noise when the signal is low, so degrading the reconstruction for low ξ values; *c.f.* with Figure 1(b) and Figure 3.



(a) The time signal, $\overline{u}(t)$ ____, and the relative noisy measured time signal $\overline{u}_m(t)$ $***$ when the noise to signal level is 10% in the ℓ^2 norm; this corresponds to a decibel rating of -19dB.

(b) The reconstructed source term $\sigma(\xi)$ _ _ _ , and the exact source term ____. Using monotone smoothing from the noisy signal of (a) with $\epsilon = 0.15$, $\delta/T = 3.5\%$.

Figure 5: The reconstruction of $\sigma(\xi)$ with considerable relative noise; from the noisy signal of (a), with monotone smoothing.



(a) The time signal, $\overline{u}(t)$ ____, and the uniform noisy measured time signal $\overline{u}_m(t)$ $***$ when the noise to signal ratio 10.0% in the ℓ^2 norm.

(b) The reconstructed source term $\sigma(\xi)$ _ _ ,and the exact source term ____. Using monotone smoothing from the noisy signal of (a) $\epsilon = 0.15$, $\delta/T = 3.5\%$

Figure 6: The reconstruction of $\sigma(\xi)$ with considerable uniform noise; from the noisy signal of (a), with monotone smoothing.

Our numerical results for the reconstruction of $\sigma(\xi)$ for a large relative noisy signal with 10% noise is illustrated in Fig. 5(b), and the signal $\overline{u}_m$ is illustrated in Fig. 5(a). If an attempt is made to obtain a reconstruction with just mollification from such a noisy signal, the reconstruction fails or produces a wildly oscillating solution. It is often found for such noisy problems that if mollification alone is used the inverse function map fails to exist.

Finally in Figure 6 we illustrate a high noise example with a uniform noisy signal with noise in the ℓ^2 norm of 10%. It is observed that for even this very noisy example the monotone smoothing regularisation yields an acceptable solution.

6 Conclusions

It has been shown that an inverse problem associated with a semi-linear scalar one-way wave equation has an analytic solution. A Lipschitz continuity result has been proven for dependence of the inverse problem upon the derivative of the data. The Lipschitz constant depends upon the lower bound of this derivative as an inverse function operation is also required. A combination of monotone approximation and mollification of the data is then applied to regularise the solution of the inverse problem.

It has been established that our reconstruction algorithm, for the inverse problem of unknown source reconstruction, is rendered well-posed by the method of monotone smoothing. Numerical evaluation of this solution, for both low and high levels of measurement noise, has been illustrated, and it has been shown that the method is robust.

The monotone technique should be useful when applied to other inverse problems that require an inverse function operation; such problems have been considered by [6, 19, 30].

Appendix A Reduction of second order equation

We consider here the second order semi-linear hyperbolic equation

$$u_{tt} - u_{xx} = \sigma(u_t), \tag{A.1}$$

and show its reduction to a semi-linear one-way wave equation as studied in the rest of this paper. It is seen that we have chosen to have the source term σ depend upon u_t similar arguments to those made here can be performed for the case when the σ term depends solely on u_x. This equation is now written in terms of the Riemann invariants of the linear second order wave equation $u_{tt} - u_{xx} = 0$. To this end, introduce two new dependent variables $u^{\pm}(z, t)$ defined by

$$\begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u_t \\ u_x \end{pmatrix} = P \begin{pmatrix} u_t \\ u_x \end{pmatrix},$$

where the transformation matrix has an inverse

$$\begin{pmatrix} u_t \\ u_x \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = P^{-1} \begin{pmatrix} u^+ \\ u^- \end{pmatrix}.$$

Substitution of these new dependent variables into (A.1) yields dynamics for the new fields $u^{\pm}(z,t)$ as

$$\mathbf{u}_t^{\pm} \pm \mathbf{u}_x^{\pm} = \begin{pmatrix} \sigma(\frac{u^+ + u^-}{2}) \\ \sigma(\frac{u^+ - u^-}{2}) \end{pmatrix}, \tag{A.2}$$

where $\mathbf{u}^{\pm} = \begin{pmatrix} u^+ & u^- \end{pmatrix}^T$. It should be observed that the two component waves still couple, through the nonlinear source term σ. Note that this would not be the case if the source term was a function of one of the invariants alone. In the special case where $\sigma \equiv$ constant we also see the equations do uncouple. Therefore it is expected that the system

$$\mathbf{u}_t^{\pm} \pm \mathbf{u}_x^{\pm} = \sigma(\frac{u^+}{2}), \tag{A.3}$$

to approximate (A.2) when both of the following holds:

1. Both $|\sigma|$ and $|\sigma'|$ are small;
2. For small time, here we consider $t < \bar{t} \ll 1$, if we specify $u^- \equiv 0$ for $t = 0$, so implying $u^+ \gg u^-$ for $t \ll 1$.

To quantify this in the $\sigma(u_t)$ case consider the identity

$$\begin{aligned} \sigma(u_t) &= \sigma(\frac{u^+ + u^-}{2}), \\ &= \sigma(\frac{u^+}{2}) + \frac{u^-}{2}\sigma'(\widetilde{u}), \quad \widetilde{u} \in [\frac{u^-}{2}, \frac{u^+ + u^-}{2}], \end{aligned}$$

with $\widetilde{u}$ time varying. Thus the error in the approximate source term of (A.3) is

$$\|\frac{u^-}{2}\sigma'(\widetilde{u})\|_\infty < \frac{1}{2}\|u^-\|_\infty \overline{M},$$

where $\|\sigma'(\widetilde{u})|_\infty < \overline{M}$.

We now need a bound on u^-, but with the condition of small time we know that $u^+ \gg u^-$ from item (2) above; so from the u^- equation of (A.3) we have the following *linear* equation

$$u_t^- - u_x^- = \sigma(\frac{u^+}{2}).$$

From an earlier nonlinear regularity result we obtained estimate (2.11), and this result was sharp. As we now have a linear equation it is clear that the estimate shows when $\bar{t} \ll 1$ that

$$\|u^-\|_\infty \leq M\bar{t},$$

where again $\bar{t}$ is the maximum time under consideration. So that the error in the right-hand-side of the equation in replacing (A.1) by the one-way wave equation

$$u_t^+ + u_x^+ = \sigma(\frac{u^+}{2}),$$

is $M\overline{M}\bar{t}/2$.

In conclusion we have found that by bounding the maximum value σ and σ' and by restricting the maximum time of applicability, that one-way wave equations will approximate these second order equations.

Appendix B Lipschitz continuity of an inverse function operation

As this is used several times in the paper it is appropriate to illustrate it here. We show here the continuity of the inverse of the function map to $\xi = \overline{u}(t)$, namely $\overline{u}^{\,-1}(\xi) = t$, and show it depends on $\inf|\overline{u}'|$.

Considering the change of $\overline{u}^{\,-1}$ from $\overline{u}$ to $\overline{u} + \delta\overline{u}$ and the corresponding change $\delta\overline{u}^{\,-1}$, it follows to first order terms that

$$\delta\overline{u}^{\,-1}(\xi) = -[\overline{u}^{\,-1}(\xi)]'\delta\overline{u}(t),$$

and note that

$$[\overline{u}^{\,-1}(\xi)]' = \frac{1}{\overline{u}'(t)},$$

by the chain rule. Hence the Fréchet differential of the inverse function can now be written

$$\delta\overline{u}^{\,-1}(\xi) = -\frac{\delta\overline{u}}{\overline{u}'(t)},$$

On taking the L^2 norm it is found that

$$\|\delta\overline{u}^{\,-1}\|_2 \leq \frac{1}{\sqrt{\inf|\overline{u}'(t)|}}\|\delta\overline{u}(t)\|_2.$$

Showing that the inverse function reconstruction from the measurement data is potentially ill-posed unless $|\overline{u}'|$ is bounded below. Obviously this will not be true for the noisy measurement $|\overline{u}_m'|$, and this is why monotone approximation is required for the measurement function in §4.

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