A survey of isoperimetric limitations on antennas

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An introduction to isoperimetric bounds on antennas

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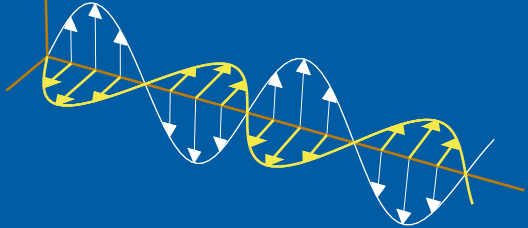

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Abstract

In this paper, physical limitations on antennas are presented based on the holomorphic properties of the forward scattering dyadic. As a direct consequence of causality and energy conservation, a forward dispersion relation for the extinction cross section is established, and isoperimetric inequalities for the partial realized gain and partial directivity are derived for antennas of arbitrary shape. Closed-form expressions for the prolate and oblate spheroids are compared with Chu's classical result for the sphere, and the effect of invoking metamaterials in the antenna design is discussed. The theory is illustrated by numerical simulations of a monopole antenna with a finite ground plane.

1 Introduction

Two questions of fundamental nature are addressed in this paper. For an arbitrary geometry, what is the upper bound on the performance of any antenna enclosed by this volume? Can electrically small broadband antennas exist unless directive properties are sacrificed for bandwidth? The history of these questions traces back to Chu and Wheeler in Refs. 1 and 9 more than half a century ago. Since then, much attention has drawn to the subject and numerous papers have been published, see Ref. 4 for a recent summary of the field. However, as far as the authors know, few successful attempts have been made to solve these problems rigorously for other geometries than the sphere. This restriction is mainly due to the failure of extending the spherical vector waves to form a set of orthogonal eigenfunctions on non-spherical surfaces. In this paper, physical limitations on antennas are presented which apply to arbitrary geometries without introducing orthogonal eigenfunctions.

The present paper is based on Refs. 2, 3, and 7, and the forward dispersion relation for the extinction cross section in Ref. 6. The theory has also successfully been applied to metamaterials in Ref. 8 to yield physical limitations on scattering and absorption by artificial materials over a frequency interval. The underlying mathematical description is influenced by the theory of dispersion relations for scattering of waves and particles in Ref. 5.

2 Physical limitations on $G_K B$ and D/Q

It is advantageous to picture the schematic antenna in Fig. 1 from a scattering point of view, *i.e.*, consider an antenna of arbitrary shape surrounded by free space and subject to a plane wave with time dependence $\mathrm{e}^{-\mathrm{i}\omega t}$ impinging in the $\hat{\boldsymbol{k}}$-direction. The material of the antenna is assumed to be lossless and satisfy the principles of reciprocity, linearity and time-translational invariance. The material properties are modeled by general anisotropic and heterogeneous constitutive relations in terms of the electric and magnetic susceptibility dyadics $\boldsymbol{\chi}_\mathrm{e}$ and $\boldsymbol{\chi}_\mathrm{m}$, respectively. The bounding volume of the antenna is naturally delimited by a reference plane at which a unique voltage and current relation is defined, see Fig. 1. Note that the present

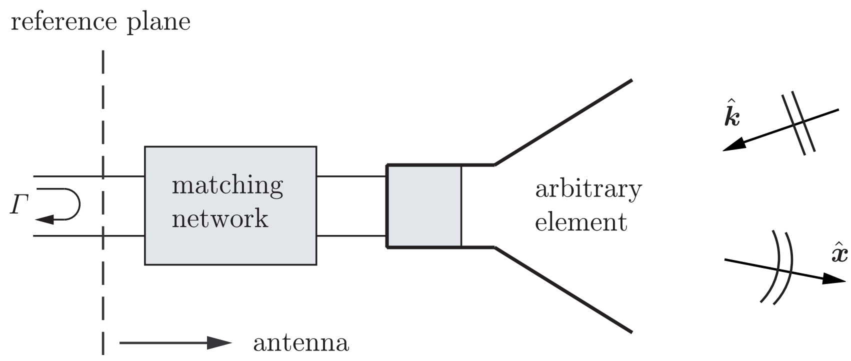

Figure 1: Illustration of a hypothetic antenna subject to a plane wave impinging in the $\hat{\boldsymbol{k}}$-direction. The incident wave is perturbed by the antenna and a scattered field is detected in the $\hat{\boldsymbol{x}}$-direction.

analysis is restricted to single port antennas with a frequency dependent scalar reflection coefficient Γ.

The scattered field caused by an incident plane wave with Fourier amplitude $\boldsymbol{E}_0$ and electric polarization $\hat{\boldsymbol{p}}_{\mathrm{e}} = \boldsymbol{E}_0/|\boldsymbol{E}_0|$ has the asymptotic behavior of an outgoing spherical wave, see Ref. 8, *i.e.*,

$$\boldsymbol{E}_{\mathrm{s}} = \frac{\mathrm{e}^{\mathrm{i}kx}}{x}\mathbf{S}(k,\hat{\boldsymbol{x}})\cdot\boldsymbol{E}_0 + \mathcal{O}(x^{-2}) \quad \text{as } x \to \infty,$$

where $\boldsymbol{x}$ denotes the position vector with respect to some origin, and $\hat{\boldsymbol{x}} = \boldsymbol{x}/x$ with $x = |\boldsymbol{x}|$. Here, $\mathbf{S}$ is independent of x and represents the scattering dyadic in the $\hat{\boldsymbol{x}}$-direction. Introduce the scattering cross section σ_{s} and the absorption cross section σ_{a} as the scattered and absorbed power divided by the incident power flow density, respectively. The principle of energy conservation then takes the form of a relation between the extinction cross section $\sigma_{\mathrm{ext}} = \sigma_{\mathrm{s}} + \sigma_{\mathrm{a}}$ and the imaginary part of the complex-valued function $\varrho = \hat{\boldsymbol{p}}_{\mathrm{e}}^* \cdot \mathbf{S}(k,\hat{\boldsymbol{k}})\cdot\hat{\boldsymbol{p}}_{\mathrm{e}}/k^2$. This relation is known as the optical theorem and states that $\sigma_{\mathrm{ext}} = 4\pi k \operatorname{Im}\varrho$ for $k \in [0,\infty)$.

Since the inverse Fourier transform of $\mathbf{S}$ is causal in the forward direction with respect to time ordered events, *i.e.*, the forward scattered field cannot precede the incident field, it can be shown that ϱ is a holomorphic function of k for $\operatorname{Im} k > 0$. Based on the optical theorem and the static limit of ϱ as $k \to 0$, Plemelj's formulae in Ref. 5 can be used to derive a forward dispersion relation for the extinction cross section. The result is

$$\int_0^\infty \frac{\sigma_{\mathrm{ext}}(k)}{k^2}\,\mathrm{d}k = \frac{\pi}{2}\sum_{i=\mathrm{e,m}} \hat{\boldsymbol{p}}_i^*\cdot\boldsymbol{\gamma}_i\cdot\hat{\boldsymbol{p}}_i, \tag{2.1}$$

where $\hat{\boldsymbol{p}}_{\mathrm{m}} = \hat{\boldsymbol{k}}\times\hat{\boldsymbol{p}}_{\mathrm{e}}$, and $\boldsymbol{\gamma}_{\mathrm{e}}$ and $\boldsymbol{\gamma}_{\mathrm{m}}$ denotes the electric and magnetic polarizability dyadics, respectively. For details on the derivation of (2.1) including definitions of the pertinent boundary value problems for $\boldsymbol{\gamma}_{\mathrm{e}}$ and $\boldsymbol{\gamma}_{\mathrm{m}}$, see Refs. 2 and 6.

The forward dispersion relation (2.1) can be used to establish upper bounds on the partial realized gain G and the relative bandwidth B of the schematic antenna in Fig. 1. In fact, for any finite interval $K \subset [0, \infty)$,

$$\int_0^\infty \frac{\sigma_{\mathrm{ext}}(k)}{k^2}\,\mathrm{d}k \geq \int_K \frac{\sigma_{\mathrm{a}}(k)}{k^2}\,\mathrm{d}k = \pi \int_K (1-|\varGamma|^2)\frac{G(k)}{k^4}\,\mathrm{d}k, \tag{2.2}$$

where $1-|\varGamma|^2$ represents the impedance mismatch of the antenna. In the last equality, it has been used that the absorption cross section is related to the partial realized gain as $\sigma_{\mathrm{a}} = \pi(1-|\varGamma|^2)G/k^2$, see Ref. 2. The estimate in (2.2) is generally not isoperimetric but can be sharpened by *a priori* information of the scattering properties of the antenna. For this purpose, introduce the quantity

$$\eta_K = \int_K \frac{\sigma_{\mathrm{a}}(k)}{k^2}\,\mathrm{d}k \bigg/ \int_K \frac{\sigma_{\mathrm{ext}}(k)}{k^2}\,\mathrm{d}k\ , \tag{2.3}$$

which is related to the absorption efficiency $\eta = \sigma_{\mathrm{a}}/\sigma_{\mathrm{ext}}$ via $\eta_K \leq \sup_{k\in K} \eta$. In particular, minimum scattering antennas defined by $\sup_{k\in K}\eta = 1/2$ contribute with at most an additional factor two on the right hand side of the inequality in (2.2).

Introduce the minimum partial realized gain $G_K = \inf_{k\in K}(1-|\varGamma|^2)G$ and the relative bandwidth $B = \int_K \mathrm{d}k/k_0$, where k_0 denotes the center wave number in K. Then the integral on the right hand side of (2.2) is estimated from below by

$$\int_K (1-|\varGamma|^2)\frac{G(k)}{k^4}\,\mathrm{d}k \geq G_K \int_K \frac{\mathrm{d}k}{k^4} = \frac{G_K B}{k_0^3}\frac{1+B^2/12}{(1-B^2/4)^3} \geq \frac{G_K B}{k_0^3}. \tag{2.4}$$

The inequality on the right hand side of (2.4) is motivated by the fact that $B \ll 1$ in many applications. Based on this observation, (2.2) and (2.4) inserted into (2.1) yields the fundamental inequality

$$G_K B \leq \frac{k_0^3}{2} \sum_{i=\mathrm{e,m}} \hat{\boldsymbol{p}}_i^* \cdot \boldsymbol{\gamma}_i \cdot \hat{\boldsymbol{p}}_i. \tag{2.5}$$

The corresponding physical limitation for the partial directivity D and the Q-factor Q is obtained from a resonance model for the absorption cross section, see Ref. 2. Under the assumption of a perfectly matched antenna at $k = k_0$, the upper bound on D/Q differs only by a factor π from (2.5), *viz.*,

$$\frac{D}{Q} \leq \frac{k_0^3}{2\pi} \sum_{i=\mathrm{e,m}} \hat{\boldsymbol{p}}_i^* \cdot \boldsymbol{\gamma}_i \cdot \hat{\boldsymbol{p}}_i. \tag{2.6}$$

Recall that G_K and D both depend on the incident direction $\hat{\boldsymbol{k}}$ and the electric polarization $\hat{\boldsymbol{p}}_{\mathrm{e}}$.

It is intriguing that it is just the static response of the antenna that bound the quantities $G_K B$ and D/Q. From the right hand side of (2.5) and (2.6), it is clear that the upper bounds on $G_K B$ and D/Q are independent of any coupling between electric and magnetic effects. Instead, electric and magnetic properties are seen to be treated on equal footing both in terms of material parameters and polarization description. For non-magnetic materials, *i.e.*, $\boldsymbol{\gamma}_{\mathrm{m}} = \mathbf{0}$, the sum on the right hand sides of (2.5) and (2.6) is simplified to only include electric quantities. Moreover, since both $\boldsymbol{\gamma}_{\mathrm{e}}$ and $\boldsymbol{\gamma}_{\mathrm{m}}$ are proportional to the volume V of the antenna, it follows that the bounds in (2.5) and (2.6) scale as $k_0^3 a^3$, where a denotes the radius of, say, the volume-equivalent sphere.

In many antenna applications, it is desirable to bound $G_K B$ and D/Q independently of both polarization states and material parameters. For this purpose, introduce the high-contrast polarizability dyadics $\boldsymbol{\gamma}_\infty$ as the limit of either $\boldsymbol{\gamma}_{\mathrm{e}}$ or $\boldsymbol{\gamma}_{\mathrm{m}}$ when the elements of $\boldsymbol{\chi}_{\mathrm{e}}$ and $\boldsymbol{\chi}_{\mathrm{m}}$ become infinite large. From the variational properties of $\boldsymbol{\gamma}_{\mathrm{e}}$ and $\boldsymbol{\gamma}_{\mathrm{m}}$ discussed in Ref. 6, it then follows that

$$\sup_{\hat{\boldsymbol{p}}_{\mathrm{e}}\cdot\hat{\boldsymbol{p}}_{\mathrm{m}}=0} G_K B \leq \frac{k_0^3}{2}(\gamma_1+\gamma_2), \qquad \sup_{\hat{\boldsymbol{p}}_{\mathrm{e}}\cdot\hat{\boldsymbol{p}}_{\mathrm{m}}=0} \frac{D}{Q} \leq \frac{k_0^3}{2\pi}(\gamma_1+\gamma_2), \tag{2.7}$$

where γ_1 and γ_2 denote the largest and second largest eigenvalue of $\boldsymbol{\gamma}_\infty$, respectively. The interpretation of (2.7) is polarization matching, *i.e.*, the polarization of the antenna coincides with the polarization of the incident wave. For non-magnetic material parameters, γ_2 vanishes in (2.7), and the upper bounds on $G_K B$ and D/Q are sharpened by at most a factor of two. Recall that γ_1 and γ_2 are easily calculated for arbitrary geometries using either the finite element method (FEM) or the method of moments (MoM).

3 Comparison with classical limitations

Closed-form expressions of γ_1 and γ_2 exist for the homogeneous ellipsoids, *viz.*, $\gamma_1 = V/L_1$ and $\gamma_2 = V/L_2$, where L_1 and L_2 denotes the smallest and second smallest depolarizing factor, respectively. The depolarizing factors satisfy $0 \leq L_j \leq 1$ and $\sum_j L_j = 1$ and are defined by

$$L_j = \frac{a_1 a_2 a_3}{2} \int_0^\infty \frac{\mathrm{d}s}{(s+a_j^2)\sqrt{(s+a_1^2)(s+a_2^2)(s+a_3^2)}}, \qquad j = 1,2,3. \tag{3.1}$$

Closed-form expressions of (3.1) in terms of the semi-axis ratio $\xi = \min_j a_j / \max_j a_j$ exist for the ellipsoids of revolution, *i.e.*, the prolate ($L_2 = L_3$) and oblate ($L_1 = L_2$) spheroids.

The eigenvalues γ_1, γ_2 and γ_3 (smallest eigenvalue $\gamma_3 = V/L_3$) are depicted in Fig. 2 for the prolate and oblate spheroids as function of ξ. The solid curves on the right hand side of Fig. 2 correspond to the combined electric and magnetic case, while the dashed curves represent pure electric material parameters. Non-magnetic material parameters with minimum scattering characteristics, *i.e.*, $\sup_{k\in K} \eta = 1/2$, is depicted by the dotted curves. In fact, the three curves for the prolate spheroid in the right figure vanish as $\xi \to 0$, while the corresponding curves for the oblate spheroid approach $16/3\pi$, $8/3\pi$, and $4/3\pi$, respectively.

A simple example of the upper bound on D/Q in (2.7) is given by the sphere of radius a for which $\gamma_1 = \gamma_2 = 4\pi a^3$. In this case, D/Q is bounded from above

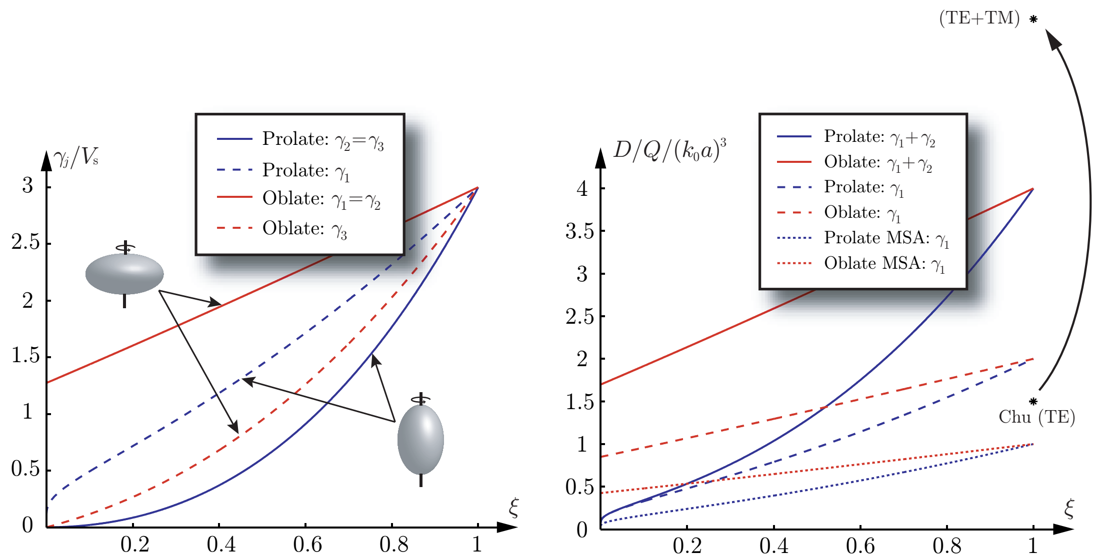

Figure 2: The eigenvalues γ_j (left figure) and the quotient D/Q (right figure) for the prolate and oblate spheroids as function of the semi-axis ratio ξ. Note the normalization with the volume V_s of the smallest circumscribing sphere.

by $4k_0^3a^3$, which is sharper than the classical limitation $6k_0^3a^3$ when both TE- and TM-polarizations are present, see Ref. 4. For omni-directional antennas with non-magnetic material parameters, the upper bound on D/Q is still slightly sharper than Chu's limit $3k_0^3a^3/2$ in Ref. 1 when minimum scattering characteristics (MSA) are assumed. Recall however that the classical results $6k_0^3a^3$ and $3k_0^3a^3/2$ are restricted to the sphere in the limit as $k_0a \to 0$, which is not the case for the theory set forth in this paper.

4 The effect of metamaterials

The fact that (2.5) and (2.6) are independent of any temporal dispersion implies that there is no difference in the upper bounds of $G_K B$ and D/Q if metamaterials are invoked in the antenna design instead of ordinary materials with identical static material parameters. In fact, it is well known that passive metamaterials are temporal dispersive since the Kramers-Kronig relations imply that $\lim_{\omega\to0+}\boldsymbol{\chi}_e(\omega)$ and $\lim_{\omega\to0+}\boldsymbol{\chi}_m(\omega)$ elementwise are non-negative in the absence of a conductivity term, see Ref. 8. When an isotropic conductivity term $i\varsigma/\omega\epsilon_0$ (scalar conductivity $\varsigma > 0$ independent of ω) is present in $\boldsymbol{\chi}_e$, the Kramers-Kronig relations is modified due to the singular behavior of $\boldsymbol{\chi}_e$ in the static limit. In the presence of a conductivity term, the analysis in Ref. 8 shows that the right hand side of (2.5) and (2.6) instead should be evaluated in the limit as the eigenvalues of $\boldsymbol{\chi}_e$ approach infinity independently of $\boldsymbol{\chi}_m$. Metamaterials may have the ability to lower the resonance frequency, but from the point of view of maximizing $G_K B$ and D/Q, such materials are believed to be of limited use.

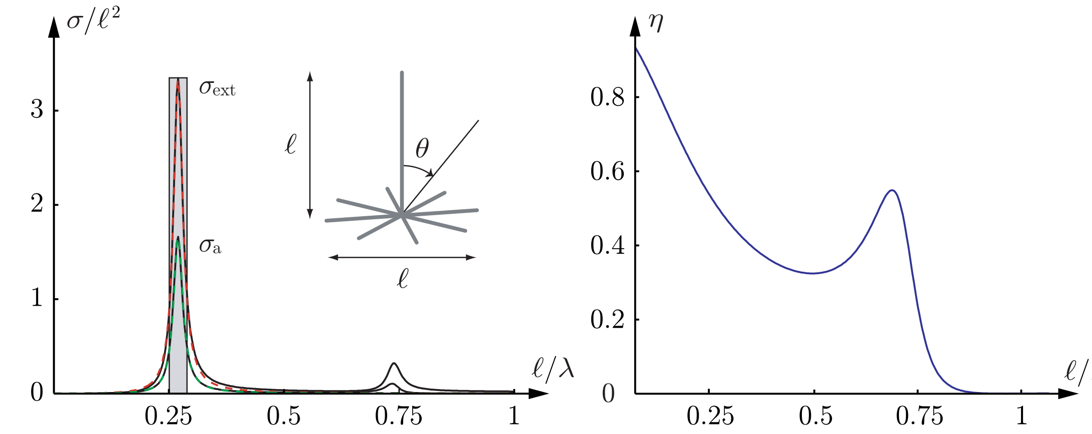

Figure 3: The extinction and absorption cross section for the monopole antenna (left figure) and the corresponding absorption efficiency (right figure). The different curves in the left figure correspond to a MoM solution (solid curves), Q-factor approximation (dashed curves), and limitation on the extinction cross section (shaded box).

5 A numerical example: the monopole antenna

The monopole antenna in Fig. 3 with a wire ground plane is used to illustrate the physical limitations introduced in Sec. 2. A monopole antenna behaves similar to a dipole antenna and the method of images can be used to analyze the antenna if the ground plane is sufficiently large, see Ref. 3. Here, a monopole antenna with height ℓ and ground plane radius $\ell/2$ is considered. The wires are cylindrical with radius $2.5 \cdot 10^{-5}\ell$. A MoM solution together with a gap feed model is used to determine the cross sections and impedance of the antenna.

The antenna is first considered as a passive scatterer loaded with $25\,\Omega$ in the gap feed. The extinction and absorption cross sections for an incident wave polarized matched at $\theta = 90^\circ$ are depicted in the left figure in Fig. 3. It is observed that the antenna is resonant for $\ell \approx 0.27\lambda$, where $\lambda = 2\pi/k$ denotes the wavelength in free space. The corresponding absorption efficiency is depicted on the right hand side of Fig. 3. It is observed that $\eta \approx 0.5$ at the resonance frequency, with $\eta_K \approx 0.5$ for $\ell/\lambda \in [0, 1]$. Note that the rather small ground plane gives a dipole-like radiation pattern at the quarter wavelength resonance.

The maximal gain, the partial gain at $\theta = 90^\circ$, and the partial realized gain at $\theta = 90^\circ$ for the antenna are depicted in the left figure in Fig. 4. At the resonance frequency, it is observed that the gain (and directivity) is 1.52 and that the radiation resistance is $25\,\Omega$. The Q-factor is estimated to $Q = 22$ by numerical differentiation of the reflection coefficient. The MoM solution is also used to determine the forward scattering properties of the antenna in terms of the extinction volume ϱ on the right hand side of Fig. 4.

The physical limitations in (2.7) require calculation of the eigenvalues γ_1 and γ_2. An electrostatic MoM simulation of the monopole antenna with a ground plane in

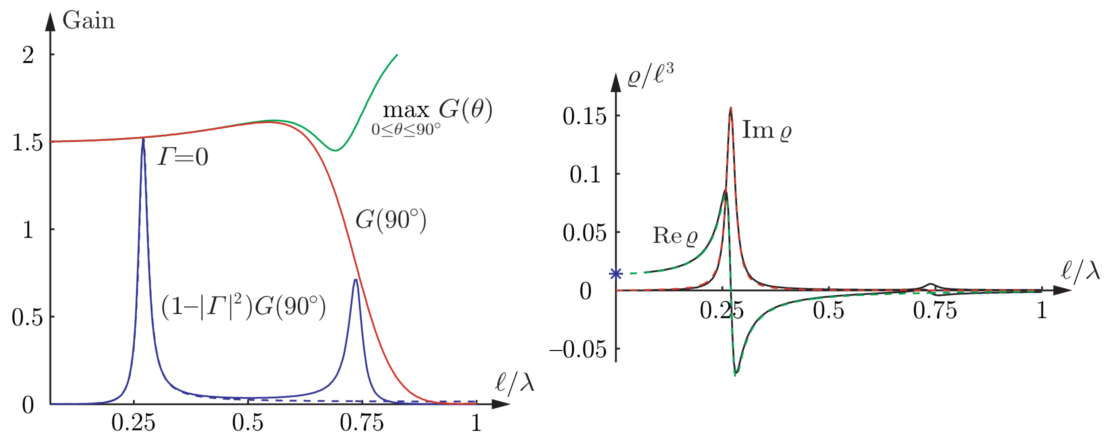

Figure 4: The maximal gain, the partial gain at $\theta = 90°$, and the partial realized gain at $\theta = 90°$ (left figure), and the extinction volume ϱ (right figure) for the monopole antenna. The different curves in the right figure correspond to a MoM solution (solid curves) and Q-factor approximation (dashed curves). The low frequency estimates of the monopole antenna with wire ground plane is indicated by the cross.

the form of a circular disk yields $\gamma_1 = 0.2\ell^3$ and hence $Q \geq 19$ if $D = 1.52$ and $\eta_K = 0.5$ are used in (2.7). Note that γ_2 vanishes from the upper bounds in (2.7) since no magnetic materials are present. As the circular ground plane contains more material than the wire ground plane it is clear that γ_1 for the monopole antenna with wire ground plane is smaller than γ_1 for the corresponding antenna with circular disk ground plane, *cf.*, the variational results in Ref. 6. The eigenvalue γ_1 for the monopole with the wire ground plane can either be determined by an electrostatic MoM solution or estimated by the forward dispersion relation (2.1). The latter method yields $\gamma_1 \geq 0.18\ell^3$, and assuming $\gamma_1 = 0.18\ell^3$ in (2.7) implies $Q \geq 22$.

In Figs. 3 and 4 it is observed that the single resonance model (dashed curves) with $Q = 22$ is a good approximation of the cross sections, extinction volume, and partial realized gain. Note also that the dipole antenna has a circumscribing sphere with $ka > 1$ and is therefore not considered electrically small according to the classical limitations in Ref. 1. In summary, the monopole antenna with wire ground plane show excellent agreement with the theory introduced in Sec. 2.

6 Conclusion

In this paper, physical limitations on reciprocal antennas of arbitrary shape are presented based on the holomorphic properties of the forward scattering dyadic. Upper bounds on $G_K B$ and D/Q are derived in terms of the electric and magnetic polarizability dyadics, $\boldsymbol{\gamma}_\mathrm{e}$ and $\boldsymbol{\gamma}_\mathrm{m}$, respectively. Since these bounds are proportional to the volume of the antenna, it is clear that for electrically small antennas, partial realized gain or partial directivity must be sacrificed for bandwidth or Q-factor.

Based on the limitations, it is also concluded that metamaterials and other exotic material models do not contribute to the upper bounds of $G_K B$ and D/Q in any larger extent than naturally formed substances.

The inequalities introduced in this paper are isoperimetric in the sense that equality in (2.5) and (2.6) hold for some physical antennas. For example, it is well known that the impedance of a cylindrical dipole antenna posses a reversed logarithmic singularity as the radius of the cylinder vanishes. In Ref. 2, this singularity is shown to coincide with the corresponding behavior of γ_1 for the prolate spheroid as $\xi \to 0$. In fact, numerical simulations of the dipole antenna in Ref. 3 show excellent agreement with the bounds presented in this paper. The present limitations are believed to be isoperimetric for a large class of antennas if *a priori* information of η_K from antenna simulations is taken into account.

The analysis in this paper generalizes in many aspects the classical results by Chu and Wheeler in Refs. 1 and 9. The main advantages of the new formulation are sixfold: 1) they hold for arbitrary geometries; 2) they are formulated both in terms of gain and bandwidth as well as directivity and Q-factor; 3) they include polarization effects with applications to diversity in MIMO communication; 4) they successfully separate electric and magnetic antenna properties in terms of the nature of the intrinsic materials; 5) they are isoperimetric; 6) *a priori* information about the scattering characteristics in the form of η_K improves the bounds.

Acknowledgment

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