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1999

CODEN:LUTEDX/(TEAT-7078)/1-17/(1999)

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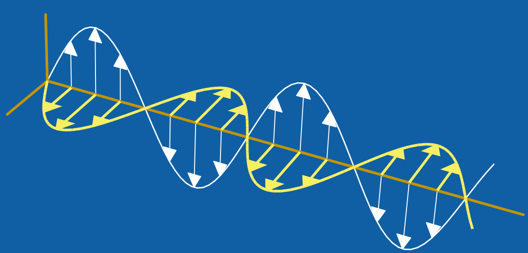

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Editor: Gerhard Kristensson

Abstract

Using simple waves and six-vector formalism, the propagation of electromagnetic waves in nonlinear, bianisotropic, nondispersive, homogeneous media is analyzed. The Maxwell equations are formulated as an eigenvalue problem, whose solutions are equivalent to the characteristic directions of the wave front. Oblique incidence of plane waves in vacuum on a half space of nonlinear material is solved, giving reflection and transmission operators for all angles of incidence and all polarizations of the incident field. A condition on Brewster angles is derived.

1 Introduction

Wave propagation in nonlinear media is a wide and quickly expanding area. In particular, the nonlinear optics field has been very prosperous [1, 3]. One of the most exciting areas is that of solitons, *i.e.*, pulses which have a very specific shape, in which the nonlinear steepening effects are precisely balanced by the dispersive broadening, thereby producing a pulse that is temporally or spatially unchanged during propagation. This delicate balance can only be understood by studying both contributing effects. In this paper, we are devoted to the nonlinear effects which occur in materials with no memory, *i.e.*, no dispersion.

Whereas the linear dispersion has been thoroughly investigated, *e.g.*, [4, 14, 20], the nonlinear properties may not have received enough attention. Some early works are summarized in [2], and especially the papers on wave propagation in nonlinear dielectrics [5, 6, 17, 21, 29] are worthy of attention. A prominent feature of nonlinear wave propagation, where the nonlinearity acts as an amplitude-dependent wave speed, is the formation of shock waves. These are discontinuous waves, which must be interpreted in a generalized way as weak solutions, see *e.g.*, [28, pp. 369–373], and the theory of these has been thoroughly studied [15, 18, 27, 31]. It is often argued that the shock waves are eliminated by the linear dispersion, see *e.g.*, [1, pp. 117–120], but since we are ignoring dispersion in this study, we expect our model to be accurate only when we are not in the vicinity of any shock formations.

An often encountered problem when studying nonlinear materials is that of finding suitable constitutive relations. In the treatise of Eringen and Maugin [9, 10], the constitutive relations for virtually every reasonable situation are presented. Some important thermodynamic restrictions are presented in [8]. The derivation of constitutive relations from a quantum mechanical point of view is presented in [3], and some theory about nonlinear dielectrics is found in [7].

This paper aims to improve the understanding of a nonlinear optical response, *i.e.*, an instantaneous nonlinear response. Earlier works, as reported above, have often made some important restrictions, such as assuming the material to be isotropic or uniaxial. Here we present a theory describing wave propagation in bianisotropic materials. We show that a generalized form of plane waves, called simple waves, can be used to analyze the wave propagation, and we reformulate the Maxwell equations as an eigenvalue problems. A brief presentation on simple waves in partial differential equations is given in [19, p. 52], and a more extensive treatment is given in [16, Chap. 3]. There are also some related results in [13, p. 47].

The paper is organized as follows: in Sections 2 and 3 we present the simple wave Ansatz and the six-vector formalism, which are the basic tools used in this paper. This is applied to the Maxwell equations in Section 4, which transforms the dynamics into an eigenvalue problem. Special notice is taken to isotropic media. In Section 5 we introduce the theory on how to classify materials. We then apply our formalism in Section 6 to the problem of a plane wave obliquely impinging on a nonlinear half space and solve the problem of finding the reflected and transmitted fields. Some results on suitable conditions on the Brewster angles are also presented, as well as a numerical example.

2 Simple wave Ansatz

Plane waves constitute a powerful tool in the analysis of wave phenomena in linear materials. The concept of plane waves transforms the problem of three spatial dimensions into a problem along the propagation direction. Simple waves are the generalization of this concept. They have previously been used in the description of nonlinear electromagnetic waves [5, 6], and are explained in basic books on partial differential equations [19, p. 52]. They also define the characteristics of the wave equation.

The simple wave Ansatz is suitable for materials which respond instantaneously to excitation, and states that the fields depend only upon a scalar parameter, which we denote ϕ. This parameter is a function of space and time. For an isotropic, linear media the simple wave Ansatz reduces to the usual phase function, $\phi(\boldsymbol{r}, t) = \boldsymbol{k} \cdot \boldsymbol{r} - \omega t$.

It is obvious that if a quantity u depends on space and time as $u(\boldsymbol{r}, t) = u(\phi(\boldsymbol{r}, t))$, the spatial gradient $\nabla\phi$ representents a propagation direction. We identify the quantity $-\frac{\nabla\phi/\phi_t}{|\nabla\phi/\phi_t|}$ as the propagation direction and $|\phi_t|/|\nabla\phi|$ as the propagation speed, where ϕ_t denotes the time derivative of ϕ. The minus sign comes from implicit differentiation of the equation $\phi(\boldsymbol{r}, t) = \text{constant}$, which is the equation of the wave front.

3 Six-vector formalism

When describing bianisotropic phenomena, it is often advantageous to use the six-vector formalism, see *e.g.*, [24]. In this approach, we make no real distinction between the electric and magnetic fields, but rather treat them as components of a single field. We define our fields as

$$\begin{cases} \boldsymbol{e} = \begin{pmatrix} \sqrt{\epsilon_0}\boldsymbol{E} \\ \sqrt{\mu_0}\boldsymbol{H} \end{pmatrix} \\ \boldsymbol{d} = \begin{pmatrix} \frac{1}{\sqrt{\epsilon_0}}\boldsymbol{D} \\ \frac{1}{\sqrt{\mu_0}}\boldsymbol{B} \end{pmatrix} \end{cases}$$

where ϵ_0 and μ_0 denote the permittivity and permeability of vacuum, respectively. The six-vector fields now both have the same dimension, *i.e.*, $\sqrt{\text{energy/volume}}$.

The scalar product between two six-vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as $\boldsymbol{a}\cdot\boldsymbol{b} = \sum_{i=1}^{6} a_i b_i$. Operations with three-vectors on six-vectors are understood in the obvious manner, *i.e.*, the scalar and cross products are

$$\boldsymbol{v}\cdot\boldsymbol{e} = \begin{pmatrix} \boldsymbol{v}\cdot\sqrt{\epsilon_0}\boldsymbol{E} \\ \boldsymbol{v}\cdot\sqrt{\mu_0}\boldsymbol{H} \end{pmatrix}, \quad \text{and} \quad \boldsymbol{v}\times\boldsymbol{e} = \begin{pmatrix} \boldsymbol{v}\times\sqrt{\epsilon_0}\boldsymbol{E} \\ \boldsymbol{v}\times\sqrt{\mu_0}\boldsymbol{H} \end{pmatrix}.$$

Using the operator $\mathbf{J} = \begin{pmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}$, which is formed from the three-dimensional spatial identity operator $\mathbf{I}$, we write the source free Maxwell equations as

$$\nabla\times\boldsymbol{e} - \frac{1}{c_0}\mathbf{J}\cdot\partial_t\boldsymbol{d} = 0,$$

where c_0 denotes the wave speed in vacuum, $1/\sqrt{\epsilon_0\mu_0}$. The spatial differential operator ∇ is treated as a three-vector, and is sometimes merged with the operator $\mathbf{J}$ to form the symmetric operator $\nabla\times\mathbf{J}$, as in [12]. This approach will be beneficial later on in this work.

4 The Maxwell equations as an eigenvalue problem

The constitutive relation for a material with no memory, *i.e.*, where the fluxes $\boldsymbol{d}$ depend only upon the present values of the field strengths $\boldsymbol{e}$, can be written

$$\boldsymbol{d}(\boldsymbol{r},t) = \boldsymbol{d}(\boldsymbol{e}(\boldsymbol{r},t)). \tag{4.1}$$

We now apply the simple wave Ansatz together with the constitutive relation,

$$\begin{cases} \boldsymbol{e}(\boldsymbol{r},t) = \boldsymbol{e}(\phi(\boldsymbol{r},t)) \\ \boldsymbol{d}(\boldsymbol{r},t) = \boldsymbol{d}(\boldsymbol{e}(\phi(\boldsymbol{r},t))). \end{cases}$$

This means that the curl operator turns into a cross product, $\nabla\times\boldsymbol{e} = \nabla\phi\times\boldsymbol{e}'$, and the time derivative becomes $\partial_t\boldsymbol{d} = \phi_t[\nabla_e\boldsymbol{d}]\cdot\boldsymbol{e}'$, where the prime denotes differentiation with respect to ϕ. The operator ∇_e denotes the field gradient operator, *i.e.*, $[\nabla_e\boldsymbol{d}]_{nm} = \frac{\partial}{\partial e_m}d_n(\boldsymbol{e})$. Since we write the linear constitutive relations as $\boldsymbol{d} = \boldsymbol{\varepsilon}\cdot\boldsymbol{e}$, where $\boldsymbol{\varepsilon}$ is a six-dyadic, we denote $[\nabla_e\boldsymbol{d}]$ by $\boldsymbol{\varepsilon}(\boldsymbol{e})$, and often suppress the argument to obtain a less cumbersome notation.

With the simple wave Ansatz, the Maxwell equations contain the generic field $\boldsymbol{e}' = \frac{d}{d\phi}\boldsymbol{e}$. However, for reasons that become more obvious below we prefer to use the time derivative, $\dot{\boldsymbol{e}} = \partial_t\boldsymbol{e} = \phi_t\boldsymbol{e}'$. This choice also becomes advantageous when implementing the equations later on. Since $\phi(\boldsymbol{r},t) = \text{constant}$ is the equation for the wave front, we identify the wave slowness $1/c$ and the propagation direction $\hat{k}$ of the simple wave by the following expressions,

$$\begin{cases} \dfrac{1}{c} = \dfrac{|\nabla\phi|}{|\phi_t|} \\ \hat{k} = -\dfrac{\nabla\phi/\phi_t}{|\nabla\phi/\phi_t|} = -\dfrac{\nabla\phi}{\phi_t}c. \end{cases}$$

Using these expressions, we write the Maxwell equations as

$$\frac{1}{c}\hat{k}\times\dot{\boldsymbol{e}} + \frac{1}{c_0}\mathbf{J}\cdot\boldsymbol{\varepsilon}\cdot\dot{\boldsymbol{e}} = 0.$$

This is an eigenvalue problem, which becomes more obvious in the form

$$\frac{c}{c_0}\dot{\boldsymbol{e}} = \boldsymbol{\varepsilon}^{-1}\cdot[\hat{k}\times\mathbf{J}]\cdot\dot{\boldsymbol{e}}, \tag{4.2}$$

which follows from $\mathbf{J}^{-1} = -\mathbf{J}$ and $\mathbf{J}\cdot[\hat{k}\times\mathbf{I}] = [\hat{k}\times\mathbf{J}]$. Observe that $[\hat{k}\times\mathbf{J}]$ is a symmetric operator. The dyadic $\boldsymbol{\varepsilon}$ is postulated to be positive definite and symmetric, and is thus invertible. In the linear case, it is possible to show that $\boldsymbol{\varepsilon}$ has to be a symmetric, positive definite dyadic in order to model passive media [12]. The assumptions made on the dyadic $\boldsymbol{\varepsilon}$ is a natural generalization of the result in the linear case.

The solution to (4.2) gives conditions on the wave speed and propagation direction in terms of the fields. In the linear case, only the directions of the field will be important, but for nonlinear materials there is also a dependence on the amplitude. For an isotropic material, where

$$\boldsymbol{\varepsilon}(\boldsymbol{e}) = \begin{pmatrix} \epsilon(E)\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mu(H)\mathbf{I} \end{pmatrix},$$

the conditions are

$$c = \frac{c_0}{\sqrt{\epsilon(E)\mu(H)}} \quad \text{and} \quad \hat{k}\cdot\dot{\boldsymbol{e}} = 0 \quad \Rightarrow \quad \dot{\boldsymbol{e}} = \begin{pmatrix} \frac{1}{\sqrt{\epsilon}}\boldsymbol{v} \\ \frac{1}{\sqrt{\mu}}\hat{k}\times\boldsymbol{v} \end{pmatrix}, \tag{4.3}$$

where the three-vector $\boldsymbol{v}$ is orthogonal to $\hat{k}$. Observe that it is the direction of the *derivatives* of the fields that are important, not the fields themselves.

For a given propagation direction $\hat{k}$ the operator $\boldsymbol{\varepsilon}^{-1}\cdot[\hat{k}\times\mathbf{J}]$ has six eigenvectors $\dot{\boldsymbol{e}}_j$, $j = 1,\dots,6$. Since the operator is not symmetric, these solutions are not guaranteed to be mutually orthogonal. We symmetrize the operator by

$$\frac{c}{c_0}(\sqrt{\boldsymbol{\varepsilon}}\cdot\dot{\boldsymbol{e}}_j) = \left[\sqrt{\boldsymbol{\varepsilon}}^{-1}\cdot[\hat{k}\times\mathbf{J}]\cdot\sqrt{\boldsymbol{\varepsilon}}^{-1}\right]\cdot(\sqrt{\boldsymbol{\varepsilon}}\cdot\dot{\boldsymbol{e}}_j),$$

where we have used the square root of the positive definite and symmetric dyadic $\boldsymbol{\varepsilon}$, which is also positive definite and symmetric. It is concluded that the eigenvectors $\sqrt{\boldsymbol{\varepsilon}}\cdot\dot{\boldsymbol{e}}_j$ are real and orthogonal, which imply that the eigenvectors $\dot{\boldsymbol{e}}_j$ are real and linearly independent. The operator $\sqrt{\boldsymbol{\varepsilon}}^{-1} \cdot [\hat{k} \times \mathbf{J}] \cdot \sqrt{\boldsymbol{\varepsilon}}^{-1}$ is a congruence transformation (see *e.g.*, [11, p. 251]) of $[\hat{k} \times \mathbf{J}]$, which has the (double) eigenvalues -1, 0 and 1. Since the signs are preserved under congruence transforms, we conclude that for a given propagation direction $\hat{k}$ there are two modes propagating in the $+\hat{k}$-direction (positive eigenvalues) and two modes propagating in the $-\hat{k}$-direction (negative eigenvalues), while two modes do not propagate with respect to $\hat{k}$ at all (zero eigenvalue). The last two can be written explicitly as $\dot{\boldsymbol{e}}_{5,6} = \binom{\pm\hat{k}}{\hat{k}}$.

5 Classification of materials

Materials are often classified as, *e.g.*, isotropic, bi-isotropic or uniaxial depending on the invariance under symmetry transformations. In our formulation, the natural way to classify the materials is by the corresponding invariance of the dyadic $\boldsymbol{\varepsilon}(\boldsymbol{e})$. This is motivated by the following way of writing the constitutive relations (4.1):

$$\boldsymbol{d}(\boldsymbol{e}) = \int_0^{\boldsymbol{e}} \boldsymbol{\varepsilon}(\boldsymbol{e}') \cdot d\boldsymbol{e}',$$

where the integral should be understood in terms of integration along a parametrized curve in $\mathbb{R}^6$. The prime is not to be confused with time differentiation, it is only denoting the integration variable. When applying a spatial transformation $\mathbf{S}$ on the field strength $\boldsymbol{e}$, we get

$$\boldsymbol{d}(\mathbf{S} \cdot \boldsymbol{e}) = \int_0^{\mathbf{S}\cdot\boldsymbol{e}} \boldsymbol{\varepsilon}(\boldsymbol{e}') \cdot d\boldsymbol{e}' = \int_0^{\boldsymbol{e}} \boldsymbol{\varepsilon}(\mathbf{S} \cdot \boldsymbol{e}'') \cdot \mathbf{S} \cdot d\boldsymbol{e}'',$$

where we have made the change of variables $\boldsymbol{e}' = \mathbf{S} \cdot \boldsymbol{e}''$. Materials are classified depending on which group of transformations $\mathbf{S}$ that satisfies $\boldsymbol{d}(\mathbf{S} \cdot \boldsymbol{e}) = \mathbf{S} \cdot \boldsymbol{d}(\boldsymbol{e})$, *i.e.*, which group of transformations that commutes with $\boldsymbol{\varepsilon}$.

Since this must hold for all transformations in the bi-isotropic case, we see that an $\boldsymbol{\varepsilon}(\boldsymbol{e})$ as

$$\boldsymbol{\varepsilon}(\boldsymbol{e}) = \begin{pmatrix} \epsilon(E,H)\mathbf{I} & \xi(E,H)\mathbf{I} \\ \zeta(E,H)\mathbf{I} & \mu(E,H)\mathbf{I} \end{pmatrix},$$

describes a bi-isotropic material, where ϵ, ξ, ζ and μ are scalar functions of the field strengths. Common restrictions on constitutive relations, [8, 12], say that $\xi = \zeta$, and if they are equal to zero, the material is said to be isotropic.

6 Oblique incidence

To demonstrate the possible application of the simple wave approach, we analyze the problem of a plane electromagnetic wave obliquely impinging from vacuum on a nonlinear half space. The problem has been studied to some extent in [5, 6], though they specialize their treatment to a uniaxial material with nonlinearity in electric field only, where the optical axis is in a special direction.

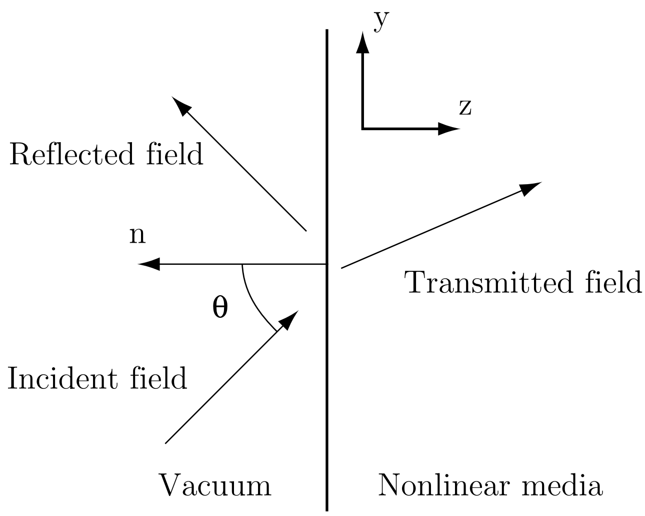

Figure 1: The geometry of the problem of oblique incidence.

6.1 Geometry and boundary conditions

The geometry of the problem is depicted in Figure 1. The incident field is a plane wave, and we make the Ansatz

$$\begin{cases} \boldsymbol{e}^i(\boldsymbol{r},t) = \boldsymbol{e}(\hat{k}^i \cdot \boldsymbol{r} - c_0 t) \\ \boldsymbol{e}^r(\boldsymbol{r},t) = \boldsymbol{e}(\hat{k}^r \cdot \boldsymbol{r} - c_0 t) \\ \boldsymbol{e}^t(\boldsymbol{r},t) = \sum \boldsymbol{e}^t_j(\phi_j(\boldsymbol{r},t)), \end{cases}$$

where c_0 denotes the wave speed in vacuum. We thus assume that the transmitted field may consist of several simple waves, as we can expect from the linear, anisotropic case. The number of those is restricted to two in Section 6.5. The usual boundary conditions apply, *i.e.*, the tangential components of the field strengths should be continuous and the normal component of the fluxes should be continuous (no sources at the interface). We write this as

$$\begin{cases} \boldsymbol{e}^i_{\|} + \boldsymbol{e}^r_{\|} = \boldsymbol{e}^t_{\|} \\ \hat{z} \cdot (\boldsymbol{d}^i + \boldsymbol{d}^r) = \hat{z} \cdot \boldsymbol{d}^t. \end{cases} \tag{6.1}$$

The latter condition is not used in the present analysis.

6.2 Reflection law and Snell's law

Since the boundary conditions (6.1) must hold for all times on the surface $z = 0$, we can differentiate them with respect to both time and y. The simple wave Ansatz implies that the operator ∂_y equals $\frac{1}{c} k_y \partial_t$, where $k_y = \hat{y} \cdot \hat{k}$. Using this result and $\boldsymbol{e}^t(\boldsymbol{r},t) = \sum \boldsymbol{e}^t_j(\phi_j(\boldsymbol{r},t))$ we write the time and y derivative of the tangential fields as

$$\begin{cases} \dot{\boldsymbol{e}}^i_\parallel + \dot{\boldsymbol{e}}^r_\parallel = \sum (\dot{\boldsymbol{e}}^t_j)_\parallel \\ \dfrac{1}{c_0} k^i_y \dot{\boldsymbol{e}}^i_\parallel + \dfrac{1}{c_0} k^r_y \dot{\boldsymbol{e}}^r_\parallel = \sum \dfrac{1}{c_j(\boldsymbol{e}^t)} k^t_{yj} (\dot{\boldsymbol{e}}^t_j)_\parallel. \end{cases}$$

These conditions are satisfied if the following holds:

$$k^i_y = k^r_y = \frac{c_0}{c_j(\boldsymbol{e}^t)} k^t_{yj}, \tag{6.2}$$

for all values of j, *cf.* phase-matching [22, p. 104]. The quotient between the wave speeds corresponds to the refractive index, and since k^i_y and k^t_{yj} are the sines of the angles of incidence and transmission, respectively, (6.2) is the well-known Snell's law. This is a purely kinematic law, so it is not surprising that it is valid also in the nonlinear case. Note that since there are several possible values for the wave speed c_j, there are several possible angles of transmission.

Since the propagation directions are normalized and there is no propagation in the x-direction, we now also have the normal reflection law for the reflected field, *i.e.*,

$$\hat{k}^r = k^i_y \hat{y} - k^i_z \hat{z}.$$

The transmitted field is more complicated, since it involves the wave speed, which may depend on the field strength.

6.3 Decomposition of the propagation direction

It seems natural to consider a decomposition of the propagation direction $\hat{k}$ in (4.2) in a y and z part. Using Snell's law and $|\hat{k}^t_j| = 1$, we find

$$\frac{c_0}{c_j} \hat{k}^t_j = \frac{c_0}{c_j} k^t_{yj} \hat{y} + \frac{c_0}{c_j} k^t_{zj} \hat{z} = k^i_y \hat{y} + \frac{c_0}{c_j} \sqrt{1 - (\frac{c_j}{c_0} k^i_y)^2}\, \hat{z}.$$

Using the eigenvalue problem (4.2) for each simple wave in the nonlinear material, we get

$$\begin{aligned} \boldsymbol{\varepsilon} \cdot \dot{\boldsymbol{e}}^t_j &= \frac{c_0}{c_j} [\hat{k}^t_j \times \mathbf{J}] \cdot \dot{\boldsymbol{e}}^t_j \\ [\boldsymbol{\varepsilon} - k^i_y \hat{y} \times \mathbf{J}] \cdot \dot{\boldsymbol{e}}^t_j &= \frac{c_0}{c_j} k^t_{zj} [\hat{z} \times \mathbf{J}] \cdot \dot{\boldsymbol{e}}^t_j \\ \frac{c_j}{c_0} \frac{1}{k^t_{zj}} \dot{\boldsymbol{e}}^t_j &= [\boldsymbol{\varepsilon} - k^i_y \hat{y} \times \mathbf{J}]^{-1} \cdot [\hat{z} \times \mathbf{J}] \cdot \dot{\boldsymbol{e}}^t_j. \end{aligned}$$

Since the operator $[\boldsymbol{\varepsilon} - k^i_y \hat{y} \times \mathbf{J}]^{-1} \cdot [\hat{z} \times \mathbf{J}]$ is independent of j, all simple waves in the nonlinear material are found from the same eigenvalue problem,

$$\lambda_j \boldsymbol{a}_j = [\boldsymbol{\varepsilon} - k^i_y \hat{y} \times \mathbf{J}]^{-1} \cdot [\hat{z} \times \mathbf{J}] \cdot \boldsymbol{a}_j, \tag{6.3}$$

where λ_j denotes the number $c_j/(c_0 k_{zj}^t)$ and $\boldsymbol{a}_j$ is shorthand for $\dot{\boldsymbol{e}}_j^t$. The corresponding problem for the vacuum fields is easily found,

$$\pm\frac{1}{k_z^i}\dot{\boldsymbol{e}}^{i,r} = [\mathbf{I} - k_y^i \hat{y}\times\mathbf{J}]^{-1}\cdot[\hat{z}\times\mathbf{J}]\cdot\dot{\boldsymbol{e}}^{i,r}, \tag{6.4}$$

where the $\pm$ comes from $k_z^r = -k_z^i$. The operator $[\mathbf{I} - k_y^i \hat{y}\times\mathbf{J}]^{-1}$ is positive definite, since $|k_y^i| < 1$. If all eigenvalues to $\boldsymbol{\varepsilon}$ are greater than one, *i.e.*, the material is denser than vacuum, the operator $[\boldsymbol{\varepsilon} - k_y^i \hat{y}\times\mathbf{J}]^{-1}$ is also positive definite.

6.4 Properties of the eigenvectors

The eigenvalue problem (6.3) is put in a symmetric form in the same manner as in Section 4. We observe that $[\boldsymbol{\varepsilon} - k_y^i \hat{y}\times\mathbf{J}]$ is positive definite and symmetric. In this section we temporarily denote this operator $\mathbf{C}$. By multiplying (6.3) with $\sqrt{\mathbf{C}}$, which is also positive definite and symmetric, we obtain

$$\begin{aligned}\lambda_j\sqrt{\mathbf{C}}\cdot\boldsymbol{a}_j &= \sqrt{\mathbf{C}}^{-1}\cdot[\hat{z}\times\mathbf{J}]\cdot\sqrt{\mathbf{C}}^{-1}\cdot\sqrt{\mathbf{C}}\cdot\boldsymbol{a}_j\\ \lambda_j\boldsymbol{u}_j &= \left[\sqrt{\mathbf{C}}^{-1}\cdot[\hat{z}\times\mathbf{J}]\cdot\sqrt{\mathbf{C}}^{-1}\right]\cdot\boldsymbol{u}_j.\end{aligned}$$

The λ_j:s are now eigenvalues to a symmetric operator, which implies that they are real. The symmetric operator $\sqrt{\mathbf{C}}^{-1}\cdot[\hat{z}\times\mathbf{J}]\cdot\sqrt{\mathbf{C}}^{-1}$ is a congruence transformation of $[\hat{z}\times\mathbf{J}]$, which has the (double) eigenvalues -1, 0 and 1. Since the signs are preserved under congruence transformations, the eigenvalues can be characterized by

$$\begin{aligned}\lambda_{1,2} &> 0\\ \lambda_{3,4} &< 0\\ \lambda_{5,6} &= 0.\end{aligned}$$

Since the $\boldsymbol{u}_j$:s are eigenvectors to a symmetric operator, they are real and mutually orthogonal. This implies that $\boldsymbol{a}_j = \sqrt{\mathbf{C}}^{-1}\cdot\boldsymbol{u}_j$ are linearly independent vectors. The eigenvectors corresponding to $\lambda_{5,6}$ can be constructed from $\boldsymbol{a}_{5,6} = \binom{\pm\hat{z}}{\hat{z}}$, which implies that $\boldsymbol{a}_{1,2,3,4}$ are the only eigenvectors needed to form the tangential fields.

The sign of the eigenvalue indicates in which direction each mode represented by an eigenvector is propagating, *i.e.*, $\boldsymbol{a}_{1,2}$ represent waves propagating in the $+z$-direction and $\boldsymbol{a}_{3,4}$ represent waves propagating in the $-z$-direction, while $\boldsymbol{a}_{5,6}$ represent waves which do not propagate with respect to z at all.

6.5 Transmission operator

Temporarily introduce the dyadic

$$\mathbf{A} = k_z^i[\mathbf{I} - \hat{z}\hat{z}]\cdot[\mathbf{I} - k_y^i\hat{y}\times\mathbf{J}]^{-1}\cdot[\hat{z}\times\mathbf{J}].$$

From (6.4) we see that $\dot{\boldsymbol{e}}_{\parallel}^{i,r} = \pm\mathbf{A}\cdot\dot{\boldsymbol{e}}_{\parallel}^{i,r}$. By multiplying the boundary condition $\dot{\boldsymbol{e}}_{\parallel}^{i} + \dot{\boldsymbol{e}}_{\parallel}^{r} = \dot{\boldsymbol{e}}_{\parallel}^{t}$ with $\mathbf{A}$ we now have

$$\dot{\boldsymbol{e}}_{\parallel}^{i} - \dot{\boldsymbol{e}}_{\parallel}^{r} = \mathbf{A}\cdot\dot{\boldsymbol{e}}_{\parallel}^{t}.$$

In the previous section, we found that only the eigenvectors $\boldsymbol{a}_{1,2,3,4}$ involve the tangential fields. Specifically, $\boldsymbol{a}_{1,2}$ correspond to waves travelling in the $+z$-direction. To this end, the transmitted tangential field is expanded as

$$\dot{\boldsymbol{e}}_{\parallel}^{t} = \sum_{j=1}^{2}\alpha_j[\mathbf{I}-\hat{z}\hat{z}]\cdot\boldsymbol{a}_j, \tag{6.5}$$

provided there are no sources in the region $z>0$, *i.e.*, no waves travelling in the $-z$-direction. We have now restricted the number of simple waves in the nonlinear material to two. From (6.3) follows

$$\begin{aligned}\mathbf{A}\cdot[\mathbf{I}-\hat{z}\hat{z}]\cdot\boldsymbol{a}_j &= k_z^i[\mathbf{I}-\hat{z}\hat{z}]\cdot[\mathbf{I}-k_y^i\hat{y}\times\mathbf{J}]^{-1}\cdot[\hat{z}\times\mathbf{J}]\cdot[\mathbf{I}-\hat{z}\hat{z}]\cdot\boldsymbol{a}_j\\ &=\lambda_j k_z^i[\mathbf{I}-\hat{z}\hat{z}]\cdot[\mathbf{I}-k_y^i\hat{y}\times\mathbf{J}]^{-1}\cdot[\boldsymbol{\varepsilon}-k_y^i\hat{y}\times\mathbf{J}]\cdot\boldsymbol{a}_j,\end{aligned}$$

where we have used $[\hat{z}\times\mathbf{J}]\cdot[\mathbf{I}-\hat{z}\hat{z}] = [\hat{z}\times\mathbf{J}]$. The operator

$$\begin{aligned}\mathbf{B} &= [\mathbf{I}-k_y^i\hat{y}\times\mathbf{J}]^{-1}\cdot[\boldsymbol{\varepsilon}-k_y^i\hat{y}\times\mathbf{J}]\\ &=\mathbf{I}+[\mathbf{I}-k_y^i\hat{y}\times\mathbf{J}]^{-1}\cdot[\boldsymbol{\varepsilon}-\mathbf{I}]\end{aligned} \tag{6.6}$$

is positive definite with eigenvalues greater than one. The boundary conditions are

$$\begin{cases}\dot{\boldsymbol{e}}_{\parallel}^{i} + \dot{\boldsymbol{e}}_{\parallel}^{r} = \displaystyle\sum_{j=1}^{2}\alpha_j[\mathbf{I}-\hat{z}\hat{z}]\cdot\boldsymbol{a}_j\\ \dot{\boldsymbol{e}}_{\parallel}^{i} - \dot{\boldsymbol{e}}_{\parallel}^{r} = \displaystyle\sum_{j=1}^{2}\alpha_j\lambda_j k_z^i[\mathbf{I}-\hat{z}\hat{z}]\cdot\mathbf{B}\cdot\boldsymbol{a}_j.\end{cases} \tag{6.7}$$

By adding these equations, we eliminate the reflected field, and obtain

$$2\dot{\boldsymbol{e}}_{\parallel}^{i} = \sum_{j=1}^{2}\alpha_j[\mathbf{I}-\hat{z}\hat{z}]\cdot[\mathbf{I}+\lambda_j k_z^i\mathbf{B}]\cdot\boldsymbol{a}_j. \tag{6.8}$$

The only unknown quantities in this equation are the coefficients α_j. If we multiply the equation by $\boldsymbol{a}_{1,2}$ from the left, we obtain a 2×2 system, which is used to extract the coefficients $\alpha_{1,2}$:

$$\begin{cases}2\boldsymbol{a}_1\cdot\dot{\boldsymbol{e}}_{\parallel}^{i} = \alpha_1\boldsymbol{a}_1\cdot[\mathbf{I}-\hat{z}\hat{z}]\cdot[\mathbf{I}+\lambda_1 k_z^i\mathbf{B}]\cdot\boldsymbol{a}_1+\alpha_2\boldsymbol{a}_1\cdot[\mathbf{I}-\hat{z}\hat{z}]\cdot[\mathbf{I}+\lambda_2 k_z^i\mathbf{B}]\cdot\boldsymbol{a}_2\\ 2\boldsymbol{a}_2\cdot\dot{\boldsymbol{e}}_{\parallel}^{i} = \alpha_1\boldsymbol{a}_2\cdot[\mathbf{I}-\hat{z}\hat{z}]\cdot[\mathbf{I}+\lambda_1 k_z^i\mathbf{B}]\cdot\boldsymbol{a}_1+\alpha_2\boldsymbol{a}_2\cdot[\mathbf{I}-\hat{z}\hat{z}]\cdot[\mathbf{I}+\lambda_2 k_z^i\mathbf{B}]\cdot\boldsymbol{a}_2.\end{cases} \tag{6.9}$$

This system is always solvable provided the following determinant is non-zero:

$$\begin{aligned}\Delta =&(\boldsymbol{a}_1\cdot[\mathbf{I}-\hat{z}\hat{z}]\cdot[\mathbf{I}+\lambda_1 k_z^i\mathbf{B}]\cdot\boldsymbol{a}_1)(\boldsymbol{a}_2\cdot[\mathbf{I}-\hat{z}\hat{z}]\cdot[\mathbf{I}+\lambda_2 k_z^i\mathbf{B}]\cdot\boldsymbol{a}_2)\\&-(\boldsymbol{a}_2\cdot[\mathbf{I}-\hat{z}\hat{z}]\cdot[\mathbf{I}+\lambda_1 k_z^i\mathbf{B}]\cdot\boldsymbol{a}_1)(\boldsymbol{a}_1\cdot[\mathbf{I}-\hat{z}\hat{z}]\cdot[\mathbf{I}+\lambda_2 k_z^i\mathbf{B}]\cdot\boldsymbol{a}_2)\\=&(\boldsymbol{a}_1\cdot\boldsymbol{v}_1)(\boldsymbol{a}_2\cdot\boldsymbol{v}_2)-(\boldsymbol{a}_2\cdot\boldsymbol{v}_1)(\boldsymbol{a}_1\cdot\boldsymbol{v}_2)\\=&\boldsymbol{a}_1\cdot(\boldsymbol{v}_1\boldsymbol{v}_2-\boldsymbol{v}_2\boldsymbol{v}_1)\cdot\boldsymbol{a}_2,\end{aligned}$$

where we have introduced the vectors $\boldsymbol{v}_{1,2}=[\mathbf{I}-\hat{z}\hat{z}]\cdot[\mathbf{I}+\lambda_{1,2}k_z^i\mathbf{B}]\cdot\boldsymbol{a}_{1,2}=\mathbf{R}_{1,2}\cdot\boldsymbol{a}_{1,2}$. The operators $\mathbf{R}_{1,2}$ are obviously positive semi-definite, where the semi-definiteness comes from the projection $[\mathbf{I}-\hat{z}\hat{z}]$. It is conjectured that these properties imply $\Delta>0$.

Using the explicit inverse of a 2×2-matrix, we can write the solution to (6.9) as

$$\begin{cases}\alpha_1=\dfrac{2}{\Delta}\Big\{(\boldsymbol{a}_2\cdot[\mathbf{I}-\hat{z}\hat{z}]\cdot[\mathbf{I}+\lambda_2 k_z^i\mathbf{B}]\cdot\boldsymbol{a}_2)(\boldsymbol{a}_1\cdot\dot{\boldsymbol{e}}_\parallel^i)\\\qquad\qquad-(\boldsymbol{a}_1\cdot[\mathbf{I}-\hat{z}\hat{z}]\cdot[\mathbf{I}+\lambda_2 k_z^i\mathbf{B}]\cdot\boldsymbol{a}_2)(\boldsymbol{a}_2\cdot\dot{\boldsymbol{e}}_\parallel^i)\Big\}\\\alpha_2=\dfrac{2}{\Delta}\Big\{(\boldsymbol{a}_1\cdot[\mathbf{I}-\hat{z}\hat{z}]\cdot[\mathbf{I}+\lambda_1 k_z^i\mathbf{B}]\cdot\boldsymbol{a}_1)(\boldsymbol{a}_2\cdot\dot{\boldsymbol{e}}_\parallel^i)\\\qquad\qquad-(\boldsymbol{a}_2\cdot[\mathbf{I}-\hat{z}\hat{z}]\cdot[\mathbf{I}+\lambda_1 k_z^i\mathbf{B}]\cdot\boldsymbol{a}_1)(\boldsymbol{a}_1\cdot\dot{\boldsymbol{e}}_\parallel^i)\Big\}.\end{cases}\tag{6.10}$$

This can be written as $\alpha_{1,2}=\frac{2}{\Delta}\boldsymbol{b}_{1,2}\cdot\dot{\boldsymbol{e}}_\parallel^i$ by introducing the vectors

$$\begin{cases}\boldsymbol{b}_1=(\boldsymbol{a}_2\cdot[\mathbf{I}-\hat{z}\hat{z}]\cdot[\mathbf{I}+\lambda_2 k_z^i\mathbf{B}]\cdot\boldsymbol{a}_2)\boldsymbol{a}_1-(\boldsymbol{a}_1\cdot[\mathbf{I}-\hat{z}\hat{z}]\cdot[\mathbf{I}+\lambda_2 k_z^i\mathbf{B}]\cdot\boldsymbol{a}_2)\boldsymbol{a}_2\\\boldsymbol{b}_2=(\boldsymbol{a}_1\cdot[\mathbf{I}-\hat{z}\hat{z}]\cdot[\mathbf{I}+\lambda_1 k_z^i\mathbf{B}]\cdot\boldsymbol{a}_1)\boldsymbol{a}_2-(\boldsymbol{a}_2\cdot[\mathbf{I}-\hat{z}\hat{z}]\cdot[\mathbf{I}+\lambda_1 k_z^i\mathbf{B}]\cdot\boldsymbol{a}_1)\boldsymbol{a}_1.\end{cases}\tag{6.11}$$

The map between $\boldsymbol{a}_{1,2}$ and $\boldsymbol{b}_{1,2}$ has the same determinant as the map between the coefficients $\alpha_{1,2}$ and the incident field, *i.e.*, Δ, which was assumed greater than zero previous in this section. This implies that the vectors $\boldsymbol{b}_{1,2}$ are linearly independent. We now formulate the relation $\dot{\boldsymbol{e}}_\parallel^t=\sum_{j=1}^2\alpha_j[\mathbf{I}-\hat{z}\hat{z}]\cdot\boldsymbol{a}_j$ as a dyadic relation between incident and transmitted fields,

$$\begin{aligned}\dot{\boldsymbol{e}}_\parallel^t=&\frac{2}{\Delta}[\mathbf{I}-\hat{z}\hat{z}]\cdot[\boldsymbol{a}_1\boldsymbol{b}_1+\boldsymbol{a}_2\boldsymbol{b}_2]\cdot\dot{\boldsymbol{e}}_\parallel^i\\=&\mathbf{T}_\parallel\cdot\dot{\boldsymbol{e}}_\parallel^i,\end{aligned}\tag{6.12}$$

where we have introduced the notation $\mathbf{T}_\parallel$ for the transmission operator acting on the tangential fields. Since the transmitted field consists of only the modes $\boldsymbol{a}_{1,2}$, the transmission operator extends to the total transmitted field:

$$\dot{\boldsymbol{e}}^t=\frac{2}{\Delta}[\boldsymbol{a}_1\boldsymbol{b}_1+\boldsymbol{a}_2\boldsymbol{b}_2]\cdot\dot{\boldsymbol{e}}_\parallel^i=\mathbf{T}\cdot\dot{\boldsymbol{e}}_\parallel^i.\tag{6.13}$$

Since the vectors $\boldsymbol{b}_{1,2}$ are linearly independent, they represent the two different polarizations of the incident field which generate the two possible modes $\boldsymbol{a}_{1,2}$ in the nonlinear material.

6.6 Reflection operator and Brewster angles

It is well known that at certain angles and polarizations of the incident field there is no reflected field at all — the Brewster angles [22, 25, 26]. From (6.7) we see that the reflected field can be written

$$2\dot{\boldsymbol{e}}_{\|}^{r} = \sum_{j=1}^{2} \alpha_j [\mathbf{I} - \hat{z}\hat{z}] \cdot [\mathbf{I} - \lambda_j k_z^i \mathbf{B}] \cdot \boldsymbol{a}_j.$$

Using $\alpha_{1,2} = \frac{2}{\Delta}\boldsymbol{b}_{1,2} \cdot \dot{\boldsymbol{e}}_{\|}^{i}$ we find the following relationship between the reflected and incident field:

$$\begin{aligned} 2\dot{\boldsymbol{e}}_{\|}^{r} =& \frac{2}{\Delta}\Big[\Big\{[\mathbf{I} - \hat{z}\hat{z}] \cdot [\mathbf{I} - \lambda_1 k_z^i \mathbf{B}] \cdot \boldsymbol{a}_1\Big\}\, \boldsymbol{b}_1 \\ &+ \Big\{[\mathbf{I} - \hat{z}\hat{z}] \cdot [\mathbf{I} - \lambda_2 k_z^i \mathbf{B}] \cdot \boldsymbol{a}_2\Big\}\, \boldsymbol{b}_2\Big] \cdot \dot{\boldsymbol{e}}_{\|}^{i} \\ =& \frac{2}{\Delta}[\mathbf{I} - \hat{z}\hat{z}] \cdot [\boldsymbol{b}_1'\boldsymbol{b}_1 + \boldsymbol{b}_2'\boldsymbol{b}_2] \cdot \dot{\boldsymbol{e}}_{\|}^{i} \\ =& 2\mathbf{R}_{\|} \cdot \dot{\boldsymbol{e}}_{\|}^{i}. \end{aligned}$$

This is the reflection operator $\mathbf{R}_{\|}$ for the tangential fields, which is represented as a factorization in the simple dyads $\boldsymbol{b}_1'\boldsymbol{b}_1$ and $\boldsymbol{b}_2'\boldsymbol{b}_2$, where $\boldsymbol{b}_{1,2}' = [\mathbf{I} - \lambda_{1,2} k_z^i \mathbf{B}] \cdot \boldsymbol{a}_{1,2}$.

Since the vectors $\boldsymbol{b}_{1,2}$ are linearly independent, we see that the Brewster angles are characterized by

$$\begin{cases} \dot{\boldsymbol{e}}_{\|}^{i} = \beta[\mathbf{I} - \hat{z}\hat{z}] \cdot \boldsymbol{b}_j \\ \ \ 0 = [\mathbf{I} - \hat{z}\hat{z}] \cdot [\mathbf{I} - \lambda_j k_z^i \mathbf{B}] \cdot \boldsymbol{a}_j \end{cases} \quad j = 1, 2, \tag{6.14}$$

where β is a scalar. This means that if the incident field is polarized along $\boldsymbol{b}_j$ and $\boldsymbol{a}_j$ is in the null space of $[\mathbf{I} - \hat{z}\hat{z}] \cdot [\mathbf{I} - \lambda_j k_z^i \mathbf{B}]$, there is no reflected field. These conditions determine the possible Brewster angles. We have

$$\begin{aligned} 0 &= [\mathbf{I} - \hat{z}\hat{z}] \cdot [\mathbf{I} - \lambda_j k_z^i \mathbf{B}] \cdot \boldsymbol{a}_j \\ &= [\mathbf{I} - \hat{z}\hat{z}] \cdot [\mathbf{I} - \frac{c_j}{c_0}\frac{k_z^i}{k_{zj}^t}(\mathbf{I} + [\mathbf{I} - k_y^i \hat{y} \times \mathbf{J}]^{-1} \cdot [\boldsymbol{\varepsilon} - \mathbf{I}])] \cdot \boldsymbol{a}_j \\ &= [\mathbf{I} - \hat{z}\hat{z}] \cdot [\mathbf{I} - \frac{c_j}{c_0}\frac{k_z^i}{k_{zj}^t}(\mathbf{I} + \frac{1}{(k_z^i)^2}[\mathbf{I} + k_y^i \hat{y} \times \mathbf{J} - (k_y^i)^2 \hat{y}\hat{y}] \cdot [\boldsymbol{\varepsilon} - \mathbf{I}])] \cdot \boldsymbol{a}_j, \end{aligned}$$

where we have introduced the explicit inverse $[\mathbf{I} - k_y^i \hat{y} \times \mathbf{J}]^{-1} = \frac{1}{(k_z^i)^2}[\mathbf{I} + k_y^i \hat{y} \times \mathbf{J} - (k_y^i)^2 \hat{y}\hat{y}]$, which can be verified by straightforward calculations. The y-component of this equation is

$$\begin{aligned} 0 &= \hat{y} \cdot \boldsymbol{a}_j - \frac{c_j}{c_0}\frac{k_z^i}{k_{zj}^t}(\hat{y} + \frac{1}{(k_z^i)^2}[\hat{y} - (k_y^i)^2 \hat{y}] \cdot [\boldsymbol{\varepsilon} - \mathbf{I}]) \cdot \boldsymbol{a}_j \\ &= \hat{y} \cdot \boldsymbol{a}_j - \frac{c_j}{c_0}\frac{k_z^i}{k_{zj}^t}(\hat{y} + \hat{y} \cdot [\boldsymbol{\varepsilon} - \mathbf{I}]) \cdot \boldsymbol{a}_j \\ &= \hat{y} \cdot \boldsymbol{a}_j - \frac{c_j}{c_0}\frac{k_z^i}{k_{zj}^t}\hat{y} \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{a}_j. \end{aligned}$$

In Section 4 it was shown that a propagating field in an isotropic material is described by $\boldsymbol{a}_j = (\frac{1}{\sqrt{\epsilon}}\boldsymbol{v}_j, \frac{1}{\sqrt{\mu}}\hat{k}\times\boldsymbol{v}_j)$, where the three-vector $\boldsymbol{v}_j$ is orthogonal to $\hat{k}$, and the only possible wave speed is $\frac{c_j}{c_0} = \frac{1}{\sqrt{\epsilon\mu}}$. In the remainder of this section, we suppress the index j, and separate the two modes in the end. The Brewster angles can now be found from the y-component defined above. By explicitly considering both electric and magnetic fields we have

$$
\begin{aligned}
\hat{y}\cdot\begin{pmatrix}\frac{1}{\sqrt{\epsilon}}\boldsymbol{v}\\ \frac{1}{\sqrt{\mu}}\hat{k}^t\times\boldsymbol{v}\end{pmatrix} &= \frac{1}{\sqrt{\epsilon\mu}}\frac{k_z^i}{k_z^t}\hat{y}\cdot\begin{pmatrix}\sqrt{\epsilon}\boldsymbol{v}\\ \sqrt{\mu}\hat{k}^t\times\boldsymbol{v}\end{pmatrix}\\
\begin{pmatrix}\frac{1}{\sqrt{\epsilon}}\hat{y}\cdot\boldsymbol{v}\\ \frac{1}{\sqrt{\mu}}\hat{k}^t\cdot(\boldsymbol{v}\times\hat{y})\end{pmatrix} &= \frac{k_z^i}{k_z^t}\begin{pmatrix}\frac{1}{\sqrt{\mu}}\hat{y}\cdot\boldsymbol{v}\\ \frac{1}{\sqrt{\epsilon}}\hat{k}^t\cdot(\boldsymbol{v}\times\hat{y})\end{pmatrix}\\
\begin{pmatrix}\frac{1}{\sqrt{\epsilon}}\hat{y}\cdot\boldsymbol{v}\\ \frac{1}{\sqrt{\mu}}k_z^t\hat{z}\cdot(\boldsymbol{v}\times\hat{y})\end{pmatrix} &= \frac{k_z^i}{k_z^t}\begin{pmatrix}\frac{1}{\sqrt{\mu}}\hat{y}\cdot\boldsymbol{v}\\ \frac{1}{\sqrt{\epsilon}}k_z^t\hat{z}\cdot(\boldsymbol{v}\times\hat{y})\end{pmatrix}.
\end{aligned}
$$

It is now obvious that one of the following sets of conditions have to be satisfied in order to satisfy the Brewster angle criterion.

$$
\begin{cases}\hat{y}\cdot\boldsymbol{v}=0\\ \sqrt{\epsilon}k_z^t = \sqrt{\mu}k_z^i\end{cases} \quad\text{or}\quad \begin{cases}\hat{z}\cdot(\boldsymbol{v}\times\hat{y})=0\\ \sqrt{\mu}k_z^t = \sqrt{\epsilon}k_z^i.\end{cases}
$$

Observe that $\hat{z}\cdot(\boldsymbol{v}\times\hat{y}) = 0$ is equivalent to $\hat{x}\cdot\boldsymbol{v} = 0$, *i.e.*, the first set of conditions corresponds to TE-polarization and the second to TM-polarization. Remember that $k_z^i = \cos\theta^i$ and $k_z^t = \cos\theta^t$, where $\theta^{i,t}$ denote the angles of incidence and transmission, respectively, and we have recovered the well known results for linear isotropic materials. Since we in general have $\theta^t < \theta^i$, only one of the above possible Brewster angles is feasible.

An interesting question is whether it always suffices to study the y-component of our original Brewster-angle-condition in (6.14). This is a problem that goes beyond the scope of this paper.

6.7 Algorithm for the direct problem

In this section we summarize the algorithm for solving the direct problem of propagating the incident field through a boundary between vacuum and a nonlinear, nondispersive, homogeneous, bianisotropic halfspace.

We have to calculate the eigenvectors $\boldsymbol{a}_{1,2}$, the eigenvalues $\lambda_{1,2}$ and the operator $\mathbf{B}$ to obtain the reflection and transmission dyadics. These quantities are determined from the relations

$$
\begin{cases}\lambda_j\boldsymbol{a}_j = [\boldsymbol{\varepsilon} - k_y^i\hat{y}\times\mathbf{J}]^{-1}\cdot[\hat{z}\times\mathbf{J}]\cdot\boldsymbol{a}_j\\ \mathbf{B} = [\mathbf{I} - k_y^i\hat{y}\times\mathbf{J}]^{-1}\cdot[\boldsymbol{\varepsilon} - k_y^i\hat{y}\times\mathbf{J}],\end{cases}
$$

i.e., we have to solve an eigenvalue problem (first row), extract the eigenvectors corresponding to positive eigenvalues, and calculate the operator $\mathbf{B}$. These calculations are evaluated at the transmitted field values at a specific time. The operators are supposed to act on time derivatives of the fields. We discretize the problem with central differences in time, and use the previously calculated values for the transmitted fields in the solution of the eigenvalue problem.

Once we have calculated the tangential fields, it is an easy task to obtain the normal components of the fields. For the transmitted fields these are already given by the transmission operator, see (6.13), and for the reflected field they are given by the relation $\hat{k}^r \cdot \dot{\boldsymbol{e}}^r = 0$, which implies $\dot{\boldsymbol{e}}_z^r = -\frac{k_y^r}{k_z^r}\dot{\boldsymbol{e}}_y^r$.

The algorithm can be summarized as follows, where the indices denote at which time level the different quantities are to be evaluated.

$$
\begin{aligned}
(\text{eigenvalue problem})_j &\Rightarrow (\lambda_{1,2})_j, (\boldsymbol{a}_{1,2})_j \\
(\mathbf{B})_j &= \mathbf{B}((\boldsymbol{e}^t)_j) \\
(\mathbf{T})_j &= \mathbf{T}((\lambda_{1,2})_j, (\boldsymbol{a}_{1,2})_j, (\mathbf{B})_j) \\
(\mathbf{R}_{\parallel})_j &= \mathbf{R}_{\parallel}((\lambda_{1,2})_j, (\boldsymbol{a}_{1,2})_j, (\mathbf{B})_j) \\
(\dot{\boldsymbol{e}}_{\parallel}^i)_j &= \frac{(\boldsymbol{e}_{\parallel}^i)_{j+1} - (\boldsymbol{e}_{\parallel}^i)_{j-1}}{2\Delta t} \\
(\boldsymbol{e}^t)_{j+1} &= (\boldsymbol{e}^t)_{j-1} + 2\Delta t(\mathbf{T})_j \cdot (\dot{\boldsymbol{e}}_{\parallel}^i)_j \\
(\boldsymbol{e}_{\parallel}^r)_{j+1} &= (\boldsymbol{e}_{\parallel}^r)_{j-1} + 2\Delta t(\mathbf{R}_{\parallel})_j \cdot (\dot{\boldsymbol{e}}_{\parallel}^i)_j \\
(\boldsymbol{e}_z^r)_{j+1} &= -\frac{k_y^r}{k_z^r}(\boldsymbol{e}_y^r)_{j+1}
\end{aligned}
$$

6.8 Numerical example

The algorithm in the previous section has been implemented for a nonlinear, anisotropic material, and the result is depicted in Figure 2. We have scaled the fields to obtain dimensionless field strengths and substantial nonlinearities for field strengths of a few units, see *e.g.*, [23, 30]. The constitutive relation is characterized by the dyadic $\boldsymbol{\varepsilon}$, which is represented in the xyz-coordinate system as

$$
\boldsymbol{\varepsilon} = \begin{bmatrix}
2+E^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 3+E^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 4+E^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
$$

Thus, the material is non-magnetic, anisotropic with principal axis in the xyz-directions, and has a nonlinear permittivity depending on the square of the electric field strength. The angle of incidence is 70°, and the incident field has the magnetic field perpendicular to the plane of incidence, *i.e.*, in the x-direction,

$$
\boldsymbol{e}^i(\boldsymbol{r}, t) = f(t - \hat{k}^i \cdot \boldsymbol{r}/c_0)\begin{pmatrix} -\hat{k}^i \times \hat{x} \\ \hat{x} \end{pmatrix}, \quad f(t) = \begin{cases} e_0\sqrt{t} & t \geq 0 \\ 0 & t < 0. \end{cases}
$$

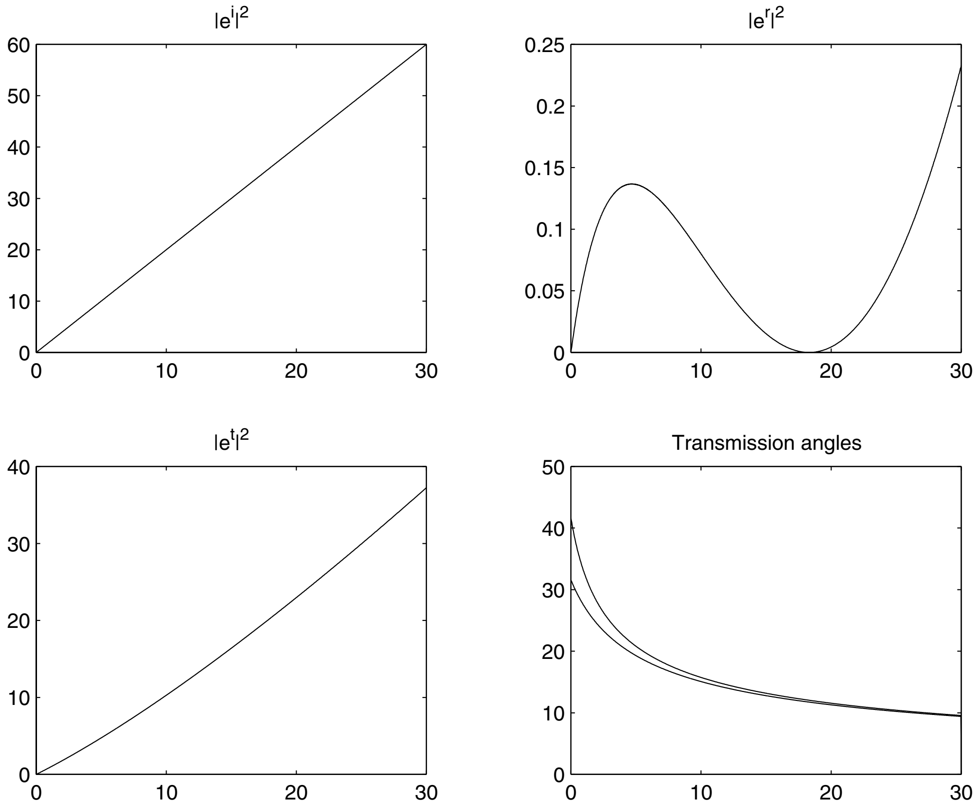

Figure 2: Oblique incidence on an anisotropic Kerr material. Observe that the horizontal scales can be used both as time and energy of the incident field. The diagrams show the squares of the incident, reflected and transmitted fields, and the two possible transmission angles.

The time dependence of the amplitude of the incident field is chosen so that its square, which is proportional to the field energy in vacuum, depends linearly on time. This implies that the horizontal scales in Figure 2 can be used both as time and energy. We see that the reflected field displays a strong dependence on the incident field energy, whereas the transmitted field has a more moderate dependence.

It is clearly seen that the Brewster angle occurs when the incident energy is approximately 18. Had the principle axis of the material not been in the xyz-directions, we would have needed another polarization of the incident field to obtain a reflected field that is zero.

The possible transmission angles start off as clearly separated, as can be expected for an anisotropic material, but become more and more equal as the incident energy increases. This can be interpreted from the material dyadic: when the electric field strength grows, the diagonal elements become essentially E^2. Thus the material becomes more and more isotropic, *i.e.*, it has only one possible angle of transmission. Observe that due to our choice of polarization of the incident field, only one of the modes is excited.

7 Conclusions

In this paper, we have introduced the concept of simple waves, as a means to analyze wave propagation problems in nonlinear materials with instantaneous response. We have applied the method to the problem of oblique incidence of a plane electromagnetic wave on a nonlinear material, and found that the direct problem can be solved for all materials and all possible polarizations of the incident wave.

The drawback of the simple wave solutions, is that they do not apply to materials with dispersion, *i.e.*, materials with memory. Our mathematical model with instantaneous nonlinearity, predicts that *all* reasonable waves eventually turn into shocks. It is often argued that the presence of linear dispersion eliminates these shocks, see *e.g.*, [1, pp. 117–120]. Therefore, we can expect our model to be accurate only when there is no shock-like behaviour and the dispersion effects are small, *i.e.*, for sufficiently smooth and slowly varying pulses. It is possible to calculate what propagation distances are necessary for the shock to develop, which means we can estimate the region of validity for our model.

The methods presented in this paper may be useful to propagate the wave front when studying wave propagation in more advanced materials. Temporal dispersion and inhomogeneous media may appear as lower order terms in the Maxwell equations, and can be treated as sources to the fields treated here.

8 Acknowledgement

The author wishes to express his sincere gratitude to Professor Gerhard Kristensson for invaluable support and encouragement during the work presented in this paper. The author's colleagues at the Department of Electromagnetic Theory are also acknowledged for valuable discussions and suggestions.

The work reported in this paper is partially supported by a grant from the Swedish Research Council for Engineering Sciences, and its support is gratefully acknowledged.

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