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Rantzer, Anders

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Total number of authors:
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LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

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On Prize Mechanisms in Linear Quadratic Team Theory

Anders Rantzer

***Abstract*— A distributed linear quadratic decision problem is considered, where several different controllers act as a team, but with access to different measurements. Previous contributions have shown how to state the optimal synthesis as a finite-dimensional convex optimization problem.**

This paper shows that the dynamic behavior can be optimized by a distributed iterative procedure, without any need for a globally available model or centralized coordination. An illustrative model with three agents is considered.

I. Introduction

Decision making when the decision makers have access to different information concerning the underlying uncertainties has been studied since the late 1950s [10], [12]. The subject is sometimes called team theory, sometimes decentralized or distributed control.

The theory was originally static, but work on dynamic aspects was initiated by Witsenhausen [19], who also pointed out a fundamental difficulty in such problems. Some special types of team problems were solved in the 1970's [17], [7], but the problem area has recently gain renewed interest. Spatial invariance was exploited in [2], [3], conditions for closed loop convexity were derived in [16], [15] and approximate methods using linear matrix inequalities were given in [9].

In our previous paper [13] a linear quadratic stochastic optimal control problem was solved for a state feedback control law with covariance constraints. Such problems have previously been solved using so-called S-procedure [11], [20]. The method gives a non-conservative extension of linear quadratic control theory to distributed control with bounds on the rate of information propagation. An output feedback version of the problem was solved in [14] and for both finite and infinite time horizons in [6].

A common feature of all synthesis methods presented so far is that the design procedure is centralized. Hence, even though the controllers are eventually supposed to operate separately, they are designed together based on a full global model of the system. In this paper, we take a different viewpoint and let the synthesis procedure be distributed. Each agent has access only to a local model and tries to optimize his control action based on a local cost function. The interaction with neighboring agents is handled through negotiations.

A. Rantzer is with Automatic Control LTH, Lund University, Box 118, SE-221 00 Lund, Sweden, rantzer at control.lth.se.

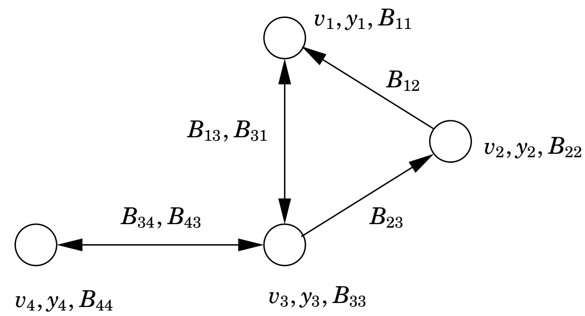

Fig. 1. The graph illustrates the interconnection structure of a team problem. The non-zero matrix elements B_{jk} specify directions of influence.

Section III introduces the classical idea of "dual decomposition" to use price dynamics for decentralized optimization. An example highlights the problem of slow transients. In section IV the method is applied to static linear quadratic team problem considered by Radner [12]. Section V extends the method to stochastic processes. It is shown that the slow transients of the original example can be removed.

II. A dynamic linear quadratic team problem

Given a set of stationary stochastic processes $(v_1, \ldots, v_J, y_1, \ldots, y_J)$ with jointly Gaussian distribution, consider the problem to find convolution operators $\kappa_1, \ldots, \kappa_J \in \ell_2[0, \infty)$ that minimize the variance

$$\mathbf{E} \sum_{j=1}^{J} \left| v_j + \sum_{k=1}^{J} B_{jk}(\kappa_k * y_k) \right|_{Q_j}^2$$

The vector v_j can be thought of as an external stochastic perturbation active in node j of a graph, while y_j is the information available to the decision maker (agent) in that node. The term $B_{jk}(\kappa_k * y_k)$ specifies the effect in node j of the decision by agent k. It is assumed that $B_{jk} = 0$ unless (j, k) is a graph edge. See Figure 1.

The purpose of the paper is to show that the optimal decision functions can be found by a distributed negotiation procedure, using the idea of dual decomposition. For large problems, such a decomposition may be crucial to reduce the computational complexity. In other cases, the procedure may be motivated by the lack of globally available models.

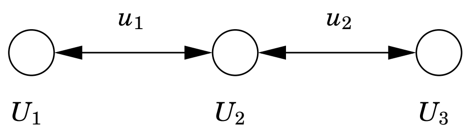

Fig. 2. In the example, we consider a team problem with three nodes in the graph. The decision variable u_1 has effects for the first and the second agent, while u_2 concerns the second and third.

III. Dual decomposition

The idea of dual decomposition has a long history [5], [8]. We use the following example to explain. Consider an optimization problem of the form

$$\max_{u_1,u_2}[U_1(w_1+u_1)+U_2(w_2-u_1+u_2)+U_3(w_3-u_2)]$$

where U_1, U_2 and U_3 are all concave. The structure of the problem is illustrated in Figure 2 and can be exploited using Lagrange variables. Under standard assumptions for strong duality [4], the problem is rewritten as

$$\min_{\lambda_{jk}}\max_{u_{jk}}\Big[U_1(w_1+u_{11})+U_2(w_2-u_{21}+u_{22})+U_3(w_3-u_{32}) \\ +\lambda_{21}(u_{21}-u_{11})+\lambda_{23}(u_{32}-u_{22})\Big]$$

For fixed λ_{jk}, the inner maximization decomposes into three separate optimization problems

$$\max_{u_{11}}\big[U_1(w_1+u_{11})-\lambda_{21}u_{11}\big]$$
$$\max_{u_{21},u_{22}}\big[U_2(w_2-u_{21}+u_{22})+\lambda_{21}u_{21}-\lambda_{23}u_{22}\big]$$
$$\max_{u_{32}}\big[U_3(w_3-u_{32})+\lambda_{23}u_{32}\big]$$

so the computations can be parallelized. The decomposition into three problems has a natural interpretation in economic terms:

The three functions U_1, U_2 and U_3 can be interpreted as the utility functions for three agents given certain allocations of a commodity. The initial endowments of the commodity are specified as w_1, w_2 and w_3. The quantity of traded commodity between the first and the second agent is u_1, while u_2 quantifies the trade between the second and third agent. When all agents try to maximize their utility, they will arrive at different opinions about the desirable values of u_j unless the trade is done at appropriate prices λ_{jk}. At the saddle point, the prices will create a consensus among all agents about the desirable values of u_j.

It turns out that the prices λ_{jk} and traded quantities can be updated in a distributed manner using the gradient method, which can be interpreted as a process of negotiation. This is known as the saddle point algorithm or Usawa's algorithm [1]. The idea is very simple: A gradient search for the saddle point

$$\min_{\lambda}\max_{x}U(\lambda,x)$$

has the dynamics

$$\dot\lambda=-(\partial U/\partial\lambda) \qquad \dot x=(\partial U/\partial x)$$

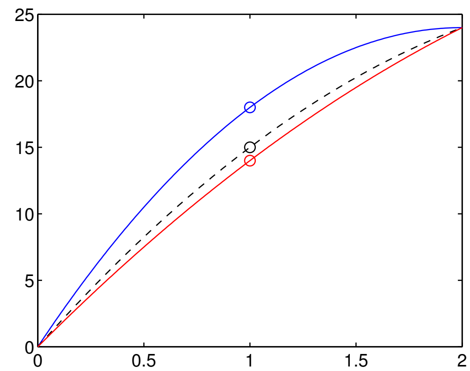

Fig. 3. The utilitiy functions for the three agents in Example 1. When all agents have a quantity of 1, there is no incentive for trade.

and the Lyapunov function $V=|\dot x|^2+|\dot\lambda|^2$ is globally decaying because

$$\begin{aligned}\dot V&=\dot x^T\ddot x+\dot\lambda^T\ddot\lambda\\ &=\dot x^T[(\partial^2U/\partial x^2)\dot x+(\partial^2U/\partial x\partial\lambda)\dot\lambda]\\ &\quad-\dot\lambda^T[(\partial^2U/\partial x\partial\lambda)\dot x+(\partial^2U/\lambda^2)\dot\lambda]\\ &=\dot x^T(\partial^2U/\partial x^2)\dot x-\dot\lambda^T(\partial^2U/\lambda^2)\dot\lambda\le 0\end{aligned}$$

When U is strictly convex-concave, the last expression is strictly negative for all non-zero $(\dot\lambda,\dot x)$. In our applications the λ-dependence is linear and not strictly convex, but a similar argument works.

Example 1 Consider three agents with utilities with

$$\begin{aligned}U_1(x_1)&=24-6(x_1-2)^2\\ U_2(x_2)&=27-3(x_2-3)^2\\ U_3(x_2)&=32-2(x_3-4)^2\end{aligned}$$

plotted in Figure 3. The endowments $w_1=w_2=w_3=1$ then give

$$\begin{aligned}&U_1(w_1+u_1)+U_2(w_2-u_1+u_2)+U_3(w_3-u_2)\\ &=47-6|u_1|^2-3|u_2-u_1|^2-2|u_2|^2\end{aligned}$$

so there is equilibrium with no trade: $u_1=u_2=0$. We will now study the effect of a perturbation to the first agent, by putting $w_1=1+v$. This gives the optimization problem

$$\begin{aligned}&\min_{u_1,u_2}\left(6|v+u_1|^2+3|u_2-u_1|^2+2|u_2|^2\right)\\ &=\max_{\lambda_j}\min_{u_{jk}}\Big(6|v+u_{11}|^2+3|u_{22}-u_{21}|^2+2|u_{32}|^2\\ &\qquad+\lambda_1(u_{11}-u_{21})+\lambda_2(u_{22}-u_{32})\Big)\\ &=\max_{\lambda_j}\min_{u_{jk}}\Big(6|v+u_{11}|^2+\lambda_1u_{11}\\ &\qquad+3|u_{22}-u_{21}|^2+\lambda_2u_{22}-\lambda_1u_{21}\\ &\qquad+2|u_{32}|^2-\lambda_2u_{32}\Big)\end{aligned}$$

For $v=0$, the optimal prices are $\lambda_1=\lambda_2=12$.

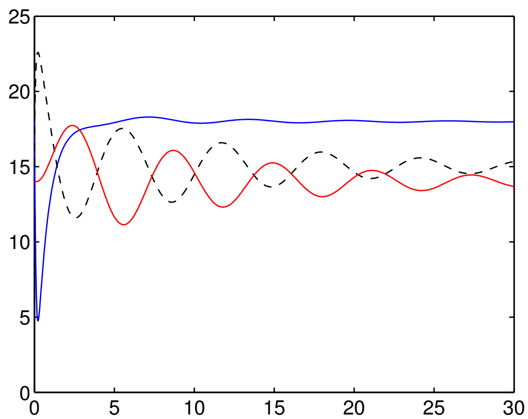

Fig. 4. The dynamics of the three utility functions when the first agent is subject to the transient perturbation $v(t) = e^{-t}$. The utility of the first agent quickly recovers, but there is a long oscillative transient for the second and third agent.

A gradient search for the saddle point can be written

$$\begin{bmatrix} \dot u_{11} \\ \dot u_{21} \\ \dot u_{22} \\ \dot u_{32} \\ \dot\lambda_1 \\ \dot\lambda_2 \end{bmatrix} = \begin{bmatrix} -12 & 0 & 0 & 0 & -1 & 0 \\ 0 & -6 & 6 & 0 & 1 & 0 \\ 0 & 6 & -6 & 0 & 0 & -1 \\ 0 & 0 & 0 & -4 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{22} \\ u_{32} \\ \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} -12v \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

As described before, the dynamics are bound to be stable. However, they easily become very oscillative. See Figure 4 for a simulation. In spite of the model simplicity, the dynamics have much in common with the classical "beer game" [18]. □

The example will be reconsidered later to demonstrate how a linear quadratic control problem can be solved to get rid of the transient oscillations.

IV. The static linear quadratic team problem

To study decision making when the agents cooperate, but have access to different information concerning the underlying uncertainties, a static linear quadratic team problem was introduced by Radner [12]:

Given stochastic variables $(v_1, \dots, v_J, y_1, \dots, y_J)$ with a joint Gaussian distribution, determine "decision functions" $\alpha_j(y_j)$ to minimize a given quadratic function of $(v_1, \dots, v_J, \alpha_1(y_1), \dots, \alpha_J(y_J))$. Radner proved that the optimal decision functions are linear

$$\alpha_j(y_j) = K_j y_j \qquad j = 1, \dots, J$$

and K_j are uniquely determined by a set of linear equations.

Consider minimization of the objective function

$$\mathbf{E} \sum_{j=1}^{J} \left| v_j + \sum_{k=1}^{J} B_{jk} K_k y_k \right|^2_{Q_j}$$

The term with index j quantifies the cost for agent j due to deviations from the desired operating point. The vector v_j specifies an external stochastic perturbation, while $B_{jk} K_k y_k$ is the effect of the decision by agent k. It is assumed that each agent is located in a node of a graph and $B_{jk} = 0$ unless (j,k) is a graph edge.

The agent in node j tries to minimize his cost by proper choice of K_j. However, the choice of K_j is also influencing the cost in neighboring nodes of the graph. Hence we need a price mechanism for the agents to compensate each other and reach consensus about the decision functions to use. For this purpose, we state the following theorem.

Theorem 1: For a graph with nodes $j \in \{1, 2, \dots, J\}$ and a set of edges $\mathcal{E}$, suppose $B_{jk} = 0$ for $(j,k) \notin \mathcal{E}$. Let the stochastic variables $v_1 \in \mathbf{R}^{n_1}, \dots, v_J \in \mathbf{R}^{n_J}$, and $y_1 \in \mathbf{R}^{p_1}, \dots, y_J \in \mathbf{R}^{p_J}$ have a joint Gaussian distribution. Then

$$\begin{aligned} &\min_{K_1,\dots,K_J} \mathbf{E} \sum_{j=1}^{J} \left| v_j + \sum_{k=1}^{J} B_{jk}(K_k y_k) \right|^2_{Q_j} \\ &= \max_{(\lambda_{jk})} \min_{(K_{jk})} \mathbf{E} \sum_{j=1}^{J} \Bigg(\left| v_j + \sum_k B_{jk}(K_{jk} y_k) \right|^2_{Q_j} \\ &\qquad + \sum_k \left[\lambda_{kj}^* K_{jj} y_j - \lambda_{jk}^* K_{jk} y_k \right] \Bigg) \end{aligned}$$

In the right hand expression, the maximization is done over (λ_{jk}) with $\lambda_{jk} = 0$ for $(j,k) \notin \mathcal{E}$, and for each j the minimization is done over K_{jk} with $(j,k) \in \mathcal{E}$. At optimality $K_{jj} = K_{jk}$ for all $(j,k) \in \mathcal{E}$.

Proof. The statement is a standard application of strong duality in finite-dimensional convex quadratic optimization. □

Theorem 1 can be used as a foundation for distributed synthesis procedures for team decision making. Notice that each agent i only needs to worry about minimizing his own cost after clearing prices with his neighbors.

The gradient method can be used as before as a distributed procedure to find the optimal prices λ_{jk}. Alternatively, the prices can be found through a sequence of negotiations: For one edge of the graph at a time, two neighbors (agent j and agent k) negotiate to update their price vector λ_{jk} and decision vectors K_{jk}, K_{jj} by solving the local min-max-problem only with respect to these variables. For this negotiation, it is not necessary to have full information about the Gaussian distribution of $(v_1, \dots, v_J, y_1, \dots, y_J)$. It is sufficient to know the cross-covariances of the variables v_j, y_j, v_k, y_k with respect to each other and with the neighboring price vectors. The negotiation is repeated over and over again for different nodes in the graph. At every iteration the saddle-point value will be non-decreasing and bounded from above by the global maximum. Hence the values will converge and the global saddle-point is obtained in the limit.

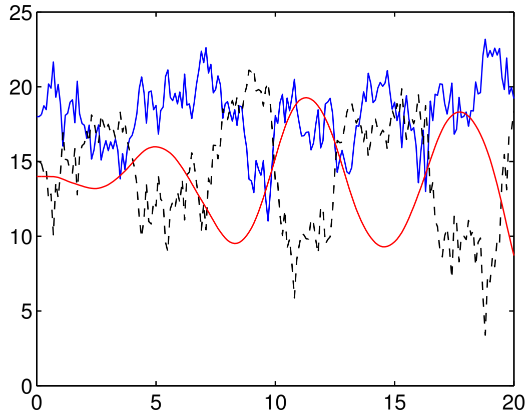

Fig. 5. **Discrete time gradient dynamics** in the three utility functions when the first agent is subject to a stochastic perturbation with spectral density $|e^{i0.1\omega} - 0.9|^{-2}/(2\pi)$. Prices and trade volumes are updated at every time step.

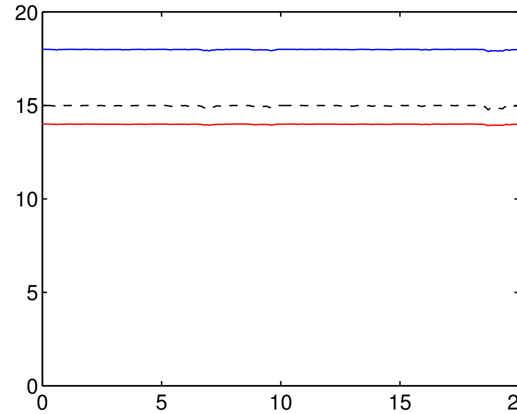

Fig. 6. **Discrete time dynamics optimized for disturbance rejection** in the three utility functions when the first agent is subject to a perturbation with spectral density $|e^{i0.1\omega} - 0.9|^{-2}/(2\pi)$. Prices and trade volumes are updated at every time step.

Example 2 We will now reconsider the three-agent example in a stochastic team decision setting. To simplify the introduction of stochastics, it is convenient to consider a discrete time version of the gradient search for a saddle-point. Let

$$\dot{x} = Ax + Bv$$

be the given continuous gradient dynamics discussed before. Then the discrete time model

$$x(t+h) = [I + Ah]x(t) + Bhv(t)$$

has similar stable dynamics for small h. The equation describes what happens when prices and trade volumes are updated every time step of length h. A simulation where v is a stationary zero mean stochastic process

$$v(t+h) = av(t) + d(t)$$

where d is white noise with unit variance and $h = 0.1$ is shown in Figure 5. A characteristic feature of the gradient dynamics is the time delay in information propagation from one node to another. In particular, $v(t)$ has no effect on $u_{22}(\tau)$ until $\tau = t + 2h$. This motivates a comparison with the following stochastic team problem.

Let

$$\begin{aligned} u_1(t) &= K_{10}v(t) + K_{11}v(t-2h) \\ u_2(t) &= K_{21}v(t-2h) \end{aligned}$$

Then

$$\begin{aligned} &\min_{u_1,u_2} \mathbf{E}\left(6|v+u_1|^2 + 3|u_2 - u_1|^2 + 2|u_2|^2\right) \\ &= \min_{K_{10},K_{11},K_{21}} \mathbf{E}\left(6|1 + K_{10} + aK_{11}|^2 \right. \\ &\qquad\qquad \left. + 3|aK_{21} - K_{10} - aK_{11}|^2 + 2|aK_{21}|^2\right) \end{aligned}$$

with minimum attained for

$$\begin{bmatrix} K_{10}/3 \\ K_{11}/3 \\ K_{21} \end{bmatrix} = -\begin{bmatrix} 1 & a & -a \\ a & 1 & -1 \\ -a & -1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 2a \\ 0 \end{bmatrix}$$

A negotiation procedure between the agents based on this model for the disturbance v will converge to the optimal decision rule. The simulation in Figure 6 is strikingly different from Figure 5 and no oscillations remain.

Notice also that the case $a = 0$ would give $K_{11} = K_{21} = 0$ at optimality. In this case v would be white noise and every attempt to compensate based on delayed information would be bound to fail. □

V. The dynamic problem again

The dynamic team problem is entirely analogous to the static one. Let $B = (B_{ik})$ and Q_j be given as before. However, now $(v_1, \ldots, v_J, y_1, \ldots, y_J)$ are assumed to be stationary stochastic processes with jointly Gaussian distribution. The objective is to find convolution operators $\kappa_1, \ldots, \kappa_J \in \ell_2[0, \infty)$ that minimize the variance

$$\mathbf{E} \sum_{j=1}^{J} \left| v_j + \sum_{k=1}^{J} B_{jk}(\kappa_k * y_k) \right|_{Q_j}^2$$

Let $\phi_{kj}(\omega)$ be the cross spectral density of (v_k, y_k) and (v_j, y_j). Let I_{jk} be the identity matrix when $j = k$ and zero otherwise. Then, in frequency domain, the problem is to find $K_1, \ldots, K_J \in \mathbf{H}_2$ minimizing

$$\sum_{j,k,l} \int_{-\pi}^{\pi} \operatorname{tr}\left(Q_j \begin{bmatrix} I_{jk} & B_{jk}K_k(e^{i\omega}) \end{bmatrix} \phi_{kj}(\omega) \begin{bmatrix} I_{jl} & B_{jk}K_l(e^{i\omega}) \end{bmatrix}^*\right) d\omega$$

This motivates an infinite-dimensional counterpart to Theorem 1.

Theorem 2: For a graph with nodes $j \in \{1, 2, \ldots, J\}$ and a set of edges $\mathcal{E}$, suppose $B_{jk} = 0$ for $(j, k) \notin \mathcal{E}$. Let $\phi(\omega) = (\phi_{jk}(\omega)) > 0$ be bounded measurable on $[-\pi, \pi]$. Then

$$\min_K \sum_{j,k,l} \int_{-\pi}^{\pi} \mathrm{tr}\left(Q_j \left[I_{jk} \quad B_{jk}K_k(e^{i\omega})\right] \phi_{kj}(\omega) \left[I_{jl} \quad B_{jk}K_l(e^{i\omega})\right]^*\right) d\omega$$

$$= \max_\Lambda \min_K \sum_{j,k,l} \int_{-\pi}^{\pi} \Big\{ \mathrm{tr}\left(Q_j \left[I_{jk} \quad B_{jk}K_{jk}\right] \phi_{kj}(\omega) \left[I_{jl} \quad B_{jk}K_{jl}\right]^*\right)$$

$$+ \mathrm{tr}[\Lambda_{jk}(e^{i\omega})^* K_{jk}(e^{i\omega})] - \mathrm{tr}[\Lambda_{kj}(e^{i\omega})^* K_{jj}(e^{i\omega})] \Big\} d\omega$$

where the minimization in the left hand expression is done over $K_1, \ldots, K_J \in \mathbf{H}_2$. In the right hand expression, the maximization is done over $\Lambda = (\Lambda_{jk})$ with $\Lambda_{jk} \in \mathbf{H}_2$ and $\Lambda_{jk} = 0$ for $(j, k) \notin \mathcal{E}$, and for each j the minimization is done over $K_{jk} \in \mathbf{H}_2$ with $(j, k) \in \mathcal{E}$. At optimality $K_j = K_{j1} = \cdots = K_{jJ}$.

Proof. The result is an application of the Hahn-Banach theorem. □

This shows that also dynamic team decision problems can be solved by a distributed negotiation procedure. For one edge of the graph at a time, two neighbors (agent j and agent k) negotiate to update their price dynamics $\Lambda_{jk}(e^{i\omega})$, and decision functions $K_{jj}(e^{i\omega})$ and $K_{jk}(e^{i\omega})$. With all other variables fixed, they only solve the max-min problem involving these three transfer functions. According to standard linear quadratic theory, such max-min problems can be solved in terms of two Riccati equations. Again, it is not necessary to have global information about the stochastic processes anywhere. Only the local spectral densities are relevant. The negotiation is repeated over and over again for different edges in the graph, giving the global saddle-point in the limit.

VI. Acknowledgment

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