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Wave splitting and imbedding equations for a spherically symmetric dispersive medium

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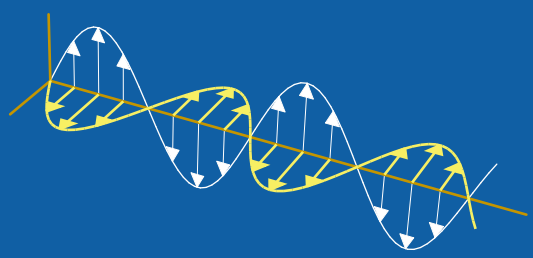

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Abstract

The direct problem of time dependent electromagnetic scattering in the dispersive sphere is solved by a wave splitting technique. The electric field is expanded in a series involving vector spherical harmonics, leading to a system of wave equations for each term. These systems are reduced to scalar wave equations for each term, which are solved via reflection operators. Some preliminary numerical results are presented.

1 Introduction

A dispersive material is characterized in the frequency domain by the frequency dependent permittivity $\epsilon(\omega)$, and in the time domain by the susceptibility kernel $g(t)$. The two functions are related by the Fourier transform

$$\frac{\epsilon(\omega) - \epsilon_0}{\epsilon_0} = \int_0^\infty g(t) e^{i\omega t}\, dt. \tag{1.1}$$

The fact that the lower integration limit is zero is a statement of the causal nature of g. Whichever form is used, the physical behavior of electromagnetic wave propagation through such material is easy to describe. Since the group and phase velocities are frequency dependent, a transient pulse, comprised of a band of frequencies, will change shape as it travels through the material.

The relation between ϵ and g also manifests itself in the relation between the electric field $\boldsymbol{E}$ and the displacement field $\boldsymbol{D}$, which for a homogeneous medium is

$$\boldsymbol{D}(\boldsymbol{r}, t) = \epsilon_0 \left(\boldsymbol{E}(\boldsymbol{r}, t) + \int_{-\infty}^{t} g(t - t') \boldsymbol{E}(\boldsymbol{r}, t')\, dt' \right). \tag{1.2}$$

The convolution integral in (1.2) is often written $g * \boldsymbol{E}$.

The problem considered in this paper is scattering of a transient wave from a homogeneous dispersive sphere in the time domain. A known incident electric field penetrates the sphere, whose susceptibility kernel $g(t)$ is known. It is desired to calculate the electric field outside the sphere at any given time. This problem is of interest in itself, but is also an important step in the solution of the more interesting inverse problem. There, the material parameters of the sphere are unknown, and it is desired to calculate them using measurements of the electric field scattered by the sphere. This inverse problem will be considered in future work.

A wave splitting technique is used to solve this problem. In this approach, the wave field is split into two parts moving in 'opposite' directions. In a spherically symmetric geometry the electric field is split into incoming and outgoing spherical waves. Wave splitting was developed originally for problems of one spatial dimension [4], [6], [3], and has been quite successful there. However, only recently has any progress been made in extending the technique to three dimensional problems, due to technical difficulties, including, for example, the determination of the form of the reflection operator. Also, in one spatial dimension, there are only two allowable directions, left and right, so the wave splitting geometry is predetermined (although

there are infinitely many possible splittings). However, in three dimensions, there is a continuum of directions in which electromagnetic waves can travel, so it is difficult to choose a splitting geometry that will effectively handle an arbitrary scattering problem. Much of the progress made has been done by considering symmetric media in various simple geometries – cylinders, spheres, and general stratified media [12], [13], [11], [10], [9]. This paper is an extension of the work by Karlsson and Kristensson [9] in which a particular wave splitting was developed for spherically symmetric geometries.

Other work has been done in the time domain on dispersive and dissipative media by wave splitting techniques. In [5] a dissipative model, in which the permittivity varies spatially but not with frequency, was used. Beezley and Krueger [1] first dealt with a frequency dependent permittivity; they solved the inverse problem for the one dimensional slab. Kreider [11], [10] extended this work to the stratified cylinder. Similarities to the slab problem are evident, and the differences due to the more complicated geometry are easily discerned.

An outline of the remainder of the paper follows. In Section 2, the mathematical formulation of the direct problem is presented, and the wave splitting is used to derive a set of equations to be solved. Section 3 contains discussion and derivations involving the so-called reflection operator, which relates the two split fields. The equations of Section 2 are modified here, and placed in a form suitable for numerical solution, which is then discussed in Section 4. An algorithm for solving the direct problem is presented. A discussion of certain numerical problems that appear is given here as well. The section concludes with several numerical examples showing the strengths and weaknesses of the algorithm. Section 5 contains a brief summary with a description of future work.

2 Problem formulation

A homogeneous, dispersive sphere of radius a rests in a homogeneous, nondispersive space. The electromagnetic propagation velocity c has the same constant value in the sphere and the surrounding space. The dispersive nature of the sphere is characterized in the time domain by the susceptibility kernel $g(t)$. The direct problem is to calculate the resulting electric field for $t > 0$ from an incident electric field $\boldsymbol{E}^i(\boldsymbol{r}, t)$ that strikes the sphere at time $t = 0$.

When the relation (1.2) is substituted into Maxwell's equations, the following wave equation governing the behavior of $\boldsymbol{E}$ results:

$$\nabla \times (\nabla \times \boldsymbol{E}) + \frac{1}{c^2}\partial_t^2(\boldsymbol{E} + g * \boldsymbol{E}) = 0. \tag{2.1}$$

Partial derivatives in t and r are denoted by ∂_t and ∂_r, respectively.

The scattering problem is now reduced to a one-dimensional problem by expanding the electric field in vector spherical harmonics, *i.e.*

$$\boldsymbol{E}(\boldsymbol{r}, t) = \sum_{\tau n} E_{\tau n}(r, t)\boldsymbol{A}_{\tau n}(\hat{r}, t), \tag{2.2}$$

where $r = |\boldsymbol{r}|$. The multi-index $n = (\ell, m, \sigma)$, with $\ell = 0, 1, 2, \ldots, \infty$, $m = 0, 1, \ldots, \ell$, and $\sigma =$ odd or even (corresponding to the odd or even parts of Y_n described below); $\tau = 1, 2, 3$, corresponding to the three vector spherical harmonic functions. The vector spherical harmonics are given by

$$\begin{aligned}
\lambda_\ell &= 1/\sqrt{\ell(\ell+1)} && (2.3)\\
\boldsymbol{A}_{1n}(\hat{r}) &= \lambda_\ell \nabla \times (\boldsymbol{r} Y_n(\theta,\phi)) && (2.4)\\
\boldsymbol{A}_{2n}(\hat{r}) &= \lambda_\ell r \nabla Y_n(\theta,\phi), && (2.5)\\
\boldsymbol{A}_{3n}(\hat{r}) &= \hat{r} Y_n(\theta,\phi), && (2.6)\\
Y_n(\theta,\phi) &= \xi_{\ell m} P_\ell^m(\cos\theta) \left\{ \begin{array}{c} \cos m\phi \\ \sin m\phi \end{array} \right\}, \quad \left\{ \begin{array}{c} \sigma = \text{even} \\ \sigma = \text{odd} \end{array} \right\}, && (2.7)\\
\xi_{\ell m} &= (2-\delta_{m,0}) \sqrt{\frac{(2\ell+1)}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}}.
\end{aligned}$$

where (r, θ, φ) are the spherical coordinates .

To obtain the equations for the $E_{\tau n}$, the series expansion (2.2) is inserted into the wave equation (2.1). After extensive simplification, they are:

$$\begin{aligned}
\frac{1}{\lambda_\ell^2 r^2} E_{1n} - \nabla^2 E_{1n} + \frac{1}{c^2} \partial_t^2 \left(E_{1n} + g * E_{1n} \right) &= 0, && (2.8)\\
\frac{1}{\lambda_\ell r} \partial_r E_{3n} - \nabla^2 E_{2n} + \frac{1}{c^2} \partial_t^2 \left(E_{2n} + g * E_{2n} \right) &= 0, && (2.9)\\
\frac{1}{\lambda_\ell^2 r^2} E_{3n} - \frac{1}{\lambda_\ell r^2} E_{2n} - \frac{1}{\lambda_\ell r} \partial_r E_{2n} + \frac{1}{c^2} \partial_t^2 \left(E_{3n} + g * E_{3n} \right) &= 0. && (2.10)
\end{aligned}$$

The divergence free condition $\nabla \cdot \boldsymbol{E} = 0$ holds because $g(t)$ has no spatial dependence. This condition implies

$$E_{2n} = \frac{\lambda_\ell}{r} \partial_r (r^2 E_{3n}), \tag{2.11}$$

which further implies that (2.9) and (2.10) are equivalent, so that the second wave equation can be replaced by (2.11).

By scaling the E_{1n} and E_{3n} fields by

$$E_{1n} = \frac{1}{r} V_{1n}, \qquad E_{3n} = \frac{1}{r^2} V_{3n}, \tag{2.12}$$

both (2.8) and (2.10) reduce to the same form

$$\frac{1}{\lambda_\ell^2 r^2} V_{\tau n} - \partial_r^2 V_{\tau n} + \frac{1}{c^2} \partial_t^2 \left(V_{\tau n} + g * V_{\tau n} \right) = 0, \quad \tau = 1, 3. \tag{2.13}$$

At this point, it is convenient to introduce dimensionless variables

$$q = r/a, \qquad s = ct/a. \tag{2.14}$$

Letting $v_\ell(q,s) = V_{\tau n}(r,t)$, (2.13) may be written as

$$\frac{1}{\lambda_\ell^2 q^2} v_\ell - \partial_q^2 v_\ell + \partial_s^2 (v_\ell + g * v_\ell) = 0, \tag{2.15}$$

with conditions

$$\begin{aligned} v_\ell(0,s) &= 0, \quad s > 0, \\ v_\ell(q,s) &= 0, \quad s \le 0,\ 0 \le q \le 1. \end{aligned}$$

The only index parameter that plays a role here is ℓ, due to the presence of λ_ℓ, so the other index parameters are suppressed until the appropriate place in Section 3.

The method used to solve (2.15) is called wave splitting. As mentioned earlier, there are many possible splittings that could be used; the one chosen for this problem was developed by Karlsson and Kristensson [9]. A summary of the relevant results of that paper is presented here.

Wave splitting. The main idea is to split the scaled electric field v_ℓ into two parts, corresponding to incoming and outgoing spherical waves in a homogeneous medium. To this end, Karlsson and Kristensson found operators Γ_ℓ^+ and Γ_ℓ^- such that

$$v_\ell(q,s) = \{\Gamma_\ell^+ f_\ell^+(q,\cdot)\}(s) + \{\Gamma_\ell^- f_\ell^-(q,\cdot)\}(s), \tag{2.16}$$

for suitable f_ℓ^+ and f_ℓ^-. In a homogeneous, dispersion-free medium, f_ℓ^+ and f_ℓ^- are outgoing and incoming spherical waves, respectively, and $\Gamma_\ell^+ f_\ell^+$ and $\Gamma_\ell^- f_\ell^-$ carry energy away from and towards the origin, respectively. The operators are initially derived for a homogeneous, dispersionless medium; the effects of dispersion and inhomogeneities appear later. The operators were found in [9] to have the following representation:

$$\begin{aligned} \{\Gamma_\ell^\pm(q) f(\cdot)\}(s) &= f(s) + \sum_{k=1}^{\ell} \frac{(\ell+k)!}{(\ell-k)!k!(k-1)!} (\pm 2q)^{-k} \times \\ &\quad \times \int_{-\infty}^{s} (s-s')^{k-1} f(s') ds'. \end{aligned} \tag{2.17}$$

An alternative representation which is often more useful is the following:

$$\{\Gamma_\ell^\pm(q) f(\cdot)\}(s) = \partial_s \int_{-\infty}^{s} P_\ell\left(1 \pm \frac{s-s'}{q}\right) f(s')\, ds', \tag{2.18}$$

where P_ℓ is the Legendre polynomial of order ℓ; note that $\Gamma_\ell^\pm$ do not depend on τ, m or σ. These operators have singular behavior at $q = 0$, which is a manifestation of the imposition of spherical coordinates on the problem. To partially alleviate this troublesome situation, Γ_ℓ^- can be replaced by a regular operator γ_ℓ given by

$$\begin{aligned} \{\gamma_\ell(q) f(\cdot)\}(s) &= \frac{1}{2}\left(\{\Gamma_\ell^-(q) f(\cdot)\}(s) - (-1)^\ell \{\Gamma_\ell^+(q) f(\cdot)\}(s-2q)\right) \\ &= \frac{1}{2}\partial_s \int_{s-2q}^{s} P_\ell\left(1 - \frac{s-s'}{q}\right) f(s')\, ds'. \end{aligned} \tag{2.19}$$

It is shown in [9] that γ_ℓ can replace Γ_ℓ^- in the sense that

$$v_\ell = \Gamma_\ell^+ f_\ell^+ + \gamma_\ell f_\ell^-, \tag{2.20}$$

and that, in homogeneous non-dispersive regions,

$$\partial_q v_\ell = (\partial_q \Gamma_\ell^+ - \partial_s \Gamma_\ell^+) f_\ell^+ + (\partial_q \gamma_\ell + \partial_s \gamma_\ell) f_\ell^-. \tag{2.21}$$

In matrix notation,

$$\begin{pmatrix} v_\ell \\ \partial_q v_\ell \end{pmatrix} = P \begin{pmatrix} f_\ell^+ \\ f_\ell^- \end{pmatrix}, \qquad P = \begin{pmatrix} \Gamma_\ell^+ & \gamma_\ell \\ \partial_q \Gamma_\ell^+ - \partial_s \Gamma_\ell^+ & \partial_q \gamma_\ell + \partial_s \gamma_\ell \end{pmatrix}, \tag{2.22}$$

$$\begin{pmatrix} f_\ell^+ \\ f_\ell^- \end{pmatrix} = P^{-1} \begin{pmatrix} v_\ell \\ \partial_q v_\ell \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} \gamma_\ell + \partial_s^{-1} \partial_q \gamma_\ell & -\partial_s^{-1} \\ \Gamma_\ell^+ - \partial_s^{-1} \partial_q \Gamma_\ell^+ & \partial_s^{-1} \Gamma_\ell^+ \end{pmatrix}. \tag{2.23}$$

These equations define the wave splitting – the scaled electric field v_ℓ is split by means of outgoing and incoming spherical waves f_ℓ^+ and f_ℓ^-. At this point, the splitting is now generalized to an inhomogeneous, dispersive medium. In those regions where g is not zero, f_ℓ^+ and f_ℓ^- are no longer pure outgoing and incoming spherical waves, but the splitting is still valid, in the sense that $v_\ell = \Gamma_\ell^+ f_\ell^+ + \gamma_\ell f_\ell^-$. Specifically, define

$$\begin{pmatrix} w_\ell^+ \\ w_\ell^- \end{pmatrix} = P^{-1} \begin{pmatrix} v_\ell \\ \partial_q v_\ell \end{pmatrix}, \qquad \begin{pmatrix} v_\ell^+ \\ v_\ell^- \end{pmatrix} = \begin{pmatrix} \Gamma_\ell^+ & 0 \\ 0 & \gamma_\ell \end{pmatrix} \begin{pmatrix} w_\ell^+ \\ w_\ell^- \end{pmatrix}, \tag{2.24}$$

where $v_\ell = v_\ell^+ + v_\ell^-$ satisfies (2.15) and in regions where c is constant, $w_\ell^\pm$ revert to outgoing and incoming spherical waves, so that $w_\ell^\pm(q,s) = w_\ell^\pm(s \mp q)$ there (making it preferable to solve the problem in terms of $w_\ell^\pm$ rather than $v_\ell^\pm$).

The immediate goal is to find equations that $w_\ell^\pm$ satisfy for a homogeneous, dispersive medium. By solving these equations, the v_ℓ can be determined, and the $E_{\tau n}$ and finally $\boldsymbol{E}$ itself can be constructed.

The wave equation (2.15) in matrix form is

$$\partial_q \begin{pmatrix} v_\ell \\ \partial_q v_\ell \end{pmatrix} = A \begin{pmatrix} v_\ell \\ \partial_q v_\ell \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ \frac{1}{\lambda_\ell^2 q^2} + \partial_s^2 (I + g*) & 0 \end{pmatrix}. \tag{2.25}$$

Letting $W = \begin{pmatrix} w_\ell^+ \\ w_\ell^- \end{pmatrix}$, this can be written

$$\partial_q (PW) = APW. \tag{2.26}$$

Some algebraic manipulation gives the desired form of the system of equations:

$$\begin{aligned} \partial_q \begin{pmatrix} w_\ell^+ \\ w_\ell^- \end{pmatrix} &= \begin{pmatrix} -\partial_s & 0 \\ 0 & \partial_s \end{pmatrix} \begin{pmatrix} w_\ell^+ \\ w_\ell^- \end{pmatrix} \\ &\quad + \begin{pmatrix} -\gamma_\ell \partial_s (g * \Gamma_\ell^+) & -\gamma_\ell \partial_s (g * \gamma_\ell) \\ \Gamma_\ell^+ \partial_s (g * \Gamma_\ell^+) & \Gamma_\ell^+ \partial_s (g * \gamma_\ell) \end{pmatrix} \begin{pmatrix} w_\ell^+ \\ w_\ell^- \end{pmatrix} \end{aligned} \tag{2.27}$$

under the conditions $w_\ell^\pm(q,0) = 0$, $q < 1$, $w_\ell^+(0,s) = 0$, $s \geq 0$.

3 Exterior field calculation

In this section, algorithms for the calculation of the near and far exterior fields are presented. The relation between w_ℓ^+ and w_ℓ^- is established in order to obtain an equation (from (2.27), and involving only the parameter ℓ) that characterizes the medium independently of the incident field. The equation is then solved for all ℓ values so that the series (2.2) can be computed, solving this direct problem. A numerical solution, of course, would involve only a finite set of ℓ values, yielding an approximate solution.

Outside the sphere, w_ℓ^- is an inward-moving spherical wave, so it is possible to set up an incident field such that its value at the boundary of the sphere is arbitrarily specified. Beginning at time $s = 0$, this incident field penetrates the sphere, whose dispersive nature gives rise to a reflected field represented by a sum of the w_ℓ^+. Karlsson and Kristensson [9] note that if the incident field is planar, then its split form contains only w_ℓ^- components. Hence, any w_ℓ^+ components present outside the sphere (outward-moving spherical waves) are due to reflection processes. This is convenient because, experimentally, one is usually interested in only the reflected field, and this set-up makes the determination of the reflected field easier.

Inside the sphere, w_ℓ^+ and w_ℓ^- lose their travelling wave interpretations. Here, the relation between the two is given by the reflection operator $\boldsymbol{R}_\ell$ [3]:

$$w_\ell^+(q,s) = \boldsymbol{R}_\ell w_\ell^-(q,s), \quad 0 \le s,\ 0 \le q < 1. \tag{3.1}$$

The exact form of $\boldsymbol{R}_\ell$ must be carefully derived. Because the definition of γ_ℓ (2.19) contains a time delay term $s - 2q$, the initial assumed form of $\boldsymbol{R}_\ell$ is

$$w_\ell^+(q,s) = \boldsymbol{R}_\ell w_\ell^-(q,s) = A_\ell(q) w_\ell^-(q, s-2q) + \int_0^s R_\ell(q, s-\sigma) w_\ell^-(q,\sigma)\, d\sigma. \tag{3.2}$$

The reflection kernel $R_\ell(q,s)$ is assumed to have jump discontinuities along $s = 0$, $s = 2q$ and $s = 4q$, due to the presence of γ_ℓ and γ_ℓ^2 in (2.27).

Substituting this expression into the system (2.27) determines A_ℓ and the values of the discontinuities, and leads to an equation of the form $\Upsilon_\ell * w_\ell^- = 0$. Since the input field w_ℓ^- is arbitrary, it is clear that $\Upsilon_\ell = 0$; this is commonly called the R-equation:

$$\begin{aligned} 0 = \Upsilon_\ell \;=\; & (\partial_q + 2\partial_s) R_\ell + g_0 R_\ell * R_\ell + B_\ell * R_\ell * R_\ell + g_0 R_\ell + 2C_\ell * R_\ell + E_\ell \\ & + g_0(2A_\ell - (-1)^\ell)\tilde{R}_\ell + 2A_\ell B_\ell * \tilde{R}_\ell + 2D_\ell * \tilde{R}_\ell + 2A_\ell \tilde{C}_\ell + \tilde{F}_\ell \\ & + 2A_\ell \tilde{\tilde{D}}_\ell + A_\ell^2 \tilde{\tilde{B}}_\ell + \tilde{\tilde{J}}_\ell, \end{aligned} \tag{3.3}$$

$$R_\ell(q,0) = -\frac{1}{8} g_0, \quad 0 < q < 1, \qquad\qquad R_\ell(0,s) = 0, \quad s \ge 0,$$

with $g_0 = g(0)$, $\tilde{R}(q,s) = R(q, s-2q), \tilde{\tilde{R}}(q,s) = R(q, s-4q)$ and

$$\begin{aligned}
A_\ell(q) &= (-1)^\ell \left(1 - e^{-g_0 q}\right)/2, \\
B_\ell &= g' + (2b_\ell + b_\ell * b_\ell) * g', \\
C_\ell &= g'/2 + (b_\ell - a_\ell - a_\ell * b_\ell) * g'/2, \\
D_\ell &= -(-1)^\ell g'/2 + (a_\ell + a_\ell * b_\ell - (-1)^\ell b_\ell) * g'/2, \\
E_\ell &= g'/4 + (a_\ell * a_\ell/4 - a_\ell/2) * g', \\
F_\ell &= -(-1)^\ell g'/2 + \left((1 + (-1)^\ell) a_\ell/2 - a_\ell * a_\ell/2\right) * g', \\
J_\ell &= g'/4 + (a_\ell * a_\ell/4 - (-1)^\ell a_\ell/2) * g', \\
a_\ell(q,s) &= \frac{1}{q} P'_\ell(1 - s/q), \\
b_\ell(q,s) &= \frac{1}{q} P'_\ell(1 + s/q).
\end{aligned}$$

The reflection kernel also has discontinuities along the lines $s = 2q$ and $s = 4q$. The first discontinuity is carried along the characteristic of the imbedding equation. The second arises from the time delayed terms that appear in the imbedding equation. These jump discontinuities are given by

$$\begin{aligned}
[R_\ell(q, 2q)] &= -\frac{1}{8} g_0 \exp(-g_0 q) + \frac{1}{2}(-1)^\ell \left\{g'(0) - (g_0/2)^2\right\} q \exp(-g_0 q), \\
[R_\ell(q, 4q)] &= \frac{1}{8} g_0 \exp(-2 g_0 q),
\end{aligned}$$

with $[R_\ell(q, f(q))] = R_\ell(q, f(q)^+) - R_\ell(q, f(q)^-)$.

It is generally acknowledged that physical arguments imply that $g(0) = 0$ [8, pp. 306-310] since $g(0) \neq 0$ corresponds to a polarisation-current that behaves like a Heaviside function at time $t = 0$. There are, however, situations (e.g. when the transient fields do not contain high frequencies) where approximate models of the medium having $g(0) \neq 0$ are relevant [2, p. 97]. Here, the more general condition $g(0) \neq 0$ is maintained, although the numerical example in Section 4 is based on a model in which $g(0) = 0$.

At this point the suppression of the subscripts of $w^\pm$ is lifted, so that m and σ may be dealt with explicitly.

The Near Field. Once (3.3) has been solved for R_ℓ (as discussed in Section 4) for the desired number of ℓ terms, the calculation of the near scattered field $\boldsymbol{E}^{\text{scat}}$ exterior to the sphere proceeds as follows:

1. Calculate w^-_{1n} and w^-_{3n} for the incident field. Specifically, for a planar delta input and $q \geq 1$ (as noted, $w^+_{\tau n} = 0$ for this field),

$$w^{-(\delta)}_{1n}(s+q-1) = -\alpha_\ell \delta_{m,1} \delta_{\sigma,\text{odd}} H(s+q-1) \tag{3.4}$$

$$w^{-(\delta)}_{3n}(s+q-1) = \beta^\ell \delta_{m,1} \delta_{\sigma,\text{even}} (s+q-1) H(s+q-1) \tag{3.5}$$

where

$$\alpha_\ell = (2c^2/a)(-1)^\ell \sqrt{(2\ell+1)\pi}, \tag{3.6}$$

$$\beta_\ell = 2ca\pi\ell(\ell+1)(-1)^\ell, \tag{3.7}$$

and $\delta_{x,y}$ is the Kronecker delta. For a general input $f(s+q-1)$, then,

$$w^-_{\tau n} = w^{-(\delta)}_{\tau n} * f. \tag{3.8}$$

The Kronecker deltas reduce the quadruple summation in (2.2) to a double summation in $\tau = 1, 2, 3$ and $\ell = 1, 2, \ldots$.

2. Calculate $w^+_{\tau n} = R_\ell * w^-_{\tau n}$ for $\tau = 1, 3$.
3. Calculate $v^+_{\tau n} = \Gamma^+_\ell w^+_{\tau n}$ for $\tau = 1, 3$.
4. Convert from normalized co-ordinates (q, s) to physical co-ordinates (r, t) via (2.14).
5. Calculate $E^{\text{scat}}_{1n} = v^+_{1n}/r$ and $E^{\text{scat}}_{3n} = v^+_{3n}/r^2$.
6. Compute $E^{\text{scat}}_{2n} = \lambda_\ell \partial_r(v^+_{3n})/r$.
7. Finally, $\boldsymbol{E}^{\text{scat}} = \sum_{\tau n} E^{\text{scat}}_{\tau n} \boldsymbol{A}_{\tau n}$.

The feasibility of practical computation involving these seven steps needs to be investigated, but in principle it provides a means of solving the scattering problem in physical co-ordinates by calculating the reflection kernel in normalized co-ordinates.

The Far Field. Calculating the far field $\boldsymbol{F}^{\text{scat}}(\hat{r}, t)$ is somewhat easier. Following Friedlander [7],

$$\boldsymbol{F}^{\text{scat}}(\hat{r}, t) = \lim_{r\to\infty} r\boldsymbol{E}^{\text{scat}}(r, t + \frac{r-a}{c_0}). \tag{3.9}$$

The form of the far scattered field $\boldsymbol{F}^{\text{scat}}$ in normalized co-ordinates is obtained by writing $\boldsymbol{E}^{\text{scat}}$ in terms of vector spherical harmonics, using (3.4-3.8) explicitly, and noting that as $q \to \infty$,

$$\begin{aligned} \Gamma^+_\ell w^+ &\to w^+, \\ \partial_q(\Gamma^+_\ell w^+) &\to \partial_q w^+. \end{aligned}$$

After simplification, this form is seen to be

$$\begin{aligned} \boldsymbol{F}^{\text{scat}}(\hat{q}, s) &= \bar{e}_\theta \sum_{\ell=1}^{\infty} \epsilon_\ell \omega_\ell(1, s) \cos\phi \left[\lambda_\ell \beta_\ell \sin\theta P^{1\prime}_\ell(\cos\theta) - \alpha_\ell \csc\theta P^1_\ell(\cos\theta)\right] \\ &\quad + \bar{e}_\phi \sum_{\ell=1}^{\infty} \epsilon_\ell \omega_\ell(1, s) \sin\phi \left[\lambda_\ell \beta_\ell \csc\theta P^1_\ell(\cos\theta) - \alpha_\ell \sin\theta P^{1\prime}_\ell(\cos\theta)\right], \end{aligned} \tag{3.10}$$

where

$$\epsilon_\ell = 2\lambda_\ell \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-1)!}{(\ell+1)!}}, \tag{3.11}$$

$$\omega_\ell(1, s) = A_\ell(1) H * f(s-2) + R_\ell * H * f(1, s). \tag{3.12}$$

These terms are easy to compute once R_ℓ is calculated and the input function f is specified, and the conversion to physical co-ordinates is achieved through (2.14).

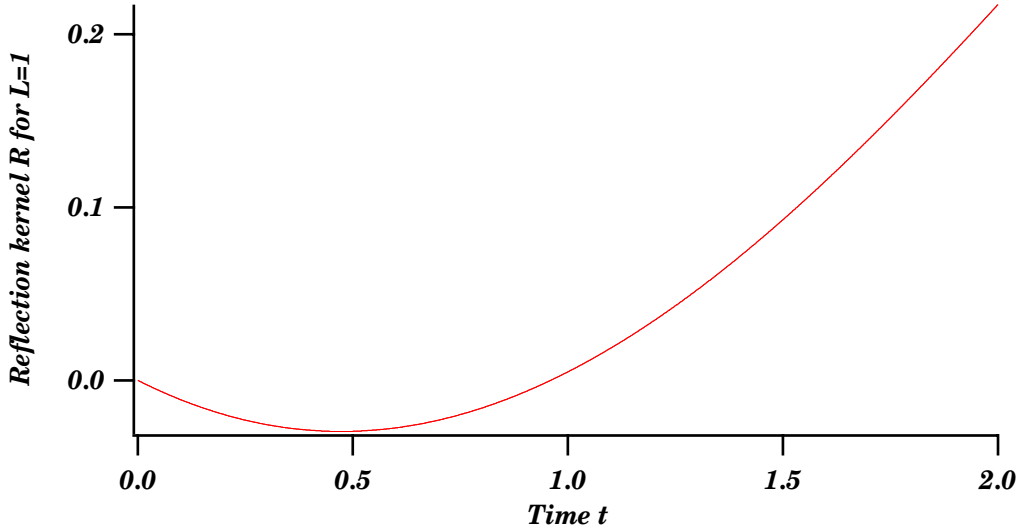

Figure 1: The reflection kernel $R_1(1,s)$ for $0 \le s \le 2$ using $N = 2048$ and $N = 4096$ spatial steps. The two graphs are indistinguishable.

4 Discretization and numerical example

This section describes the method used to solve (3.3) numerically. For the sake of clarity, two assumptions are made:

a) $g(0) = g_0 = 0$,

b) $s < 2q$ (only the first round trip is considered).

Under these conditions, (3.3) reduces to

$$0 \;=\; (\partial_q + 2\partial_s)R_\ell + B_\ell * R_\ell * R_\ell + 2C_\ell * R_\ell + E_\ell. \tag{4.1}$$

The expressions for B_ℓ, C_ℓ, and E_ℓ do not change, but now, $A_\ell(q) \equiv 0$ and the initial and boundary conditions become $R_\ell(q,0) = 0$, $0 < q < 1$, $R_\ell(0,0) = 0$. Note that the computational domain is the triangle $0 < s < 2q$ for $0 < q < 1$, so the boundary condition applies only at $s = 0$.

Equation (4.1) is discretized by finite differences. In the computational region $T = \{(q,s) : 0 < q \le 1, 0 < s < 2, s \le 2q\}$, a regular grid is established; fix N and let $h = 1/N$, and $\Delta q = h$, $\Delta s = 2h$, so that $q_i = ih$, $s_j = 2jh$ for $i = 1, 2, \ldots N$, $j = 0, 1, 2, \cdots, i$. The functions can now be discretized as

$$F_{ij} = F(q_i, s_j),$$

where F represents any of R_ℓ, B_ℓ, C_ℓ, E_ℓ, a_ℓ or b_ℓ. Convolutions are evaluated using the Trapezoid Rule

$$(f * g)_{ij} = \frac{2/N}{2}\left(f_{i0}g_{ij} + 2f_{i1}g_{i,j-1} + \cdots + 2f_{i,j-1}g_{i1} + f_{ij}g_{i0}\right).$$

The derivatives in (4.1) are approximated by forward and backward differences;

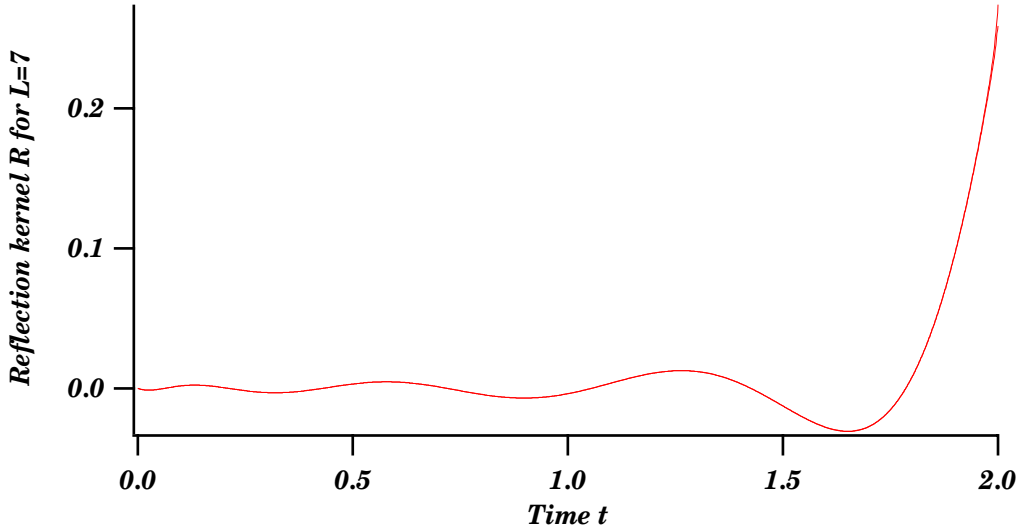

Figure 2: The reflection kernel $R_7(1,s)$ for $0 \le s \le 2$ using $N = 2048$ and $N = 4096$ spatial steps. The two graphs are indistinguishable. As L increases, R_L becomes more oscillatory; this is caused by oscillations in a_L and b_L, which appear in (4.2).

the subsequent equations are averaged to give the discrete equation

$$\begin{aligned} \frac{2}{h}\left(R_{i+1,j+1} - R_{ij}\right) &= -(B*R*R)_{i+1,j+1} - (B*R*R)_{ij} - 2(C*R)_{i+1,j+1} \\ &\quad - 2(C*R)_{ij} - E_{i+1,j+1} - E_{ij}. \end{aligned} \tag{4.2}$$

For clarity, the subscript ℓ has been suppressed. Equation (4.2) is solved by calculating along vertical strips, starting at $q_0 = 0$ and moving to the right, so that R_ℓ is calculated at (in order) (q_0, s_0), (q_1, s_0), (q_1, s_1), (q_2, s_0), (q_2, s_1), (q_2, s_2), (q_3, s_0), and so on. The desired output is the discrete approximation to $R_\ell(q_N, s)$ for various ℓ values. These functions are then used to calculate the near or far electric fields, as indicated in Section 3.

For small q, the coefficients B_ℓ, C_ℓ and E_ℓ are not well behaved. First, a_ℓ and b_ℓ are not defined for $q = 0$. However, the initial and boundary conditions on R_ℓ ($R_\ell(q,0) = 0$ and $R_\ell(0,s) = 0$) insure that (4.2) need not be solved at $q = 0$.

A more serious numerical problem occurs for small q values. The terms a_ℓ and b_ℓ oscillate considerably in this region, because they are Legendre polynomial derivatives. In a_ℓ, P'_ℓ is evaluated at argument in $[-1, 1]$, so although it oscillates, it remains fairly well-behaved. However, in b_ℓ, P'_ℓ is evaluated at argument in $[1, 3]$; b_ℓ varies considerably in this interval, even for small to moderate ℓ values. Consequently, the discrete equation does not produce R_ℓ well for small q. This inaccuracy is propagated along the characteristic and appears in $R_\ell(1,s)$ for $s \approx 2$. For small s, the algorithm does provide a good approximation to $R_\ell(1,s)$. This problem appears to be caused by the vector spherical harmonic representation of $\boldsymbol{E}$ rather than the consequence of some physical effect, so it should prove possible to resolve.

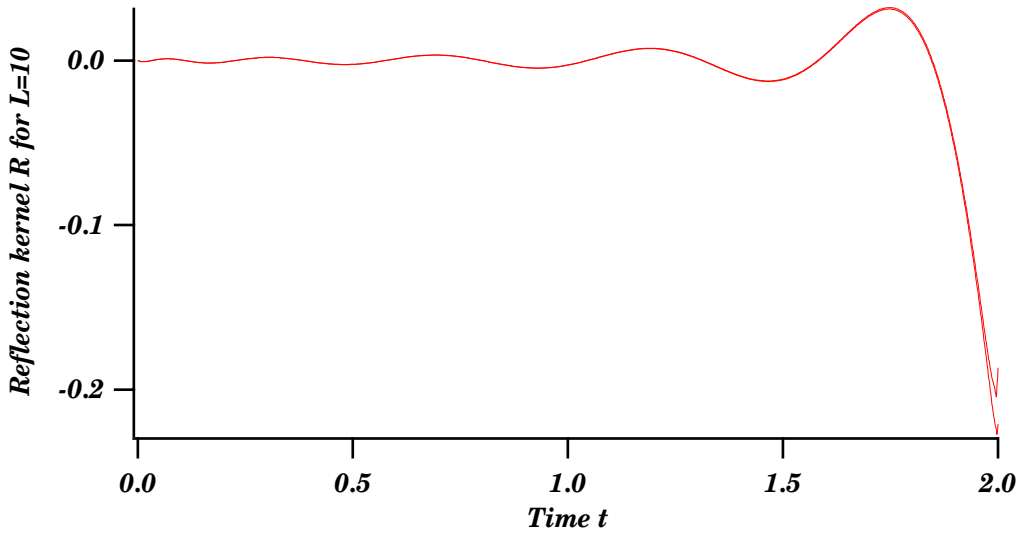

Figure 3: The reflection kernel $R_{10}(1,s)$ for $0 \le s \le 2$ using $N = 2048$ and $N = 4096$ spatial steps. Slight discretization error due to oscillations in a_L and b_L is evident.

One possible solution to this problem is to linearize the equation for small q (say, $q < q_L$). A simple asymptotic analysis provides the linearized form of (4.2):

$$\frac{2}{h}\left(R_{i+1,j+1} - R_{ij}\right) = -E_{i+1,j+1} - E_{ij}. \tag{4.3}$$

Note that the more troublesome term b_ℓ does not appear in this equation. Although detailed analysis is incomplete, numerical experimentation indicates that the solution $R_\ell(1,s)$ is fairly insensitive to the choice of q_L, except for s very close to 2.

The example below displays some preliminary numerical results.

Example. Consider the dispersive sphere of radius 1 with susceptibility kernel $g(s) = s\exp(.1s)$. A delta pulse penetrates the sphere along the incident angle $\theta = 0$.

Reflection kernels $R_1(1,s)$, $R_7(1,s)$ and $R_{10}(1,s)$ are shown in Figures 1, 2 and 3. In Figure 1, $R_1(1,s)$ is shown for $0 \le s \le 2$ with $N = 2048$ and $N = 4096$ spatial steps. The two graphs are indistinguishable; N is large enough that (4.2) is solved to convergence. In Figure 2, $R_7(1,s)$ is shown with $N = 2048$ and $N = 4096$. Again, the two graphs are indistinguishable. In fact, for $L = 1, \ldots, 9$, the graphs of $R_L(1,s)$ using $N = 2048$ and $N = 4096$ are indistinguishable. For $L = 10$, the oscillations in a_L and b_L cause a slight discretization error, as evident in Figure 3, where $R_{10}(1,s)$ is displayed for $N = 2048$ and $N = 4096$. Clearly, as L increases, R_L becomes more oscillatory, leading to larger discretization errors. This increasing oscillation will limit the number of terms that can be used to approximate the far scattered field. A study of the calculation of the far scattered field is currently underway.

5 Conclusion

The direct problem of time dependent electromagnetic scattering is solved by a wave splitting technique. A series expansion in vector spherical harmonics for the vector electric field is developed, leading to a system of wave equations for each term in the series. These systems are reduced to scalar wave equations, which are solved via reflection operators.

The numerical implementation of the solution technique is not yet complete; certain coefficients in the discrete equation are highly oscillatory, leading to inaccuracies in the solution. This limits the number of terms that can be used to approximate the electric field.

More detailed numerical studies are currently underway. Once the numerical problem mentioned above is resolved, and the direct problem for one round trip time completed, the direct problem will be considered with no restrictions on the time interval. At this point, it will be appropriate to study the corresponding inverse problem, to recover the unknown $g(t)$ from far field or near field measurements. Eventually, the method will be extended to spatially inhomogeneous dispersive spheres. A comparison with Weston's splitting techniques might help to illuminate some of the mathematical problems inherent in wave splitting in three dimensions.

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