# Constitutive relations, dissipation and reciprocity for the Maxwell equations in the time domain

Karlsson, Anders; Kristensson, Gerhard

1989

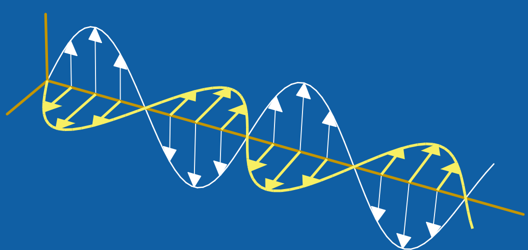
CODEN:LUTEDX/(TEAT-7005)/1-36/(1989)

Revision No. 2: August 1999

# Constitutive relations, dissipation and reciprocity for the Maxwell equations in the time domain

**Anders Karlsson and Gerhard Kristensson**

Department of Electroscience
Electromagnetic Theory
Lund Institute of Technology
Sweden



Anders Karlsson and Gerhard Kristensson

Department of Electroscience
Electromagnetic Theory
Lund Institute of Technology
P.O. Box 118
SE-221 00 Lund
Sweden

Editor: Gerhard Kristensson

**Abstract**

The main goal of this paper is to establish general constitutive relations for the electromagnetic fields $\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{B}$ and $\boldsymbol{H}$ in a time domain setting. The four basic assumptions of the medium are linearity, invariance to time translations, causality and continuity. These four assumptions imply that the constitutive relations are convolutions of Riemann-Stieltjes type. A review of the classification of media in bianisotropic, biisotropic, anisotropic and isotropic media, respectively, is made. Dissipation and reciprocity are defined and the constraints these concepts make on the constitutive relations are analyzed in detail. Furthermore, an appropriate form of time reversal and functions of positive type are introduced and some consequences of these concepts are showed.

# 1 Introduction

The Maxwell equations for the macroscopic electromagnetic fields are well known

$$\begin{cases} \nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B} \\ \nabla \times \boldsymbol{H} = \boldsymbol{J} + \partial_t \boldsymbol{D}. \end{cases} \tag{1.1}$$

These equations, however, are not complete. Six more equations, the constitutive relations, have to be added relating the electric field $\boldsymbol{E}$, the magnetic induction $\boldsymbol{B}$, the displacement field $\boldsymbol{D}$ and the magnetic field $\boldsymbol{H}$ to each other. These constitutive relations are completely independent of the Maxwell equations. The Maxwell equations involve only the fields and their sources. The constitutive relations, however, are concerned with the equations of motion of the constituents of the medium in an electromagnetic field [16]. Traditionally, these constitutive relations are described as a relation at fixed frequency. The intensified interest in transient phenomena, however, especially wave propagation properties in more complex media, motivates a fresh look at these problems from a different starting point.

The constitutive relations in its most general form are usually given as a relationship between the pairs of fields $\{\boldsymbol{D}, \boldsymbol{H}\}$ and $\{\boldsymbol{E}, \boldsymbol{B}\}$. Other combinations between different pairs of fields are also frequently used [25]. The constitutive relations used in this paper can formally be written as a general functional dependence

$$\begin{cases} \boldsymbol{D} = \boldsymbol{D}\left(\{\boldsymbol{E}, \boldsymbol{B}\}\right) \\ \boldsymbol{H} = \boldsymbol{H}\left(\{\boldsymbol{E}, \boldsymbol{B}\}\right). \end{cases} \tag{1.2}$$

If space is empty the vacuum relations between the fields hold, i.e.

$$\begin{cases} \boldsymbol{D} = \epsilon_0 \boldsymbol{E} \\ \boldsymbol{H} = \boldsymbol{B}/\mu_0, \end{cases} \tag{1.3}$$

where $\epsilon_0$ and $\mu_0$ are the vacuum permittivity and permeability, respectively. The difference between the non-vacuum relations and the vacuum ones reflects the presence of a medium.

The most frequently used constitutive relations in the literature deal with the case of no coupling between the electric and the magnetic fields. Electric polarization and magnetization are then two separate phenomena and the constitutive relations separate into two functionals; one relating the electric fields, $\boldsymbol{E}$ and $\boldsymbol{D}$, to each other and a separate one relating the magnetic fields, $\boldsymbol{B}$ and $\boldsymbol{H}$. There are, however, several classes of materials that do show magneto-electric behavior, and these are modelled by a coupling between the electric and magnetic fields in the constitutive relations. The general name for these constitutive relations having a coupling between the electric and the magnetic fields is bianisotropic media [25].

Magnetic crystals that lack the symmetry of spatial inversion and the symmetry of time reversal, i.e. all spin directions reversed, can show magneto-electric behavior. Examples of these materials are $Cr_2O_3$ and related oxides [10, 29, 30]. More examples, such as moving dielectric media, are found in O'Dell [31], who refers to these media as magneto-electric media. In O'Dell [31] (see also [1] and [24]) the questions related to the absence of a thermodynamical equilibrium in magneto-electric media are also addressed. Other types of media that show magneto-electric behavior are the chiral media. A newborn interest in these media is noted by the extensive new literature in the field, see e.g. [2, 3, 11, 12, 20, 21, 27].

Several constitutive relations have been suggested as models for the magneto-electric medium. An early suggestion is due to Born [5]

$$\left\{\begin{array}{l} \boldsymbol{D} = \epsilon\left\{\boldsymbol{E} + \eta\nabla\times\boldsymbol{E}\right\} \\ \boldsymbol{B} = \mu\boldsymbol{H}. \end{array}\right.$$

The magneto-electric effects are here modelled by the constant $\eta$. Since the magneto-electric effect usually is very small, this constant is small compared to other relevant quantities. Inserted into the Maxwell equations these constitutive relations imply

$$\eta\epsilon\mu\partial_t^2\nabla\times\boldsymbol{E} + \nabla\times(\nabla\times\boldsymbol{E}) + \epsilon\mu\partial_t^2\boldsymbol{E} = \boldsymbol{0}.$$

Later, Condon [7] used

$$\left\{\begin{array}{l} \boldsymbol{D} = \epsilon\boldsymbol{E} - \eta\partial_t\boldsymbol{H} \\ \boldsymbol{B} = \mu\boldsymbol{H} + \eta\partial_t\boldsymbol{E}, \end{array}\right.$$

where again the magneto-electric effects are modelled by the constant $\eta$. These constitutive relations lead to

$$\eta^2\partial_t^4\boldsymbol{E} + 2\eta\partial_t^2\nabla\times\boldsymbol{E} + \nabla\times(\nabla\times\boldsymbol{E}) + \epsilon\mu\partial_t^2\boldsymbol{E} = \boldsymbol{0}.$$

A third example of constitutive relations for magneto-electric media are those due to Fedorov [4, 13, 14][1]

$$\left\{\begin{array}{l} \boldsymbol{D} = \epsilon\left\{\boldsymbol{E} + \eta\nabla\times\boldsymbol{E}\right\} \\ \boldsymbol{B} = \mu\left\{\boldsymbol{H} + \eta\nabla\times\boldsymbol{H}\right\}. \end{array}\right.$$

[1] These constitutive relations are also used in the frequency domain. $\eta$ is then a function of frequency [28, p. 362].

The corresponding partial differential equation is then

$$\eta^2\epsilon\mu\partial_t^2\nabla\times(\nabla\times\boldsymbol{E})+2\eta\epsilon\mu\partial_t^2\nabla\times\boldsymbol{E}+\nabla\times(\nabla\times\boldsymbol{E})+\epsilon\mu\partial_t^2\boldsymbol{E}=\boldsymbol{0}.$$

These three examples of models all lead to partial differential equations where the coefficient multiplying the principal part of the equations is small in some sense. This leads to drastic changes in the wave propagation properties as this constant varies.

The basic problem in this paper is to investigate the form of the functionals in (1.2) under certain assumptions that are physically sound. The analysis will be given in a time dependent formulation and the results are of importance in the treatment of the transient behavior of the fields on a macroscopic scale. The underlying microscopic theory is not addressed in this paper.

In elasticity the time domain formulation of the corresponding constitutive relations has been investigated by several authors. Some of the results in the elastic case are given in [8, 9, 17, 18]. In the electromagnetic community, however, much less attention has been given to the time domain formulation of the constitutive relations. The constitutive relations in the electromagnetic case are usually stated as relations between the appropriate fields for a fixed frequency [25]. A Fourier transformation then transforms the fixed frequency result to the time domain. However, in the analysis of the transient behavior of the fields, especially the short time behavior near a wave front, the investigation of the problem as a time domain problem is more appropriate. Causality and time invariance are naturally built into the formulation, whereas in the fixed frequency formulation, these properties have to be added to the constitutive relations at a later stage.

Some of the mathematical notations used in this paper is introduced in Section 2. The general form of the constitutive relations used in this paper is defined in Section 3. The definitions of the concepts of dissipation, mirror process and reciprocity are introduced in Sections 4, 5 and 6, respectively. These sections also contain some consequences on the constitutive relations as functions of time. The main ideas in this paper are presented in these sections. The mathematical details in the derivations of the results have been collected in a series of appendices.

# 2 Mathematical notations

Denote by $V$ a region (bounded and open) in space, bounded by the closed surface $S$ with outward pointing normal $\hat{n}$. The region $V$ is assumed to be filled with a medium, characterized by some non-trivial constitutive relations. Outside $V$ the vacuum relations in (1.3) are assumed to hold.

All fields in this paper are real-valued functions of the spatial variable $\boldsymbol{r}$ and the time $t$. The dependence of the fields on the spatial variable is usually suppressed throughout the paper since it is not of primary importance for the analysis carried out below. For this reason only the time dependence of the equations will be displayed. In general, no explicit regularity constraints are imposed on the fields as functions of the spatial variables except very mild conditions on differentiability[2].

**Definition 2.1.** *A function $f(t)$, defined on $[a,b]$, is in $C^n$ on $[a,b]$, denoted $C^n[a,b]$, if $f(t)$ is continuous on $[a,b]$ and $n$ times continuously differentiable on $[a,b]$.*

**Definition 2.2.** *A function $f(t)$, defined on $(-\infty,\infty)$, is in $H^n$ if $f(t) = 0$ on $(-\infty,0)$, and $f(t)$ is in $C^n$ on $[0,\infty)$.*

**Definition 2.3.** *The class $\mathcal{C}^n$ is defined as all vector-valued functions that belong to $H^n \cap C^n$ on $(-\infty,\infty)$, that is, all vector-valued functions with $n$ times continuously differentiable components on $(-\infty,\infty)$ as a function of the time variable that are identically zero on $(-\infty,0)$.*

The Riemann convolution of two functions $\varphi$ and $\psi$ defined on $[0,\infty)$ and $(-\infty,\infty)$, respectively, is defined as

$$\vartheta(t) = \int_{-\infty}^{t} \varphi(t-t')\psi(t')\,dt',$$

provided $\vartheta$ exists for all $t \in (-\infty,\infty)$. The function $\vartheta$ so defined on $(-\infty,\infty)$ is the Riemann convolution of $\varphi$ and $\psi$ and it is denoted

$$\vartheta(t) = (\varphi * \psi)(t).$$

The Stieltjes convolution of two functions $\varphi$ and $\psi$ defined on the intervals $[0,\infty)$ and $(-\infty,\infty)$, respectively, is defined as the Riemann-Stieltjes integral

$$\vartheta(t) = \int_{-\infty}^{t} \varphi(t-t')d\psi(t'),$$

provided $\vartheta$ exists for all $t \in (-\infty,\infty)$. The function $\vartheta$ so defined on $(-\infty,\infty)$ is the Stieltjes convolution of $\varphi$ and $\psi$ and it is denoted

$$\vartheta(t) = (\varphi \odot d\psi)(t).$$

If $\varphi \in H^0$ and $\psi \in H^1$, then

$$\vartheta(t) = \psi(0^+)\varphi(t) + (\varphi * \partial_t\psi)(t), \quad t \geq 0,$$

and $\vartheta \in H^0$.

# 3 Constitutive relations

## 3.1 General form

The constitutive relations relate the electric displacement field $\boldsymbol{D}$ and the magnetic field $\boldsymbol{H}$ with the electric field $\boldsymbol{E}$ and the magnetic induction $\boldsymbol{B}$. In this paper the formal constitutive relations is a transformation

$$\begin{Bmatrix} \boldsymbol{D} \\ \boldsymbol{H} \end{Bmatrix} = L \begin{Bmatrix} \boldsymbol{E} \\ \boldsymbol{B} \end{Bmatrix}.$$

[2]See also the comments in the paragraph below Definition 3.1, and footnote 6.

The transformation $L$ associates with each pair of fields $\{\boldsymbol{E}, \boldsymbol{B}\}$ a pair of fields $\{\boldsymbol{D}, \boldsymbol{H}\}$. For physical reasons the transformation $L$ is limited to be a *linear dispersive law* defined in the following definition.

**Definition 3.1.** *A transformation $L$ is said to be a linear dispersive law if it to every pair $\{\boldsymbol{E}, \boldsymbol{B}\}$ that belongs to the class $\mathcal{C}^0$ associates a pair of fields $\{\boldsymbol{D}, \boldsymbol{H}\}$ given by*

$$\begin{Bmatrix} \boldsymbol{D} \\ \boldsymbol{H} \end{Bmatrix} = L \begin{Bmatrix} \boldsymbol{E} \\ \boldsymbol{B} \end{Bmatrix},$$

*and that satisfies the conditions 1–4 below. Here $\{\boldsymbol{D}, \boldsymbol{H}\}$ and $\{\boldsymbol{D}', \boldsymbol{H}'\}$ are defined by*

$$\begin{Bmatrix} \boldsymbol{D} \\ \boldsymbol{H} \end{Bmatrix} = L \begin{Bmatrix} \boldsymbol{E} \\ \boldsymbol{B} \end{Bmatrix}, \quad \begin{Bmatrix} \boldsymbol{D}' \\ \boldsymbol{H}' \end{Bmatrix} = L \begin{Bmatrix} \boldsymbol{E}' \\ \boldsymbol{B}' \end{Bmatrix},$$

*where $\{\boldsymbol{E}, \boldsymbol{B}\}$ and $\{\boldsymbol{E}', \boldsymbol{B}'\}$ both belong to the class $\mathcal{C}^0$.*

1. *The transformation is linear, i.e. for every pair of real numbers $\alpha$, $\beta$*
   $L\left[\alpha \begin{Bmatrix} \boldsymbol{E} \\ \boldsymbol{B} \end{Bmatrix} + \beta \begin{Bmatrix} \boldsymbol{E}' \\ \boldsymbol{B}' \end{Bmatrix}\right] = \alpha L \begin{Bmatrix} \boldsymbol{E} \\ \boldsymbol{B} \end{Bmatrix} + \beta L \begin{Bmatrix} \boldsymbol{E}' \\ \boldsymbol{B}' \end{Bmatrix}$.
2. *The transformation is invariant to time translations, i.e. for every fixed time[3] $\tau > 0$ the relation $\begin{Bmatrix} \boldsymbol{E}'(t) \\ \boldsymbol{B}'(t) \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{E}(t-\tau) \\ \boldsymbol{B}(t-\tau) \end{Bmatrix}$ for all $t \in (-\infty, \infty)$ implies*
   $\begin{Bmatrix} \boldsymbol{D}'(t) \\ \boldsymbol{H}'(t) \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{D}(t-\tau) \\ \boldsymbol{H}(t-\tau) \end{Bmatrix}$ *for all $t \in (-\infty, \infty)$.*
3. *The transformation satisfies causality, i.e. for every fixed $t$ such that $\begin{Bmatrix} \boldsymbol{E} \\ \boldsymbol{B} \end{Bmatrix} = 0$ on $(-\infty, t]$ implies $\begin{Bmatrix} \boldsymbol{D} \\ \boldsymbol{H} \end{Bmatrix} = 0$ on $(-\infty, t]$.*
4. *The transformation is continuous, i.e. for every fixed $\tau$ and every $\epsilon > 0$ there exists a $\delta(\epsilon, \tau) > 0$ such that* $\max\{|\boldsymbol{E}(t)|, |\boldsymbol{B}(t)|\} < \delta(\epsilon, \tau)$ *for all $t \in (-\infty, \tau]$ implies* $\max\{|\boldsymbol{D}(\tau)|, |\boldsymbol{H}(\tau)|\} < \epsilon$.

The fields $\boldsymbol{E}$ and $\boldsymbol{B}$ in this definition are assumed to be related by the induction law $\boldsymbol{B}(t) = -\int_{-\infty}^{t} \nabla \times \boldsymbol{E}(t')\, dt'$. The fields are therefore not completely independent of each other. This assumption implies certain regularity restrictions on the fields as functions of the spatial variables, e.g. $\boldsymbol{E}$ is continuously differentiable as a function of $\boldsymbol{r}$ in $V$. Furthermore, it is assumed that all electric fields in class $\mathcal{C}^0$ can be generated in the medium. Although, the electric field $\boldsymbol{E}$ and the magnetic induction $\boldsymbol{B}$ are connected by the relation above, it is assumed that each component of the electric

[3] $\tau < 0$ is not of interest here since then $\{\boldsymbol{E}'(t), \boldsymbol{B}'(t)\} \notin \mathcal{C}^0$.

field and the magnetic induction can be generated independently at a specific fixed position $\boldsymbol{r}$. This assumption implies that, e.g the electric field $\boldsymbol{E}$ can be chosen arbitrarily at $\boldsymbol{r}$, while the magnetic induction $\boldsymbol{B}$ is zero there. This can, of course, just be true at that specific location. At other points in the medium both $\boldsymbol{E}$ and $\boldsymbol{B}$ fields exist. Further considerations on this matter are found in Appendix I.

The postulate of invariance to time translations implies that effects due to aging of the medium are excluded. The last postulate demands, in view of linearity, that two pairs of $\{\boldsymbol{E}, \boldsymbol{B}\}$, which are approximately close to each other in the interval $(-\infty, \tau]$, give two pairs of $\{\boldsymbol{D}, \boldsymbol{H}\}$, which are approximately close at time $\tau$. This continuity requirement can be replaced by a stronger one as shown by the following theorem.

**Theorem 3.1.** *Let $L$ be linear dispersive law. Then, for every fixed $\tau$ and every $\epsilon > 0$ there exists a $\delta(\epsilon, \tau) > 0$ such that $\max\{|\boldsymbol{E}(t)|, |\boldsymbol{B}(t)|\} < \delta(\epsilon, \tau)$ for all $t \in (-\infty, \tau]$ implies that $\max\{|\boldsymbol{D}(t)|, |\boldsymbol{H}(t)|\} < \epsilon$ for all $t \in (-\infty, \tau]$.*

**Theorem 3.2.** *Let $L$ be a linear dispersive law. Then $L$ is a map from $\mathcal{C}^0$ to $\mathcal{C}^0$.*

The proofs of these two theorems are presented in Appendix B.

The general constitutive relationship considered in this paper between the two pairs of fields $\{\boldsymbol{E}, \boldsymbol{B}\}$ and $\{\boldsymbol{D}, \boldsymbol{H}\}$, respectively, is assumed to satisfy the basic postulates made in Definition 3.1. The first three postulates are general and should hold for all non-aging, linear media that respond causally to electromagnetic disturbances. The fourth one, however, is more specific to the medium under consideration and reflects the equation of motion of the constituents of the medium. This last postulate is not the most general one that can be formulated, but general enough to accommodate most electromagnetic phenomena observed in material media[4].

The constitutive relations as expressed in the linear dispersive law can be represented as a Riemann-Stieltjes integral. The following theorem is proved in Appendix C.

**Theorem 3.3.** *To every linear dispersive law there exist four uniquely defined tensor-valued functions[5] $\mathcal{G}_{ij}(t)$, $\mathcal{F}_{ij}(t)$, $\mathcal{K}_{ij}(t)$ and $\mathcal{L}_{ij}(t)$ of time $t$ defined on $(-\infty, \infty)$ with the following properties[6]:*

1. *$\mathcal{G}_{ij}(t) = 0$, $\mathcal{F}_{ij}(t) = 0$, $\mathcal{K}_{ij}(t) = 0$ and $\mathcal{L}_{ij}(t) = 0$ on $(-\infty, 0)$.*
2. *$\mathcal{G}_{ij}(t)$, $\mathcal{F}_{ij}(t)$, $\mathcal{K}_{ij}(t)$ and $\mathcal{L}_{ij}(t)$ are of bounded variation on every closed subinterval of $(-\infty, \infty)$.*
3. *$\mathcal{G}_{ij}(t)$, $\mathcal{F}_{ij}(t)$, $\mathcal{K}_{ij}(t)$ and $\mathcal{L}_{ij}(t)$ are continuous on the right on $(-\infty, \infty)$, i.e. $\mathcal{G}_{ij}(t) = \mathcal{G}_{ij}(t+)$ and similarly for the other tensors.*

[4]More general constitutive relations, such as instantaneous response to the time derivatives of the field $\boldsymbol{E}$, or $\boldsymbol{B}$, can be obtained by considering a sequence of linear dispersive laws as defined in Definition 3.1.

[5]The convention of summation over repeated indices is used in this paper, and the Latin letters run through the integers $1, 2, 3$.

[6]The regularity of these tensors as functions of the spatial variables is assumed to be, e.g., continuously differentiable.

*4. for every pair of fields $\{\boldsymbol{E},\boldsymbol{B}\}$ and $\{\boldsymbol{D},\boldsymbol{H}\}$ associated through $L$*

$$\begin{cases} D_i(t) = (E_j \odot d\mathcal{G}_{ij})(t) + (B_j \odot d\mathcal{K}_{ij})(t) \\ H_i(t) = (E_j \odot d\mathcal{L}_{ij})(t) + (B_j \odot d\mathcal{F}_{ij})(t). \end{cases} \tag{3.1}$$

*5. $\mathcal{G}_{ij}(t)$ and $\mathcal{F}_{ij}(t)$ are tensors of second rank and $\mathcal{K}_{ij}(t)$ and $\mathcal{L}_{ij}(t)$ are pseudotensors of second rank.*

*Conversely, every set of tensor-valued functions defined on $(-\infty,\infty)$, satisfying the properties 1–5 above, generates in the sense of (3.1), a linear dispersive law.*

## 3.2 Special forms and classification

Restrictions are now introduced on the tensor-valued functions $\mathcal{G}_{ij}(t)$, $\mathcal{F}_{ij}(t)$, $\mathcal{K}_{ij}(t)$ and $\mathcal{L}_{ij}(t)$. It is assumed that these functions, as functions of $t$, belong to $H^2$ on $(-\infty,\infty)$. The constitutive relations in (3.1) can then be written as

$$\begin{cases} D_i(t) = a_{ij}E_j(t) + b_{ij}B_j(t) + (G_{ij} * E_j)(t) + (K_{ij} * B_j)(t) \\ H_i(t) = c_{ij}E_j(t) + d_{ij}B_j(t) + (L_{ij} * E_j)(t) + (F_{ij} * B_j)(t), \end{cases} \tag{3.2}$$

or

$$\begin{cases} D_i(t) = \partial_t(\mathcal{G}_{ij} * E_j)(t) + \partial_t(\mathcal{K}_{ij} * B_j)(t) \\ H_i(t) = \partial_t(\mathcal{L}_{ij} * E_j)(t) + \partial_t(\mathcal{F}_{ij} * B_j)(t). \end{cases}$$

In (3.2) the tensor-valued functions $G_{ij}(t)$, $F_{ij}(t)$, $K_{ij}(t)$ and $L_{ij}(t)$ are the time derivatives of $\mathcal{G}_{ij}(t)$, $\mathcal{F}_{ij}(t)$, $\mathcal{K}_{ij}(t)$ and $\mathcal{L}_{ij}(t)$ for $t>0$, respectively, and the coefficients $a_{ij}$, $b_{ij}$, $c_{ij}$ and $d_{ij}$ are the values of $\mathcal{G}_{ij}(t)$, $\mathcal{F}_{ij}(t)$, $\mathcal{K}_{ij}(t)$ and $\mathcal{L}_{ij}(t)$ at $t=0^+$, respectively, i.e. $G_{ij}(t) = \partial_t\mathcal{G}_{ij}(t)$ and $a_{ij} = \mathcal{G}_{ij}(0^+)$ and so on. $G_{ij}(t)$, $F_{ij}(t)$, $a_{ij}$ and $d_{ij}$ are tensors of second rank and $K_{ij}(t)$, $L_{ij}(t)$, $b_{ij}$ and $c_{ij}$ are pseudotensors of second rank.

As functions of the spatial variables the tensors and coefficients above are assumed to be smooth, but generalizations to the case of discontinuities can be made. It is physically motivated, see Section 4, that the diagonal elements $a_{ii}$ and $d_{ii}$ are non-negative everywhere. It is also convenient to restrict the fields $\{\boldsymbol{E},\boldsymbol{B}\}$ to be in class $\mathcal{C}^1$, i.e. to have continuously differentiable components as a function of time $t$.

Media modelled by these constitutive relations will be referred to as *bianisotropic* media and the tensor-valued functions $G_{ij}(t)$, $F_{ij}(t)$, $K_{ij}(t)$ and $L_{ij}(t)$ are called the *generalized susceptibility kernels* and they belong to $H^1$ as functions of time. The tensors $a_{ij}$, $b_{ij}$, $c_{ij}$ and $d_{ij}$ are referred to as the *generalized susceptibility tensors*. Media referred to as chiral or magneto-electric ones are special cases of this more general concept. The notation bianisotropic is general and reflects the macroscopic symmetries of the medium and is therefore used throughout this paper.

The constitutive relations in (3.2) describe the equation of motion of the constituents of the medium in the electromagnetic field. These equations give the additional information to solve the Maxwell equations in (1.1). The terms characterized by the four generalized susceptibility tensors $a_{ij}$, $b_{ij}$, $c_{ij}$ and $d_{ij}$ represent the instantaneous reaction to the fields $\{\boldsymbol{E}, \boldsymbol{B}\}$. The convolution parts of the constitutive relations, represented by the kernels $G_{ij}(t)$, $F_{ij}(t)$, $K_{ij}(t)$ and $L_{ij}(t)$, however, model the dispersion in the medium, i.e. the fields $\{\boldsymbol{D}, \boldsymbol{H}\}$ depend on the earlier history of the fields $\{\boldsymbol{E}, \boldsymbol{B}\}$. In this model the convolution parts also represent dissipation in the medium. The effects of dissipation is further developed in Section 4.

Several authors have suggested constitutive relations in the time domain similar to (3.2). Toupin and Revlin [34] assume the following constitutive relations

$$\begin{cases} D_i(t) = \sum_{\nu=0}^{p} \left( a_{ij}^{\nu} \partial_t^{\nu} E_j(t) + b_{ij}^{\nu} \partial_t^{\nu} B_j(t) \right) + (G_{ij} * E_j)(t) + (K_{ij} * B_j)(t) \\ H_i(t) = \sum_{\nu=0}^{p} \left( c_{ij}^{\nu} \partial_t^{\nu} E_j(t) + d_{ij}^{\nu} \partial_t^{\nu} B_j(t) \right) + (L_{ij} * E_j)(t) + (F_{ij} * B_j)(t), \end{cases}$$

which is more general than the one assumed in this paper (see also [35] and [36]). Media modelled by these constitutive relations are called hemihedral anisotropic media by the authors.

The medium is referred to as a *biisotropic* medium if the the tensors $a_{ij}$, $b_{ij}$, $c_{ij}$, $d_{ij}$, $G_{ij}(t)$, $F_{ij}(t)$, $K_{ij}(t)$ and $L_{ij}(t)$ in (3.2) are isotropic tensors of second rank, i.e. they are proportional to $\delta_{ij}$[7]. That is, the constitutive relations for a biisotropic medium are

$$\begin{cases} \boldsymbol{D}(t) = a\boldsymbol{E}(t) + b\boldsymbol{B}(t) + (G * \boldsymbol{E})(t) + (K * \boldsymbol{B})(t) \\ \boldsymbol{H}(t) = c\boldsymbol{E}(t) + d\boldsymbol{B}(t) + (L * \boldsymbol{E})(t) + (F * \boldsymbol{B})(t). \end{cases} \tag{3.3}$$

The constants $a$ and $d$ and the functions $G(t)$ and $F(t)$, are scalars, while the constants $b$ and $c$ and the functions $K(t)$ and $L(t)$, are pseudoscalars, respectively.

An *anisotropic* medium and an *isotropic* medium are similarly defined as media satisfying

$$\begin{cases} D_i(t) = a_{ij} E_j(t) + (G_{ij} * E_j)(t) \\ H_i(t) = d_{ij} B_j(t) + (F_{ij} * B_j)(t), \end{cases} \tag{3.4}$$

and

$$\begin{cases} \boldsymbol{D}(t) = a\boldsymbol{E}(t) + (G * \boldsymbol{E})(t) \\ \boldsymbol{H}(t) = d\boldsymbol{B}(t) + (F * \boldsymbol{B})(t), \end{cases} \tag{3.5}$$

respectively.

[7] An isotropic tensor is a tensor having the same set of components in all rotated Cartesian coordinate systems.

The Maxwell equations are in general

$$\begin{cases} \nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B} \\ \nabla \times \boldsymbol{H} = \boldsymbol{J} + \partial_t \boldsymbol{D}. \end{cases}$$

The current density $\boldsymbol{J}$ is assumed to consist of two parts

$$\boldsymbol{J} = \boldsymbol{J}_f + \boldsymbol{J}_{ind}.$$

The first part, $\boldsymbol{J}_f$, referred to as the forced or impressed part, is assumed to have its support outside the medium of interest, i.e. outside $V$. These currents excite the medium and could be sources of electric or magnetic type as well as sources of non-electric or non-magnetic origin. The other part, $\boldsymbol{J}_{ind}$, is the induced part of the current density, and is governed by the equation of motion of the constituents of the medium. It can therefore be assumed that these currents are included in the constitutive relations. This assumption is no loss of generality since a new displacement field, $\boldsymbol{D}'$, satisfying

$$\boldsymbol{D}'(t) = \int_{-\infty}^{t} \boldsymbol{J}_{ind}(t')\, dt' + \boldsymbol{D}(t),$$

can be defined and where this new displacement field satisfies the constitutive relations in (3.2). The current $\boldsymbol{J}$ is, therefore, assumed to be absorbed in the displacement current $\partial_t \boldsymbol{D}$ and the Maxwell equation inside the medium is

$$\begin{cases} \nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B} \\ \nabla \times \boldsymbol{H} = \partial_t \boldsymbol{D}. \end{cases} \tag{3.6}$$

# 4 Dissipation

In the presence of dispersion the electromagnetic energy cannot be defined as a thermodynamic quantity [28, p. 272]. The dispersion implies that energy can be absorbed in the medium. The effects of absorption are due to the presence of the generalized susceptibility kernels $G_{ij}(t)$, $F_{ij}(t)$, $K_{ij}(t)$ and $L_{ij}(t)$. The generalized susceptibility tensors $a_{ij}$, $b_{ij}$, $c_{ij}$ and $d_{ij}$ give the instantaneous contribution to the $\boldsymbol{D}$ and $\boldsymbol{H}$ fields and these tensors are related to the energy density of the electromagnetic field and not to the absorption of energy in the medium.

The analysis of dissipation in this section is based upon a macroscopic description of the fields, i.e. the Maxwell equations and the constitutive relations, and the causes of the dissipation on the microscopic level are out of the scope of this paper. The dissipation is therefore modelled by the specific form of the constitutive relations.

The fields are assumed to satisfy the Maxwell equations without any source term, see (3.6), (the induced part of the current density is absorbed in $\partial_t \boldsymbol{D}$), i.e.

$$\begin{cases} \nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B} \\ \nabla \times \boldsymbol{H} = \partial_t \boldsymbol{D}, \end{cases}$$

and the constitutive relations in (3.2), i.e.

$$\begin{cases} D_i(t) = a_{ij}E_j(t) + b_{ij}B_j(t) + (G_{ij} * E_j)(t) + (K_{ij} * B_j)(t) \\ H_i(t) = c_{ij}E_j(t) + d_{ij}B_j(t) + (L_{ij} * E_j)(t) + (F_{ij} * B_j)(t). \end{cases}$$

The Maxwell equations imply the Poynting theorem

$$\nabla \cdot \boldsymbol{S} + \boldsymbol{E} \cdot \partial_t \boldsymbol{D} + \boldsymbol{H} \cdot \partial_t \boldsymbol{B} = 0,$$

where

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}.$$

For fields in $\mathcal{C}^1$ integrate the Poynting theorem over an arbitrary volume $V_{\boldsymbol{r}}$ centered around $\boldsymbol{r}$ and bounded by the surface $S_{\boldsymbol{r}}$ (outward directed normal $\hat{n}$) and use the constitutive relations (3.2) and integrate over time. The result is

$$\mathcal{E}(\tau) = \iiint_{V_{\boldsymbol{r}}} \{w_{em}(\tau) + w_d(\tau)\}\, dv,$$

where $dv$ is the volume measure and

$$\begin{aligned} \mathcal{E}(\tau) &= -\iint_{S_{\boldsymbol{r}}} \int_0^\tau \boldsymbol{S}(t) \cdot \hat{n}\, dt\, dS, \\ w_{em}(\tau) &= \int_0^\tau \{E_i(t)\,[a_{ij}\partial_t E_j(t) + b_{ij}\partial_t B_j(t)] \\ &\quad + \partial_t B_i(t)\,[c_{ij}E_j(t) + d_{ij}B_j(t)]\}\, dt, \qquad (4.1) \\ w_d(\tau) &= \int_0^\tau \{E_i(t)\,[\partial_t(G_{ij} * E_j)(t) + \partial_t(K_{ij} * B_j)(t)] \\ &\quad + \partial_t B_i(t)\,[(L_{ij} * E_j)(t) + (F_{ij} * B_j)(t)]\}\, dt. \qquad (4.2) \end{aligned}$$

In this expression $\mathcal{E}(\tau)$ is the total energy entered through the surface $S_{\boldsymbol{r}}$ up to time $\tau$. The quantity $w_{em}(\tau)$ is the sum of the electric and magnetic field energy densities and $w_d(\tau)$ is the dissipated energy density.

**Definition 4.1.** *A bianisotropic medium is dissipative at a point $\boldsymbol{r}$ in the region $V$ if and only if for all $\tau > 0$ the total energy $\mathcal{E}(\tau) \geq 0$ for every electromagnetic field $\{\boldsymbol{E}, \boldsymbol{B}\}$ in $\mathcal{C}^1$ in every volume $V_{\boldsymbol{r}}$, such that $V_{\boldsymbol{r}} \subset V$, around the point $\boldsymbol{r}$. The bianisotropic medium is dissipative in $V$ if and only if it is dissipative at all points in $V$.*

Physically, this definition states that the total electro-magnetic energy entering through a sphere $S_{\boldsymbol{r}}$ is always non-negative at all times. Or stated differently, no net production of electro-magnetic energy inside $S_{\boldsymbol{r}}$ is possible—the medium is passive. Furthermore, it is a local property in space. Thus, in principle, parts of the medium can be dissipative, others not.

Since the integrand is continuous in the spatial variables this definition is equivalent to

$$w_{em}(\tau) + w_d(\tau) \geq 0, \quad \tau > 0, \tag{4.3}$$

for all fields $\in \mathcal{C}^1$.

The dissipation property implies that the constitutive relations must satisfy certain symmetries. The following theorem is proved in Appendix D.

**Theorem 4.1.** *Let the bianisotropic medium be dissipative at a point* $\boldsymbol{r}$*. Then the constitutive relations satisfy*

$$\begin{cases} a_{ij} = a_{ji} \\ b_{ij} = -c_{ji} \\ d_{ij} = d_{ji}, \end{cases}$$

*and the symmetric tensors* $a_{ij}$ *and* $d_{ij}$ *are non-negative definite. The quantity* $w_{em}$ *is*

$$w_{em}(\tau) = \frac{1}{2} a_{ij} E_i(\tau) E_j(\tau) + \frac{1}{2} d_{ij} B_i(\tau) B_j(\tau).$$

*Furthermore, the matrix*

$$\begin{pmatrix} c_0^2 G_{ij}(0) & c_0 K_{ij}(0) \\ -c_0 L_{ij}(0) & -F_{ij}(0) \end{pmatrix},$$

*is non-negative definite.*

In a dissipative medium the symmetric tensors $a_{ij}$ and $d_{ij}$ cannot be arbitrary real numbers. Since the symmetric tensors $a_{ij}$ and $d_{ij}$ are non-negative definite it implies that the tensor elements must satisfy

$$\begin{cases} a_{ii} \geq 0 \\ \\ d_{ii} \geq 0 \end{cases}, i = 1, 2, 3, \qquad \begin{cases} a_{ij}^2 \leq a_{ii} a_{jj} \\ \\ d_{ij}^2 \leq d_{ii} d_{jj}. \end{cases}, i, j = 1, 2, 3, \quad i \neq j,$$

These conditions are necessary for the tensors $a_{ij}$ and $d_{ij}$ to be non-negative definite. Similarly, the generalized susceptibility kernels $G_{ij}$ and $F_{ij}$ satisfy

$$\begin{cases} G_{ii}(0) \geq 0 \\ \\ F_{ii}(0) \leq 0 \end{cases}, i = 1, 2, 3,$$

and

$$\begin{cases} (G_{ij}(0) + G_{ji}(0))^2 \leq 4 G_{ii}(0) G_{jj}(0) \\ (F_{ij}(0) + F_{ji}(0))^2 \leq 4 F_{ii}(0) F_{jj}(0) \\ (K_{ij}(0) - L_{ji}(0))^2 \leq -4 G_{ii}(0) F_{jj}(0). \end{cases}, i, j = 1, 2, 3,$$

A further restriction of the form of the constitutive relations is given by the concept of positive functions, which are reviewed in Appendix E[8]. Results with applications to viscoelastic media are found in [8]. The following theorem is proved in Appendix E.

**Theorem 4.2.** *If a bianisotropic dissipative medium has generalized susceptibility kernels that are continuous at time $t = 0$, i.e. $G_{ij}(0) = K_{ij}(0) = L_{ij}(0) = F_{ij}(0) = 0$*[9]*, then*[10]

$$\begin{pmatrix} c_0^2(G'_{ij}(0) \pm G'_{ij}(t)) & c_0(K'_{ij}(0) \pm K'_{ij}(t)) \\ -c_0(L'_{ij}(0) \pm L'_{ij}(t)) & -(F'_{ij}(0) \pm F'_{ij}(t)) \end{pmatrix},$$

*and*

$$\begin{pmatrix} c_0^2 G'_{ij}(0) & c_0 K'_{ij}(0) \\ -c_0 L'_{ij}(0) & -F'_{ij}(0) \end{pmatrix},$$

*are non-negative definite matrices.*

# 5 Mirror process

The following definition is appropriate for an analysis of the medium under time reversal. Similar ideas have been used to characterize viscoelastic media [9].

**Definition 5.1.** *The mirror process of the fields $\{\boldsymbol{E}(t), \boldsymbol{B}(t)\} \in \mathcal{C}^0$, satisfying*

$$\begin{cases} \boldsymbol{E}(t) = \boldsymbol{0} \\ \boldsymbol{B}(t) = \boldsymbol{0}, \end{cases}$$

*for all $t \geq \tau > 0$, is the fields $\{\boldsymbol{E}^*(t), \boldsymbol{B}^*(t)\}$ defined by*

$$\begin{cases} \boldsymbol{E}^*(t) = \boldsymbol{E}(\tau - t) \\ \boldsymbol{B}^*(t) = -\boldsymbol{B}(\tau - t). \end{cases}$$

Note that the fields $\{\boldsymbol{E}^*(t), \boldsymbol{B}^*(t)\} \in \mathcal{C}^0$. The mirror process of a pair $\{\boldsymbol{E}(t), \boldsymbol{B}(t)\}$ is essentially a pair $\{\boldsymbol{E}^*(t), \boldsymbol{B}^*(t)\}$ for which the time evolution is reversed.

Let $w_d^*(\tau)$ denote the dissipated energy density corresponding to the mirror process $\{\boldsymbol{E}^*(t), \boldsymbol{B}^*(t)\}$. If $w_d^*(\tau) = w_d(\tau)$, then the dissipative energy is the same as in its mirror process. This statement is not equivalent to invariance under time reversal since the fields $\{\boldsymbol{D}^*(t), \boldsymbol{H}^*(t)\}$ are, in general, not related in the same simple way to $\{\boldsymbol{D}(t), \boldsymbol{H}(t)\}$ as $\{\boldsymbol{E}^*(t), \boldsymbol{B}^*(t)\}$ to $\{\boldsymbol{E}(t), \boldsymbol{B}(t)\}$. The following theorem is proved in Appendix D and provides a further classification of the symmetries of the constitutive relations.

[8] The relation between a function of positive type and its Fourier transform is briefly analyzed in footnote 13 on page 26.

[9] This assumption is closely related to the assumption of a non-zero mass of the particles carrying charge, see also [19, pp. 309-310].

[10] The prime denotes differentiation with respect to time, i.e. $G'_{ij}(t) = \partial_t G_{ij}(t)$ etc.

**Theorem 5.1.** *A bianisotropic medium, defined by the constitutive relations in (3.2), satisfies*

$$\begin{cases} G'_{ij}(t) = G'_{ji}(t) \\ K_{ij}(t) = L_{ji}(t) \\ F'_{ij}(t) = F'_{ji}(t), \end{cases} \tag{5.1}$$

*if and only if* $w_d^*(\tau) = w_d(\tau)$ *for all* $\tau > 0$ *and all fields* $\{\boldsymbol{E}, \boldsymbol{B}\} \in \mathcal{C}^1$ *such that* $\{\boldsymbol{E}(\tau), \boldsymbol{B}(\tau)\} = \{\boldsymbol{0}, \boldsymbol{0}\}$.

The constitutive relations for a medium that is dissipative and for which $w_d^*(\tau) = w_d(\tau)$ are therefore

$$\begin{cases} D_i(t) = a_{ij}E_j(t) + b_{ij}B_j(t) + (G_{ij} * E_j)(t) + (K_{ij} * B_j)(t) \\ \\ H_i(t) = -b_{ji}E_j(t) + d_{ij}B_j(t) + (K_{ji} * E_j)(t) + (F_{ij} * B_j)(t), \end{cases} \tag{5.2}$$

where the tensors $a_{ij}, d_{ij}, G'_{ij}$ and $F'_{ij}$ are symmetrical tensors.

# 6 Reciprocity

The reciprocity theorem for simple harmonic time dependence was first derived in 1895 by Lorentz. In principle, reciprocity for a general time dependence can be obtained from this solution by Fourier analysis. The definition and the analysis, however, presented in this paper, do not rely on any fixed frequency results and holds for a larger class of time dependence. Instead the definition of reciprocity is defined as a property in physical space-time. Similar treatments can be found in e.g. [6, 15, 23, 37]. Moreover, reciprocity is defined as a local property in space. Thus, in principle, parts of the medium could be reciprocal, others not.

**Definition 6.1.** *A medium is defined to be reciprocal at a point* $\boldsymbol{r}$ *in the region* $V$ *if and only if*

$$\iint_{S_{\boldsymbol{r}}} \left\{ \epsilon_{ijk}(E_j^a * H_k^b)(\tau) + \epsilon_{ijk}(H_j^a * E_k^b)(\tau) \right\} \hat{n}_i \, dS = 0,$$

*holds for all* $\tau$ *and all electromagnetic fields* $\{\boldsymbol{E}^a, \boldsymbol{B}^a\}$ *and* $\{\boldsymbol{E}^b, \boldsymbol{B}^b\}$ *in* $\mathcal{C}^1$ *and for every closed surface* $S_{\boldsymbol{r}}$, *such that* $S_{\boldsymbol{r}} \subset V$, *around the point* $\boldsymbol{r}$. *The medium is reciprocal in* $V$ *if and only if it is reciprocal at all points in* $V$.

In this definition $\epsilon_{ijk}$ is the Levi-Civita symbol. Note that the surface integral in Definition 6.1 has the same value on both sides of the surface $S$ irrespective of whether the fields have jump discontinuities on this surface due to a jump discontinuity in the material parameters or not (the electric and the magnetic fields have continuous tangential components on such a surface).

The following theorem is derived in Appendix F and gives a necessary and sufficient condition for reciprocity.

**Theorem 6.1.** *The medium is reciprocal at a point* $\boldsymbol{r}$ *if and only if the constitutive relations are symmetric at* $\boldsymbol{r}$*, i.e.*

$$\begin{cases} a_{ij} = a_{ji} \\ b_{ij} = c_{ji} \\ d_{ij} = d_{ji} \end{cases} \qquad \begin{cases} G_{ij}(t) = G_{ji}(t) \\ K_{ij}(t) = L_{ji}(t) \\ F_{ij}(t) = F_{ji}(t). \end{cases}$$

The constitutive relations for a reciprocal medium are

$$\begin{cases} D_i(t) = a_{ij}E_j(t) + b_{ij}B_j(t) + (G_{ij} * E_j)(t) + (K_{ij} * B_j)(t) \\ \\ H_i(t) = b_{ji}E_j(t) + d_{ij}B_j(t) + (K_{ji} * E_j)(t) + (F_{ij} * B_j)(t), \end{cases} \tag{6.1}$$

where the tensors $a_{ij}, d_{ij}, G_{ij}$ and $F_{ij}$ are symmetrical tensors.

An immediate consequence of this theorem is that a dissipative biisotropic medium with $b \neq 0$ is not reciprocal. The most general constitutive relations for a biisotropic, dissipative medium that is reciprocal are

$$\begin{cases} \boldsymbol{D}(t) = a\boldsymbol{E}(t) + (G * \boldsymbol{E})(t) + (K * \boldsymbol{B})(t) \\ \\ \boldsymbol{H}(t) = d\boldsymbol{B}(t) + (K * \boldsymbol{E})(t) + (F * \boldsymbol{B})(t). \end{cases}$$

Another result of this theorem is that a reciprocal medium, due to Theorem 5.1, necessarily has generalized susceptibility kernels satisfying the mirror process $w_d^*(\tau) = w_d(\tau)$. The opposite conclusion is, however, not always true. If, however, $G_{ij}(0) = G_{ji}(0)$, $F_{ij}(0) = F_{ji}(0)$, $a_{ij} = a_{ji}$, $b_{ij} = c_{ji}$ and $d_{ij} = d_{ji}$, then the converse conclusion is also true. The common case of a dissipative medium for which $b_{ij} = c_{ij} = 0$ and $G_{ij}(0) = F_{ij}(0) = 0$, satisfies this symmetry.

# Appendix A Covariant formulation of the Maxwell equations

For the sake of a condensed notation and short derivation in the proofs of the theorems in this paper it is convenient to work with the covariant formulation of the Maxwell equations and the constitutive relations in space-time.

Define two antisymmetric tensors[11] $F_{\mu\nu}$ and $G_{\mu\nu}$ of rank 2 as

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_z & -B_y & E_x/c_0 \\ -B_z & 0 & B_x & E_y/c_0 \\ B_y & -B_x & 0 & E_z/c_0 \\ -E_x/c_0 & -E_y/c_0 & -E_z/c_0 & 0 \end{pmatrix}$$

[11]These tensors should not be confused with the susceptibility kernels $G_{ij}(t)$ and $F_{ij}(t)$ in (3.2). The Greek letter indices run through the integers $1, 2, 3, 4$ and the Latin letters run through the integers $1, 2, 3$. The convention of summation over repeated indices is used throughout this paper.

and

$$G_{\mu\nu} = \begin{pmatrix} 0 & H_z & -H_y & c_0 D_x \\ -H_z & 0 & H_x & c_0 D_y \\ H_y & -H_x & 0 & c_0 D_z \\ -c_0 D_x & -c_0 D_y & -c_0 D_z & 0 \end{pmatrix},$$

where $c_0$ is the speed of light in vacuum. Both tensors are functions of the 4-vector

$$x^\mu = (x, y, z, c_0 t).$$

The 4-current is defined as

$$J^\mu = (J_x, J_y, J_z, c_0 \rho),$$

and the gradient in four space is

$$\partial_\mu = (\partial_x, \partial_y, \partial_z, c_0^{-1} \partial_t).$$

Introduce the metric tensor $g_{\mu\nu}$ which is diagonal, with diagonal elements $g_{11} = g_{22} = g_{33} = 1$ and $g_{44} = -1$. The metric tensor is used to lower and raise the indices of tensors.

The Maxwell equations in (1.1) in the covariant form are

$$\begin{cases} \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} + \partial_\sigma F_{\mu\nu} = 0 \\ \\ \partial_\nu G^{\mu\nu} = J^\mu. \end{cases} \tag{A.1}$$

The constitutive relations defined in (3.2) is also written in a covariant form

$$G_{\mu\nu} = a_{\mu\nu}{}^{\sigma\lambda} F_{\sigma\lambda} + b_{\mu\nu}{}^{\sigma\lambda} * F_{\sigma\lambda}, \tag{A.2}$$

where the tensors $a_{\mu\nu}{}^{\sigma\lambda}$ and $b_{\mu\nu}{}^{\sigma\lambda}$ are antisymmetric in the first pair and the second pair of indices, respectively, e.g. $a_{\mu\nu}{}^{\sigma\lambda} = -a_{\nu\mu}{}^{\sigma\lambda}$ etc. In terms of the tensor-valued functions in (3.2) these tensors are

$$\begin{cases} a_{ij} = 2a_{i4}{}^{j4}/c_0^2 \\ b_{ij} = \epsilon_{jkl} a_{i4}{}^{kl}/c_0 \\ c_{ij} = \epsilon_{ikl} a_{kl}{}^{j4}/c_0 \\ d_{ij} = \frac{1}{2}\epsilon_{ikl}\epsilon_{jmn} a_{kl}{}^{mn} \end{cases} \qquad \begin{cases} G_{ij} = 2b_{i4}{}^{j4}/c_0^2 \\ K_{ij} = \epsilon_{jkl} b_{i4}{}^{kl}/c_0 \\ L_{ij} = \epsilon_{ikl} b_{kl}{}^{j4}/c_0 \\ F_{ij} = \frac{1}{2}\epsilon_{ikl}\epsilon_{jmn} b_{kl}{}^{mn}, \end{cases}$$

where $\epsilon_{ijk}$ is the Levi-Civita symbol.

It should be noted, that the constitutive relations given in (A.2) are not given in a covariant form, since time convolutions are involved. It is, however, not the purpose of this paper to give such a covariant formulation. The purpose of this appendix is to utilize the compact notation of the field tensors $F_{\mu\nu}$ and $G_{\mu\nu}$ and the compact way of writing the constitutive relations in (A.2). All the analyses found in this paper are carried out in one fixed space-time coordinate system. In the main text above, the explicit fields $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$ and $\boldsymbol{H}$ are used to stress the fact that the formulation is not written in a covariant form.

# Appendix B Some mathematical proofs

**Proof of Theorem 3.1:** Let $\tau$ be a fixed point in $(-\infty, \infty)$ and take $\epsilon > 0$. Then, there exists a $\delta(\epsilon, \tau) > 0$ such that $\max\{|\boldsymbol{E}(t)|, |\boldsymbol{B}(t)|\} < \delta(\epsilon, \tau)$ for all $t \leq \tau$ and $\max\{|\boldsymbol{D}(\tau)|, |\boldsymbol{H}(\tau)|\} < \epsilon$. Assume there exists a $t' < \tau$ such that

$$\max\{|\boldsymbol{D}(t')|, |\boldsymbol{H}(t')|\} \geq \epsilon.$$

Define a new pair of fields by

$$\begin{Bmatrix} \boldsymbol{E}'(t) \\ \boldsymbol{B}'(t) \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{E}(t - \tau + t') \\ \boldsymbol{B}(t - \tau + t') \end{Bmatrix},$$

for all $t \in (-\infty, \infty)$. It is clear that $\{\boldsymbol{E}', \boldsymbol{B}'\} \in \mathcal{C}^0$. Then $\{\boldsymbol{D}', \boldsymbol{H}'\} = L\{\boldsymbol{E}', \boldsymbol{B}'\}$, due to invariance to time translations, satisfies

$$\begin{Bmatrix} \boldsymbol{D}'(t) \\ \boldsymbol{H}'(t) \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{D}(t - \tau + t') \\ \boldsymbol{H}(t - \tau + t') \end{Bmatrix}.$$

Since $\max\{|\boldsymbol{E}'(t)|, |\boldsymbol{B}'(t)|\} < \delta(\epsilon, \tau)$ for all $t \leq \tau$, continuity implies that

$$\max\{|\boldsymbol{D}'(\tau)|, |\boldsymbol{H}'(\tau)|\} = \max\{|\boldsymbol{D}(t')|, |\boldsymbol{H}(t')|\} < \epsilon.$$

This contradicts the assumption made above and the theorem is proved. ■

**Proof of Theorem 3.2:** Let $\{\boldsymbol{E}, \boldsymbol{B}\}$ belong to the class $\mathcal{C}^0$ and

$$\begin{Bmatrix} \boldsymbol{D} \\ \boldsymbol{H} \end{Bmatrix} = L \begin{Bmatrix} \boldsymbol{E} \\ \boldsymbol{B} \end{Bmatrix}.$$

It is immediately clear from property 3 in Definition 3.1 that $\begin{Bmatrix} \boldsymbol{D} \\ \boldsymbol{H} \end{Bmatrix} = 0$ on $(-\infty, 0]$. It remains to prove that $\begin{Bmatrix} \boldsymbol{D} \\ \boldsymbol{H} \end{Bmatrix}$ is continuous on $(-\infty, \infty)$.

Let $\tau$ be a fixed point in $(-\infty, \infty)$ and $\epsilon > 0$ and for $\alpha > 0$ define

$$\begin{Bmatrix} \boldsymbol{E}_\alpha(t) \\ \boldsymbol{B}_\alpha(t) \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{E}(t) - \boldsymbol{E}(t - \alpha) \\ \boldsymbol{B}(t) - \boldsymbol{B}(t - \alpha) \end{Bmatrix}$$

$$\begin{Bmatrix} \boldsymbol{D}_\alpha(t) \\ \boldsymbol{H}_\alpha(t) \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{D}(t) - \boldsymbol{D}(t - \alpha) \\ \boldsymbol{H}(t) - \boldsymbol{H}(t - \alpha) \end{Bmatrix}.$$

Obviously $\{\boldsymbol{E}_\alpha(t), \boldsymbol{B}_\alpha(t)\}$ belongs to $\mathcal{C}^0$ and

$$\begin{Bmatrix} \boldsymbol{D}_\alpha \\ \boldsymbol{H}_\alpha \end{Bmatrix} = L \begin{Bmatrix} \boldsymbol{E}_\alpha \\ \boldsymbol{B}_\alpha \end{Bmatrix},$$

by Definition 3.1.

First show continuity on the left. By Definition 3.1, property 4, there exists a $\delta(\epsilon, \tau) > 0$ such that

$$\max\{|\boldsymbol{E}_\alpha(t)|, |\boldsymbol{B}_\alpha(t)|\} < \delta \text{ for all } t \leq \tau \Rightarrow \max\{|\boldsymbol{D}_\alpha(\tau)|, |\boldsymbol{H}_\alpha(\tau)|\} < \epsilon.$$

Since $\{\boldsymbol{E}(t), \boldsymbol{B}(t)\}$ is uniformly continuous on $(-\infty, \tau]$ there exists a $\eta(\delta)$ such that

$$\max\{|\boldsymbol{E}_\alpha(t)|, |\boldsymbol{B}_\alpha(t)|\} < \delta \text{ for all } t \leq \tau \text{ if } \alpha < \eta.$$

Thus, $\{\boldsymbol{D}, \boldsymbol{H}\}$ is continuous on the left, i.e.

$$\max\{|\boldsymbol{D}_\alpha(\tau)|, |\boldsymbol{H}_\alpha(\tau)|\} < \epsilon \text{ for all } \alpha < \eta.$$

Continuity on the right is proved with similar arguments. Restrict $\alpha$ to the interval $[0, \alpha_0]$. Theorem 3.1 shows that there exists a $\delta(\epsilon)$ such that

$$\max\{|\boldsymbol{E}_\alpha(t)|, |\boldsymbol{B}_\alpha(t)|\} < \delta \text{ for all } t \leq \tau + \alpha_0,$$

implies

$$\max\{|\boldsymbol{D}_\alpha(\tau+\alpha)|, |\boldsymbol{H}_\alpha(\tau+\alpha)|\} < \epsilon.$$

Furthermore, $\{\boldsymbol{E}(t), \boldsymbol{B}(t)\}$ is uniformly continuous on $(-\infty, \tau+\alpha_0]$ and there exists a $\eta(\delta)$ such that

$$\max\{|\boldsymbol{E}_\alpha(t)|, |\boldsymbol{B}_\alpha(t)|\} < \delta \text{ for all } t \leq \tau + \alpha_0 \text{ if } \alpha < \eta.$$

Thus, there exists an $\eta(\epsilon) \in [0, \alpha_0]$ such that

$$\begin{aligned}&\max\{|\boldsymbol{D}_\alpha(\tau+\alpha)|, |\boldsymbol{H}_\alpha(\tau+\alpha)|\}\\ &= \max\{|\boldsymbol{D}(\tau+\alpha) - \boldsymbol{D}(\tau)|, |\boldsymbol{H}(\tau+\alpha) - \boldsymbol{H}(\tau)|\} < \epsilon \text{ for all } \alpha < \eta,\end{aligned}$$

and the right continuity is proved. ∎

# Appendix C Representation of the linear dispersive law

This appendix contains a proof of Theorem 3.3. Similar proofs are found in [26] and [18]. It is convenient to split the proof into parts and the following lemma and scalar version of the theorem are helpful in proving Theorem 3.3.

**Lemma C.1.** *To every continuous linear functional $A$ on the space $C = \{f : f \in C^0[a, b], f(b) = 0\}$ there exists a unique real-valued function $G$ with the following properties.*

1. *$G$ is of bounded variation on $[a, b]$.*
2. *$G$ is continuous on the right in $(a, b)$, i.e $G(t) = G(t+)$, for all $a < t < b$.*
3. *$G(a) = 0$,*

*and such that*

$$Af = \int_a^b f(t)\, dG(t),$$

*for all $f \in C$. Furthermore, the norm of the functional is*

$$\|A\| = [\,Var\ G]_a^b\,.$$

**Proof:** Let $A$ be a continuous linear functional on $C$. By the Hahn-Banach theorem [33, p. 186], there exists a continuous linear functional $B$ defined on $C^0[a,b]$ such that $Af = Bf$ for all $f \in C$. Furthermore, the norms are equal, i.e. $\|A\| = \|B\|$. Riesz representation theorem [33, p. 200], (see also [32]), then proves the existence of a unique function $G$ that is of bounded variation on $[a,b]$ and continuous on the right in the interior of the interval $[a,b]$. Furthermore, $G(a) = 0$ and the norm $\|A\| = \|B\| = [\mathrm{Var}\ G]_a^b$. ∎

**Theorem C.1.** *Let $L$ be a transformation defined on $H^0$ satisfying*

1. *The transformation is linear, i.e. for every pair of real numbers $\alpha$, $\beta$ $L[\alpha f_1 + \beta f_2] = \alpha L f_1 + \beta L f_2$.*
2. *The transformation is invariant to time translations, i.e. for every fixed time $\tau > 0$ the relation $f_1(t) = f(t-\tau)$ for all $t \in (-\infty, \infty)$ implies $g_1(t) = g(t-\tau)$, where $g = Lf$ and $g_1 = Lf_1$.*
3. *The transformation satisfies causality, i.e. for every fixed $t$ such that $f = 0$ on $(-\infty, t]$ implies $g = 0$ on $(-\infty, t]$, where $g = Lf$.*
4. *The transformation is continuous, i.e. for every fixed $\tau$ and every $\epsilon > 0$ there exists a $\delta(\epsilon,\tau) > 0$ such that $|f(t)| < \delta(\epsilon,\tau)$ for all $t \in (-\infty, \tau]$ implies $|g(\tau)| < \epsilon$, where $g = Lf$.*

*Then $L$ is a map from $H^0$ to $H^0$ and, furthermore, there exists a unique function $G(t)$ such that*

1. *$G(t) = 0$, on $(-\infty, 0)$.*
2. *$G(t)$ is of bounded variation on every closed subinterval of $(-\infty, \infty)$.*
3. *$G(t)$ is continuous on the right on $(-\infty, \infty)$, i.e. $G(t) = G(t+)$.*
4. *for every $f$ and $g$ associated through $L$*

$$g(t) = (f \odot dG)(t).$$

*Conversely, every function $G$ defined on $(-\infty, \infty)$, satisfying the properties 1–4 above, generates a linear mapping from $H^0$ to $H^0$.*

**Proof:** By techniques similar to the ones used in the proof of Theorem 3.2 it is clear that the transformation $L$ maps $H^0$ into $H^0$. Let $\alpha > 0$ and $\tau > 0$ be fixed real numbers. For every $f \in H^0$ define a function $F(t)$ defined on $[-\alpha, \tau]$ by

$$F(t) = f(\tau - t), \quad t \in [-\alpha, \tau]. \tag{C.1}$$

The function $F \in C = \{f : f \in C^0[-\alpha, \tau], f(\tau) = 0\}$. Define a linear functional $A$ on $C$ by

$$AF = (Lf)(\tau),$$

where $F$ is defined as in (C.1). This linear functional is properly defined due to causality. To see this, take two functions $f_1$ and $f_2$ in $H^0$ such that they both define the same $F$ in (C.1). Then $f(t) = f_1(t) - f_2(t) = 0$ for all $t \leq \tau + \alpha$ and by causality $(Lf)(\tau) = 0$, i.e. $(Lf_1)(\tau) = (Lf_2)(\tau)$. The linear functional $A$ is also continuous by property 4. The Lemma C.1 now proves the existence of a unique function $G_{\alpha,\tau}$ with the following properties.

1. $G_{\alpha,\tau}$ is of bounded variation on $[-\alpha, \tau]$,
2. $G_{\alpha,\tau}$ is continuous on the right in $(-\alpha, \tau)$,
3. $G_{\alpha,\tau}(-\alpha) = 0$,

and such that

$$AF = \int_{-\alpha}^{\tau} F(t)\, dG_{\alpha,\tau}(t),$$

for all $F \in C$. Stated differently,

$$(Lf)(\tau) = \int_{-\alpha}^{\tau} f(\tau - t)\, dG_{\alpha,\tau}(t),$$

for all $f \in H^0$.

To prove that $G_{\alpha,\tau}(t) = 0$ for $t < 0$, let $0 < \beta < \alpha$, and define

$$f_n(t) = \begin{cases} 0, & t < \tau + \beta - \frac{1}{n}, \\ 1, & t \geq \tau + \beta, \end{cases}$$

and where $f_n$ is linear between zero and one in the interval $[\tau + \beta - \frac{1}{n}, \tau + \beta)$. Then, due to causality, $(Lf_n)(\tau) = 0$ for sufficiently large $n$ and, furthermore,

$$(Lf_n)(\tau) = G_{\alpha,\tau}(-\beta) - G_{\alpha,\tau}(-\alpha) + O(n^{-1}) = G_{\alpha,\tau}(-\beta) + O(n^{-1}) = 0.$$

Thus $G_{\alpha,\tau}(-\beta) = 0$ for all $0 < \beta < \alpha$, which implies that $G_{\alpha,\tau}(t) = 0$ at all points $-\alpha \leq t < 0$. The function $G$, therefore, does not depend on $\alpha$ and the limit in the integration could for convenience be taken as $-\infty$, i.e.

$$(Lf)(\tau) = \int_{-\infty}^{\tau} f(\tau - t)\, dG_{\tau}(t) = (f \odot dG_{\tau})(\tau),$$

for all $f \in H^0$.

The remaining property of $G$ to prove is that $G$ is independent of the parameter $\tau$. For each $f \in H^0$ and $\tau > 0$ define a new function $f_1(t) = f(t-\tau)$. For this function holds

$$(Lf_1)(t+\tau) = \int_{-\infty}^{t+\tau} f_1(t+\tau-t')\, dG_{t+\tau}(t') = \int_{-\infty}^{t} f(t-t')\, dG_{t+\tau}(t').$$

Then, by invariance to time translations,

$$\int_{-\infty}^{t} f(t-t')\, dG_t(t') = (Lf)(t) = (Lf_1)(t+\tau) = \int_{-\infty}^{t} f(t-t')\, dG_{t+\tau}(t').$$

Uniqueness of the function $G_t$ then implies that $G_t$ is independent of $t$ and

$$(Lf)(t) = \int_{-\infty}^{t} f(t-t')\, dG(t') = (f \odot dG)(t).$$

To prove the converse part of the theorem assume $G$ is a function satisfying the properties in the theorem and let $G$ define a transformation $L$ by

$$(Lf)(t) = \int_{-\infty}^{t} f(t-t')\, dG(t') = (f \odot dG)(t).$$

It is obvious that this transformation is linear and causal. To see that the transformation is invariant to time translations, define for $\tau > 0$ $f_1(t) = f(t-\tau)$, where $f \in H^0$. Then,

$$\begin{aligned}(Lf_1)(t) &= \int_{-\infty}^{t} f(t-\tau-t')\, dG(t') \\ &= \int_{-\infty}^{t-\tau} f(t-\tau-t')\, dG(t') = (Lf)(t-\tau).\end{aligned}$$

The continuity is proved by the inequality

$$|(Lf)(\tau)| = |(f \odot dG)(\tau)| \le \|f\| \,[\mathrm{Var}\ G]_0^\tau < \epsilon,$$

if $\|f\| = \max_{0 \le t \le \tau} |f(t)| < \delta$ where $\delta = \epsilon / [\mathrm{Var}\ G]_0^\tau$. ∎

**Proof of Theorem 3.3:** Consider a linear linear dispersive law and its transformation $L$. Due to linearity there exists a set of linear transformations $L_{\mu\nu}{}^{\sigma\lambda}$ such that in covariant form

$$G_{\mu\nu} = L_{\mu\nu}{}^{\sigma\lambda} F_{\sigma\lambda},$$

for a fixed system of coordinates. Each component in the set of transformations $L_{\mu\nu}{}^{\sigma\lambda}$ satisfies the assumption of the definition of a linear dispersive law, Definition 3.1. Thus, it suffices to consider one component of the transformation. An application of Theorem C.1 proves the existence of uniquely defined functions $\mathcal{G}_{ij}(t)$, $\mathcal{F}_{ij}(t)$, $\mathcal{K}_{ij}(t)$ and $\mathcal{L}_{ij}(t)$ satisfying properties 1-4 in Theorem 3.3 such that

$$\begin{cases} D_i(t) = (E_j \odot d\mathcal{G}_{ij})(t) + (B_j \odot d\mathcal{K}_{ij})(t) \\ \\ H_i(t) = (E_j \odot d\mathcal{L}_{ij})(t) + (B_j \odot d\mathcal{F}_{ij})(t). \end{cases}$$

The converse part of the theorem is also clear by Theorem C.1.

The final part of the theorem is to prove that $\mathcal{G}_{ij}(t)$, $\mathcal{F}_{ij}(t)$, $\mathcal{K}_{ij}(t)$ and $\mathcal{L}_{ij}(t)$ have the correct transformation properties. Denote by $a_{ij}$ a transformation (proper rotation or spatial inversion) from one set of coordinates to another, i.e.

$$x'_i = a_{ij} x_j,$$

where

$$a_{ik}a_{jk} = a_{ki}a_{kj} = \delta_{ij}, \qquad \det[a_{ij}] = \pm 1.$$

The vectors $\boldsymbol{E}$ and $\boldsymbol{D}$ and the pseudovectors $\boldsymbol{B}$ and $\boldsymbol{H}$ transform as

$$\begin{cases} E'_i = a_{ij} E_j \\ D'_i = a_{ij} D_j, \end{cases} \tag{C.2}$$

and

$$\begin{cases} B'_i = \det[a_{ij}] a_{ij} B_j \\ H'_i = \det[a_{ij}] a_{ij} H_j, \end{cases} \tag{C.3}$$

respectively. The transformation

$$\begin{cases} D_i(t) = (E_j \odot d\mathcal{G}_{ij})(t) + (B_j \odot d\mathcal{K}_{ij})(t) \\ \\ H_i(t) = (E_j \odot d\mathcal{L}_{ij})(t) + (B_j \odot d\mathcal{F}_{ij})(t), \end{cases}$$

can then, by use of the transformation properties of the fields in (C.2) and (C.3), be written as

$$\begin{cases} D'_i(t) = (E'_l \odot d(a_{ik}a_{lj}\mathcal{G}_{kj}))(t) + (B'_l \odot d(\det[a_{ij}] a_{ik}a_{lj}\mathcal{K}_{kj}))(t) \\ \\ H'_i(t) = (E'_l \odot d(\det[a_{ij}] a_{ik}a_{lj}\mathcal{L}_{kj}))(t) + (B'_l \odot d(a_{ik}a_{lj}\mathcal{F}_{kj}))(t). \end{cases}$$

Compare this expression with the corresponding one in the transformed coordinate system

$$\begin{cases} D'_i(t) = (E'_j \odot d\mathcal{G}'_{ij})(t) + (B'_j \odot d\mathcal{K}'_{ij})(t) \\ \\ H'_i(t) = (E'_j \odot d\mathcal{L}'_{ij})(t) + (B'_j \odot d\mathcal{F}'_{ij})(t). \end{cases}$$

Uniqueness of the functions $\mathcal{G}'_{ij}(t)$, $\mathcal{F}'_{ij}(t)$, $\mathcal{K}'_{ij}(t)$ and $\mathcal{L}'_{ij}(t)$ then shows that

$$\begin{cases} \mathcal{G}'_{ij}(t) = a_{ik}a_{jl}\mathcal{G}_{kl}(t) \\ \mathcal{F}'_{ij}(t) = a_{ik}a_{jl}\mathcal{F}_{kl}(t) \\ \mathcal{K}'_{ij}(t) = \det[a_{ij}] a_{ik}a_{jl}\mathcal{K}_{kl}(t) \\ \mathcal{L}'_{ij}(t) = \det[a_{ij}] a_{ik}a_{jl}\mathcal{L}_{kl}(t), \end{cases}$$

which confirms the correct transformation properties. ∎

# Appendix D Dissipation and symmetries of the constitutive relations

**Proof of Theorem 4.1:** The energy density terms in (4.1) and (4.2) written in covariant form are

$$w_{em}(\tau) = \frac{1}{2} a^{ij\sigma\lambda} F_{ij}(\tau) F_{\sigma\lambda}(\tau) - \frac{1}{2} a^{\mu\nu\sigma\lambda} \int_0^\tau F_{\mu\nu}(t) \partial_t F_{\sigma\lambda}(t)\, dt, \tag{D.1}$$

and

$$w_d(\tau) = \frac{1}{2} F_{ij}(\tau) \left( b^{ij\sigma\lambda} * F_{\sigma\lambda} \right)(\tau) - \frac{1}{2} \int_0^\tau F_{\mu\nu}(t) \partial_t (b^{\mu\nu\sigma\lambda} * F_{\sigma\lambda})(t)\, dt,$$

Denote by $\mathcal{C}_0^1$ the subspace in $\mathcal{C}^1$ fields[12] that satisfy $F_{\mu\nu}(\tau) = 0$. The definition of dissipation applied to this subspace is

$$-\frac{1}{2} a^{\mu\nu\sigma\lambda} \int_0^\tau F_{\mu\nu}(t) \partial_t F_{\sigma\lambda}(t)\, dt - \frac{1}{2} \int_0^\tau F_{\mu\nu}(t) \partial_t (b^{\mu\nu\sigma\lambda} * F_{\sigma\lambda})(t)\, dt \geq 0, \tag{D.2}$$

for all fields $F_{\mu\nu} \in \mathcal{C}_0^1$. Select two arbitrary components of $F_{\mu\nu} \in \mathcal{C}_0^1$ and denote

$$\begin{cases} f_1(t) = F_{\mu\nu}(t) \\ f_2(t) = F_{\sigma\lambda}(t). \end{cases}$$

Equation (D.2) then simplifies after integration by parts to (no summation over $\mu$ and $\nu$)

$$\begin{aligned} &-2 \left\{ a^{\mu\nu\sigma\lambda} - a^{\sigma\lambda\mu\nu} \right\} \int_0^\tau f_1(t) f_2'(t)\, dt \\ &\quad -2 \int_0^\tau \left\{ f_1(t)(b^{\mu\nu\sigma\lambda} * f_2')(t) + f_2(t)(b^{\sigma\lambda\mu\nu} * f_1')(t) \right. \\ &\quad \left. + f_1(t)(b^{\mu\nu\mu\nu} * f_1')(t) + f_2(t)(b^{\sigma\lambda\sigma\lambda} * f_2')(t) \right\} dt \geq 0. \end{aligned}$$

Let $f_1$ and $f_2$ have support in $[\xi - \epsilon/2, \xi + \epsilon/2]$, where $\xi \in (0, \tau)$ and $\epsilon$ taken so small that $[\xi - \epsilon/2, \xi + \epsilon/2] \in (0, \tau)$. Use the mean value theorem of integral calculus and let $\epsilon \to 0$. The result is

$$-2 \left\{ a^{\mu\nu\sigma\lambda} - a^{\sigma\lambda\mu\nu} \right\} \xi_1 \xi_2 \geq 0,$$

for all real $\xi_1$ and $\xi_2$. This can, however, only be true if

$$a^{\mu\nu\sigma\lambda} = a^{\sigma\lambda\mu\nu},$$

or equivalently

$$\begin{cases} a_{ij} = a_{ji} \\ b_{ij} = -c_{ji} \\ d_{ij} = d_{ji}. \end{cases}$$

[12] The notion that $F_{\mu\nu} \in \mathcal{C}^1$ is an obvious extension of Definition 2.3, that each component of $F_{\mu\nu}$ belongs to $H^1 \cap C^1$ on $(-\infty, \infty)$.

The electromagnetic field energy density $w_{em}$ in (D.1) then simplifies to

$$\begin{aligned} w_{em}(\tau) &= \frac{1}{4}a^{ijkl}F_{ij}(\tau)F_{kl}(\tau) - a^{i4j4}F_{i4}(\tau)F_{j4}(\tau) \\ &= \frac{1}{2}a_{ij}E_i(\tau)E_j(\tau) + \frac{1}{2}d_{ij}B_i(\tau)B_j(\tau), \end{aligned}$$

for $F_{\mu\nu} \in \mathcal{C}^1$.

To prove that $a_{ij}$ and $d_{ij}$ are non-negative definite, choose $F_{\mu\nu} \in \mathcal{C}^1$ such that $\text{supp}F_{\mu\nu} \in [\tau - \epsilon, \tau]$ and denote $\xi_{\mu\nu} = F_{\mu\nu}(\tau)$. In the limit $\epsilon \to 0$ equation (4.3) reduces to

$$\frac{1}{4}a^{ijkl}\xi_{ij}\xi_{kl} - a^{i4j4}\xi_{i4}\xi_{j4} \geq 0,$$

for all $\xi_{\mu\nu}$, or equivalently

$$\left\{ \begin{array}{l} a_{ij}\xi_i\xi_j \geq 0 \\ \\ d_{ij}\xi_i\xi_j \geq 0. \end{array} \right. \text{, for all real } \xi_i, \xi_j,$$

This proves that the tensors $a_{ij}$ and $d_{ij}$ are non-negative definite.

The statement about the generalized susceptibility kernels at time $t = 0$ is proved by using (D.2), which with the symmetry in $a^{\mu\nu\sigma\lambda}$, simplifies to

$$-\frac{1}{2}\int_0^\tau F_{\mu\nu}(t)\partial_t(b^{\mu\nu\sigma\lambda} * F_{\sigma\lambda})(t)\,dt \geq 0,$$

for all fields in $\mathcal{C}_0^1$. Let $F_{\mu\nu}(t) = \xi_{\mu\nu}f(t)$, where $f \in C^1[0, \tau]$ with $f(0) = f(\tau) = 0$ where the constant tensor $\xi_{\mu\nu}$ is antisymmetric. Denote by $B(t) = -\xi_{\mu\nu}b^{\mu\nu\sigma\lambda}(t)\xi_{\sigma\lambda}$. The inequality then is

$$B(0)\int_0^\tau f^2(t)\,dt + \int_0^\tau f(t)\left((\partial_t B) * f\right)(t)\,dt \geq 0. \tag{D.3}$$

Let $\text{supp}f \in [\xi - \epsilon, \xi + \epsilon] \in (0, \tau)$ and take the limit $\epsilon \to 0$ which implies $B(0) \geq 0$, i.e.

$$-\xi_{\mu\nu}b^{\mu\nu\sigma\lambda}(0)\xi_{\sigma\lambda} \geq 0,$$

for all $\xi_{\mu\nu}$. As a matrix relation in a six dimensional Euclidean space (the antisymmetric tensor $\xi_{\mu\nu}$ has six independent components) this inequality is equivalent to that the matrix

$$\begin{pmatrix} c_0^2 G_{ij}(0) & c_0 K_{ij}(0) \\ -c_0 L_{ij}(0) & -F_{ij}(0) \end{pmatrix},$$

is non-negative definite in this six dimensional Euclidean space. This completes the proof of Theorem 4.1. ∎

**Proof of Theorem 5.1:** The dissipated energy density in (4.2) written in covariant form reads

$$w_d(\tau) = -\frac{1}{2}\int_0^\tau F_{\mu\nu}(t)\partial_t(b^{\mu\nu\sigma\lambda} * F_{\sigma\lambda})(t)\,dt, \tag{D.4}$$

for all fields $F_{\mu\nu} \in \mathcal{C}_0^1$. The field tensor corresponding to the mirror process defined in Definition 5.1 is $F^*_{\mu\nu}(t) = -\Lambda_\mu{}^{\mu'}\Lambda_\nu{}^{\nu'}F_{\mu'\nu'}(\tau - t)$, where the time reversal transformation $\Lambda_\mu{}^\nu$ is diagonal, with diagonal elements $\Lambda_1{}^1 = \Lambda_2{}^2 = \Lambda_3{}^3 = 1$ and $\Lambda_4{}^4 = -1$. The corresponding dissipated energy density reads accordingly

$$\begin{aligned} w_d^*(\tau) &= -\frac{1}{2}\int_0^\tau F^*_{\mu\nu}(t)\partial_t(b^{\mu\nu\sigma\lambda} * F^*_{\sigma\lambda})(t)\,dt \\ &= -\frac{1}{2}\int_0^\tau F_{\mu\nu}(\tau - t)\partial_t\int_0^t \tilde{b}^{\mu\nu\sigma\lambda}(t - t')F_{\sigma\lambda}(\tau - t')\,dt'\,dt \\ &= -\frac{1}{2}\int_0^\tau F_{\mu\nu}(t)\partial_t(\tilde{b}^{\sigma\lambda\mu\nu} * F_{\sigma\lambda})(t)\,dt, \end{aligned}$$

where the tensor-valued function $\tilde{b}^{\mu\nu\sigma\lambda}(t)$ is defined as

$$\tilde{b}^{\mu\nu\sigma\lambda}(t) = \Lambda_{\mu'}{}^\mu\Lambda_{\nu'}{}^\nu\Lambda_{\sigma'}{}^\sigma\Lambda_{\lambda'}{}^\lambda b^{\mu'\nu'\sigma'\lambda'}(t).$$

The difference $w_d(\tau) - w_d^*(\tau)$ is

$$w_d(\tau) - w_d^*(\tau) = \frac{1}{2}\int_0^\tau F_{\mu\nu}(t)\partial_t(B^{\mu\nu\sigma\lambda} * F_{\sigma\lambda})(t)\,dt, \tag{D.5}$$

where $B^{\mu\nu\sigma\lambda}(t) = \tilde{b}^{\sigma\lambda\mu\nu}(t) - b^{\mu\nu\sigma\lambda}(t)$.

Select two arbitrary components of $F_{\mu\nu} \in \mathcal{C}_0^1$ and denote

$$\begin{cases} f_1(t) = F_{\mu\nu}(t) \\ f_2(t) = F_{\sigma\lambda}(t). \end{cases}$$

If $w_d(\tau) = w_d^*(\tau)$ for all $F_{\mu\nu} \in \mathcal{C}_0^1$ then the integral on the right hand side is zero and the summation over the components simplifies to

$$\int_0^\tau dt\int_0^t dt' f_1(t)B^{\mu\nu\sigma\lambda}(t - t')f_2'(t') = \int_0^\tau dt\int_t^\tau dt' f_1(t)B^{\sigma\lambda\mu\nu}(t' - t)f_2'(t'). \tag{D.6}$$

Let supp$f_1 \in [t_0 - \epsilon/2, t_0 + \epsilon/2]$, where $t_0 \in (0, \tau)$. In the limit $\epsilon \to 0$ this equality becomes

$$\int_0^{t_0} B^{\mu\nu\sigma\lambda}(t_0 - t')f_2'(t')\,dt' = \int_{t_0}^\tau B^{\sigma\lambda\mu\nu}(t' - t_0)f_2'(t')\,dt',$$

for all $f_2 \in C^1[0, \tau]$ such that $f_2(0) = f_2(\tau) = 0$. Take the limit as $t_0 \to \tau$. By continuity

$$\int_0^\tau B^{\mu\nu\sigma\lambda}(\tau - t')f'(t')\,dt' = 0,$$

for all $f \in C^1[0,\tau]$ such that $f(0) = f(\tau) = 0$. Integration by parts gives

$$\int_0^\tau \partial_t B^{\mu\nu\sigma\lambda}(\tau - t) f(t)\, dt = 0.$$

This equation shows that $\partial_t B^{\mu\nu\sigma\lambda}(t) = 0$, $t \in [0,\tau]$, or equivalently

$$\begin{cases} G'_{ij}(t) = G'_{ji}(t) \\ K'_{ij}(t) = L'_{ji}(t) \\ F'_{ij}(t) = F'_{ji}(t), \end{cases}$$

where prime denotes differentiation with respect to time $t$.

The constant $B^{\mu\nu\sigma\lambda}(t) = B^{\mu\nu\sigma\lambda}(0)$ can be determined only when the indices $\mu\nu\sigma\lambda$ is a mixture between space and time indices. To see this, insert $B^{\mu\nu\sigma\lambda}(t) = B^{\mu\nu\sigma\lambda}(0)$ in (D.6). The result is

$$\left(B^{\mu\nu\sigma\lambda}(0) + B^{\sigma\lambda\mu\nu}(0)\right) \int_0^\tau f_1(t) f_2(t)\, dt = 0,$$

for all $f_i \in C^1[0,\tau]$ such that $f_i(0) = f_i(\tau) = 0$, $i = 1, 2$, which implies that $B^{\mu\nu\sigma\lambda}(0) + B^{\sigma\lambda\mu\nu}(0) = 0$, which implies that $K_{ij}(t) = L_{ji}(t)$, but nothing new for the generalized susceptibility kernels $G_{ij}$ and $F_{ij}$. This completes the first part of the theorem.

The only if part of the theorem is easy. Assume that

$$\begin{cases} G'_{ij}(t) = G'_{ji}(t) \\ K_{ij}(t) = L_{ji}(t) \\ F'_{ij}(t) = F'_{ji}(t), \end{cases}$$

or $\partial_t B^{\mu\nu\sigma\lambda}(t) = 0$. The assumption implies that

$$B^{\mu\nu\sigma\lambda}(0) + B^{\sigma\lambda\mu\nu}(0) = 0.$$

Then equation (D.5) implies

$$\begin{aligned} w_d(\tau) - w_d^*(\tau) &= B^{\mu\nu\sigma\lambda}(0) \frac{1}{2} \int_0^\tau F_{\mu\nu}(t) F_{\sigma\lambda}(t)\, dt \\ &= \frac{1}{4} \left(B^{\mu\nu\sigma\lambda}(0) + B^{\sigma\lambda\mu\nu}(0)\right) \int_0^\tau F_{\mu\nu}(t) F_{\sigma\lambda}(t)\, dt = 0, \end{aligned}$$

and the proof is completed. ∎

# Appendix E Positive functions

The definition and some properties of positive functions are reviewed in this appendix.

**Definition E.1.** *A real function $K(t) \in H^0$ is of positive type if for every real function $f$ in $C^0$ on* $[0,\infty)$

$$\int_0^\tau \int_0^\tau K(t-t')f(t)f(t')\,dt\,dt' \geq 0, \text{ for all } \tau \in [0,\infty).$$

The following theorem is a minor variant of a theorem proved in [38, p. 271].

**Theorem E.1.** *Let $K(t) \in H^0$ and define a new function $\mathcal{K}(t) = K(|t|)$ for all $t \in (-\infty,\infty)$. Then $K(t)$ is of positive type if and only if for every finite sequence of non-negative distinct real numbers $\{t_i\}_{i=0}^n$ the quadratic form*

$$\mathcal{K}(t_i - t_j),$$

*is non-negative definite, i.e.*

$$\sum_{i=0}^n \sum_{j=0}^n \mathcal{K}(t_i - t_j)\xi_i\xi_j \geq 0, \text{ for all real numbers } \{\xi_i\}_{i=0}^n.$$

An application of this theorem when $n=0$ and $n=1$ gives

$$\begin{cases} K(0) \geq 0 \\ |K(t)| \leq K(0), \text{ for all } t \in [0,\infty). \end{cases}$$

**Proof of Theorem 4.2:** From (D.3) it is immediately clear that

$$\int_0^\tau f(t)\left((\partial_t B) * f\right)(t)\,dt \geq 0,$$

or

$$\int_0^\tau \int_0^\tau f(t)[B'(t-t') + B'(t'-t)] * f(t')\,dt\,dt' \geq 0,$$

for all $f \in C^1[0,\tau]$ such that $f(0) = f(\tau) = 0$ and where $B(t) = -\xi_{\mu\nu}b^{\mu\nu\sigma\lambda}(t)\xi_{\sigma\lambda}$[13]. An application of Theorem E.1 then implies that[14]

$$B'(0) = -\xi_{\mu\nu}b'^{\mu\nu\sigma\lambda}(0)\xi_{\sigma\lambda} \geq 0$$
$$|B'(t)| = |\xi_{\mu\nu}b'^{\mu\nu\sigma\lambda}(t)\xi_{\sigma\lambda}| \leq B'(0) = -\xi_{\mu\nu}b'^{\mu\nu\sigma\lambda}(0)\xi_{\sigma\lambda},$$

[13] The relation between a function of positive type and its Fourier transform is presented in e.g. [17]. Bochner's Theorem implies the existence of a non-decreasing, bounded function $\alpha(\omega)$ for a continuous function of positive type, such that

$$\partial_t B(t) = \int_{-\infty}^{\infty} \cos\omega t\,d\alpha(\omega).$$

If, furthermore, $\partial_t B(t)$ is absolutely integrable on $(-\infty,\infty)$, then $\alpha$ is smooth and

$$\frac{d\alpha}{d\omega} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \partial_t B(t)\cos\omega t\,dt.$$

This implies that the Fourier transform of $B(t)$ has a non-negative imaginary part.

[14] The assumptions of vanishing values at the endpoints of the interval $[0,\tau]$ and $f \in C^1$ do not affect the proof of Theorem E.1.

for all $\xi_{\mu\nu}$ or equivalently

$$-\xi_{\mu\nu}b'^{\mu\nu\sigma\lambda}(0)\xi_{\sigma\lambda} \geq 0$$
$$-\left(\xi_{\mu\nu}b'^{\mu\nu\sigma\lambda}(0)\xi_{\sigma\lambda} \pm \xi_{\mu\nu}b'^{\mu\nu\sigma\lambda}(t)\xi_{\sigma\lambda}\right) \geq 0,$$

for all $\xi_{\mu\nu}$. This is equivalent to

$$\begin{pmatrix} c_0^2(G'_{ij}(0) \pm G'_{ij}(t)) & c_0(K'_{ij}(0) \pm K'_{ij}(t)) \\ -c_0(L'_{ij}(0) \pm L'_{ij}(t)) & -(F'_{ij}(0) \pm F'_{ij}(t)) \end{pmatrix},$$

and

$$\begin{pmatrix} c_0^2 G'_{ij}(0) & c_0 K'_{ij}(0) \\ -c_0 L'_{ij}(0) & -F'_{ij}(0) \end{pmatrix},$$

being non-negative definite matrices. ∎

# Appendix F Reciprocity and symmetries of the constitutive relations

This appendix contains the mathematical details of the derivation of the necessary and sufficient conditions for reciprocity. As a preparation to the proof of Theorem 6.1 the following two theorems are proved.

**Theorem F.1.** *Let $\tau > 0$, and $b \in C^0[0,\tau]$. If*

$$\int_0^\tau f(t)(b * f')(t)\, dt = 0, \tag{F.1}$$

*for all $f \in C^1[0,\tau]$, such that $f(0) = f(\tau) = 0$. Then $b(t) = 0$.*

**Proof of Theorem F.1:** Let $t_0$ be an arbitrary number in $(0,\tau)$ and let $f(t) = f_1(t) + f_2(t)$, where $f_1, f_2 \in C^1[0,\tau]$, $f_1(0) = f_2(0) = f_1(\tau) = f_2(\tau) = 0$ and $\mathrm{supp} f_1 \in [0, t_0]$ and $\mathrm{supp} f_2 \in [t_0, \tau]$. Repeated use of (F.1) implies that

$$\int_{t_0}^\tau f_2(t)g(t)\, dt = 0, \tag{F.2}$$

where $g(t) = \int_0^{t_0} b(t-t')f_1'(t')\, dt'$. Equation (F.2) can only be satisfied for arbitrary $f_2$ provided $g(t) = 0$, $t_0 < t < \tau$. To see this, assume that $g(t_1) \neq 0$, for some $t_1 \in (t_0, \tau)$. By continuity $g(t) \neq 0$ in a neighborhood of $t_1$. Then choose a positive function $f_2$ with support in this neighborhood. This leads to a contradiction and $g(t_1) = 0$. Thus, $g(t) = 0$ for $t_0 < t < \tau$ or

$$\int_0^{t_0} b(t-t')f_1'(t')\, dt' = 0, \quad t_0 < t < \tau,$$

for arbitrary $f_1 \in C^1[0,\tau]$, $f_1(0) = 0$ such that $\mathrm{supp} f_1 \in [0, t_0]$. This equation shows that

$$\int_0^{t_0} b(t-t')f_1(t')\,dt' = C, \quad t_0 < t < \tau,$$

where $C$ is a constant (depending on $f_1$). Thus

$$\int_0^{t_0} \left(b(t-t') - b(t_0-t')\right) f_1(t')\,dt' = 0, \quad t_0 < t < \tau,$$

for arbitrary $f_1 \in C^1[0,\tau]$, $f_1(0) = 0$ such that $\mathrm{supp} f_1 \in [0, t_0]$. Similar arguments to the ones above now show that $b(t) = b = \text{constant}, t \in (0,\tau)$. Equation (F.1) is then

$$b \int_0^{\tau} f^2(t)\,dt = 0,$$

for all $f \in C^1[0,\tau]$, such that $f(0) = f(\tau) = 0$. Thus $b = 0$. ∎

**Theorem F.2.** *Let $\tau > 0$, $a$ a constant and $b \in C^0[0,\tau]$. If*

$$a(f_1 * f_2)(\tau) + (f_1 * (b * f_2))(\tau) = 0, \tag{F.3}$$

*where $f_i(t)$, $i = 1,2$ are arbitrary functions in $C^0[0,\tau]$ such that $f_i(0) = 0$, $i = 1,2$. Then $a = 0$ and $b(t) = 0$.*

**Proof of Theorem F.2:** Let $\epsilon > 0$, $t_0 \in (\epsilon, \tau - \epsilon)$ and $\mathrm{supp} f_1 \in [t_0 - \epsilon, t_0 + \epsilon]$. Apply the mean value theorem of integral calculus to (F.3) with $f_2(t) = f_1(\tau - t)$.

$$af_1^2(t_0 + \theta_1\epsilon) + \epsilon(1-\theta_2)f_1(t_0 + \theta_2\epsilon)b(\epsilon(\theta_3 - \theta_2))f_1(t_0 + \theta_3\epsilon) = 0,$$

where $\theta_i \in [-1,1]$, $i = 1,2$ and $\theta_3 \in [\theta_2, 1]$. This equality can only be true if $a = 0$ (let $\epsilon$ be sufficiently small and choose $f_1(t_0)$ different from zero). Thus,

$$\int_0^{\tau} f_1(\tau - t)(b * f_2)(t)\,dt = 0,$$

where $f_i(t)$, $i = 1,2$ are arbitrary functions in $C^0[0,\tau]$ such that $f_i(0) = 0$, $i = 1,2$. The theorem is now proved by Theorem F.1 by restricting $f_1(t) = f(\tau - t) \in C^1$, $f(0) = f(\tau) = 0$ and $f_2(t) = f'(t)$. ∎

**Proof of Theorem 6.1:** Define the 3-vector $g^i(ab)$ for any field $F_{\mu\nu} \in \mathcal{C}^1$ as

$$g^i(ab) = F_{4\sigma}(a) * G^{\sigma i}(b) = \epsilon_{ijk} E_j^a * H_k^b / c_0,$$

where the arguments $a$ and $b$ indicate two different source configurations, respectively. Straightforward calculations using (A.1) and (A.2) show that

$$\begin{aligned} c_0\partial_i g^i(ab) \;=\; & -\frac{1}{2}\Lambda_\mu{}^{\mu'}\Lambda_\nu{}^{\nu'}\left\{a^{\mu\nu\sigma\lambda}F_{\mu'\nu'}(a) * (\partial_t F_{\sigma\lambda}(b))\right. \\ & \left. + F_{\mu'\nu'}(a) * \left(b^{\mu\nu\sigma\lambda} * (\partial_t F_{\sigma\lambda}(b))\right)\right\} + c_0 F_{4\mu}(a) * J^\mu(b), \end{aligned}$$

where the time reversal transformation ${\Lambda_\mu}^\nu$ is diagonal, with diagonal elements ${\Lambda_1}^1 = {\Lambda_2}^2 = {\Lambda_3}^3 = 1$ and ${\Lambda_4}^4 = -1$.

Interchanging the $a$ and $b$ and subtraction imply

$$\begin{aligned}
&c_0\left(\partial_i g^i(ab) - \partial_i g^i(ba)\right) = \\
&\quad -\frac{1}{2}\tilde{a}^{\mu\nu\sigma\lambda} F_{\mu\nu}(a) * (\partial_t F_{\sigma\lambda}(b)) \\
&\quad -\frac{1}{2}F_{\mu\nu}(a) * \left(\tilde{b}^{\mu\nu\sigma\lambda} * (\partial_t F_{\sigma\lambda}(b))\right) \\
&\quad + c_0 F_{4\mu}(a) * J^\mu(b) - c_0 F_{4\mu}(b) * J^\mu(a),
\end{aligned} \tag{F.4}$$

where the tensors $\tilde{a}^{\mu\nu\sigma\lambda}$ and $\tilde{b}^{\mu\nu\sigma\lambda}$ are defined as

$$\tilde{a}^{\mu\nu\sigma\lambda} = {\Lambda_{\mu'}}^\mu {\Lambda_{\nu'}}^\nu a^{\mu'\nu'\sigma\lambda} - {\Lambda_{\sigma'}}^\sigma {\Lambda_{\lambda'}}^\lambda a^{\sigma'\lambda'\mu\nu},$$

and

$$\tilde{b}^{\mu\nu\sigma\lambda}(t) = {\Lambda_{\mu'}}^\mu {\Lambda_{\nu'}}^\nu b^{\mu'\nu'\sigma\lambda}(t) - {\Lambda_{\sigma'}}^\sigma {\Lambda_{\lambda'}}^\lambda b^{\sigma'\lambda'\mu\nu}(t).$$

The source density $J^\mu$ is here interpreted as the forced or impressed part of the sources and thus zero inside the medium. Reciprocity, as it is defined in Definition 6.1, then implies

$$\begin{aligned}
&\iiint_{V_{\boldsymbol{r}}} \tilde{a}^{\mu\nu\sigma\lambda} F_{\mu\nu}(a) * (\partial_t F_{\sigma\lambda}(b))\ dv \\
&+ \iiint_{V_{\boldsymbol{r}}} F_{\mu\nu}(a) * \left(\tilde{b}^{\mu\nu\sigma\lambda} * (\partial_t F_{\sigma\lambda}(b))\right)\ dv = 0,
\end{aligned}$$

where $V_{\boldsymbol{r}}$ is an arbitrary region around $\boldsymbol{r}$. Due to continuity in the spatial variables the integrand must be zero, i.e.

$$\tilde{a}^{\mu\nu\sigma\lambda} F_{\mu\nu}(a) * (\partial_t F_{\sigma\lambda}(b)) + F_{\mu\nu}(a) * \left(\tilde{b}^{\mu\nu\sigma\lambda} * (\partial_t F_{\sigma\lambda}(b))\right) = 0.$$

This equality, however, is studied in Theorem F.2, e.g. take[15]

$$\begin{aligned}
&f_1(t) = F_{\mu\nu}(a)(t) \\
&f_2(t) = \partial_t F_{\sigma\lambda}(b)(t) \\
&a = \tilde{a}^{\mu\nu\sigma\lambda} \\
&b(t) = \tilde{b}^{\mu\nu\sigma\lambda}(t).
\end{aligned}$$

Thus, the following symmetries hold

$${\Lambda_{\mu'}}^\mu {\Lambda_{\nu'}}^\nu a^{\mu'\nu'\sigma\lambda} = {\Lambda_{\sigma'}}^\sigma {\Lambda_{\lambda'}}^\lambda a^{\sigma'\lambda'\mu\nu},$$

[15] Note that the $a$ and $b$ configurations are independent. Thus, only one component of $F_{\mu\nu}(a)$ and $F_{\mu\nu}(b)$, respectively, is different from zero.

and

$$\Lambda_{\mu'}{}^{\mu}\Lambda_{\nu'}{}^{\nu}b^{\mu'\nu'\sigma\lambda}(t) = \Lambda_{\sigma'}{}^{\sigma}\Lambda_{\lambda'}{}^{\lambda}b^{\sigma'\lambda'\mu\nu}(t).$$

or in the notation of (3.2)

$$\begin{cases} a_{ij} = a_{ji} \\ b_{ij} = c_{ji} \\ d_{ij} = d_{ji} \end{cases} \qquad \begin{cases} G_{ij}(t) = G_{ji}(t) \\ K_{ij}(t) = L_{ji}(t) \\ F_{ij}(t) = F_{ji}(t). \end{cases}$$

The if part of the theorem is easy. The assumption of the symmetries in the constitutive relations imply that

$$\Lambda_{\mu'}{}^{\mu}\Lambda_{\nu'}{}^{\nu}a^{\mu'\nu'\sigma\lambda} = \Lambda_{\sigma'}{}^{\sigma}\Lambda_{\lambda'}{}^{\lambda}a^{\sigma'\lambda'\mu\nu},$$

and

$$\Lambda_{\mu'}{}^{\mu}\Lambda_{\nu'}{}^{\nu}b^{\mu'\nu'\sigma\lambda}(t) = \Lambda_{\sigma'}{}^{\sigma}\Lambda_{\lambda'}{}^{\lambda}b^{\sigma'\lambda'\mu\nu}(t).$$

This symmetry and (F.4) with vanishing source density $J^{\mu}$ prove that $g^{i}(ab) = g^{i}(ba)$ and the theorem is proved. ∎

# Appendix G Discontinuous solutions

Solutions to the Maxwell equations that are discontinuous can conveniently be defined as solutions to integral equations in space-time [22]. These integral equations may be obtained from the covariant formulation of the Maxwell equations in Appendix A, see (A.1).

Let $\Omega$ be a region (open and bounded) in space-time and let $\Gamma$ be its boundary. An application of the divergence theorem in 4-space then implies that

$$\begin{cases} \iiint_{\Gamma} \{\hat{n}_{\mu}F_{\nu\sigma} + \hat{n}_{\nu}F_{\sigma\mu} + \hat{n}_{\sigma}F_{\mu\nu}\}\, dS = 0 \\[2ex] \iiint_{\Gamma} \hat{n}_{\nu}G^{\mu\nu}\, dS = \iiiint_{\Omega} J^{\mu}\, dv, \end{cases} \tag{G.1}$$

where $dS$ and $dv$ are the surface and volume elements in 4-space, respectively, and where $\hat{n}_{\nu}$ are the components of the normal unit vector of $\Gamma$ in 4-space directed away from $\Omega$. Solutions of these equations are called weak solutions[16]. These solutions may be discontinuous along hypersurfaces in space-time.

Suppose a weak solution is discontinuous along a hypersurface $\phi(x, y, z, t) = 0$. The form of this hypersurface $\phi = 0$ is now investigated.

Assume that the hypersurface $\phi(x, y, z, t) = 0$ divides the region $\Omega$ into two regions $\Omega_1$ and $\Omega_2$. Denote by $\Gamma_i$ the part of $\Gamma$ that is the boundary of $\Omega_i$, $i = 1, 2$, and denote by $\Gamma_0$ the portion of $\phi = 0$ that lies in $\Omega$. Thus $\Gamma_1 + \Gamma_2$ constitutes the entire surface $\Gamma$, see Figure 1.

[16] Note that still weaker solutions can be formulated but this is not necessary for the analysis carried out in this appendix.

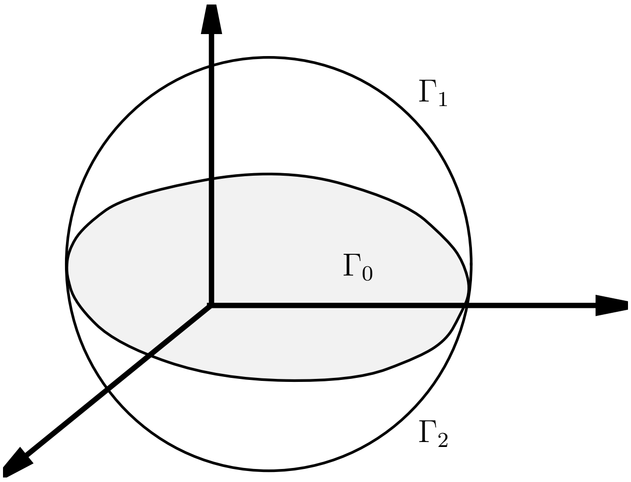



Figure 1.

Apply (G.1) to the regions $\Omega, \Omega_1$ and $\Omega_2$, respectively. The result is

$$\begin{cases} \iiint_{\Gamma_0} \{\hat{n}_\mu[F_{\nu\sigma}] + \hat{n}_\nu[F_{\sigma\mu}] + \hat{n}_\sigma[F_{\mu\nu}]\}\, dS = 0 \\ \\ \iiint_{\Gamma_0} \hat{n}_\nu[G^{\mu\nu}]\, dS = 0, \end{cases}$$

where the brackets indicate the difference of the values of the tensors $F_{\mu\nu}$ and $G^{\mu\nu}$ on both sides of the hypersurface $\phi = 0$. Since $\Omega$ can be chosen arbitrarily small and enclose an arbitrarily small section of $\phi = 0$ the integrands must be zero, i.e. on $\phi = 0$

$$\begin{cases} \hat{n}_\mu[F_{\nu\sigma}] + \hat{n}_\nu[F_{\sigma\mu}] + \hat{n}_\sigma[F_{\mu\nu}] = 0 \\ \\ \hat{n}_\nu[G^{\mu\nu}] = 0. \end{cases}$$

In terms of the fields $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{D}$ and $\boldsymbol{H}$ these equations are

$$\begin{cases} \nabla\phi \times [\boldsymbol{E}] = -(\partial_t\phi)[\boldsymbol{B}] \\ \nabla\phi \cdot [\boldsymbol{B}] = 0 \\ \nabla\phi \times [\boldsymbol{H}] = (\partial_t\phi)[\boldsymbol{D}] \\ \nabla\phi \cdot [\boldsymbol{D}] = 0. \end{cases} \tag{G.2}$$

In a biisotropic medium without dispersion, i.e. with the constitutive relations

$$\begin{cases} \boldsymbol{D}(t) = a\boldsymbol{E}(t) + b\boldsymbol{B}(t) \\ \\ \boldsymbol{H}(t) = c\boldsymbol{E}(t) + d\boldsymbol{B}(t), \end{cases}$$

and where $a$ and $b$ are assumed to be non-zero constants, the jump discontinuities in the fields satisfy (the generalized susceptibility tensors are assumed to be smooth functions of the space variables)

$$\begin{cases} [\boldsymbol{D}(t)] = a[\boldsymbol{E}(t)] + b[\boldsymbol{B}(t)] \\ \\ [\boldsymbol{H}(t)] = c[\boldsymbol{E}(t)] + d[\boldsymbol{B}(t)]. \end{cases}$$

This equation and (G.2) imply

$$\begin{cases} \{a(\partial_t\phi)^2 - d(\nabla\phi)^2\}[\boldsymbol{B}(t)] = (b+c)(\partial_t\phi)\nabla\phi \times [\boldsymbol{B}(t)] \\ \\ \{a(\partial_t\phi)^2 - d(\nabla\phi)^2\}[\boldsymbol{E}(t)] = (b+c)(\partial_t\phi)\nabla\phi \times [\boldsymbol{E}(t)]. \end{cases}$$

A necessary condition on the surface $\phi = 0$ to support a discontinuity in the fields is therefore

$$a(\partial_t\phi)^2 = d(\nabla\phi)^2.$$

Furthermore, if $b \neq -c$ there can be no discontinuity in the tangential components of $\boldsymbol{E}$ and $\boldsymbol{B}$ on the surface $\phi = 0$.

In a biisotropic medium with dispersion the constitutive relations are

$$\begin{cases} \boldsymbol{D}(t) = a\boldsymbol{E}(t) + b\boldsymbol{B}(t) + (G * \boldsymbol{E})(t) + (K * \boldsymbol{B})(t) \\ \\ \boldsymbol{H}(t) = c\boldsymbol{E}(t) + d\boldsymbol{B}(t) + (L * \boldsymbol{E})(t) + (F * \boldsymbol{B})(t). \end{cases}$$

The scalars $a$ and $d$ (functions of the space coordinates) are again assumed to be non-zero. If $\partial_t\phi \neq 0$ then the jump discontinuities in the fields satisfy

$$\begin{cases} [\boldsymbol{D}(t)] = a[\boldsymbol{E}(t)] + b[\boldsymbol{B}(t)] \\ \\ [\boldsymbol{H}(t)] = c[\boldsymbol{E}(t)] + d[\boldsymbol{B}(t)], \end{cases}$$

and the result is identical to the non-dispersive case. If $\partial_t\phi = 0$ then it is easy to show that the medium can support no discontinuities at all.

# Appendix H The wave equation

The wave equation for some special cases of the media is given in this appendix.

For a homogeneous biisotropic media, i.e. constitutive relations

$$\begin{cases} \boldsymbol{D}(t) = a\boldsymbol{E}(t) + b\boldsymbol{B}(t) + (G * \boldsymbol{E})(t) + (K * \boldsymbol{B})(t) \\ \\ \boldsymbol{H}(t) = c\boldsymbol{E}(t) + d\boldsymbol{B}(t) + (L * \boldsymbol{E})(t) + (F * \boldsymbol{B})(t), \end{cases}$$

the wave equation is

$$\begin{aligned} &d\nabla \times (\nabla \times \boldsymbol{E}) + \partial_t^2 \left\{ \left[1 + d\tilde{F}*\right] [a\boldsymbol{E} + G * \boldsymbol{E}] \right\} \\ &\quad -\partial_t \left\{ \left[1 + d\tilde{F}*\right] [(b+c)\nabla \times \boldsymbol{E} + (L+K) * (\nabla \times \boldsymbol{E})] \right\} = 0, \end{aligned}$$

where the resolvent $\tilde{F}$ of $F$ satisfies

$$d\tilde{F} + d^{-1}F + F * \tilde{F} = 0.$$

Another example is a bianisotropic medium with constitutive relations

$$\begin{cases} D_i(t) = aE_i(t) + b_{ij}B_j(t) + (G * E_i)(t) + (K_{ij} * B_j)(t) \\ \\ H_i(t) = c_{ij}E_j(t) + dB_i(t) + (L_{ij} * E_j)(t), \end{cases}$$

where

$$\begin{cases} b_{ij} = b\xi_{ij} \\ c_{ij} = c\xi_{ij} \\ K_{ij} = K\xi_{ij} \\ L_{ij} = L\xi_{ij}, \end{cases}$$

and

$$\xi_{ij} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

These constitutive relations are the appropriate ones for a moving isotropic dispersive dielectric. The wave equation for this medium is

$$\begin{aligned} & d(\partial_i\partial_j E_j - \partial_j\partial_j E_i) + \partial_t^2 \left[aE_i + G * E_i\right] \\ & \quad -\partial_t \xi_{ij} \left[b\epsilon_{jkl}\partial_k E_l + K * (\epsilon_{jkl}\partial_k E_l)\right] \\ & \quad -\partial_t \left[c\epsilon_{ijk}\partial_j \xi_{kl} E_l + L * (\epsilon_{ijk}\partial_j \xi_{kl} E_l)\right] = 0. \end{aligned}$$

The wave equation for a general bianisotriopic medium is (no assumption on homogeneity of the medium is made)

$$\begin{aligned} & \epsilon_{ijk}\partial_j \left[d_{kl}\epsilon_{lmn}\partial_m E_n + F_{kl} * (\epsilon_{lmn}\partial_m E_n)\right] - \partial_t \epsilon_{ijk}\partial_j \left[c_{kl}E_l + L_{kl} * E_l)\right] \\ & \quad - \partial_t \left[b_{ij}\epsilon_{jkl}\partial_k E_l + K_{ij} * (\epsilon_{jkl}\partial_k E_l)\right] + \partial_t^2 \left[a_{ij}E_j + G_{ij} * E_j\right] = 0. \end{aligned}$$

This equation can also be rewritten in a compact vector-dyadic notation as

$$\begin{aligned} & \nabla \times \left[\mathbf{d} \cdot (\nabla \times \boldsymbol{E}) + \mathbf{F} * (\nabla \times \boldsymbol{E})\right] - \partial_t \nabla \times \left[\mathbf{c} \cdot \boldsymbol{E} + \mathbf{L} * \boldsymbol{E}\right] \\ & \quad - \partial_t \left[\mathbf{b} \cdot (\nabla \times \boldsymbol{E}) + \mathbf{K} * (\nabla \times \boldsymbol{E})\right] + \partial_t^2 \left[\mathbf{a} \cdot \boldsymbol{E} + \mathbf{G} * \boldsymbol{E}\right] = \mathbf{0}. \end{aligned}$$

# Appendix I On the arbitrariness of the components of the field tensor

That each component of the field tensor $F_{\mu\nu}$ can be treated separately is an assumption already discussed in Section 3 and the following simple non mathematical argument can be of interest.

There is no loss of generality in choosing the point $\boldsymbol{r}$ as the origin. Assume the sources (outside $V$) can be arranged such that the electric field in a neighborhood of the origin is

$$\boldsymbol{E}_1(\boldsymbol{r},t) = c_0 f(t + z/c - t_0)\hat{e}_1,$$

or

$$\boldsymbol{E}_2(\boldsymbol{r},t) = c_0 f(t - z/c - t_0)\hat{e}_1,$$

where $c$ is a constant and $f(t)$ is in $H^1 \cap C^0$. The speed of light in vacuum is denoted by $c_0$. It is obvious that these fields are in $\mathcal{C}^0$ for a suitable choice of $t_0$. The associated magnetic inductions are

$$\boldsymbol{B}_1(\boldsymbol{r},t) = -\frac{c_0}{c} f(t + z/c - t_0)\hat{e}_2,$$

and

$$\boldsymbol{B}_2(\boldsymbol{r},t) = \frac{c_0}{c} f(t - z/c - t_0)\hat{e}_2,$$

respectively. If the two fields are added and evaluated at the origin the only non-vanishing components of $F_{\mu\nu}$ are

$$F_{14}(t) = -F_{41}(t) = 2f(t - t_0).$$

Subtraction of the two fields gives only contributions to the components $F_{13} = -F_{31}$. For the other components hold similar results. In this paper it is assumed that the medium and the sources generating the fields $F_{\mu\nu}$ are such that each components of $F_{\mu\nu}$ can be chosen independently.

# References


[1] Birss R.R. and R.G. Shrubsall, "The Propagation of Electromagnetic Waves in Magnetoelectric Crystals," *Phil. Mag.*, **15**, 687-700 (1967).

[2] Bassiri S., N. Engheta and C.H. Papas, "Dyadic Green's Function and Dipole Radiation in Chiral Media," *Alta Frequenza*, **LV**, 83-88 (1986).

[3] Bassiri S., C.H. Papas and N. Engheta, "Electromagnetic Wave Propagation through a Dielectric-Chiral Interface and through a Chiral Slab," *J. Opt. Soc. Am.*, **A5**, 9 (1988).

[4] Bokut' B.V. and F.I. Fedorov, "On the theory of optical activity in crystals. III. General equations of normals," *Opt. Spectrosc.*, **6**, 342 (1959).

[5] Born M., "Über die natürliche optische Aktivität von Flüssigkeiten und Gasen," *Physik. Zeitschr.*, **XVI**, 251–258 (1915).

[6] Cheo B.R., "A Reciprocity Theorem for Electromagnetic Fields with General Time Dependence," *IEEE Trans. on Antennas and Propagation*, AP-**13**, 278-284 (1965).

[7] Condon E.U., "Theories of Optical Rotatory Power," *Rev. Mod. Phys.*, **9**, 432–457 (1937).

[8] Day W.A., "On the monotonicity of the relaxation functions of viscoelastic materials," *Proc. Camb. Phil. Soc.*, **67**, 503-508 (1970).

[9] Day W.A., "Time-Reversal and the Symmetry of the Relaxation Function of a Linear Viscoelastic Material," *Arch. Rat. Mech. Anal.*, **40**, 155-159 (1971).

[10] Dzyaloshinskiǐ I.E., "On the magneto-electrical effect in antiferromagnets," *Soviet Physics JETP*, **10**, 628-629 (1960).

[11] Engheta N. and D.L. Jaggard, "Electromagnetic Chirality and its applications," *IEEE Antennas and Propagation Society Newsletter*, October 1988, 6-12.

[12] Engheta N. and A.R. Mickelson, "Transition Radiation Caused by a Chiral Plate," *IEEE Trans. on Antennas and Propagation*, AP-**30**, 1213-1216 (1982).

[13] Fedorov F.I., "On the theory of optical activity in crystals. I. The law of conservation of energy and the optical activity tensors," *Opt. Spectrosc.*, **6**, 49 (1959).

[14] Fedorov F.I., "On the theory of optical activity in crystals. II. Crystals of cubic symmetry and plane classes of central symmetry," *Opt. Spectrosc.*, **6**, 237 (1959).

[15] Goubau G., "A reciprocity theorems for non-periodic fields," *IRE Trans. on Antennas and Propagation*, AP-**8**, 339-342 (1960).

[16] deGroot S.R., *The Maxwell equations*, (North-Holland, Amsterdam 1969).

[17] Gurtin M.E. and I. Herrera, "On dissipation inequalities and linear viscoelasticity," *Quart. Appl. Math.*, **23**, 235-245 (1965).

[18] Gurtin M.E. and E. Sternberg, "On the Linear Theory of Viscoelasticity," *Arch. Rat. Mech. Anal.*, **11**, 291-356 (1962).

[19] Jackson J.D., *Classical Electrodynamics*, (Wiley, New York 1975).

[20] Jaggard D.L., A.R. Mickelson and C.H. Papas, "On Electromagnetic Waves in Chiral Media," *Appl. Phys.*, **18**, 211-216 (1979).

[21] Jaggard D.L., X. Sun and N. Engheta, "Canonical Sources and Duality in Chiral Media," *IEEE Trans. on Antennas and Propagation*, AP-**36**, 1007-1013 (1988).

[22] Kline M. and I.W. Kay, *Electromagnetic Theory and Geometrical Optics*, (Wiley, New York 1965).

[23] Kong J.A., "Reciprocity Relationships for Bianisotropic Media," *Proc. IEEE*, **58**, 1966-1967 (1970).

[24] Kong J.A., "Theorems of Bianisotropic Media," *Proc. IEEE*, **60**, 1036-1046 (1972).

[25] Kong J.A., *Electromagnetic Wave Theory*, (Wiley, New York 1986).

[26] König H. and J. Meixner, "Lineare Systeme und lineare Transformationen," *Mathematische Nachrichten*, **19**, 265-322 (1958).

[27] Lakhtakia A., V.K. Varadan and V.V. Varadan, *Time-Harmonic Electromagnetic Fields in Chiral Media*, Lecture Notes in Physics, vol. 335, (Springer, New York 1989).

[28] Landau L.D., E.M. Lifshitz and L.P. Pitaevskiĭ, *Electrodynamics of Continuous Media*, (Pergamon, Oxford 1984).

[29] O'Dell T.H., "Measurements of the Magneto-electric Susceptibility of Polycrystalline Chromium Oxide," *Phil. Mag.*, **13**, 921-933 (1966).

[30] O'Dell T.H., "Magnetoelectrics — a new class of materials," *Electron. and Power*, **11**, 266-267 (1965).

[31] O'Dell T.H., *The Electrodynamics of Magneto-electric Media*, (North-Holland Publishing Company, Amsterdam 1970).

[32] Riesz F. and B. Sz.-Nagy, *Functional analysis*, (Frederick Ungar Publishing Company, New York 1955).

[33] Taylor A.E., *Introduction to functional analysis*, (Wiley, New York 1966).

[34] Toupin R.A. and R.S. Revlin, "Linear Functional Electromagnetic Constitutive Relations and Plane Waves in a Hemihedral Isotropic Material," *Arch. Rat. Mech. Anal.*, **6**, 188-197 (1960).

[35] Volterra V., *Theory of functionals and of integral and integro-differential equations*, (New York, Dover 1959).

[36] Volterra V., "Sur les équations intégro-différentielles et leurs applications," *Acta Math.*, **35**, 295-356 (1912).

[37] Welch W.J., "Reciprocity Theorems for Electromagnetic Fields Whose Time Dependence Is Arbitrary," *IRE Trans. on Antennas and Propagation*, AP-**8**, 68-73 (1960).

[38] Widder D.V., *The Laplace Transform*, (Princeton, Princeton 1946).