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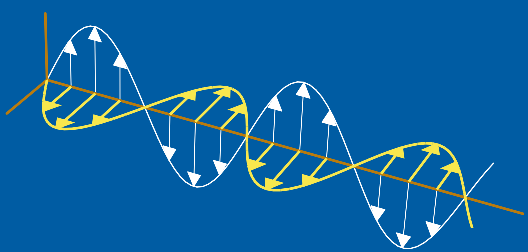

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Abstract

The fundamental limits of the gain and efficiency of an antenna are explored. The antenna is confined in a sphere and all of the currents are assumed to run in a material with a conductivity that is a function of the radial coordinate. The analysis is based on the expansion of the electromagnetic fields in terms of vector spherical harmonics. Explicit expressions for the limits of gain and efficiency are derived for different types of antennas.

1 Introduction

Small antennas suffer from physical limitations that reduce their bandwidth and increase return losses and ohmic losses. The limitations are caused by the large reactive electromagnetic fields in the vicinity of the antenna. The reactive fields are related to the reactive currents in the antenna, and these reactive currents causes ohmic losses in the metal. It is important to realize that the reactive electromagnetic fields and currents are consequences of Maxwell's equations and hence inevitable. This paper investigates fundamental limits of the ohmic losses in an antenna and of the gain of an antenna. The method for the investigation is based upon expansions of the electromagnetic fields in terms of spherical vector waves. A similar method was used in a classic paper by Chu [2] on the fundamental limits of the $Q-$value of omni-directional antennas. The results by Chu were generalized to non-axially symmetric antennas by Harrington [6]. There are a number of other papers that focus on the fundamental limits of antennas and a summary of the main results can be found in [5] and [3].

The objective of the paper is to give measures of the efficiency of antennas that can be used by antenna designers. It is possible to estimate the power efficiency of a design if one can compare it with the physical limit. If a certain power efficiency of an antenna is required the physical limits give the bound for the size of the antenna. This bound indicates the realistic size of the antenna.

2 Prerequisites

The following problem is analyzed in the paper: Consider an antenna that is circumferenced by a sphere of radius a. Outside the sphere there is vacuum and the electromagnetic fields satisfy Maxwell's equations. The current densities are confined in a sphere and run in a metal with conductivity $\sigma(r)$ and relative permittivity $\varepsilon_r = 1$. The volume of the sphere is denoted V_a. The frequency is fixed at f. What are the physical limits for the efficiency and the gain of such an antenna?

The time convention $e^{\mathrm{j}\omega t}$ is adopted in the paper. The efficiency is defined as

$$\eta_{\text{eff}} = \frac{P_{\text{rad}}}{P_{\text{rad}} + P_{\text{ohm}}} \tag{2.1}$$

where P_{rad} is the radiated power and P_{ohm} is the power dissipated in the antenna, due to ohmic losses. The Ohm's law $\boldsymbol{J} = \sigma \boldsymbol{E}$ holds and the ohmic loss is

$$P_{\text{ohm}} = \frac{1}{2} \int_{V_a} \frac{1}{\sigma(r)} |\boldsymbol{J}(\boldsymbol{r})|^2 dv \tag{2.2}$$

The far field amplitude $\boldsymbol{F}(\theta, \phi)$ of the antenna is related to the far field by

$$\boldsymbol{F}(\theta, \phi) = \lim_{kr \to \infty} \boldsymbol{E}(\boldsymbol{r}) kr e^{\mathrm{j}kr} \tag{2.3}$$

The radiated power is

$$P_{\text{rad}} = \frac{1}{2\eta_0 k^2} \int_0^{2\pi} \int_0^{\pi} |\boldsymbol{F}(\theta, \phi)|^2 \sin\theta d\phi d\theta$$

The definition of the directivity, D, and gain, G, are

$$\begin{aligned} D &= \frac{2\pi |\boldsymbol{F}(\theta, \phi)|^2_{\text{max}}}{k^2 \eta_0 P_{\text{rad}}} \\ G &= D \eta_{\text{eff}} \end{aligned} \tag{2.4}$$

where max denotes the maximum wrt θ and ϕ. The wave number $k = \omega\sqrt{\varepsilon_0 \mu_0}$ and the wave impedance $\eta_0 = \sqrt{\mu_0/\varepsilon_0}$ refer to vacuum.

3 General antennas

In the region exterior to the sphere, the electric field is expanded in spherical vector waves, $\boldsymbol{u}_{\tau\kappa ml}(\boldsymbol{r})$, also referred to as partial waves. These waves satisfy Maxwell's equations and constitute a complete set of vector valued functions on a spherical surface. The details of the spherical vector waves are given in appendix A. The expansion reads

$$\boldsymbol{E}(\boldsymbol{r}) = \sum_{l=1}^{\infty} \sum_{m=0}^{l} \sum_{\kappa=\text{e/o}} \sum_{\tau=1}^{2} a_{\tau\kappa ml} \boldsymbol{u}_{\tau\kappa ml}(\boldsymbol{r}). \tag{3.1}$$

The corresponding magnetic field is given by the induction law

$$\begin{aligned} \boldsymbol{H}(\boldsymbol{r}) &= \frac{\mathrm{j}}{\omega\mu} \sum_{l=1}^{\infty} \sum_{m=0}^{l} \sum_{\kappa=\text{e/o}} \sum_{\tau=1}^{2} a_{\tau\kappa ml} \nabla \times \boldsymbol{u}_{\tau\kappa ml}(\boldsymbol{r}) \\ &= \frac{\mathrm{j}k}{\omega\mu} \sum_{l=1}^{\infty} \sum_{m=0}^{l} \sum_{\kappa=\text{e/o}} \sum_{\tau=1}^{2} a_{\tau\kappa ml} \boldsymbol{u}_{\tau'\kappa ml}(\boldsymbol{r}), \end{aligned} \tag{3.2}$$

where $\tau' = 3 - \tau$. Here $\tau = 1, 2$ is the index for the two different wave types (TE and TM), $\kappa = \text{e}$ for waves that are even with respect to the azimuthal angle ϕ and $\kappa = \text{o}$ for the waves that are odd w.r.t. to ϕ, $l = 1, 2 \ldots$ is the index for the polar angle, and $m = 0, \ldots l$ is the index for the azimuthal angle. For $m = 0$ only the partial waves with $\kappa = \text{e}$ are non-zero, *cf.*, Eq. (A.2). The expansion in Eq. (3.1) covers all possible types of time harmonic sources inside V_{int}.

3.1 Classification

Antennas that radiate partial waves with $\tau = 1$ are referred to as magnetic antennas, since the reactive part of their radiated complex power is positive, *i.e.*, inductive. Antennas radiating partial waves with $\tau = 2$ are referred to as electric antennas, since they are capacitive when they are small compared to the wavelength.

The expansion coefficients $a_{\tau\kappa ml}$ in the expansion (3.1) can theoretically be altered independently of each other. Hence, each partial wave corresponds to an independent port of the antenna. The maximum number of ports, or channels, an antenna can use is then equal to the maximum number of partial waves the antenna can radiate.

The following classification of antennas is used in this paper:

Partial wave antenna An antenna that radiates only one partial wave $(\tau\kappa ml)$. The antenna has one port.

Magnetic multipole antenna of order l An antenna that radiates partial waves with $\tau = 1$ and index l. The maximum number of ports is $N_{l\text{port}} = 2l + 1$.

Electric multipole antenna of order l An antenna that radiates partial waves with $\tau = 2$ and index l. The maximum number of ports is $N_{l\text{port}} = 2l + 1$.

Magnetic antenna of order $l_{\mathbf{max}}$ An antenna that radiates partial waves with $\tau = 1$ and with $l = 1, \dots l_{\text{max}}$. The maximum number of ports is $N_{\text{port}} = l_{\text{max}}(l_{\text{max}} + 2)$.

Electric antenna of order $l_{\mathbf{max}}$ An antenna that radiates partial waves with $\tau = 2$ and with $l = 1, \dots l_{\text{max}}$. The maximum number of ports is $N_{\text{port}} = l_{\text{max}}(l_{\text{max}} + 2)$.

Combined antenna of order $l_{\mathbf{max}}$ An antenna that radiates partial waves with $\tau = 1, 2$ and $l = 1, \dots l_{\text{max}}$. The maximum number of ports is $N_{\text{port}} = 2l_{\text{max}}(l_{\text{max}} + 2)$.

4 Efficiency

Consider first a partial wave antenna of magnetic type, $\tau = 1$. Due to the orthogonality of the vector spherical harmonics, Eq. (A.4), and Eqs. (A.7) and (A.8), the current density in the sphere has to be proportional to the vector wave function $\boldsymbol{A}_{1\kappa ml}(\theta, \phi)$,

$$\boldsymbol{J}(r, \theta, \phi) = \sigma(r) f(r) \boldsymbol{A}_{1\kappa ml}(\theta, \phi) \tag{4.1}$$

The optimization problem is to find $f(r)$ such that the efficiency is maximized. The ohmic losses are

$$P_{\text{ohm}} = \frac{1}{2} \int_0^a \sigma(r)(f(r))^2 r^2 dr \tag{4.2}$$

due to the orthonormality of the vector wave functions, cf appendix A

From Eq. (A.8) and the asymptotic expressions for the Hankel functions, Eq. (A.6) it follows that the current density in Eq. (4.1) gives rise to the far field amplitude

$$\boldsymbol{F}(\theta,\phi) = -k\omega\mu_0 \int\limits_0^a \sigma(r) j_l(kr) f(r) r^2 dr \mathrm{j}^{l+1} \boldsymbol{A}_{1\kappa ml}(\theta,\phi) \tag{4.3}$$

The corresponding radiated power is

$$P_{\mathrm{rad}} = \frac{1}{2k^2\eta_0} k^2\omega^2\mu_0^2 \left(\int_0^a \sigma(r) j_l(kr) f(r) r^2 dr \right)^2 = \frac{1}{2} k\omega\mu_0 \left(\int_0^a \sigma(r) j_l(kr) f(r) r^2 dr \right)^2 \tag{4.4}$$

The efficiency is given by

$$\eta_{\mathrm{eff}} = \left(1 + \frac{1}{k\omega\mu_0} \frac{\int_0^a \sigma(r)(f(r))^2 r^2 dr}{\left(\int_0^a \sigma(r) j_l(kr) f(r) r^2 dr \right)^2} \right)^{-1} \tag{4.5}$$

It is seen that this function is minimal when $f(r) = j_l(kr)$ and hence the maximal efficiency for a magnetic partial wave antenna of order l is

$$\eta_{\mathrm{eff}} = \left(1 + \frac{k}{\eta_0 \int\limits_0^{ka} \sigma(x/k)(j_l(x))^2 x^2 dx} \right)^{-1} \tag{4.6}$$

When the electric type partial wave antenna is considered the current density is

$$\boldsymbol{J}(\boldsymbol{r}) = \mathrm{j}\sigma(r) \nabla \times (f(r) \boldsymbol{A}_{1\kappa ml}(\boldsymbol{r})) \tag{4.7}$$

The corresponding far field amplitude reads

$$\boldsymbol{F}(\theta,\phi) = -k\omega\mu_0 \mathrm{j}^{l+1} \int_{V_a} \sigma(r) \boldsymbol{v}_{2\kappa ml}(\boldsymbol{r}') \cdot (\nabla' \times f(r) \boldsymbol{A}_{1\kappa ml}(\theta',\phi')) dv' \boldsymbol{A}_{2\kappa ml}(\theta,\phi) \tag{4.8}$$

The resulting efficiency is

$$\eta_{\mathrm{eff}} = \left(1 + \frac{\eta_0}{\omega^2\mu_0^2} \frac{\int_{V_a} \sigma(r)(\nabla \times f(r) \boldsymbol{A}_{1\kappa ml}(\theta,\phi))^2 dv}{\left(\int_{V_a} \sigma(r)(\nabla \times j_l(kr) \boldsymbol{A}_{1\kappa ml}(\theta,\phi)) \cdot (\nabla \times f(r) \boldsymbol{A}_{1\kappa ml}(\theta,\phi)) dv \right)^2} \right)^{-1} \tag{4.9}$$

By assuming that $f(r) = j_l(kr) + \alpha h(r)$ and finding the minimum of this function, it is seen that $\alpha = 0$. Hence the most efficient antenna of electric type has the efficiency

$$\eta_{\mathrm{eff}} = \left(1 + \frac{k}{\eta_0 \int\limits_0^{ka} \sigma(x/k) \left(\left(j_l'(x) + \frac{1}{x} j_l(x) \right)^2 + l(l+1) \left(\frac{1}{x} j_l(x) \right)^2 \right) x^2 dx} \right)^{-1} \tag{4.10}$$

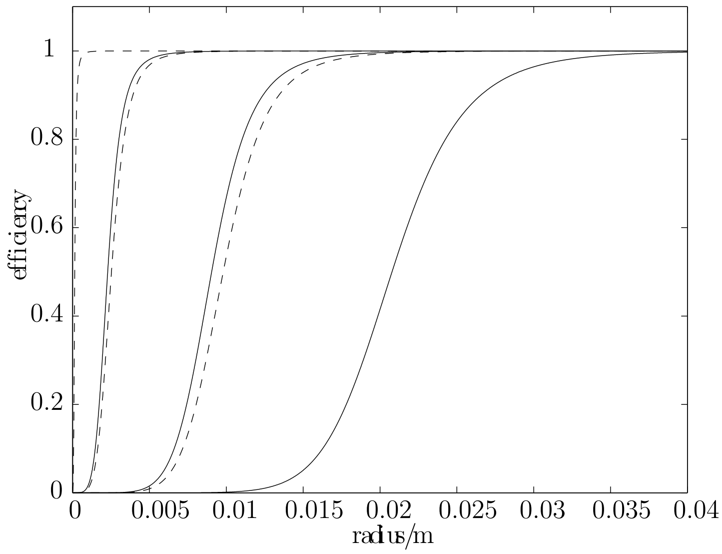

Figure 1: Efficiency for magnetic (solid line) and electric (dashed line) partial wave antenna with $l = 1$ (left curve), $l = 2$ (middle) and $l = 3$ (right) when $\sigma = 10^7$ S/m and $f = 1$ GHz. Notice that the electric partial wave antenna of order l has almost the same efficiency as the magnetic partial wave antenna of order $l - 1$.

By introducing the dimensionless quantities

$$B_{1l} = \frac{\eta_0}{k} \int_0^{ka} \sigma(x/k)(j_l(x))^2 x^2 dx$$
$$B_{2l} = \frac{\eta_0}{k} \int_0^{ka} \sigma(x/k) \left(\left(j_l'(x) + \frac{1}{x} j_l(x) \right)^2 + l(l+1) \left(\frac{1}{x} j_l(x) \right)^2 \right) x^2 dx$$

the efficiency reads

$$\eta_{\tau\text{eff}} = \frac{B_{\tau l}}{B_{\tau l} + 1}$$

where $\tau = 1$ for the magnetic antenna and $\tau = 2$ for the electric antenna. In the case of constant conductivity, $\sigma(r) = \sigma$, the integrals can be solved analytically

$$B_{1l} = \frac{\eta_0 \sigma a}{2} \left((kaj_l')^2 + kaj_l(ka)j_l'(ka) + ((ka)^2 - l(l+1))(j_l(ka))^2 \right)$$
$$B_{2l} = \eta_0 \sigma a j_l(ka) \left(j_l(ka) + kaj_l'(ka) \right) + B_{1l}$$

The explicit expressions for the corresponding electric field, the far field amplitude, the radiated and the dissipated powers are given in appendix B.

Finally, consider a combined multipole antenna with fixed far field amplitude for each of the multipoles. The efficiency of this antenna is optimized when the efficiency of each multipole is optimized. Thus the radial dependence of the current density of each multipole of index l is given by $f_l \sim j_l(kr)$ in Eqs. (4.1) and (4.7). As higher order multipoles are added to an antenna, the efficiency decreases. From Eqs. (4.6) and (4.10) and figure 1 it is seen that there are breakpoints for the efficiency when $B_{\tau l} = 1$. If ka is below this value it is very power consuming to add the multipole of index l. On the other hand, if ka is above the value then the efficiency is only slightly degraded by the addition of the multipole. The curves in figure 1 are valuable for an antenna designer that, *e.g.*, intends to design an antenna with a certain number of ports.

5 Gain

The optimal directivity of a multipole antenna of order l is $D_{\text{opt}} = N_{\text{port}}/2 = (2l+1)/2$, *cf.*, [6] and [7]. The corresponding optimal gain is

$$G_{\tau l} = D_{\text{opt}}\eta_{\tau\text{eff}} = \frac{2l+1}{2}\frac{B_{\tau l}}{B_{\tau l}+1} \tag{5.1}$$

Notice that $G_l \to N_{\text{port}}/2 = (2l+1)/2$ as $ka \to \infty$ and $G_{\tau l}$ is very close to $N_{\text{port}}/2$ once ka passes the breakpoint given by $B_{\tau l} = 1$.

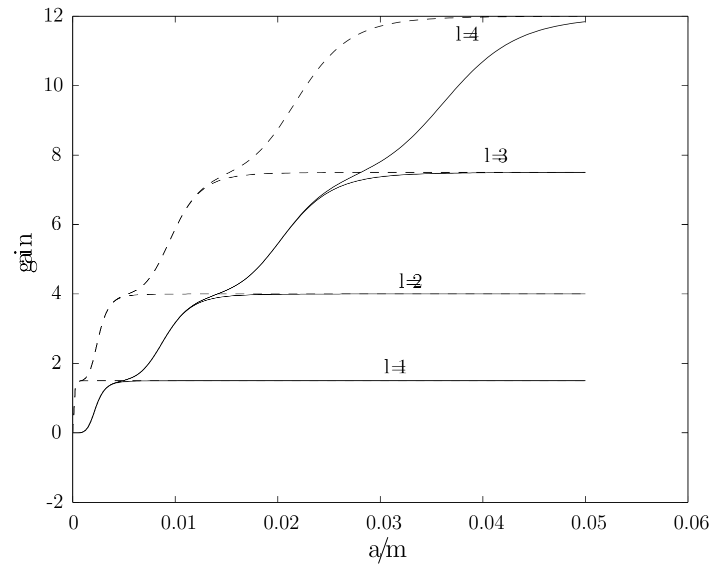

Figure 2: Optimal gain for an electric (dashed line) or magnetic (solid line) antenna of order $l_{\max} = 1, 2, \ldots 4$, when $\sigma = 1 \cdot 10^7$ S/m. The frequency is $f = 1$ GHz. Asymptotically the gain approaches the maximum directivity $D_{\text{opt}} = N_{\text{port}}/2 = l_{\max}(l_{\max}+2)/2$.

The optimal gain of an electric or magnetic antenna of order $l_{\max}$ is somewhat harder to find. However it turns out that the antenna with the optimal gain has a

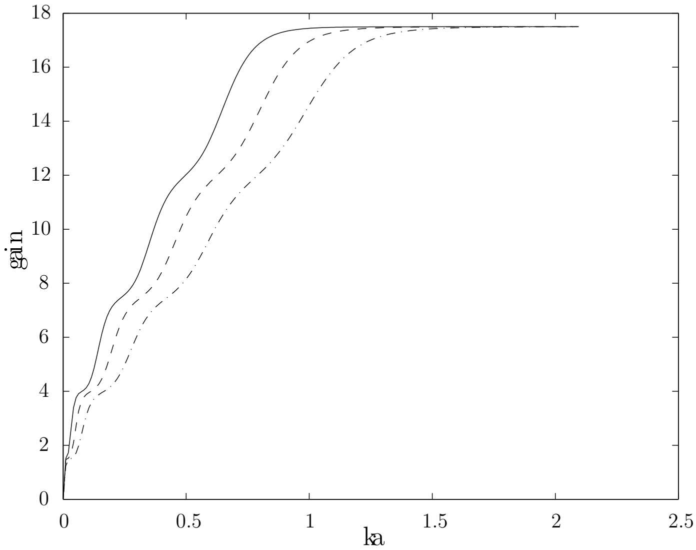

Figure 3: Optimal gain for an electric antenna of order $l_{\max} = 5$, when $\sigma = 1 \cdot 10^7$ S/m as a function of ka. The frequency is $f = 10$ GHz (dash-dot line), $f = 1$ GHz (dashed line) and $f = 100$ MHz (solid line). Notice that the maximum gain does not scale with frequency if the conductivity is kept constant.

gain that is the sum of the optimal gains of the multipole antennas. Thus

$$G_\tau = \sum_{l=1}^{l_{\max}} G_{\tau l} = \sum_{l=1}^{l_{\max}} \frac{2l+1}{2} \frac{B_{\tau l}}{B_{\tau l} + 1} \tag{5.2}$$

The proof is given in appendix B. Also here $G \to N_{\text{port}}/2$ as $ka \to \infty$, *cf.*, figure 2. The efficiency of the order $l_{\max}$ antenna is

$$\eta_\tau = \frac{G_\tau}{l_{\max}(l_{\max} + 2)} \tag{5.3}$$

The optimal gain of a combined antenna of order $l_{\max}$ is simply $G = G_1 + G_2$.

6 Concluding remarks

The currents that give the most optimal antennas in this paper were chosen independently of Maxwell's equations. Needless to say, the real currents that can be created inside a spherical volume have to satisfy Maxwell's equations and will suffer from induction and capacitive coupling that lead to effects that are hard to tamper with, *e.g.*, the skin effect. Thus it is not possible to realize the optimal currents. Anyway, the physical limits of antennas give the antenna designer indications on the achievable efficiency, gain and bandwidth for an antenna of a certain size and frequency. The limits also serve as a measures of the quality of a design. If the values of efficiency, gain and bandwidth are far from the physical limits it might be worthwhile to redesign the antenna. This paper gives no rules of thumb on what can be considered as far from the physical limits, that is left to the designers to explore. It is quite straightforward to write a computer program that illustrates the current densities in Eq. (B.8) in two-dimensional graphs. From such graphs a designer can get ideas on how to construct an antenna with high gain. It is seen that an antenna that is large compared to the wavelength should have its currents close to the surface of the sphere in order to maximize the gain whereas a an antenna that is small compared to the wavelength should have its currents distributed over the entire volume. The amplitude and phase of these currents can be obtained from a graph of the optimal current density.

Appendix A   Vector waves and Green dyadic

The definition of spherical vector waves can be found in different textbooks, e.g. [4] and [6]. In this paper they are defined using vector spherical harmonics, *cf.*, [1]

$$\begin{aligned} \boldsymbol{A}_{1\kappa ml}(\theta,\phi) &= \frac{1}{\sqrt{l(l+1)}}\nabla\times(\boldsymbol{r}Y_{\kappa ml}(\theta,\phi)) \\ \boldsymbol{A}_{2\kappa ml}(\theta,\phi) &= \frac{1}{\sqrt{l(l+1)}}r\nabla Y_{\kappa ml}(\theta,\phi) \\ \boldsymbol{A}_{3\kappa ml}(\theta,\phi) &= \hat{r}Y_{ml}(\theta,\phi). \end{aligned} \tag{A.1}$$

The following definition of the spherical harmonics is used:

$$Y_{\kappa ml}(\theta,\phi)=\sqrt{\frac{\varepsilon_m}{2\pi}}\sqrt{\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)\begin{pmatrix}\cos m\phi\\ \sin m\phi\end{pmatrix} \tag{A.2}$$

where $\varepsilon_m = 2-\delta_{m0}$ and κ, m, l take the values

$$\kappa=\begin{pmatrix}\mathrm{e}\\ \mathrm{o}\end{pmatrix},\quad m=0,\,1,\,2,\,\dots,\,l,\quad l=0,\,1,\dots \tag{A.3}$$

In the current application the index l will never take the value 0, since there are no monopole antennas. The vector spherical harmonics constitute an orthogonal set of vector functions on the unit sphere

$$\int_\Omega \boldsymbol{A}_{\tau n}(\theta,\phi)\cdot\boldsymbol{A}_{\tau' n'}(\theta,\phi)d\Omega=\delta_{\tau\tau'}\delta_{nn'} \tag{A.4}$$

where the integration is over the unit sphere and where $n=\kappa ml$. The outgoing divergence-free spherical vector waves are defined by

$$\begin{cases} \boldsymbol{u}_{1n}(\boldsymbol{r}) = h_l(kr)\boldsymbol{A}_{1n}(\theta,\phi) \\ \boldsymbol{u}_{2n}(\boldsymbol{r}) = \dfrac{1}{k}\nabla\times(h_l(kr)\boldsymbol{A}_{1n}(\theta,\phi)) \\ \qquad\quad = h_l'(kr)\boldsymbol{A}_{2n}(\theta,\phi)+\dfrac{1}{kr}h_l(kr)(\boldsymbol{A}_{2n}(\theta,\phi)+\sqrt{l(l+1)}\boldsymbol{A}_{3n}(\theta,\phi)) \end{cases} \tag{A.5}$$

where $h_l(kr) = h_l^{(2)}(kr)$ is the spherical Hankel function of the second kind. The asymptotic behavior in the far zone of the spherical Hankel functions is

$$h_l^{(2)}(kr) \to \mathrm{j}^{l+1}\frac{e^{-\mathrm{j}kr}}{kr} \quad \text{when}|k|r \to \infty \tag{A.6}$$

The regular wave function $\boldsymbol{v}_{\tau n}(\boldsymbol{r})$ are obtained by replacing the spherical Hankel functions h with the corresponding spherical Bessel functions.

The Green dyadic is given by

$$\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}') = -\mathrm{j}\sum_n \boldsymbol{v}_n(\boldsymbol{r}_<)\boldsymbol{u}_n(\boldsymbol{r}_>) \tag{A.7}$$

where $v_n(\boldsymbol{r})$ is the regular wave function. In a homogeneous space with current density $\boldsymbol{J}$ the electric field is given by

$$\boldsymbol{E}(\boldsymbol{r}) = -\mathrm{j}\omega\mu_0 k\int_{V_a} \boldsymbol{G}\cdot\boldsymbol{J}dv \tag{A.8}$$

Appendix B Optimal gain of an electric or magnetic antenna of order $l_{\max}$.

For a given far field amplitude the most efficient current distribution for each partial wave is the same as for a multipole antenna of order l. Thus the current densities read

$$\boldsymbol{J}(\boldsymbol{r}) = \sigma(r)\sum_{\tau=1}^{2}\sum_n \gamma_{\tau n}\mathrm{j}^{-l+\tau}\boldsymbol{v}_{\tau n}(\boldsymbol{r})$$

where n is the multi-index $n = \kappa ml$, $\gamma_{\tau n}$ are the so far unknown amplitudes of the currents and the factor $\mathrm{j}^{-l-\tau}$ has been introduced for convenience. The corresponding far field amplitude, the radiated power, and the dissipated power read

$$\begin{aligned}
\boldsymbol{F}(\theta,\phi) &= \sum_{\tau=1}^{2}\sum_n \gamma_{\tau n}B_{\tau l}\boldsymbol{A}_{\tau n}(\theta,\phi)\\
P_{\text{rad}} &= \frac{1}{2\eta_0 k^2}\sum_{\tau=1}^{2}\sum_n (\gamma_{\tau n}B_{\tau l})^2\\
P_{\text{ohm}} &= \frac{1}{2\eta_0 k^2}\sum_{\tau=1}^{2}\sum_n \gamma_{\tau n}^2 B_{\tau l}
\end{aligned} \tag{B.1}$$

The gain is given by

$$G = \frac{2\pi|\boldsymbol{F}(\theta,\phi)|^2_{\max}}{k^2\eta_0(P_{\text{rad}}+P_{\text{ohm}})} \tag{B.2}$$

That results in the following expression

$$G = \frac{4\pi\left(\sum_{\tau=1}^{2}\sum_n \gamma_{\tau n}B_{\tau l}\boldsymbol{A}_{\tau n}(\theta,\phi)\right)^2_{\max}}{\sum_{\tau=1}^{2}\sum_n \gamma_{\tau n}^2\left(B_{\tau l}^2 + B_{\tau l}\right)} \tag{B.3}$$

where max is with respect to θ and ϕ. At this stage one can use the same technique as in [6] or [7] to find the maximal gain. Let the direction of maximum gain be $\hat{\boldsymbol{z}}$, i.e., $\theta = 0$ and the polarization be $\hat{\boldsymbol{x}}$. Then

$$G = \frac{4\pi \left(\sum_{\tau=1}^{2} \sum_{n} \gamma_{\tau n} B_{\tau l} |\hat{\boldsymbol{x}} \cdot \boldsymbol{A}_{\tau n}(0, \phi)| \right)^2_{\max}}{\sum_{\tau=1}^{2} \sum_{n} \gamma_{\tau n}^2 \left(B_{\tau l}^2 + B_{\tau l} \right)} \tag{B.4}$$

where

$$\begin{aligned} |\hat{\boldsymbol{x}} \cdot \boldsymbol{A}_{1n}(0, \phi)| &= \delta_{m1} \delta_{\sigma \mathrm{o}} \sqrt{\frac{2l+1}{8\pi}} \\ |\hat{\boldsymbol{x}} \cdot \boldsymbol{A}_{2n}(0, \phi)| &= \delta_{m1} \delta_{\sigma \mathrm{e}} \sqrt{\frac{2l+1}{8\pi}} \end{aligned} \tag{B.5}$$

That means that only $m = 1$ terms should be in the sum.

The extreme value of G is when $\frac{\partial G}{\partial \gamma_{\tau l}} = 0$ for all l. That leads to the relations

$$\gamma_{\tau l} = \sqrt{\frac{2l+1}{3}} \frac{B_{11}+1}{B_{\tau l}+1} \gamma_{11} = \sqrt{\frac{2l+1}{3}} \frac{B_{21}+1}{B_{\tau l}+1} \gamma_{21} \tag{B.6}$$

and the gain

$$G = \sum_{\tau=1}^{2} \sum_{l=1}^{l_{\max}} \frac{2l+1}{2} \frac{B_{\tau l}}{B_{\tau l}+1} \tag{B.7}$$

If only electric or magnetic antennas are used than the sum in τ is omitted.

The optimal current density is

$$\boldsymbol{J}(r, \theta, \phi) = \sum_{\tau=1}^{2} \sum_{l=1}^{l_{\max}} \mathrm{j}^{-l+\tau} \gamma_{11} \sqrt{\frac{2l+1}{3}} \frac{B_{11}+1}{B_{\tau l}+1} \left(\boldsymbol{v}_{1\mathrm{o}1l}(r, \theta, \phi) \delta_{\tau 1} + \boldsymbol{v}_{2\mathrm{e}1l}(r, \theta, \phi) \delta_{\tau 2} \right) \tag{B.8}$$

The regular vector waves $\boldsymbol{v}_{\tau\kappa ml}(\boldsymbol{r})$ are given in appendix A. These current densities result in a far field that is maximal in the direction $\theta = 0$ and with the electric field polarized in the x-direction. The corresponding far field amplitude and the electric field are given by

$$\begin{aligned} \boldsymbol{F}(\theta, \phi) &= \sum_{\tau=1}^{2} \sum_{l=1}^{l_{\max}} \gamma_{11} \sqrt{\frac{2l+1}{3}} \frac{B_{11}+1}{B_{\tau l}+1} B_{\tau l} \left(\boldsymbol{A}_{1\mathrm{o}1l}(r, \theta, \phi) \delta_{\tau 1} + \boldsymbol{A}_{2\mathrm{e}1l}(r, \theta, \phi) \delta_{\tau 2} \right) \\ \boldsymbol{E}(r, \theta, \phi) &= \sum_{\tau=1}^{2} \sum_{l=1}^{l_{\max}} \mathrm{j}^{-l+\tau+2} \gamma_{11} \sqrt{\frac{2l+1}{3}} \frac{B_{11}+1}{B_{\tau l}+1} B_{\tau l} \left(\boldsymbol{u}_{1\mathrm{o}1l}(r, \theta, \phi) \delta_{\tau 1} + \boldsymbol{u}_{2\mathrm{e}1l}(r, \theta, \phi) \delta_{\tau 2} \right) \end{aligned} \tag{B.9}$$

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