Physical bounds on small antennas as convex optimization problems

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***Abstract*—Convex optimization is used to determine the upper bound on G/Q for arbitrarily shaped antennas. The new formulation generalizes previous bounds and can include power dissipated in the antenna. The results are illustrated with a numerical example for planar rectangles.**

I. Introduction

Chu used the stored and radiated energies outside a sphere, with radius a, circumscribing the antenna [1] to determine physical bounds on the Q-factor and the directivity Q-factor quotient, D/Q, see also [2]. The physical bounds on D/Q were generalized to arbitrary size and shape in [3], [4], [5] for $Q \gg 1$. Corresponding bounds on the Q-factor are investigated in the limit of small antennas $ka \ll 1$ in [6], [7] and for finite sizes in [8]. The bounds in [3], [4], [5], [6], [7] are similar for the case of small dipole antennas composed of non-magnetic materials.

In [9] optimal currents and physical bounds on D/Q are formulated as an optimization problem using the expressions for the stored energies by Vandenbosch [10]. The optimization problem is solved with a Lagrangian formulation. The results are valid for antennas composed of non-magnetic materials and they verify the corresponding results in [3], [4], [5].

Here, convex optimization [11] is used to reformulate the optimization problem in [9]. This generalizes the optimization problem to include both the stored electric and magnetic energies. Moreover, it is shown that a finite conductivity can be included in the formulation. The convex optimization problem is only valid for stored energies that are positive semidefinite. This limits its validity for electrically larger structures as shown in [9]. The theoretical results are illustrated by a numerical example for planar structures.

II. G/Q for antennas

The partial gain, $G(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})$, is defined as

$$G(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}}) = 4\pi\frac{P(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{P_{\mathrm{rad}}+P_{\mathrm{loss}}}, \tag{1}$$

where $P(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})$ denotes the radiation intensity in the direction $\hat{\boldsymbol{k}}$ with polarization $\hat{\boldsymbol{e}}$, P_{rad} is the total radiated power, and P_{loss} is the absorbed power in the antenna structure. The quality factor, Q, is defined as

$$Q = \frac{2\mathrm{c}_0 kW}{P_{\mathrm{rad}}+P_{\mathrm{loss}}}, \tag{2}$$

where $W = \max\{W_{\mathrm{e}}, W_{\mathrm{m}}\}$ denotes the maximum of the stored electric and magnetic energies, k the wavenumber, and c_0 the speed of light in free space. Combine (1) and (2) to express the gain Q-factor quotient as

$$\frac{G(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{Q} = \frac{2\pi P(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})}{\mathrm{c}_0 kW}. \tag{3}$$

Use the radiation vector projected on $\hat{\boldsymbol{e}}$, *i.e.*,

$$\hat{\boldsymbol{e}}^*\cdot\boldsymbol{F}(\hat{\boldsymbol{k}}) = \int_V \hat{\boldsymbol{e}}^*\cdot\boldsymbol{J}(\boldsymbol{r})\mathrm{e}^{\mathrm{j}k\hat{\boldsymbol{k}}\cdot\boldsymbol{r}}\,\mathrm{d}V \tag{4}$$

to express the radiation intensity in the electric current density $\boldsymbol{J}$ for the direction $\hat{\boldsymbol{k}}$ and polarization $\hat{\boldsymbol{e}}$ as $P(\hat{\boldsymbol{k}},\hat{\boldsymbol{e}}) = \frac{\zeta_0 k^2}{32\pi^2}|\hat{\boldsymbol{e}}^*\cdot\boldsymbol{F}(\hat{\boldsymbol{k}})|^2$, where ζ_0 denotes the free space impedance, the superscript $*$ denotes the complex conjugate, and the time convention $\mathrm{e}^{\mathrm{j}\omega t}$ is used.

Follow the approach in [9] and use the results by Vandenbosch [10], to write the stored electric energy as $W_{\mathrm{e}} = \widetilde{W}^{(\mathrm{e})}_{\mathrm{vac}} = \frac{\mu_0}{16\pi k^2}w^{(\mathrm{e})}$, where

$$w^{(\mathrm{e})}(\boldsymbol{J}) = \int_V\int_V \nabla_1\cdot\boldsymbol{J}_1\nabla_2\cdot\boldsymbol{J}_2^*\frac{\cos(kR_{12})}{R_{12}} - \frac{k}{2}\big(k^2\boldsymbol{J}_1\cdot\boldsymbol{J}_2^* - \nabla_1\cdot\boldsymbol{J}_1\nabla_2\cdot\boldsymbol{J}_2^*\big)\sin(kR_{12})\,\mathrm{dV}_1\,\mathrm{dV}_2, \tag{5}$$

and $\boldsymbol{J}_1 = \boldsymbol{J}(\boldsymbol{r}_1)$, $\boldsymbol{J}_2 = \boldsymbol{J}(\boldsymbol{r}_2)$, $R_{12} = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$ and μ_0 is the permeability of free space. The corresponding stored magnetic energy is $W_{\mathrm{m}} = \widetilde{W}^{(\mathrm{m})}_{\mathrm{vac}} = \frac{\mu_0}{16\pi k^2}w^{(\mathrm{m})}$, where

$$w^{(\mathrm{m})}(\boldsymbol{J}) = \int_V\int_V k^2\boldsymbol{J}_1\cdot\boldsymbol{J}_2^*\frac{\cos(kR_{12})}{R_{12}} - \frac{k}{2}\big(k^2\boldsymbol{J}_1\cdot\boldsymbol{J}_2^* - \nabla_1\cdot\boldsymbol{J}_1\nabla_2\cdot\boldsymbol{J}_2^*\big)\sin(kR_{12})\,\mathrm{dV}_1\,\mathrm{dV}_2. \tag{6}$$

Expand the current density in basis functions

$$\boldsymbol{J}(\boldsymbol{r}) \approx \sum_{n=1}^{N} J_n\boldsymbol{\psi}_n(\boldsymbol{r}) \tag{7}$$

and introduce the matrix $\mathbf{w}_{\mathrm{e}}$ with elements

$$w^{(\mathrm{e})}_{mn} = \int_V\int_V \nabla_1\cdot\boldsymbol{\psi}_{m1}\nabla_2\cdot\boldsymbol{\psi}_{n2}\frac{\cos(kR_{12})}{R_{12}} - \frac{k}{2}\big(k^2\boldsymbol{\psi}_{m1}\cdot\boldsymbol{\psi}_{n2} - \nabla_1\cdot\boldsymbol{\psi}_{m1}\nabla_2\cdot\boldsymbol{\psi}_{n2}\big)\sin(kR_{12})\,\mathrm{dV}_1\,\mathrm{dV}_2, \tag{8}$$

and similar matrices for the stored magnetic energy and the radiated power.

It is convenient to decompose the current into its real and imaginary parts and collect the expansion coefficients in a column matrix, *i.e.*,

$$\mathbf{J}^{\mathrm{T}} = [\operatorname{Re} J_1, ..., \operatorname{Re} J_N, \operatorname{Im} J_1, ..., \operatorname{Im} J_N]. \tag{9}$$

This gives the normalized stored electric energy as

$$w^{(\mathrm{e})}(\boldsymbol{J}) \approx \sum_{mn} J_m^* w_{mn}^{(\mathrm{e})} J_n = \mathbf{J}^{\mathrm{T}} \begin{pmatrix} \mathbf{w}_{\mathrm{e}} & \mathbf{0} \\ \mathbf{0} & \mathbf{w}_{\mathrm{e}} \end{pmatrix} \mathbf{J} = \mathbf{J}^{\mathrm{T}} \mathbf{W}_{\mathrm{e}} \mathbf{J} \tag{10}$$

and similarly for the stored magnetic energy, $w^{(\mathrm{m})}(\boldsymbol{J}) \approx \mathbf{J}^{\mathrm{T}}\mathbf{W}_{\mathrm{m}}\mathbf{J}$, and the radiated power, $P_{\mathrm{rad}}(\boldsymbol{J}) \approx \mathbf{J}^{\mathrm{T}}\mathbf{P}\mathbf{J}$, where $\mathbf{W}_{\mathrm{e}}$, $\mathbf{W}_{\mathrm{m}}$, and $\mathbf{P}$ are real-valued symmetric matrices.

III. Convex optimization

We use convex optimization [11] to determine fundamental bounds on the antenna performance and their corresponding optimal current densities. We assume that $\mathbf{W}_{\mathrm{e}}$, $\mathbf{W}_{\mathrm{m}}$, and $\mathbf{P}$ are positive semidefinite for sufficiently small structures, see also [9] for examples of indefinite $\mathbf{W}_{\mathrm{e}}$. In [9], the D/Q quotient is maximized for the case with $w^{(\mathrm{e})} \geq w^{(\mathrm{m})}$ using a Lagrangian formulation. To instead obtain a convex optimization problem we rewrite the quotient G/Q as a constrained optimization problem.

We follow [9] and note that G/Q is invariant for multiplicative scalings $\boldsymbol{J} \to \alpha\boldsymbol{J}$ with arbitrary complex valued $\alpha \neq 0$. It is hence sufficient to consider real-valued quantities $\hat{\boldsymbol{e}}^* \cdot \boldsymbol{F} \approx \mathbf{F}^{\mathrm{T}}\mathbf{J}$, see (4). Moreover, maximization of $P \sim |\mathbf{J}^{\mathrm{T}}\mathbf{F}|^2$ can be replaced by minimization of $-\mathbf{F}^{\mathrm{T}}\mathbf{J}$. This gives the convex optimization problem

$$\begin{cases} p = \min_{\mathbf{J}}\{-\mathbf{F}^{\mathrm{T}}\mathbf{J}\} \\ \mathbf{J}^{\mathrm{T}}\mathbf{W}_{\mathrm{e}}\mathbf{J} \leq 1 \\ \mathbf{J}^{\mathrm{T}}\mathbf{W}_{\mathrm{m}}\mathbf{J} \leq 1 \end{cases} \tag{11}$$

if $\mathbf{W}_{\mathrm{e}}$ and $\mathbf{W}_{\mathrm{m}}$ are positive semidefinite. This is a quadratically constrained linear program (QCLP) with the upper bound for $G/Q \leq p^2$. There are many alternative convex formulations, *e.g.*, the Lagrange dual or using that the maximum of two convex functions is convex to minimize the stored energy.

The radiation efficiency can be included in the optimization formulation. Consider for simplicity a thin metallic sheet modeled as a resistive sheet with the constitutive relation $\boldsymbol{E} = R\boldsymbol{J}_{\mathrm{s}}$, where $R = 1/(\sigma d)$ is the surface resistance, d the sheet thickness, σ the conductivity, and $\boldsymbol{J}_{\mathrm{s}}$ the surface current. The absorbed power is

$$P_{\mathrm{loss}} = \int_V \boldsymbol{E} \cdot \boldsymbol{J}_{\mathrm{s}} \,\mathrm{dS} = R \int_V |\boldsymbol{J}_{\mathrm{s}}|^2 \,\mathrm{dS} \approx R\mathbf{J}^{\mathrm{T}}\mathbf{D}\mathbf{J}. \tag{12}$$

We can formulate many convex optimization problems that include losses.

IV. Numerical example

We consider antennas confined to a planar rectangle to illustrate the physical bounds. The bound on G/Q and its corresponding Q are depicted in Fig. 1 for rectangles with side lengths ℓ_1 and ℓ_2 solved using CVX [12].

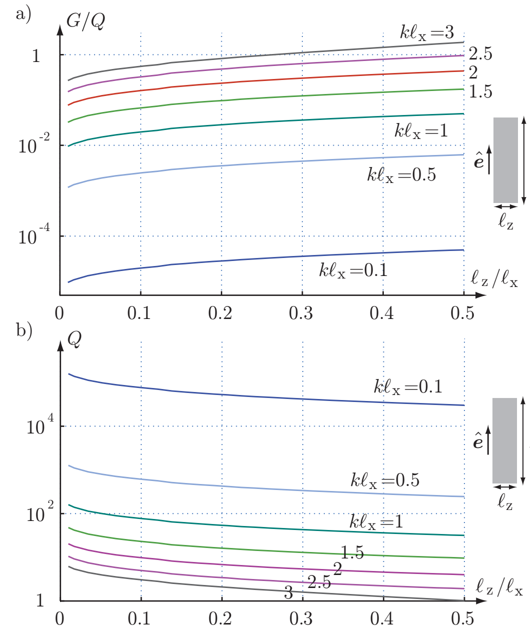

Fig. 1. Upper bounds on G/Q and Q for antennas confined to planar rectangles for $k\ell_{\mathrm{x}} = \{0.1, 1, 1.5, 2, 2.5, 3\}$.

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