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# Wave propagators for transient waves in periodic media

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**Abstract**

One-dimensional propagation of transient EM-waves in periodic media is studied. The media are periodic in the direction of propagation and can be of finite or infinite length. Wave propagators, that map a transient field from one point in the medium to another, are introduced. A number of useful relations for the propagators are presented. Some of these relations are used in the determination of explicit expressions for the short time behavior of a transient wave as it propagates in a periodic medium. The theory is exemplified by several numerical examples.

# 1   Introduction

The wave propagator is a concept that has been used for different wave propagation problems, mainly in the frequency domain [4]. In this paper the propagator is a time-domain operator which propagates a pulse from one point to another in a one-dimensional medium. The approach is based upon results that have been developed during the last 15 years in time-domain invariant imbedding theory, wave splitting methods and the Green functions approach, cf [2] and [10]. The time domain propagators are in this paper applied to periodic media. Wave propagation in these media has been studied in the frequency domain, eg with Flouquet theory and related methods. A review of frequency domain methods for periodic media is found in [3]. Often the determination of pass bands and stop bands for a periodic medium is the main objective of a frequency domain analysis. In a time domain analysis other aspects of wave propagation can be considered and a time domain analysis then serves as a complement rather than an alternative to frequency domain methods.

The algorithm for transient wave propagation in a semi-infinite periodic medium are easily obtained by utilizing basic proparties of the propagators. Once the propagator for one period is known, the propagator for an arbitrary number of periods is known. Also in the case of a medium with a finite number of periods the periodicity can be utilized. It turns out that if the propagator for a medium consisting of one period is known then the propagator for a medium with an arbitrary number of slabs can be constructed in an efficient way. The short time behavior of a wave that propagates in a periodic half-space is also analyzed in the paper. From the definition of the propagator it is seen that the precursor for a periodic medium resembles the first precursor of a homogeneous dispersive medium.

The outline of the paper is as follows: In section 2 the basic equations are presented and the concept of wave splitting is introduced. In section 3 the propagators are presented and in section 4 equations for the corresponding propagator kernels are stated. The periodic half space is analyzed in section 5 and the theory for short time behavior of the transient wave, the precursor, is presented in the same section. The finite periodic slab is analyzed in section 6. Finally, in section 7 several numerical examples are presented.

# 2 The wave equation and wave splitting

One dimensional wave propagation of electromagnetic waves is considered in a linear isotropic non-magnetic medium. The medium can be dissipative and dispersive. If $E$ is a transverse component of the electric field, the wave equation has the following general form

$$\partial_z^2 E(z,t) - \frac{1}{c_0^2}\left(\varepsilon(z)\partial_t^2 E(z,t) + F[E]\right) = 0 \tag{2.1}$$

Here $c_0$ is the speed of light in vacuum, $\varepsilon(z)$ is the permittivity and $F[E]$ is the linear functional

$$\begin{aligned}
F[E] &= \int_{-\infty}^t \chi(z, t-\tau)\partial_\tau^2 E(z,\tau)d\tau \\
&= \chi(z,0)\partial_t E(z,t) + (\partial_t\chi(z,0))\, E(z,t) + \int_{-\infty}^t \left(\partial_t^2\chi(z,t-\tau)\right) E(\tau)d\tau
\end{aligned}$$

The kernel $\chi(t)$ is the susceptibility kernel that relates the displacement field to the electric field by

$$D(z,t) = \varepsilon_0\varepsilon(z)E(z,t) + \varepsilon_0 \int_{-\infty}^t \chi(t-t')E(t')dt'$$

In a matrix form the wave equation reads

$$\partial_z \begin{pmatrix} E \\ \partial_z E \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{c_0^2}\left(\varepsilon(z)\partial_t^2 + F[\cdot]\right) & 0 \end{pmatrix} \begin{pmatrix} E \\ \partial_z E \end{pmatrix} = A \begin{pmatrix} E \\ \partial_z E \end{pmatrix}$$

The wave splitting is defined as the following change of basis

$$\begin{pmatrix} E^+ \\ E^- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -c(z)\partial_t^{-1} \\ 1 & c(z)\partial_t^{-1} \end{pmatrix}\begin{pmatrix} E \\ \partial_z E \end{pmatrix} = P\begin{pmatrix} E \\ \partial_z E \end{pmatrix} \tag{2.2}$$

where

$$\partial_t^{-1}E(z,t) = \int_{-\infty}^t E(z,t')dt'$$

The wave equation, expressed in terms of the split fields $E^\pm$, now takes the form

$$\partial_z \begin{pmatrix} E^+ \\ E^- \end{pmatrix} = (\partial_z P)P^{-1}\begin{pmatrix} E^+ \\ E^- \end{pmatrix} + PAP^{-1}\begin{pmatrix} E^+ \\ E^- \end{pmatrix}$$

where

$$P^{-1} = \begin{pmatrix} 1 & 1 \\ -\frac{1}{c(z)}\partial_t & \frac{1}{c(z)}\partial_t \end{pmatrix}$$

Thus

$$\partial_z \begin{pmatrix} E^+ \\ E^- \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} E^+ \\ E^- \end{pmatrix} \tag{2.3}$$

$$\alpha = -\frac{1}{c(z)}\partial_t - \frac{c(z)}{2c_0^2}\partial_t^{-1}F[\cdot] + \frac{1}{2c(z)}\partial_z c(z) \tag{2.4}$$

$$\beta = -\frac{c(z)}{2c_0^2}\partial_t^{-1}F[\cdot] - \frac{1}{2c(z)}\partial_z c(z) \tag{2.5}$$

$$\gamma = \frac{c(z)}{2c_0^2}\partial_t^{-1}F[\cdot] - \frac{1}{2c(z)}\partial_z c(z) \tag{2.6}$$

$$\delta = \frac{1}{c(z)}\partial_t + \frac{c(z)}{2c_0^2}\partial_t^{-1}F[\cdot] + \frac{1}{2c(z)}\partial_z c(z) \tag{2.7}$$

It is seen that

$$\partial_t^{-1}F[E] = \int_{-\infty}^{t}\chi(z, t-\tau)\partial_\tau E(z,\tau)d\tau$$

In a homogeneous non-dispersive medium, i.e., where $c(z)$ is constant and $F[E] \equiv 0$, the system of equations for $E^{\pm}$ is diagonal. Then $E^+$ is a right moving wave and $E^-$ is a left moving wave. The wave splitting is not unique and there are other possible splittings, cf [5]. However, a necessary condition for a splitting is that the principal part of the hyperbolic wave operator is split as in Eq. (2.2).

# 3   The wave propagators

The propagators are the key for the time-domain analysis of wave propagation in a periodic medium. From the definition of the propagators a number of relations, which are useful for a periodic medium, can be derived.

Consider a one-dimensional medium occupying the interval $0 < z < L$. To the left ($z < 0$) and to the right ($z > L$) of the medium there is vacuum. The medium can be semi-infinite, i.e., $L$ may be infinite. It is assumed that the phase velocity is continuously differentiable and that the susceptibility kernel $\chi(z,t)$ is a continuous function of $z$ in the region $0 \le z \le L$. There are ways to handle discontinuities in the phase velocity and its derivative, cf [9], but for simplicity it is not considered in this paper. An incident wave impinges at $z = 0$ from the left at time $t = 0$. In the slab the total field and its time derivative are zero for negative times. For positive times the total field is the sum of the split fields, i.e.,

$$E(z,t) = E^+(z,t) + E^-(z,t)$$

The wave propagators $\mathcal{G}^{\pm}$ map the field $E^+$ at a point $z_0$ where $0 \le z_0 \le L$ to a point $z_1$ where $0 \le z_1 \le L$. They are defined as

$$E^+(z_1, t + \ell(z_0, z_1)) = \mathcal{G}^+(z_0, z_1)E^+(z_0, t) \tag{3.1}$$

$$E^-(z_1, t + \ell(z_0, z_1)) = \mathcal{G}^-(z_0, z_1)E^+(z_0, t) \tag{3.2}$$

where $\ell(z_0, z_1)$ is the travel time for the wave front from $z_0$ to $z_1$, i.e.,

$$\ell(z_0, z_1) = \int_{z_0}^{z_1}\frac{1}{c(z)}dz \tag{3.3}$$

Normally it is assumed that $z_1 > z_0$, so that the wave is propagated forward in time and space. However, the definitions are valid also for $z_1 < z_0$.

The propagators can be shown from arguments based upon causality and invariance under time translation to have the representations

$$E^+(z_1, t + \ell(z_0, z_1)) = a(z_0, z_1)E^+(z_0, t) + [G^+(z_0, z_1, \cdot) * E^+(z_0, \cdot)](t) \quad (3.4)$$
$$E^-(z_1, t + \ell(z_0, z_1)) = [G^-(z_0, z_1, \cdot) * E^+(z_0, \cdot)](t) \quad (3.5)$$

The shorthand notation for convolution

$$[G^+(z_0, z_1, \cdot) * E^+(z_0, \cdot)](t) = \int_0^t G^+(z_0, z_1, t - t')E^+(z_0, t')dt'$$

is adopted in these expressions. It should be noticed that wave front time is used in the representations (3.4) and (3.5). This means that at every point $z$, time is put to zero when the wave front passes. The lower integration limit in the convolution is zero, since there is no field in the slab for negative times and the upper limit is $t$ due to causality. The factor $a(z_0, z_1)$ is the attenuation of the wave front when it travels from $z_0$ to $z_1$. The explicit expression for this factor reads

$$a(z_0, z_1) = \sqrt{\frac{c(z_1)}{c(z_0)}} \exp\left(-\frac{1}{2c_0^2} \int_{z_0}^{z_1} c(z)\chi(z, 0)\, dz\right) \quad (3.6)$$

The expression is derived in the appendix.

The following rule is implied from the definition of the propagators, Eqs. (3.1) and (3.2),

$$\mathcal{G}^\pm(z, z'') = \mathcal{G}^\pm(z', z'')\mathcal{G}^+(z, z') \quad (3.7)$$

The kernels $G^\pm$ then obey

$$G^+(z, z'', t) = a(z, z')G^+(z', z'', t) + a(z', z'')G^+(z, z', t) \quad (3.8)$$
$$+ [G^+(z, z', \cdot) * G^+(z', z'', \cdot)](t)$$
$$G^-(z, z'', t) = a(z, z')G^-(z', z'', t) + [G^-(z', z'', \cdot) * G^+(z, z', \cdot)](t) \quad (3.9)$$

It is also seen from Eq. (3.7) that $\mathcal{G}^+(z', z)$ is the inverse operator of $\mathcal{G}^+(z, z')$, i.e., $\mathcal{G}^+(z, z')\mathcal{G}^+(z', z)E^+(z, t) = E^+(z, t)$. The kernel $G^+(z', z, t)$ is then the resolvent of the propagator kernel $G^+(z, z', t)$, i.e.,

$$a(z, z')G^+(z', z, t) + a(z', z)G^+(z, z', t) + [G^+(z, z', \cdot) * G^+(z', z, \cdot)](t) = 0$$

From the definition of the propagators $\mathcal{G}^\pm(z_0, z_1)$, $z_0 < z_1$, it follows that they are independent of the half-space $z < z_0$.

The boundary values of the propagator kernels $G^\pm(z, z', t)$ are related to the Green kernels used in the Green functions technique, cf [10], and to the reflection and transmission kernels used in the invariant imbedding technique, cf [8], in the

following manner:

$$G^{\pm}(0, z, t) = \quad \text{the Green kernels used in the Green functions} \tag{3.10}$$
$$\text{technique} \tag{3.11}$$
$$G^{+}(z, L, t) = \quad T(z, t) = \text{the transmission kernel for the imbedded} \tag{3.12}$$
$$\text{subslab}\, [z, L]$$
$$G^{-}(z, z, t) = \quad R(z, t) = \text{the reflection kernel for the imbedded} \tag{3.13}$$
$$\text{subslab}\, [z, L]$$
$$G^{+}(0, L, t) = \quad T(t) = \text{the transmission kernel for the slab} \tag{3.14}$$
$$G^{-}(0, 0, t) = \quad R(t) = \text{the reflection kernel for the slab} \tag{3.15}$$

As a special case of Eq. (3.9) it is seen that for $z'' = z'$

$$G^{-}(z, z') = R(z', t) + \left[ R(z', \cdot) * G^{+}(z, z', \cdot) \right](t) \tag{3.16}$$

# 4 Equations for the propagator kernels

In this section four partial differential equations are presented for the two propagator kernels $G^{+}(z, z', t)$ and $G^{-}(z, z', t)$. The derivation of these equations are found in the appendix. The variables $\eta$ and $\zeta$ are introduced as two internal points in the slab, such that $0 \leq \eta \leq \zeta \leq L$. The space and time derivatives of the material parameters are denoted by subindices, thus $\partial_t \chi(t) = \chi_t(t)$ and $\partial_z c(z) = c_z(z)$. The first two equations are obtained by differentiating the representations in Eqs. (3.4) and (3.5) with respect to $\zeta$. By using the dynamics for the split fields, Eq. (2.3), two equations are obtained as

$$\partial_\zeta G^{+}(\eta, \zeta, t) = \frac{c_\zeta(\zeta)}{2c(\zeta)}(G^{+}(\eta, \zeta, t) - G^{-}(\eta, \zeta, t)) \tag{4.1}$$
$$- \frac{c(\zeta)}{2c_0^2}\left( a(\eta, \zeta)\chi_t(\zeta, t) + \chi(\zeta, 0)(G^{+}(\eta, \zeta, t) + G^{-}(\eta, \zeta, t)) \right.$$
$$+ [\chi_t(\zeta, \cdot) * (G^{+}(\eta, \zeta, \cdot) + G^{-}(\eta, \zeta, \cdot))](t))$$
$$\partial_\zeta G^{-}(\eta, \zeta, t) - \frac{2}{c(\zeta)}\partial_t G^{-}(\eta, \zeta, t) = \frac{c_\zeta(\zeta)}{2c(\zeta)}\left( G^{-}(\eta, \zeta, t) - G^{+}(\eta, \zeta, t) \right) \tag{4.2}$$
$$+ \frac{c(\zeta)}{2c_0^2}\left( a(\eta, \zeta)\chi_t(\zeta, t) + \chi(\zeta, 0)(G^{+}(\eta, \zeta, t) + G^{-}(\eta, \zeta, t)) \right.$$
$$+ [\chi_t(\zeta, \cdot) * (G^{+}(\eta, \zeta, \cdot) + G^{-}(\eta, \zeta, \cdot))](t))$$

The initial condition for the kernel $G^{-}$ reads

$$G^{-}(\eta, \zeta, 0) = \frac{1}{4}a(\eta, \zeta)\left( c_\zeta(\zeta) - \left( \frac{c(\zeta)}{c_0} \right)^2 \chi(\zeta, 0) \right) \tag{4.3}$$

The two equations (4.1) and (4.2) together with the initial condition above and the boundary condition for $G^+$, which is presented in the next two sections, form a complete set of equations for the propagator kernels $G^\pm(\eta, \zeta)$.

The alternative set if equations are obtained by differentiating Eqs. (3.4) and (3.5) with respect to $\eta$ and using the dynamics, Eq. (2.3),

$$\partial_\eta G^+(\eta, \zeta, \ t) = \frac{c_\eta(\eta)}{2c(\eta)} \left( a(\eta, \zeta)R(\eta, t) - G^+(\eta, \zeta, t) + [G^+(\eta, \zeta, \cdot) * R(\eta, \cdot)](t) \right) \tag{4.4}$$

$$+ \frac{c(\eta)}{2c_0^2} \left( \chi(\eta, 0)(a(\eta, \zeta)R(\eta, t) + G^+(\eta, \zeta, t) + [G^+(\eta, \zeta, \cdot) * R(\eta, \cdot)](t)) \right.$$
$$+ a(\eta, \zeta)\chi_t(\eta, t) + \left[ \chi_t(\eta, \cdot) * (a(\eta, \zeta)R(\eta, \cdot) + G^+(\eta, \zeta, \cdot)) \right](t)$$
$$+ [\chi_t(\eta, \cdot) * G^+(\eta, \zeta, \cdot) * R(\eta, \cdot)](t))$$

$$\partial_\eta G^-(\eta, \zeta, t) = \frac{c_\eta(\eta)}{2c(\eta)} \left( [G^-(\eta, \zeta, \cdot) * R(\eta, \cdot)](t) - G^-(\eta, \zeta, t) \right) \tag{4.5}$$

$$+ \frac{c(\eta)}{2c_0^2} \left( \chi(\eta, 0)([G^-(\eta, \zeta, \cdot) * R(\eta, \cdot)](t) + G^-(\eta, \zeta, t)) \right.$$
$$+ \left[ \chi_t(\eta, \cdot) * (G^-(\eta, \zeta, \cdot) + [G^-(\eta, \zeta, \cdot) * R(\eta, \cdot)](\cdot)) \right](t))$$

As pointed out in the previous section $R(\eta, t) = G^-(\eta, \eta, t)$ is the reflection kernel for the imbedded slab $[\eta, L]$. Thus $R(\eta, t)$ is needed in order to solve Eqs. (4.4) and (4.5). The equation for $R(\eta, t)$ is obtained by adding Eq. (4.2) to Eq. (4.5) and then let $\eta = \zeta$

$$2R_\eta(\eta, t) - \frac{4}{c(\eta)} R_t \ (\eta, t) = \frac{c_\eta(\eta)}{c(\eta)} [R(\eta, \cdot) * R(\eta, \cdot)](t) \tag{4.6}$$

$$+ \frac{c(\eta)}{c_0^2} \left( \chi_t(\eta, t) + 2\chi(\eta, 0)R(\eta, t) + \chi(\eta, 0)[R(\eta, \cdot) * R(\eta, \cdot)](t) \right.$$
$$+ 2[\chi_t * R](\eta, t) + [\chi_t * R * R](\eta, t))$$

The initial condition for $R(\eta, t)$ is obtained from Eq. (4.3)

$$R(\eta, 0) = \frac{1}{4} \left( c_\eta(\eta) - \left( \frac{c(\eta)}{c_0} \right)^2 \chi(\eta, 0) \right)$$

In order to solve Eq. (4.4), given the reflection kernel, a boundary condition for $G^+$ is needed this is presented in the next two sections. The initial condition for $G^-$, which is needed for the solution of Eq. (4.2), is given by Eq. (4.3).

It is worth noticing that when $\zeta = L$ $G^+(\eta, L, t)$ is the transmission kernel for the imbedded slab $(\eta, L)$ and Eq. (4.4) is the same equation as the one used for the transmission kernel in the invariant imbedding technique, cf [8]

In order to solve Eqs. (4.1), (4.2), (4.4)-(4.6) boundary conditions for the kernels are needed. These conditions are discussed in the next two sections.

# 5 The periodic half-space

Consider that the medium introduced in the previous section is periodic and semi-infinite. The medium is partitioned into identical subslabs $S_i$, $i = 1, 2, ...$ of length $q$, with interfaces at $z_i = iq$, $i = 0, 1, 2...$. The phase velocity and the susceptibility kernel $\chi(z, t)$ are periodic functions of $z$ so that $c(z) = c(z + q)$ and $\chi(z, t) = \chi(z + q, t)$ when $z \geq 0$.

From Eq. (3.7) it is seen that for a semi-infinite periodic medium

$$\mathcal{G}^+(0, z_j) = (\mathcal{G}^+(0, z_1))^j \tag{5.1}$$

and furthermore for $z_j \leq z \leq z_{j+1}$

$$\mathcal{G}^+(0, z) = \mathcal{G}^+(z_j, z)\mathcal{G}^+(0, z_j)$$
$$\mathcal{G}^-(0, z) = \mathcal{G}^-(z, z)\mathcal{G}^+(0, z)$$

It is in this case enough to determine the propagators $\mathcal{G}^\pm$ in the first subslab $0 \leq z \leq q$, in order to determine the field at any point in the semi-infinite half space. There are two different ways to numerically determine $\mathcal{G}^\pm$ for one period. The two are comparable in efficiency. The first one is to solve Eqs. (4.1) and (4.2) as in [10]. The other way is to determine the kernels $G^+$ and $R$ in the first subslab, and then construct $G^-$ from Eq. (3.16). To obtain $G^+$ and $R$ it is appropriate to first determine the reflection kernel $R(z, t)$ for $0 \leq z \leq q$ from Eq. (4.6). The boundary condition for the reflection kernel is

$$R(q, t) = R(0, t)$$

due to the periodicity. The boundary condition means that the equation for $R$, Eq. (4.6), breeds itself with the boundary condition. Once the reflection kernel is known, the kernel $G^+(0, z, t)$ can be determined. This is done by solving Eq. (4.4) in the interval $0 \leq z \leq q$ with the boundary condition

$$G^+(q, q) = 0$$

The periodicity of the reflection kernel implies that the propagator $\mathcal{G}^+(z_i, z_j)$ not only is the propagator for the field $E^+(z, t)$ from $z_i$ to $z_j$ but also for the total field $E(z, t)$ as well as for the field $E^-(z, t)$. This is seen by

$$E^+(z_j, t + \ell(z_i, z_j)) = \mathcal{G}^+(z_i, z_j)E^+(z_i, t)$$
$$E^-(z_j, t + \ell(z_i, z_j)) = \mathcal{G}^+(z_i, z_j)\mathcal{G}^-(z_j, z_j)E^+(z_i, t)$$
$$E^-(z_i, t) = \mathcal{G}^-(z_i, z_i)E^+(z_i, t)$$

Since $\mathcal{G}^-(z_i, z_i) = \mathcal{G}^-(z_j, z_j)$ and $E(z, t) = E^+(z, t) + E^-(z, t)$ it follows that

$$E^-(z_j, t + \ell(z_i, z_j)) = \mathcal{G}^+(z_i, z_j)\mathcal{G}^-(z_i, z_i)E^+(z_i, t) = \mathcal{G}^+(z_i, z_j)E^-(z_i, t)$$
$$E(z_j, t + \ell(z_i, z_j)) = \mathcal{G}^+(z_i, z_j)E(z_i, t)$$

The total field $E(z, t)$ is understood to be a field which is generated by an incident wave from the left. Since the propagator $\mathcal{G}^+(z_i, z_j)$ is a propagator for the entire field it must be independent of the splitting.

# 6  The precursor in a semi-infinite slab

As a wave propagates in a semi-infinite periodic medium a precursor develops. Since the precursor is the short time behavior of the wave, it consists of the high frequencies of the wave. An expression for the precursor can be found from Eqs. (3.4) and (3.5). First the homogeneous dispersive medium is analyzed. Since there is no $z-$dependence in this case, it means that the period of the medium is infinitesimal. Let $q = dz$ be an infinitesimal piece of the medium and let $N = z/dz$. It then follows from Eq. (5.1) that

$$\mathcal{G}^+(0, z) = (\mathcal{G}^+(0, dz))^N$$

For small enough times the following Maclaurin expansion is a good approximation to the propagator

$$\mathcal{G}^+\ (0, z) \approx ((a(0, dz)\delta(t) + G^+(0, dz, 0))*)^{(N-1)}(a(0, dz)\delta(t) + G^+(0, dz, 0))$$
$$= ((a(0, dz)\delta(t) + dzG_z^+(0, 0, 0))*)^{(N-1)}(a(0, dz)\delta(t) + dzG_z^+(0, 0, 0))$$

where $G_z(0, 0, 0) = \partial_z G(0, z, 0)|_{z=0}$ denotes the $z-$derivative of the Green function at $z = 0$ and $t = 0$.

The asymptotic expression for the propagator for small $t$ is found by using the binomial expansion. The relation

$$(G_z^+(0, 0, 0)*)^k G_z^+(0, 0, 0) = \frac{(G_z^+(0, 0, 0)t)^k}{k!}$$

and the limiting value

$$\lim_{N\to\infty} \binom{N}{k}(dz)^k = \lim_{N\to\infty} \frac{(Ndz)^k}{k!} = \frac{z^k}{k!}$$

for any fixed $k$ are then used. It is seen that

$$\mathcal{G}^+(0, z) \approx a(0, z)\left(\delta(t) + zG_z^+(0, 0, 0)\sum_{k=0}^{\infty} \frac{(tzG_z^+(0, 0, 0))^k}{(k+1)!k!}\right)$$
$$= a(0, z)\left(\delta(t) + zG_z^+(0, 0, 0)\frac{J_1(2\sqrt{-tzG_z^+(0, 0, 0)})}{\sqrt{-tzG_z^+(0, 0, 0)}}\right)$$

where $J_1$ is the Bessel function of order one. This is the expression for the first precursor (the Sommerfeld precursor). The expression is well known and is normally derived by the stationary phase method, cf [6] and [11]. The value of $G_z^+(0, 0, 0)$ for a homogeneous dispersive medium is easily obtained from Eqs. (4.1) and (4.3)

$$G_z(0, 0, 0) = \frac{c(0)}{8c_0^2}\left(\chi(0)^2\left(\frac{c(0)}{c_0}\right)^2 - 4\chi_t(0)\right)$$

Next consider a periodic medium with period $q$. As above the following expansion is a good approximation to the wave propagator for small enough times

$$\mathcal{G}^+ (0, Nq) \approx ((a(0,q)\delta(t) + G^+(0,q,0))*)^{(N-1)}(a(0,q)\delta(t) + G^+(0,q,0))$$

$$= a(0, Nq)\left(\delta(t) + \frac{G^+(0,q,0)}{a(0,q)}\sum_{k=1}^{N}\binom{N}{k}\frac{(a^{-1}(0,q)G^+(0,q,0)t)^{k-1}}{(k-1)!}\right)$$

As $N \to \infty$ then

$$\mathcal{G}^+(0, Nq) \to a(0, Nq)\left(\delta(t) + NG^+(0,q,0)\frac{J_1(2\sqrt{-a^{-1}(0,q)tNG^+(0,q,0)})}{\sqrt{-a(0,q)tNG^+(0,q,0)}}\right) \quad (6.1)$$

Thus, far into the medium the precursor for the periodic medium resembles the precursor for a homogeneous dispersive medium. The initial value of the kernel $G^+(0, q, 0)$ is obtained from Eq. (4.1). The expression is

$$G^+(0, q, 0) = \frac{a(0,q)}{8}\int_0^q \frac{1}{c(z)}\left((\chi(z,0))^2\left(\frac{c(z)}{c_0}\right)^4\right.$$

$$\left. -4\left(\frac{c(z)}{c_0}\right)^2\chi_t(z,0) - \left(\frac{c_z(z)}{c(z)}\right)^2\right)dz \quad (6.2)$$

If $G^+(0, q, 0) \leq 0$ then the argument of the Bessel function is real and the Green kernel is oscillating. From Eq. (6.2) it is seen that this is always the case for a non-dispersive medium. If $G^+(0, q, 0) > 0$ the argument of the Bessel function is imaginary and the Green kernel is exponentially growing for small times.

# 7   The finite periodic slab

The analysis and results presented in this section are similar to the ones presented in [7] for a medium partitioned into several subslabs. In a periodic medium the subslabs are identical and that leads to considerable simplifications.

Assume that the periodic medium is finite and consists of $N$ identical subslabs. Since the length of the medium is denoted $L$, it follows that $q = L/N$. It is clear that the propagators $\mathcal{G}^\pm(z_i, z_{i+1})$ are now dependent on the index $i$. The algorithm described in this section utilizes the propagators for a subslab imbedded in vacuum. Once these propagators are known, the propagators for the entire slab can be formed. Consider a subslab $S_i$ imbedded in vacuum. For a right moving wave incident at the surface $z_i$ at time $t = 0$, the reflected and transmitted fields have the following representations, cf Eqs. (3.4) and (3.5),

$$E^+(z_{i+1}, t + \ell) = \mathcal{T}_0 E^+(z_i, t) = aE^+(z_i, t) + [T_0(\cdot) * E^+(z_i, \cdot)](t)$$
$$E^-(z_i, t) = \mathcal{R}_0^+ E^+(z_i, t) = [R_0^+(\cdot) * E^+(z_i, \cdot)](t)$$

Here $\ell = \ell(z_i, z_{i+1})$ is the time it takes for the wave front to travel through a subslab, and $a = a(z_i, z_{i+1})$ is the attenuation of the wavefront when it travels through a

subslab, cf Eqs. (3.3) and (3.6). The reflection and transmission operators for an incident left moving wave that impinges at $z_{i+1}$ at time $t = 0$ are defined analogously

$$E^-(z_i, t + \ell) = \mathcal{T}_0 E^-(z_{i+1}, t) = a E^-(z_{i+1}, t) + [T_0(\cdot) * E^-(z_{i+1}, \cdot)](t)$$
$$E^+(z_{i+1}, t) = \mathcal{R}_0^- E^-(z_{i+1}, t) = [R_0^-(\cdot) * E^-(z_{i+1}, \cdot)](t)$$

Notice that the transmission operators for incidence from the left and from the right are identical since the medium is reciprocal. The reflection and transmission kernels can be found by solving either the Green function equations, Eqs. (4.1) and (4.2), or the invariant imbedding equations (4.4) and (4.6) for the region $0 \le z \le q$. The boundary conditions used for the Green function equations are

$$G_0^+(0, 0, t) = 0$$
$$G_0^-(0, q, t) = 0$$

and for the imbedding equations

$$G_0^+(q, q, t) = 0$$
$$R_0^+(q, t) = 0$$

Going back to the entire slab, the split fields at a point $z_{i+1}$ is related to the field $E^+(z_i)$ by Eqs. (3.4) and (3.5), i.e.,

$$E^+(z_{i+1}) = \mathcal{G}^+(z_i, z_{i+1}) E^+(z_i)$$
$$E^-(z_{i+1}) = \mathcal{G}^-(z_i, z_{i+1}) E^+(z_i)$$

The following representation also holds

$$\mathcal{G}^+(z_i, z_{i+1}) E^+(z_i) = \mathcal{T}_0 E^+(z_i) + \mathcal{R}_0^- E^-(z_{i+1}) \tag{7.1}$$

where

$$E^-(z_{i+1}) = \mathcal{G}^-(z_{i+1}, z_{i+1}) E^+(z_{i+1}) = \mathcal{G}^-(z_{i+1}, z_{i+1})(\mathcal{T}_0 E^+(z_i) + \mathcal{R}_0^- E^-(z_{i+1})) \tag{7.2}$$

This operator equation may be solved formally for $E^-(z_{i+1})$ giving

$$E^-(z_{i+1}) = (1 - \mathcal{G}^-(z_{i+1}, z_{i+1}) \mathcal{R}_0^-)^{-1} \mathcal{G}^-(z_{i+1}, z_{i+1}) \mathcal{T}_0 E^+(z_i) \tag{7.3}$$

Two expressions for the Green operators $\mathcal{G}^\pm(z_i, z_{i+1})$ have then been derived

$$\mathcal{G}^+(z_i, z_{i+1}) = \mathcal{T}_0 + \mathcal{R}_0^-(1 - \mathcal{G}^-(z_{i+1}, z_{i+1}) \mathcal{R}_0^-)^{-1} \mathcal{G}^-(z_{i+1}, z_{i+1}) \mathcal{T}_0 \tag{7.4}$$
$$\mathcal{G}^-(z_i, z_{i+1}) = (1 - \mathcal{G}^-(z_{i+1}, z_{i+1}) \mathcal{R}_0^-)^{-1} \mathcal{G}^-(z_{i+1}, z_{i+1}) \mathcal{T}_0 \tag{7.5}$$

The operator $(1 - \mathcal{G}^-(z_{i+1}, z_{i+1}) \mathcal{R}_0^-)^{-1}$ has the explicit representation

$$(1 - \mathcal{G}^-(z_{i+1}, z_{i+1}) \mathcal{R}_0^-)^{-1} E(t) = E(t) + [K_{i+1}(\cdot) * E(\cdot)](t) \tag{7.6}$$

The resolvent kernel $K_i(t)$ satisfies the Volterra equation of the second kind

$$K_i(t) - [R(z_i, \cdot) * R_0^-(\cdot)](t) - [R(z_i, \cdot) * R_0^-(\cdot) * K_i(\cdot)](t) = 0, \tag{7.7}$$

This is found by operating with $(1 - \mathcal{G}^-(z_{i+1}, z_{i+1})\mathcal{R}_0^-)$ on Eq. (7.6) and using $G^-(z, z, t) = R(z, t)$, cf Eq. (3.13). The explicit expressions for the kernels $G^\pm(z_i, z_{i+1}, t)$ follow from Eqs. (7.4) and (7.5)

$$
\begin{aligned}
G^+(z_i, z_{i+1}, t) \ =& \ T_0(t) + a \left( [R_0^-(\cdot) * R(z_{i+1}, \cdot)](t) \right. \\
&+ [R_0^-(\cdot) * K_{i+1}(\cdot) * R(z_{i+1}, \cdot)](t) \left.\right) + [R_0^-(\cdot) * R(z_{i+1}, \cdot) * T_0(\cdot)](t) \\
&+ [R_0^-(\cdot) * K_{i+1}(\cdot) * R(z_{i+1}, \cdot) * T_0(\cdot)](t)
\end{aligned} \tag{7.8}
$$

$$
\begin{aligned}
G^-(z_i, z_{i+1}, t) \ =& \ a \left( R(z_{i+1}, t) + [R(z_{i+1}, \cdot) * K_{i+1}(\cdot)](t) \right) \\
&+ [R(z_{i+1}, \cdot) * T_0(\cdot)](t) + [R(z_{i+1}, \cdot) * K_{i+1}(\cdot) * T_0(\cdot)](t)
\end{aligned} \tag{7.9}
$$

The transmitted field from an incident field $E^+(0, t)$ is obtained from Eqs. (3.8), (3.9), (7.7)-(7.9). To exemplify this, the scheme for the determination of the reflection and transmission kernels for a medium which consists of $N = 2^n$ identical symmetric subslabs is discussed. Since the subslabs are symmetric it follows that $R_0^+ = R_0^- = R_0$. As a first step, the reflection and transmission kernels $R_0$ and $T_0$ are calculated for the subslab imbedded in vacuum. In the last slab, $S_N$, it is seen that

$$
\begin{aligned}
G^+(z_{N-1}, z_N, t) \ &= \ T_0(t) \\
R(z_{N-1}, t) \ &= \ R_0(t)
\end{aligned}
$$

Since $R(z_{N-1}, t)$ is known, the kernel $K_{N-1}$ can be obtained from Eq. (7.7). The kernels $G^\pm(z_{N-2}, z_{N-1}, t)$ are then obtained from Eqs. (7.8) and (7.9), whereupon the kernel $G^+(z_{N-2}, L, t) = T(z_{N-2}, t)$ is obtained from Eq. (3.8). If one can obtain the reflection kernel $R(z_{N-2}, z_{N-2}, t)$ then one may view the medium as composed of $2^{n-1}$ subslabs of width $2z_1 = 2q$ and where the reflection and transmission kernels for each subslab imbedded in vacuum are known. To find the expression for $R(z_{N-2}, t)$, it is first noticed that

$$
E^-(z_{N-2}) = \mathcal{R}(z_{N-2})E^+(z_{N-2}) = \mathcal{T}_0 E^-(z_{N-1}) + \mathcal{R}_0 E^+(z_{N-2}) \tag{7.10}
$$

where $E^-(z_{N-1})$ is found from Eq. (7.3). Thus

$$
\begin{aligned}
R(z_{N-2}, t) \ =& \ R_0(t) + a^2 R(z_{N-1}, t) + a^2 [K_{N-1}(\cdot) * R(z_{N-1}, \cdot)](t) \\
&+ 2a[T_0(\cdot) * R(z_{N-1}, \cdot)](t) + [T_0(\cdot) * R(z_{N-1}, \cdot) * T_0(\cdot)](t) \\
&+ 2a[R(z_{N-1}, \cdot) * K_{N-1} * T_0(\cdot)](t) \\
&+ [T_0(\cdot) * K_{N-1}(\cdot) * R(z_{n-1}, \cdot) * T_0(\cdot)](t)
\end{aligned}
$$

By doing the procedure above once more $R(z_{N-4}, t)$ and $T(z_{N-4}, t)$ are obtained. Finally, after $n - 1$ iterations the reflection and transmission kernels for the entire slab, $R(t)$ and $T(t)$, are obtained.

# 8   Numerical examples

In this section two examples of wave propagation in a periodic half space are presented. In the first example the medium is dielectric and non-dispersive. The
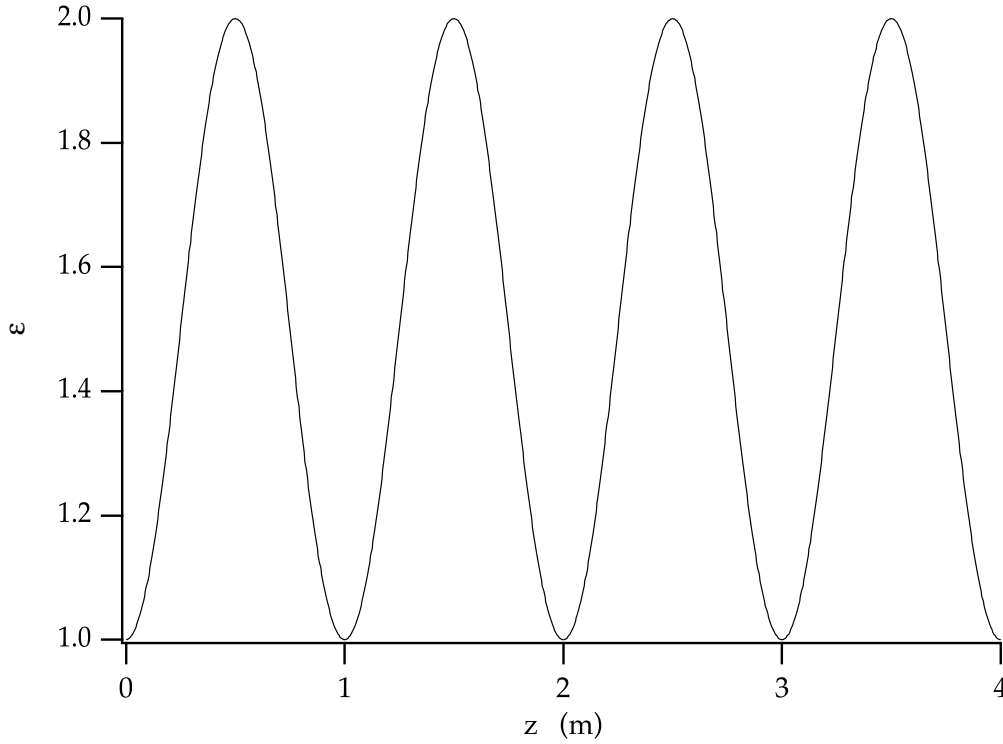
**Figure 1**: The permittivity $\varepsilon(z) = 1 + \sin^2 \pi z$ for the periodic dielectric medium.

permittivity is depicted in Figure 1 and is given by

$$\varepsilon(z) = 1 + \sin^2 \pi z$$

where the unit for $z$ is meter. One period of the medium then has a length of one meter. In Figure 2 the propagator kernel $G^+(0, z, t)$ is shown at $z = 1$, 4 and 8 meters. The maximum time $t = 6.5 \cdot 10^{-8}$ s corresponds to the time it takes for the wavefront to travel 16 periods into the medium. The periodic behavior of the kernel reflects the periodicity of the medium, which at these $z$-values induces a dominant frequency in the kernel but almost no higher frequencies. In Figure 3 $G^+(0, z, t)$ is shown at $z = 64$ and 128 meters. This far into the medium higher frequencies are superimposed on the dominant frequency. Also it is seen that for short times the precursor starts to build up. In Figure 4 $G^+(0, z, t)$ at $z = 1024$ meters is compared to the asymptotic expression for the precursor given by Eq. (6.1). The precursor is now clearly seen and the two curves agree well. The agreement improves with depth. Notice that in this graph the maximum time is reduced to $t = 8 \cdot 10^{-9}$ s, which is the time it takes for the wavefront to travel two periods into the medium. The CPU time to produce the results in the figures is some seconds on a work station.

In the second example wave propagation in a dispersive periodic medium of finite length is considered. The medium is a Lorentz medium, see e.g. [6], with the
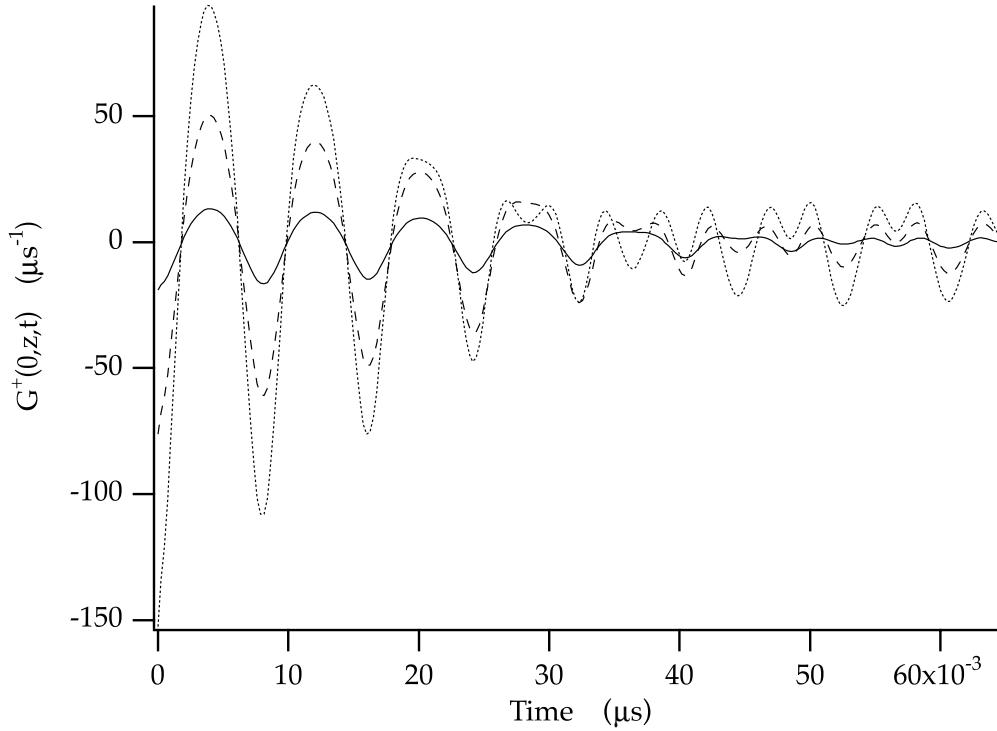
**Figure 2**: The propagator kernel $G^+(0, z, t)$ for the dielectric medium at $z = 1$ m (solid curve), $z = 4$ m (dashed curve) and $z = 8$ m (dotted curve).

following susceptibility kernel

$$\chi(z, t) = \omega_p^2 \frac{\sin \omega_0(z)t}{\omega_0(z)} \tag{8.1}$$

The plasma frequency has the constant value $\omega_p = 4 \times 10^8$ rad/s and the resonant frequency is given by

$$\omega_0 = \frac{1}{2}\omega_m(\cos(2\pi z) + 2) \tag{8.2}$$

where the unit for $z$ is meter and $\omega_m = 3.61 \times 10^8$ rad/s. Thus, also for this medium one period is one meter long. The length of the medium is 512 periods, and on both sides of the medium there is vacuum. The reflection kernel of the medium is shown in Figure 5. In Figure 6 the transmission kernel and the asymptotic expression for $G^+$, Eq. (6.1), are compared. There is a small difference even for small times since the asymptotic expression of $G^+$ assumes a semi-infinite medium whereas the transmission kernel is for a finite medium. In Figure 7 the kernel $G^+$ is shown at the midpoint $z = 256$ meter and it is compared with the asymptotic expression in Eq. (6.1). In this case the two curves are very similar for short times. Since time in this figure is less than the arrival time for the reflection from the back of the medium, there is no difference between $G^+$ for this medium and $G^+$ for a semi-infinite medium.
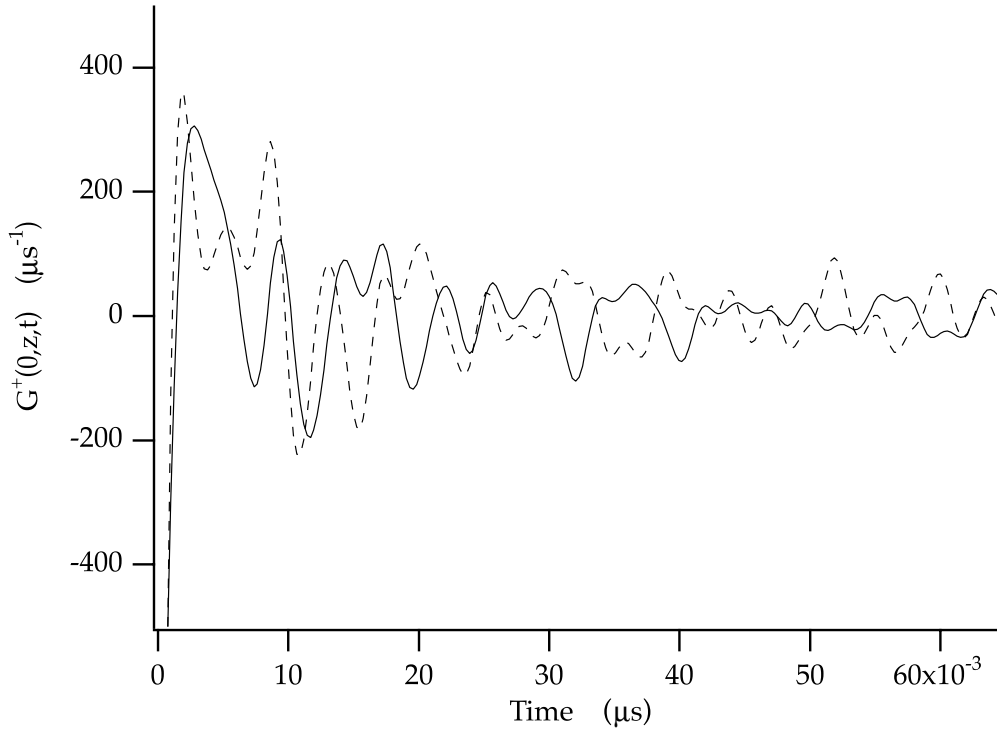
**Figure 3**: The propagator kernel $G^+(0, z, t)$ for the dielectric medium at $z = 64$ m (solid curve) and $z = 128$ m (dashed curve).

## 9   Conclusions

It has been shown that wave propagation in periodic media can be treated systematically in the time domain by utilizing properties of the propagators. An important feature of the propagators is that they are independent of the fields. Thus, once the propagators are known one may apply it to any incident pulse. Since it is not appropriate to study pulse propagation in the frequency domain, the current time-domain method serves as a complement to existing frequency domain methods.

Possible extensions of the present work is to consider the periodic slab made out of more complex materials, such as chiral, bianisotropic and anisotropic media. It is also possible to study wave propagation in media which are not periodic but are composed of subslabs arranged according to some pattern.

## Appendix A

In this appendix a quite detailed derivation of Eq. (4.1) is given for the case $F[E] \equiv 0$. It is straightforward to generalize the derivation to the case when $F$ is not zero, but since the equations then becomes lengthy this case is omitted here. Then the derivation of the other three equations for the wave propagators, Eqs. (4.2), (4.4) and (4.5) are indicated.
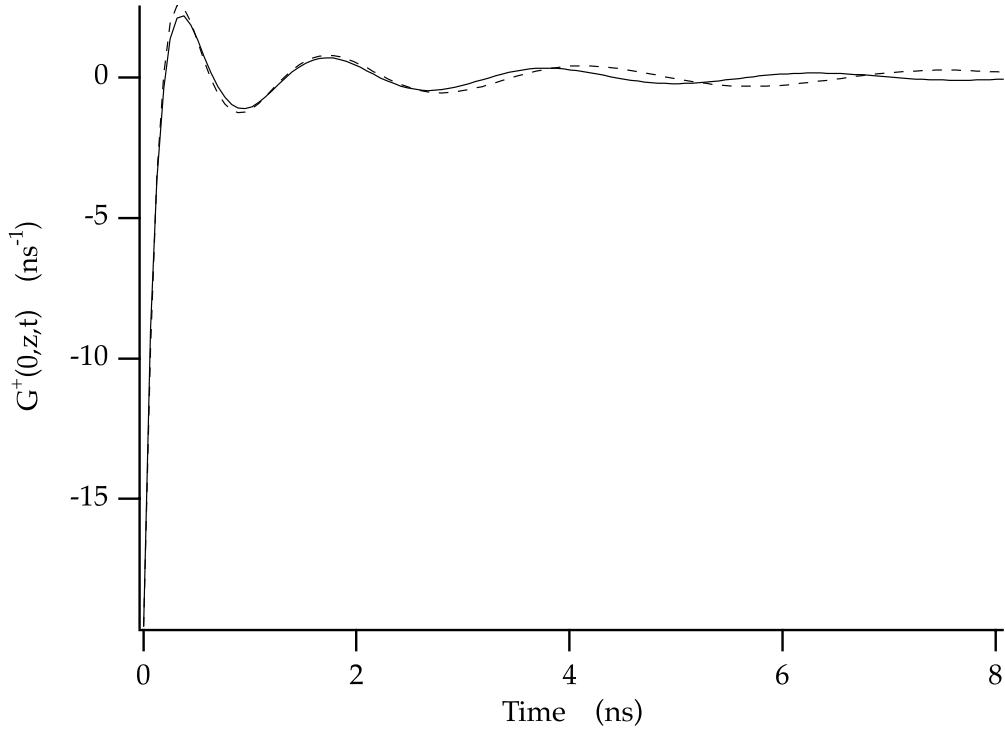
**Figure 4**: The propagator kernel $G^+(0, z, t)$ for the dielectric medium at $z = 1024$ m (solid curve) and the asymptotic value of $G^+$ given by Eq. (6.1) at the same $z$ value (dashed curve).

The first step in the derivation of Eq. (4.1) is to differentiate Eq. (3.4) wrt $\zeta$. The derivative of the right hand side reads
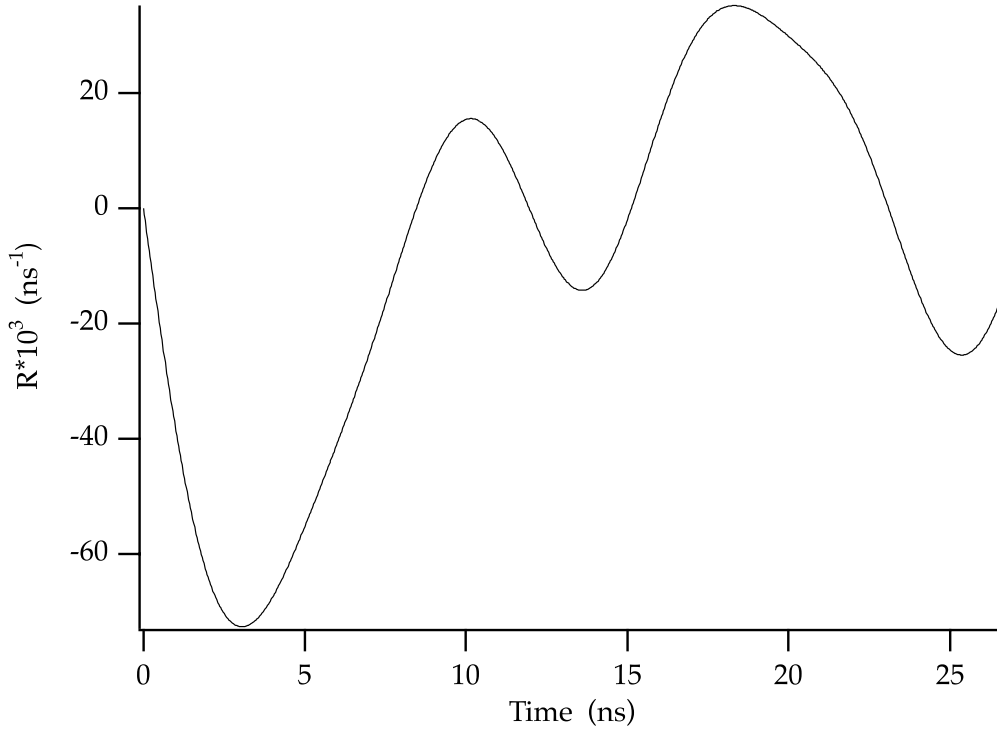
$$\frac{d}{d\zeta}(a(\eta, \zeta) \ E^+(\eta, t) + [G^+(\eta, \zeta, \cdot) * E^+(\eta, \cdot)](t))$$

$$= a_\zeta(\eta, \zeta)E^+(\eta, t) + [G_\zeta^+(\eta, \zeta, \cdot) * E^+(\eta, \cdot)](t)$$

The $\zeta-$derivative of the left hand side reads

$$\frac{d}{d\zeta}E^+(\zeta, t + \ell(\eta, \zeta)) = \left(\partial_\zeta + \frac{1}{c(\zeta)}\partial_t\right) E^+(\zeta, t + \ell(\eta, \zeta)) \tag{A.1}$$

$$= (\alpha(\zeta) + \frac{1}{c(\zeta)}\partial_t)E^+(\zeta, t + \ell) + \beta(\zeta)E^-(\zeta, t + \ell)$$

where the dynamics for $E^\pm$, Eq. (2.3), has been used. The fields $E^\pm(\zeta, t + \ell)$ are expressed in terms of $E^+(\eta, t)$ by Eqs. (3.4) and (3.5). The explicit expressions for the operators $\alpha$ and $\beta$ are given by Eqs. (2.4) and (2.5). Using these expressions, the $\zeta-$ derivative of Eq. (3.4) reads

$$\frac{c_\zeta(\zeta)}{2c(\zeta)} \left(a(\eta, \zeta)E^+(\eta, t) + [G^+(\eta, \zeta, \cdot) * E^+(\eta, \cdot)](t) - [G^-(\eta, \zeta, \cdot) * E^+(\eta, \cdot)](t)\right)$$

$$= a_\zeta(\eta, \zeta)E^+(\eta, t) + [G_\zeta^+(\eta, \zeta, \cdot) * E^+(\eta, \cdot)](t)$$

**Figure 5**: The reflection kernel for the Lorentz medium given by Eqs. (8.1) and (8.2).

Since $E^+$ is an arbitrary field the following equations are implied

$$a_\zeta(\eta, \zeta) = \left( \frac{c_\zeta(\zeta)}{2c(\zeta)} - \frac{c(\zeta)}{2c_0^2}\chi(\zeta, 0) \right) a(\eta, \zeta) \tag{A.2}$$

$$G_\zeta^+(\eta, \zeta, t) = \frac{c_\zeta(\zeta)}{2c(\zeta)} \left( G^+(\eta, \zeta, t) - G^-(\eta, \zeta, t) \right) \tag{A.3}$$

The boundary condition for $a(\eta, \zeta)$ is $a(\eta, \eta) = 1$ and thus Eq. (A.2) is consistent with Eq. (3.6). Since $F[E] \equiv 0$ it is seen that Eq. (A.3) is identical with Eq. (4.1).

Equation (4.2) is derived in the same manner as above. Thus Eq. (3.5) is differentiated wrt $\zeta$ which, by utilizing the dynamics, gives

$$\begin{aligned}
\gamma(\zeta) \left( a(\eta, \zeta)E^+(\eta, t) \right. &+ [G^+(\eta, \zeta, \cdot) * E^+(\eta, \cdot)](t) + \delta(\zeta)[G^-(\eta, \zeta, \cdot) * E^+(\eta, \cdot)](t) \\
&+ \frac{1}{c(\zeta)} \left( G^-(\eta, \zeta, 0)E^+(\eta, t) + [G_t^-(\eta, \zeta, \cdot) * E^+(\eta, \cdot)](t) \right) \\
&= [G_\zeta^-(\eta, \zeta, \cdot) * E^+(\eta, \cdot)](t)
\end{aligned}$$

Since $E^+$ is an arbitrary incident field the Eq. (4.2) and the initial condition, Eq. (4.3), follow.

The two remaining equations (4.4) and (4.5) are derived by differentiating Eqs.
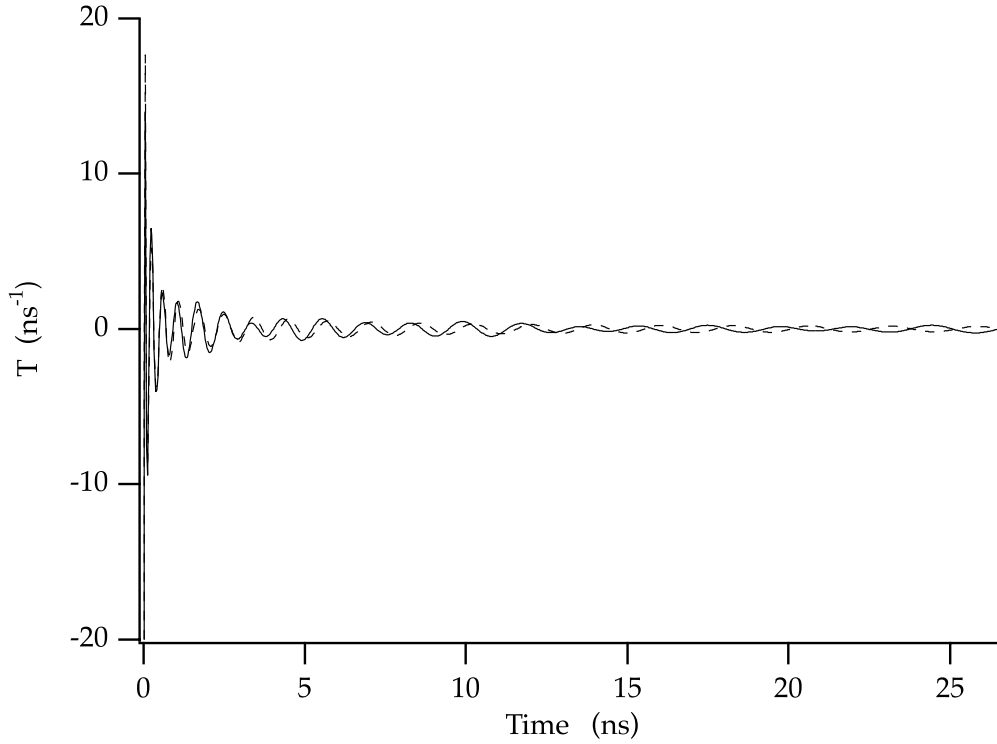
**Figure 6**: The transmission kernel (solid line) and the asymptotic Green kernel given by Eq. (6.1) (dashed line) for the Lorentz medium given by Eqs. (8.1) and (8.2).

(3.4) and (3.5) wrt $\eta$. The $\eta-$derivative of Eq.(3.4) can be written as

$$-\frac{1}{c(\eta)} \ \left(a(\eta,\zeta)E_t^+(\eta,t) + G^+(\eta,\zeta,0)E^+(\eta,t) + [G_t^+(\eta,\zeta,\cdot) * E^+(\eta,\cdot)](t)\right) \ \text{(A.4)}$$

$$= a_\eta(\eta,\zeta)E^+(\eta,t) + a(\eta,\zeta)\left(\alpha(\eta)E^+(\eta,t) + \beta(\eta)E^-(\eta,t)\right)$$
$$+ [G_\eta^+(\eta,\zeta,\cdot) * E^+(\eta,\cdot)](t) + [G^+ * \left(\alpha(\eta)E^+(\eta,t) + \beta(\eta)E^-(\eta,t)\right)](t)$$

This equation is expressed solely in terms of the field $E^+(\eta,t)$ if the relation

$$E^-(\eta,t) = [R(\eta,\cdot) * E^+(\eta,\cdot)](t)$$

is used. Equation (4.4) now follow when the explicit expressions for the operators $\alpha$ and $\beta$ are inserted into Eq. (A.4)

Finally the derivation of Eq.(4.5) and the initial condition in Eq. (4.3) are sketched. The $\eta-$derivative of Eq.(3.5) is written as

$$-\frac{1}{c(\eta)} \ \left(G^-(\eta,\zeta,0)E^+(\eta,t) + [G_t^-(\eta,\zeta,\cdot) * E^+(\eta,\cdot)](t)\right)$$

$$= [G_\eta^-(\eta,\zeta,\cdot) * E^+(\eta,\cdot)](t) + [G^-(\eta,\zeta,\cdot) * (\alpha(\eta)E^+(\eta,\cdot) + \beta(\zeta)E^-(\eta,\cdot)](t)$$

In the last equation $E^-(\eta,t)$ is rewritten as $E^-(\eta,t) = [R(\eta,\cdot) * E^+(\eta,\cdot)](t)$. By inserting the explicit expression, Eqs. (2.4) and (2.5), for the operators $\alpha$ and $\beta$, Eq. (4.5) follows.
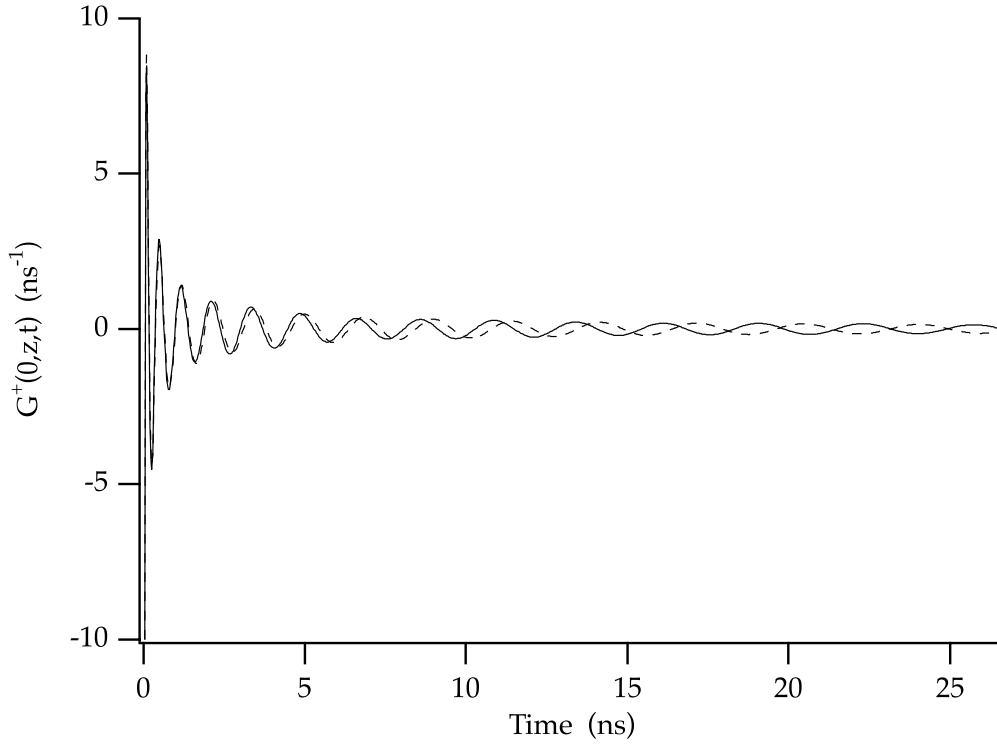
**Figure 7**: The propagator kernel $G^+$ and the asymptotic kernel in Eq. (6.1) at $z = 256$ m, for the Lorentz medium given by Eqs. (8.1) and (8.2).

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