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Transient electromagnetic waves and inversion techniques

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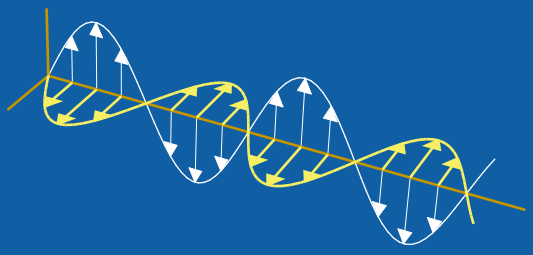

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Abstract

This paper reviews an efficient inversion technique for complex media utilizing transient electromagnetic scattering data. The approach to solve the scattering problems (direct and inverse) relies on a wave splitting technique and an invariant imbedding technique. The analysis is illustrated by a numerical example.

1 Introduction

In this paper a review of some results on propagation of transient electromagnetic waves in spatially inhomogeneous, lossy slabs of finite length is presented. The sources of the problem are assumed to be located outside the slab, and they generate a transient, transversely polarized wave that impinges normally on the slab. Furthermore, the permittivity and the conductivity profiles are assumed to vary with depth only, and the scattering problem is thus one-dimensional. The slab extends from $[0, L]$, and outside the slab the medium is lossless and homogeneous, see Figure 1. A more extensive treatment of this scattering problem is found in Refs 5,6,9. Several related problems are found in Ref. 2. Reconstructions of profiles from experimental data are found in Ref. 3.

The direct and inverse scattering problems for the continuous permittivity profiles are reviewed (the corresponding scattering problems for discontinuous permittivity profiles are given in Refs 7–9). The direct scattering problem is to calculate the scattering kernels from known permittivity and conductivity profiles. In the inverse problem, these profiles are calculated from finite time traces of scattering data. The input data set consists of finite time traces of reflection and transmission data.

2 Basic time domain equations

The medium, which is inhomogeneous wrt the depth z, is assumed to be non-magnetic and non-dispersive. Extensions to dispersive media are found in Refs 1,4. The losses of the medium are modeled by a DC conductivity function $\sigma(z)$. The appropriate constitutive relations are therefore

$$\begin{cases} \boldsymbol{D}(\boldsymbol{r},t) = \epsilon_0\epsilon(z)\boldsymbol{E}(\boldsymbol{r},t) \\ \boldsymbol{J}(\boldsymbol{r},t) = \sigma(z)\boldsymbol{E}(\boldsymbol{r},t) \\ \boldsymbol{B}(\boldsymbol{r},t) = \mu_0\boldsymbol{H}(\boldsymbol{r},t) \end{cases}$$

where $\epsilon(z)$ is the relative permittivity of the slab, and ϵ_0 and μ_0 are the permittivity and the permeability of vacuum, respectively. The medium is assumed to be homogeneous and lossless outside a slab of thickness L, see Figure 1.

The electric field are assumed to vary with z and t only, and, furthermore, it has only a transverse component $E(z,t)$. The Maxwell equations and the constitutive

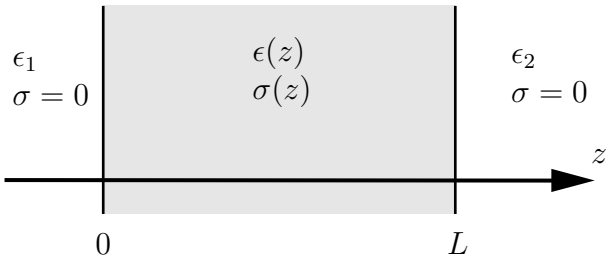

Figure 1: Geometry of the lossy slab. The media outside the slab are assumed to be homogeneous, permittivity ϵ_1 and ϵ_2, respectively, and lossless, i.e. $\sigma = 0$.

relations imply that the electric field $E(z,t)$ satisfy the wave equation

$$\frac{\partial^2 E}{\partial z^2}(z,t) - \frac{1}{c^2(z)}\frac{\partial^2 E}{\partial t^2}(z,t) - \mu_0\sigma(z)\frac{\partial E}{\partial t}(z,t) = 0 \tag{2.1}$$

where the wave front speed $c(z)$ is

$$c(z) = c_0/\sqrt{\epsilon(z)}$$

and c_0 is the speed of light in vacuum. Note that there are two varying coefficients in this wave equation, *viz*, the wave front speed $c(z)$ (or equivalently, the permittivity $\epsilon(z)$) and the conductivity $\sigma(z)$.

The direct scattering problem is to compute the reflected and the transmitted fields if the material parameters, $\epsilon(z))$ and $\sigma(z)$, are known. The inverse problem is the reverse; to infer information about the material parameters, $\epsilon(z))$ and $\sigma(z)$, from the reflected and the transmitted fields.

3 Wave splitting

The electric field that satisfies the wave equation, (2.1), can be written as a system of first order equations in the variable z.

$$\frac{\partial}{\partial z}\begin{pmatrix} E \\ E_z \end{pmatrix}(z,t) = \begin{pmatrix} 0 & 1 \\ c^{-2}(z)\frac{\partial^2}{\partial t^2} + \mu_0\sigma(z)\frac{\partial}{\partial t} & 0 \end{pmatrix}\begin{pmatrix} E \\ E_z \end{pmatrix}(z,t)$$

where subscript on the field, e.g., $E_z(z,t)$, denotes differentiation wrt z.

The dependent variables in this system of equations, $E(z,t)$ and $E_z(z,t)$, are now transformed, by the wave splitting transform, into two new dependent variables $E^{\pm}(z,t)$.

$$E^{\pm}(z,t) = \frac{1}{2}\left\{E(z,t) \mp c(z)\int_{-\infty}^{t} E_z(z,t')dt'\right\}$$

This transformation transforms the equation into the following system of PDEs:

$$\frac{\partial}{\partial z}\begin{pmatrix} E^+ \\ E^- \end{pmatrix}(z,t) + \frac{1}{c(z)}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix} E^+ \\ E^- \end{pmatrix}(z,t) = \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{pmatrix}\begin{pmatrix} E^+ \\ E^- \end{pmatrix}(z,t) \tag{3.1}$$

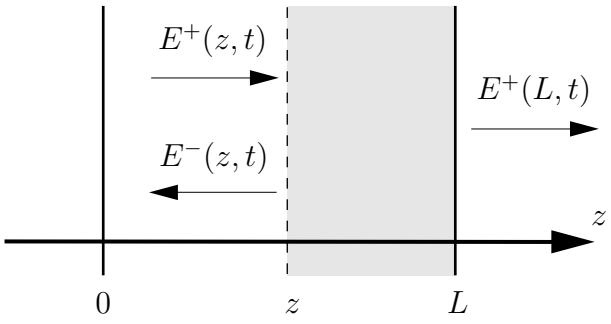

Figure 2: The imbedding geometry of the slab.

where the coefficients $\alpha(z)$, $\beta(z)$, $\gamma(z)$ and $\delta(z)$ are

$$\begin{cases} \alpha(z) = \dfrac{1}{2}\left(\dfrac{c'(z)}{c(z)} - c(z)\mu_0\sigma(z)\right) \\ \beta(z) = -\dfrac{1}{2}\left(\dfrac{c'(z)}{c(z)} + c(z)\mu_0\sigma(z)\right) \end{cases} \qquad \begin{cases} \gamma(z) = -\dfrac{1}{2}\left(\dfrac{c'(z)}{c(z)} - c(z)\mu_0\sigma(z)\right) \\ \delta(z) = \dfrac{1}{2}\left(\dfrac{c'(z)}{c(z)} + c(z)\mu_0\sigma(z)\right) \end{cases}$$

In a homogeneous, lossless region these functions are identically zero, and the system of PDEs decouple into right- and left-moving waves, i.e. $c(z) = c =$ constant and $\sigma = 0$ and

$$\begin{cases} E^+(z,t) = f(z-ct) \\ E^-(z,t) = g(z+ct) \end{cases}$$

These new dependent variables, $E^{\pm}(z,t)$, are therefore the natural dependent variables to work with in the solution of the scattering problem, since they generalize the concept of reflected and transmitted fields inside the inhomogeneous medium.

4 Scattering operator formulation

This section presents the scattering operator formulation of the scattering problem in an invariant imbedding geometry.

Consider a subsection $[z, L]$ of the physical region $[0, L]$, see Figure 2. Mathematically, the original problem, $[0, L]$, is imbedded in a family of problems where the left edge of the slab, z, is the parameter that is varied. The value $z = 0$ corresponds to the full slab problem, and the other limit value, $z = L$, corresponds to the absence of the slab.

The fields $E^+(z,t)$ and $E^-(z,t)$, defined at the position z, are related to each other by the reflection operator. In the same way the fields $E^+(z,t)$ and $E^+(L,t)$ are related to each other by the transmission operator. These operators are represented

as integral operators, where the scattering kernels $R^+(z,t)$ and $T(z,t)$ are defined.

$$\begin{cases} E^-(z,t) = \displaystyle\int_{-\infty}^{t} R^+(z,t-t')E^+(z,t')dt' \\ E^+(L,t+\tau(z)) = \Gamma(z)\left\{E^+(z,t) + \displaystyle\int_{-\infty}^{t} T(z,t-t')E^+(z,t')dt'\right\} \end{cases} \tag{4.1}$$

In these expressions the wave front factor $\Gamma(z)$ and the time delay $\tau(z)$ are used. Their definitions are

$$\begin{cases} \Gamma(z) = \sqrt{\dfrac{c(L)}{c(z)}}\exp\left\{-\dfrac{\mu_0}{2}\displaystyle\int_z^L \sigma(z')c(z')\,dz'\right\} \\ \tau(z) = \displaystyle\int_z^L \frac{dz'}{c(z')} \end{cases}$$

The kernels $R^+(z,t)$ and $T(z,t)$ are identically zero for negative time due to causality, i.e. $R^+(z,t) = T(z,t) = 0$ for $t < 0$.

As the parameter z varies from $z = 0$ to $z = L$, the corresponding kernels $R^+(z,t)$ and $T(z,t)$ vary from their physical values at $z = 0$, $R^+(0,t)$ and $T(0,t)$, to zero values at $z = L$, since the subsection $[z, L]$ then vanishes.

From (4.1) and (3.1) it is possible to derive an imbedding equation for the reflection kernel $R(z,t)$, for details see Ref. 5. The result is $(t > 0)$

$$\begin{aligned} R_z^+(z,t) - \frac{2}{c(z)}R_t^+(z,t) &= c(z)\mu_0\sigma(z)R^+(z,t) \\ &+ \frac{1}{2}\left(\frac{c'(z)}{c(z)} + c(z)\mu_0\sigma(z)\right)\int_0^t R^+(z,t-t')R^+(z,t')dt' \end{aligned} \tag{4.2}$$

The reflection kernel, $R^+(z,t)$ has the initial value [5]

$$R^+(z,0^+) = \frac{1}{4}\left(c'(z) - c^2(z)\mu_0\sigma(z)\right) \tag{4.3}$$

5 The resolvent kernel of $T(z,t)$

The crucial quantity for the solution of the inverse problem is the resolvent of the transmission kernel $T(z,t)$. This kernel is denoted by $W(z,t)$ and satisfies the resolvent equation.

$$T(z,t) + W(z,t) + \left(T(z,\cdot) * W(z,\cdot)\right)(t) = 0$$

The kernel $W(z,t)$ satisfies an imbedding equation similar to (4.2). This imbedding equation is $(t > 0)$, see Ref. 5

$$W_z(z,t) = -\frac{1}{2}\left(\frac{c'(z)}{c(z)} + c(z)\mu_0\sigma(z)\right)\left\{R^+(z,t) + \int_0^t W(z,t-t')R^+(z,t')dt'\right\} \tag{5.1}$$

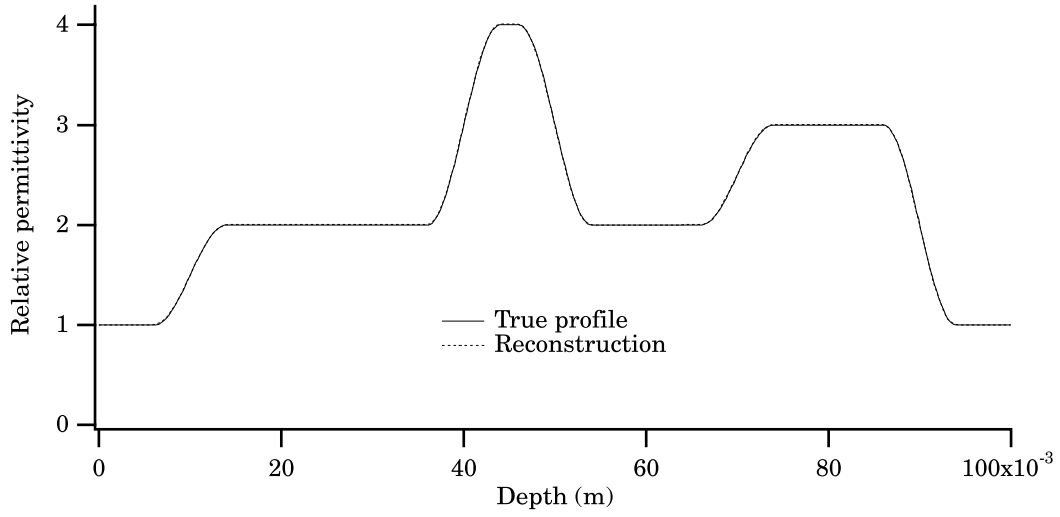

Figure 3: Reconstruction of the permittivity profile.

Notice the explicit presence of the reflection kernel $R^+(z,t)$ in the right hand side of the equation.

The kernel $W(z,t)$ has compact support in $[0, 2\tau(z)]$, for a proof, see Ref. 5. The length of time interval $[0, 2\tau(z)]$ is the time it takes for the wave front to travel through the slab, reflect at the right edge at $z = L$, and travel back to z again. This time interval is usually referred to as one round trip through the subsection $[z, L]$. The compact support of the kernel $W(z,t)$ and the resolvent equation can be used to extend transmission data from one round trip to arbitrary time intervals [5].

In Section 4 the reflection kernel for reflection from the left, $R^+(z,t)$, was introduced. Reflection from the right is represented by a similar reflection kernel, $R^-(z,t)$. This kernel has a finite jump discontinuity after one round trip, $2\tau(z)$, in the imbedding medium $[z, L]$. The finite jump discontinuity is related to the $W(z,t)$ kernel and the material parameters at z, see Ref. 5. The explicit result is

$$\begin{aligned} &\int_0^{2\tau(z)} W(z, 2\tau(z) - t')R^-(0,t')dt' + R^-(0, 2\tau(z)^-) \\ &\qquad = -\frac{1}{4}\left(c'(z) + c^2(z)\mu_0\sigma(z)\right) \exp\left\{-\mu_0 \int_z^L c(z')\sigma(z')\,dz'\right\} \end{aligned} \tag{5.2}$$

Notice that (4.3) contains information about $c'(z) - c^2(z)\mu_0\sigma(z)$, but (5.2) contains information about $c'(z) + c^2(z)\mu_0\sigma(z)$. By clever use of these two quantities, the two parameters, wave front speed $c(z)$ (or equivalently the permittivity $\epsilon(z)$), and the conductivity $\sigma(z)$, can be constructed from scattering data. This reconstruction is presented in the next section.

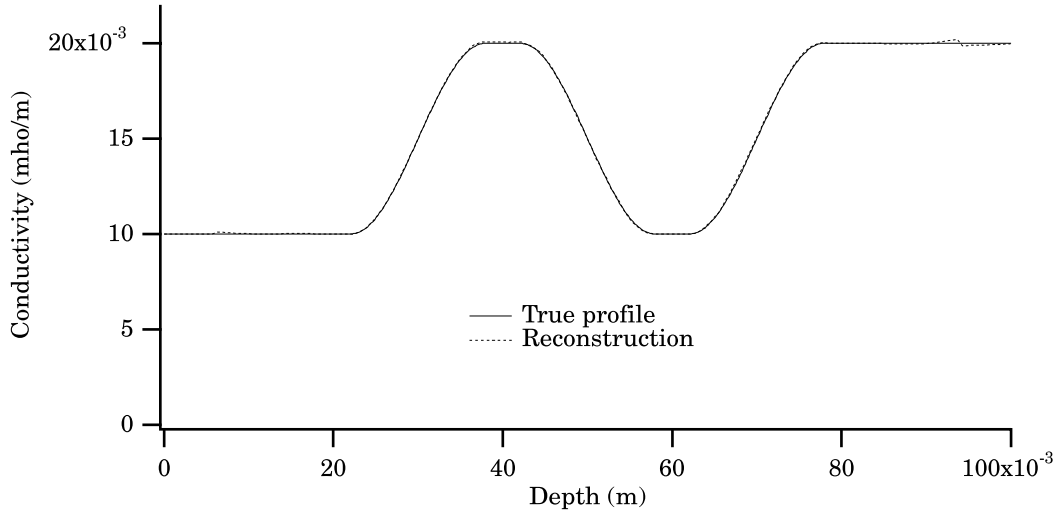

Figure 4: Reconstruction of the conductivity profile.

6 Inversion algorithm

In this section the inversion algorithm of this paper is presented. The input data set needed for the algorithm is given in Table 1. Three scattering kernels are required and three constants. The scattering kernels are measure during a finite time interval, $[0, 2\tau(0)]$, which is one round trip through the entire slab. The first constant $\exp\left\{-\mu_0 \int_0^L \sigma(z')c(z')\,dz'\right\}$ can be obtained from transmission data, cf. $\Gamma(0)$; the other two are obtained from the medium outside the unknown slab.

Data set for inversion
$R^{\pm}(0,t), \quad 0 < t < 2\tau(0)$
$T(0,t), \quad 0 < t < 2\tau(0)$
$\exp\left\{-\mu_0 \int_0^L \sigma(z')c(z')\,dz'\right\}$
$c(0)$ and $c(L)$

Table 1: Data requirements for inversion of the profiles $\epsilon(z)$ and $\sigma(z)$.

The inversion algorithm using the invariant imbedding equations then proceeds as follows starting at $z_0 = 0$:

1) Use (5.1) to explicitly step $W(z,t)$ from the current grid line z_0 to the next grid line $z_1 = z_0 + h$ for $0 < t < 2\tau(z_1)$.

2) Use (4.2) to implicitly step a portion of $R^+(z,t)$ forward in the z-direction to the grid point $(z_1, 0)$.

3) To estimate $c(z_1)$ and $\sigma(z_1)$, set $z = z_1$ in (4.3) and (5.2). These coefficients are found by solving a set of non-linear equations. This is straightforwardly done by a Newton iteration scheme.

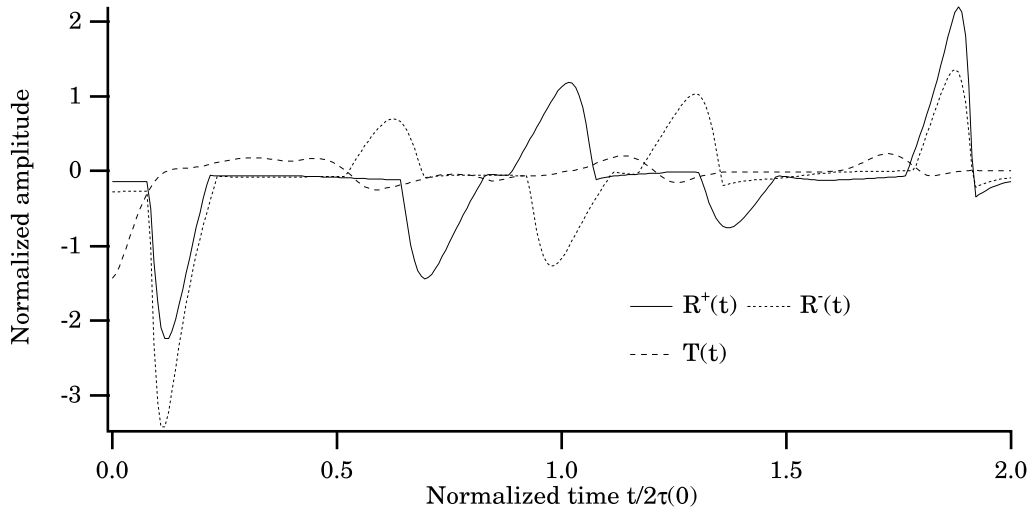

Figure 5: The reflection and transmission data set used in the reconstruction of the example given in Figures 3 and 4.

4) Use (4.2) to implicitly step the remaining $R^+(z_0, s)$ data forward in the z-direction to the set of grid points at z_1 for $0 < t < 2\tau(z_1)$.

5) Repeat steps 1-4 to move one grid line deeper into the medium.

The accuracy of this algorithm can be considerably improved through the use of iteration. All that needs to be done is to add the following steps after step 4:

4a) Use (5.1) to implicitly step $W(z, t)$ from the current grid line z_0 to the next grid line z_1 for $0 < t < 2\tau(z_1)$. This can be done since $c(z)$, $\sigma(z)$ and $R^+(z, t)$ are known (to some degree of accuracy) on the grid lines z_0 and z_1.

4b) Go to step 2.

7 Numerical example

In this section a numerical example is given that illustrates the performance of the algorithm. The original permittivity and the conductivity profiles as well as the reconstructed profiles are found in Figures 3 and 4. The input scattering data to these reconstructions are given in Figure 5. 257 data points are used to represent the data in the inversion algorithm, and two iterations have been used to improve the reconstruction. As clearly is seen from Figures 3 and 4, the reconstructions are very good and stable. Several additional examples, which illustrates the performance with noise-corrupted data, are presented in Ref. 9.

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