Distributed Performance Analysis of Heterogeneous Systems

Rantzer, Anders

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Anders Rantzer

***Abstract*— In the study of distributed systems, it is of fundamental interest to understand how specifications on local components and interconnections influence global properties of the system. In this paper, we consider linear continuous time systems described by sparse matrices. Properties of interest are stability and quadratic performance specifications such as passivity and input-output gain. In particular, for systems with sparsity structure corresponding to a chordal graph, we show that sparse performance conditions can be expressed without conservatism.**

I. Introduction

Analysis of large-scale control systems has received considerable attention in the control literature. Early contributions based on dissipative theory given in [9], [5], [8] were later generalized using integral quadratic constraints [4]. In these contributions, stability and performance for a large-scale interconnected system is analyzed based on constraints on its local components. However, the criteria are conservative. In particular, failure to satisfy the stability conditions for some interconnection does not necessarily imply that the system is unstable.

The aim of this paper is to introduce a new approach for componentwise analysis of large-scale systems. For interconnected systems with "chordal" graph structure, the method is non-conservative, so the interconnection is stable if and only if the component-wise conditions can be satisfied. The exact definition of chordal graphs will be given later, but includes tree structured graphs as a special case.

Example 1 Consider a water distribution system consisting of n series-connected reservoirs illustrated in Figure 1 with the state space model below.

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ x_4(t+1) \end{bmatrix} = \begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & 0 & * \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_n(t) \end{bmatrix} + \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \\ w_4(t) \end{bmatrix}$$

The system matrix is lower triangular, reflecting the fact that the water only flows in one direction. Hence disturbances down-stream will have no effect up-stream. However, suppose that the water level in each researvoir is controlled based on measurements in the neighboring reservoirs. Then information can flow also up-stream and the triangular structure is destroyed. However, the matrix is still banded. □

A. Rantzer is with Automatic Control LTH, Lund University, Box 118, SE-221 00 Lund, Sweden, rantzer at control.lth.se.

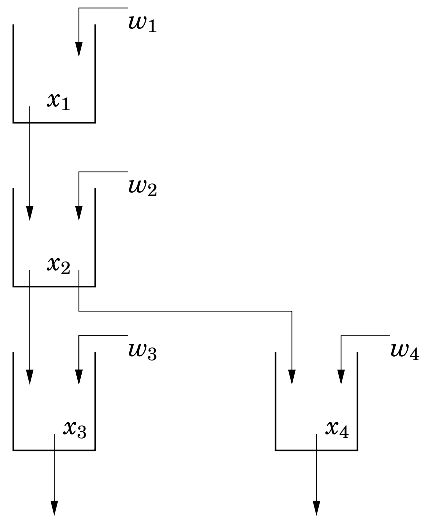

Fig. 1. Water reservoirs connected in a tree structure. A tree is a special case of chordal graph, so the theory of this paper can be applied without conservatism.

In the next section, the main idea is explained for systems defined by banded matrices. Stability of systems with chordal graph structure is treated in section IV, while the final section extends the method beyond stability to verify properties like passivity and input-output gain.

II. Positivity and Stability of Banded Matrices

The main idea behind the results of the paper dates back at least to the 1960s. It is illustrated in Figure 4 and stated rigorously in the following proposition:

Proposition 1: Suppose that the matrix $M \in \mathbf{R}^{n\times n}$, with coefficients M_{ij}, is symmetric and banded, i.e. there exists $d < (n-1)/2$ such that $M_{ij} = 0$ for $|i-j| > d$. If M is positive semi-definite, then there exist positive semi-definite matrices $M^k = (M^k_{ij})$, $k = 1+d, \dots, n-d$ such that $M = M^{1+d} + \dots + M^{n-d}$ and $M^k_{ij} = 0$ for $\min\{|i-k|, |j-k|\} > d$.

Proposition 1 is an immediate consequence of the following lemma [3]. See Figure 2.

Lemma 1: Suppose that $M = (M_{ij}) \in \mathbf{R}^{n\times n}$ is symmetric and banded, i.e. there exists $d < n-1$ such that $M_{ij} = 0$ for $|i-j| > d$. Then there exists a lower triangular matrix $L = (L_{ij})$ such that $LL^T = M$ and $L_{ij} = 0$ for $|i-j| > d$.

Given the positivity test of Proposition 1, the step to stability verification is not long:

Proposition 2: Assume $A \in \mathbf{R}^{n\times n}$ is banded, i.e. there exists $d_1 \leq 0 \leq d_2$ such that $A_{ij} = 0$ for $i-j < d_1$ and for $i-j > d_2$. Then A is Hurwitz if and only if there exist rational matrix functions $M^k(\cdot,\cdot) = (M^k_{ij})$ with

$$(sI-A)^*(sI-A) = M^1(\bar{s},s) + \cdots + M^{n-d}(\bar{s},s) \quad (1)$$

where $M^k(\bar{s},s)$ is positive definite for Re $s \geq 0$, while $M^k_{ij} = 0$ for $\min\{|i-k|,|j-k|\} > d_2 - d_1$.
A detailed proof of Proposition 2 will be given in section IV.

Remark 1. At first sight, it may look like the equality condition (1) makes Proposition 2 very sensitive to perturbations in the system coefficients A_{ij}. However, the equality can be recoverded by just updating the local M^k-matrix. This works as long as the matrix remains positive definite for s in the right half plane.

Proposition 2 may be used for distributed system verification as follows: Suppose that a system has been designed to behave according to the dynamics described by the matrix A and matrix functions M^k have been computed to satisfy the conditions of Proposition 2. Suppose that each node has access to local model data A_{ij} and M^k for the neighboring nodes. Then the i:th row of condition (1) can be verified in node i before the system is put into operation. If a component changes and the local dynamics is updated, a local update of M^i could be sufficient to maintain equality in the i:th row. However, if the positivity of M^i is lost, it could be necessary to recompute all the performance certificates M^i, or conclude that stability has been lost.

III. Graph Notation and Preliminaries

Consider an undirected graph $\mathbb{G} = (\mathbb{V},\mathbb{E})$ with nodes $\mathbb{V} = \{1,2,\ldots,J\}$ and edges $\mathbb{E}$. A *cycle* in the graph is a sequence of pairwise distinct nodes $(n_1,\ldots,n_s)$ such that $(n_1,n_2),(n_2,n_3),\ldots,(n_{s-1},n_s),(n_s,n_1) \in \mathbb{E}$.

A *chord* of the graph cycle $(n_1,\ldots,n_s)$ is an edge $(n_i,n_j) \in \mathbb{E}$, where $1 \leq i < j \leq s$, $(i,j) \neq (1,s)$ and $|i-j| \geq 2$. The number s is referred to as the *length* of the cycle. The graph is called *chordal* if every cycle of length ≥ 4 has a chord. See the upper part of Figure 3.

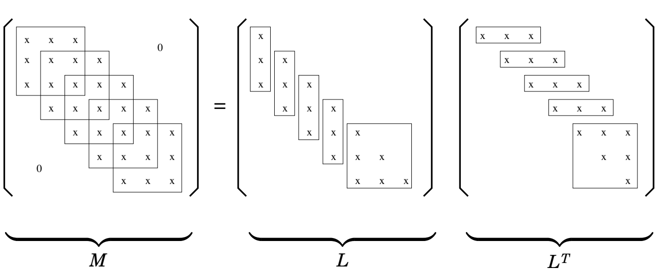

Fig. 2. Illustration of Lemma 1. A banded positive semi-definite matrix has a banded Cholesky-factorization. Multiplying the first column of L with the first row of L^T gives the first term on the right hand side of Figure 4. The other terms are obtained analogously.

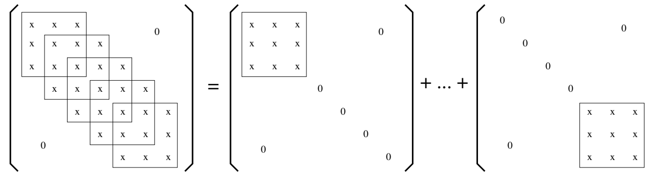

Fig. 4. Illustration of Proposition 1 and Proposition 2. A banded positive semi-definite matrix can be decomposed as a sum of positive semi-definite matrices, where each term is non-zero only in a small square. As a consequence, the banded matrix A is Hurwitz stable if and only if the matrix $(sI-A)^*(sI-A)$ can be decomposed this way for every s with Re $s \geq 0$.

A *clique* of the graph $(\mathbb{V},\mathbb{E})$ is a subset $\mathbb{C}$ of $\mathbb{V}$ having the property that $(i,j) \in \mathbb{E}$ for all $i,j \in \mathbb{C}$. An *ordering* of the graph is a bijection $\alpha : \mathbb{V} \to \{1,2,\ldots,J\}$. The ordering α is a *perfect elimination ordering* if for every $i \in \mathbb{V}$, the set

$$\{j \mid (i,j) \in \mathbb{E},\ \alpha(i) < \alpha(j)\}$$

is a clique. The following result is cited from [7]:

Proposition 3: A graph has a perfect elimination ordering if and only if it is chordal.

Chordal graphs have appeared in the literature in relation to the problem of reordering rows and columns of a sparse matrix in such a way that no nonzero elements are created in the course of Gaussian elimination [7]. They also appear in the search for positive definite completions of partial Hermitean matrices [2]. The decomposition (1) can be viewed as a dual of the second problem.

Given a graph node i, define $[i]$ as the set of all j such that $(i,j) \in \mathbb{E}$. For each node i of the graph, we assign a vector $x_i \in \mathbf{C}^{n_i}$. The concatenation of all vectors is $x \in \mathbf{C}^n$. Similarly $x_{[i]}$ is the concatenation of only those vectors x_j where $(i,j) \in \mathbb{E}$. The dimension of $x_{[i]}$ is denoted $n_{[i]}$.

For a matrix $A \in \mathbf{R}^{n\times n}$ with blocks $A_{ij} \in \mathbf{R}^{n_i \times n_j}$, let $A_{[i]} \in \mathbf{R}^{n\times n_{[i]}}$ be the submatrix consisting of blocks A_{kj} with $j \in [i]$. Let $E_{[i]} \in \mathbf{R}^{n\times n_{[i]}}$ be the corresponding submatrix of the identity matrix $I \in \mathbf{R}^{n\times n}$. Similarly, $E_i \in \mathbf{R}^{n\times n_i}$ is the block column matrix where all blocks are zero except the i:th block, which is identity.

IV. Main result

We are now ready to give generalizations of the results in section II beyond banded matrices. The following generalization of Lemma 1, known since the 1970s [6], [1], indicates that this should be possible.

Lemma 2: Given a chordal graph $\mathbb{G} = (\mathbb{V},\mathbb{E})$, suppose that $M = (M_{ij}) \in \mathbf{R}^{n\times n}$ is symmetric and $M_{ij} = 0$ for $(i,j) \notin \mathbb{E}$. Then there exists a lower triangular matrix $L = (L_{ij})$ such that $LL^T = M$ and $L_{ij} = 0$ for $(i,j) \notin \mathbb{E}$.

In order to verify that $M^i(\cdot,\cdot)$ in Proposition 2 and its generalizations can be chosen as rational functions, we

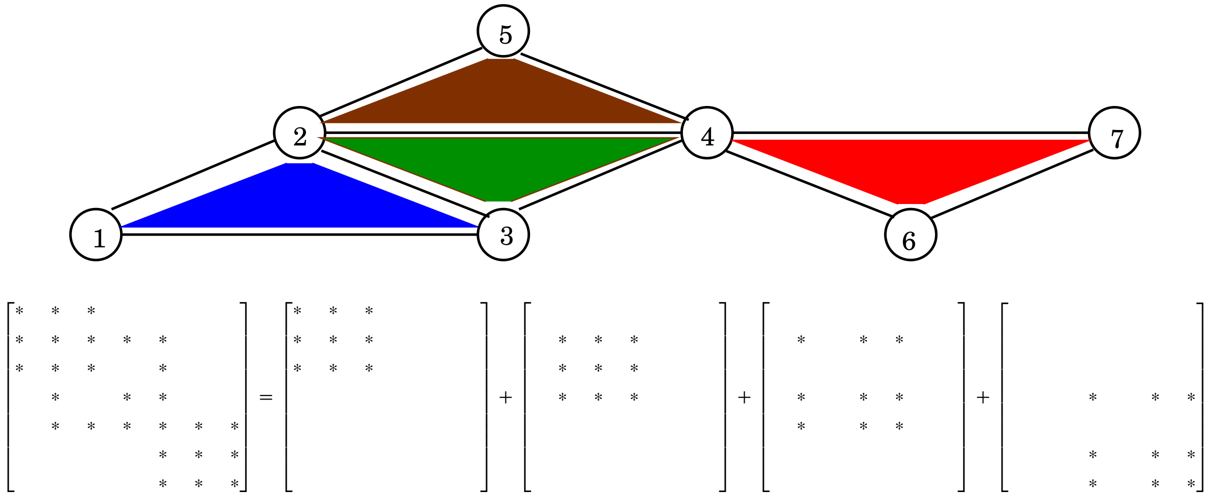

Fig. 3. The diagram is an illustration of a chordal graph, characterized by the property that every cycle of length ≥ 4 has a chord. The numbering is a perfect elimination ordering and the four maximal cliques are marked as triangles. The brackets represent sparse matrices, where each star is a non-zero number. The sparsity structure of the matrix on the left corresponds to the given graph. Each matrix term on the right corresponds to a maximal clique of the graph.

state the following main result, with an independent proof.

Theorem 1: Given a chordal graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, let $M(\cdot,\cdot)$ be a rational matrix with blocks $M_{ij}(s)$ that are non-zero only for $(i,j) \in \mathbb{E}$. Suppose that $M(\bar{s}, s)$ is Hermitean positive definite for Re $s \geq 0$. Then there exist rational matrix functions $M^1(\cdot,\cdot), \ldots, M^J(\cdot,\cdot)$ such that for each i and Re $s \geq 0$ the matrix $M^i(\bar{s}, s)$ is identically zero outside the submatrix ${E_{[i]}}^T M^i(\bar{s}, s) E_{[i]}$ which is positive definite, and

$$M(\bar{s}, s) = \sum_{i=1}^{J} M^i(\bar{s}, s) \tag{2}$$

For illustration of (2), see the lower part of Figure 1.

Remark 2. Notice that (2) holds if and only if

$${E_{[i]}}^T M(\bar{s}, s) E_{[i]} = \sum_{j \in [i]} {E_{[i]}}^T M^j(\bar{s}, s) E_{[i]} *$$

for all i. Hence the condition has the desired sparsity and can be verified node by node.

Proof. Theorem 1 will be proved by induction over J. The statement is trivial for $J = 1$. Suppose that the statement holds for $J = n$ and we want to prove it for $J = n + 1$. Consider a chordal graph with $n + 1$ nodes. After a permutation of the nodes, by Proposition 3, we may assume without restriction that the set

$$\langle i \rangle := \{ j \mid (i,j) \in \mathbb{E},\ i < j \}$$

is a clique for every i. In particular, the graph $\mathbb{G}_1$, obtained by removing the first node from $\mathbb{G}$, is also chordal. Partition the matrix $M(\bar{s}, s)$ as

$$M(\bar{s}, s) = \left[\begin{array}{c|c} M_{1,1} & N_{12} \\ \hline N_{21} & N_{22} \end{array}\right]$$
$$= \left[\begin{array}{c|ccc} M_{1,1} & M_{1,n} & \ldots & M_{1,n+1} \\ \hline M_{2,1} & M_{2,2} & \ldots & M_{2,n+1} \\ \vdots & & \ddots & \vdots \\ M_{n+1,1} & M_{n+1,2} & \ldots & M_{n+1,n+1} \end{array}\right]$$

Then $M(\bar{s}, s) = M^1(\bar{s}, s) + M^2(\bar{s}, s)$ where

$$M^1(\bar{s}, s) = \begin{bmatrix} M_{1,1} & N_{12} \\ N_{21} & N_{21} M_{1,1}^{-1} N_{12} \end{bmatrix} + \epsilon (E_{[1]} E_{[1]}^T - E_1 E_1^T)$$
$$M^2(\bar{s}, s) = \begin{bmatrix} 0 & 0 \\ 0 & N_{22} - N_{21} M_{1,1}^{-1} N_{12} \end{bmatrix} - \epsilon (E_{[1]} E_{[1]}^T - E_1 E_1^T)$$

The matrix $M^1(\bar{s}, s)$ vanishes outside the submatrix ${E_{[1]}}^T M^1(\bar{s}, s) E_{[1]}$, which is positive definite for $\epsilon > 0$. For sufficiently small ϵ, also the lower right corner of $M^2(\bar{s}, s)$ is positive definite. The fact that $\langle 1 \rangle$ is a clique implies that the lower right corner of $M^2(\bar{s}, s)$ has the same sparsity structure as N_{22}, namely the structure corresponding to the chordal graph $\mathbb{G}_1$. Hence, the desired decomposition follows from the induction assumption. □

A proof of Proposition 2 follows immediately:

Proof of Proposition 2. The (if)-statement is the simple direction is trivial. The (only if)-statement follows from Theorem 1 with $M(\bar{s}, s) = (sI - A)^*(sI - A)$. □

V. Beyond Stability

Theorem 1 is not only useful for verification of stability, but also for other types of performance measures. To illustrate this, consider the following condition criteria for gain and passivity.

Proposition 4: The following statements about input-to-state gain are equivalent:

(i) The matrix A is Hurwitz stable and

$$\int_0^\infty |x(t)|^2 dt \le \gamma^2 \int_0^\infty |u(t)|^2 dt$$

for solutions to $\dot{x}(t) = Ax(t) + Bu(t)$, $x(0) = 0$.

(ii)

$$\begin{bmatrix} (sI-A)^*(sI-A) & B \\ B^T & \gamma^2 I \end{bmatrix}$$

is positive semi-definite for Re $s \ge 0$.

In addition, the following statements are equivalent characterizations of passivity:

(iii) The matrix A is Hurwitz stable and

$$\int_0^\infty y(t)^T u(t) dt + \frac{1}{2\gamma^2} \int_0^\infty |w(t)|^2 dt \ge 0$$

when $\dot{x}(t) = Ax(t) + Bu(t) + w(t)$, $y = Cx$, $x(0) = 0$.

(iv)

$$\begin{bmatrix} (sI-A)^*(sI-A) & \gamma^2 C^T - (sI-A)^* B \\ \gamma^2 C - B^T(sI-A) & B^T B \end{bmatrix}$$

is positive semi-definite for Re $s \ge 0$.

Hence, Theorem 1 gives distributed non-conservative tests for gain and passivity respectively in interconnected systems such that the matrices in (ii) and (iv) hava sparsity structure corresponding to a chordal graph.

VI. Acknowledgment

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