Sum rule for the transmission cross section of apertures in thin opaque screens

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Abstract

Extraordinary transmission through sub-wavelength apertures is usually observed in a narrow bandwidth range and the transmission outside this range is low in agreement with classical results. The analysis presented here is based on the Babinet's principle and hence scattering by the complementary structure, where the apertures are replaced by finite scatterers. It is shown that the transmission cross section of a set of apertures in an opaque screen satisfies a sum rule that relates the transmission cross section integrated over all wavelengths with the polarizability of the complementary structure as defined by Babinet's principle. The theoretical results are illustrated with numerical examples.

1 Introduction

Transmission of electromagnetic energy through sub-wavelength apertures in an opaque screen has recently attracted much interest [1, 3, 9, 12, 13]. It is well known that the transmission of an electromagnetic wave, with wavelength λ, through a single circular aperture with radius a is proportional to $(a/\lambda)^4$ as $a/\lambda \to 0$. The transmission cross section, σ_{t}, is hence negligible if $\lambda \gg a$. Extraordinary transmission, $\sigma_{\mathrm{t}} \gg a^2$ can *e.g.*, occur for periodic arrays of apertures, corrugated surroundings of an aperture, cavity antennas, and for resonant apertures. This is often associated with surface plasmon resonances [1, 3, 4, 9, 12, 13] but it also resembles frequency selective surfaces and aperture antennas [10].

In this letter, it is shown that the transmission cross section, σ_{t}, for apertures in a thin opaque screen satisfies a sum rule. The sum rule relates $\sigma_{\mathrm{t}}(\lambda)$ integrated over all wavelengths, λ, to the long wavelength (*i.e.*, static) polarizability of the complementary structure as defined by the Babinet's principle [11, 17]. This restricts the extraordinary transmission with large σ_{t} for specific wavelengths to be narrow banded, *i.e.*, the enhanced transmission must be compensated by a reduced transmission as the total integral is determined by the polarizability and is hence fixed. Moreover, variational principles of the polarizability dyadics [7, 15] are used to show that the integrated transmission is monotonic in the aperture size, *i.e.*, the total transmission over all wavelengths is not smaller for larger apertures.

The sum rule used here is the integrated extinction [15] that has been extensively verified and shown to have several intriguing applications in scattering and antenna theory [5, 6, 16]. It is derived from the holomorphic properties of the forward scattering dyadic that determines the extinction cross section via the optical theorem. It is solely based on the principles of causality, time translational invariance, and passivity [14, 15, 18].

2 Sum rules

Consider an infinite perfectly conducting plate, S_{e}, with an aperture S_{m}, see Fig. 1. The field satisfies the boundary condition $\hat{\boldsymbol{z}} \times \boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{0}$ for $\boldsymbol{r} \in S_{\mathrm{e}}$. The Babinet's

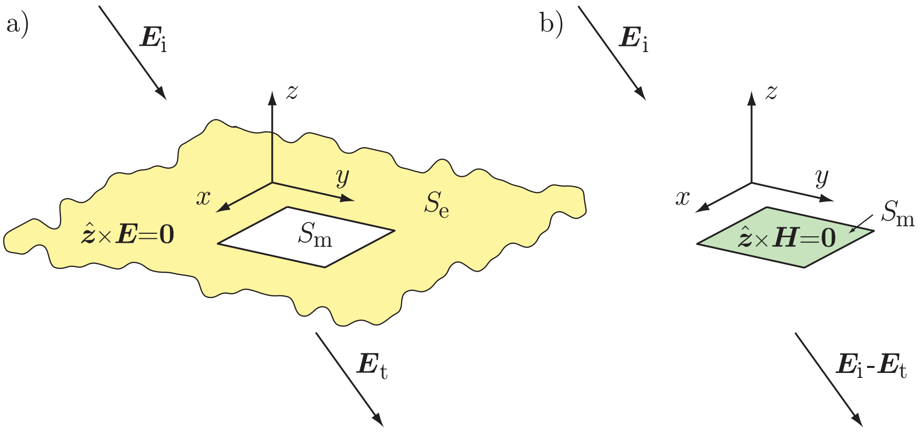

Figure 1: Illustration of the Babinet's principle where the field, $\boldsymbol{E}_\mathrm{t}$, transmitted though the aperture S_m in a) is determined by the scattered field, $\boldsymbol{E}_\mathrm{s}$, of the complementary structure in b), *i.e.*, $\boldsymbol{E}_\mathrm{t} = -\boldsymbol{E}_\mathrm{s}$.

principle states that the transmission pattern of the apertures in the planar opaque screen is identical to the scattering pattern of the complementary structure [11, 17], see Fig. 1. The transmitted field, $\boldsymbol{E}_\mathrm{t}$, can hence be determined as $\boldsymbol{E}_\mathrm{t} = -\boldsymbol{E}_\mathrm{s}$ where $\boldsymbol{E}_\mathrm{s}$ is the scattered field from the complementary magnetic surface where $\hat{\boldsymbol{z}} \times \boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{0}$. In conclusion, instead of analyzing the transmitted field through an aperture in S_e, the scattered field by the complementary magnetic surface S_m is analyzed [11].

The surface S_m does not absorb power so the absorptions cross section of S_m is zero and the extinction cross section, σ_ext, equals the scattering cross section. Moreover, the scattered field radiates equally in the positive and negative half spaces. The transmitted power, P_t, of an incident plane wave is, hence, the incident power flux times half the extinction cross section, *i.e.*,

$$P_\mathrm{t} = -\operatorname{Re}\frac{1}{2}\int_{S_\mathrm{m}} \hat{\boldsymbol{z}}\cdot\boldsymbol{E}_\mathrm{t}\times\boldsymbol{H}_\mathrm{t}^{*}\,\mathrm{dS} = \frac{1}{2\eta_0}\sigma_\mathrm{t}|\boldsymbol{E}_\mathrm{i}|^2, \tag{2.1}$$

where the transmission cross section is $\sigma_\mathrm{t} = \sigma_\mathrm{ext}/2$ and η_0 denotes the free space impedance.

Consider an incident plane wave propagating in the $\hat{\boldsymbol{k}}$-direction with linear polarization $\hat{\boldsymbol{e}}$, *i.e.*, $\boldsymbol{E}_\mathrm{i} = \hat{\boldsymbol{e}}E_0\mathrm{e}^{-\mathrm{i}\hat{\boldsymbol{k}}\cdot\boldsymbol{r}/\lambda}$, where $E_0 \in \mathbb{R}$ and $\hat{\boldsymbol{e}}$ denotes a unit vector orthogonal to $\hat{\boldsymbol{k}}$. The sum rule is derived from the forward scattering via the optical theorem $\sigma_\mathrm{ext} = \operatorname{Im} h$, where $h(\lambda) = 2\lambda\hat{\boldsymbol{e}}\cdot\mathbf{S}(\lambda,\hat{\boldsymbol{k}})\cdot\hat{\boldsymbol{e}}$ is a Herglotz function [14] and $\mathbf{S}$ denotes the forward scattering dyadic [15]. The long-wavelength limit of h is

$$h(\lambda) = \big(\hat{\boldsymbol{e}}\cdot\boldsymbol{\gamma}_\mathrm{e}\cdot\hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}}\times\hat{\boldsymbol{e}})\cdot\boldsymbol{\gamma}_\mathrm{m}\cdot(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{e}})\big)2\pi/\lambda + \mathcal{O}(\lambda^{-2}) \tag{2.2}$$

as $\lambda \to \infty$, where $\boldsymbol{\gamma}_\mathrm{e}$ and $\boldsymbol{\gamma}_\mathrm{m}$ denote the electric and magnetic polarizability dyadics, respectively [8, 15]. The corresponding short-wavelength limit is $2A$, where A is the geometrical cross section of the object, *cf.*, the extinction paradox [2]. Applying the

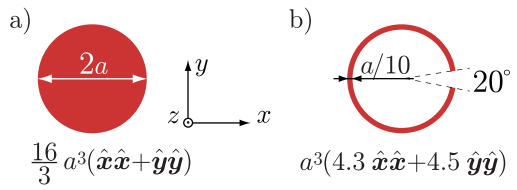

Figure 2: High contrast polarizability dyadics, $\boldsymbol{\gamma}_\infty$, for the a) solid disc and b) split ring resonator.

Cauchy integral formula to a large semi circle in the upper complex half plane gives the integrated extinction [15].

Now, the relation $\sigma_{\mathrm{t}} = \sigma_{\mathrm{ext}}/2$ defines a sum rule for the transmission cross section, *i.e.*,

$$\frac{2}{\pi^2}\int_0^\infty \sigma_{\mathrm{t}}(\lambda;\hat{\boldsymbol{k}},\hat{\boldsymbol{e}})\,\mathrm{d}\lambda = \hat{\boldsymbol{e}}\cdot\boldsymbol{\gamma}_{\mathrm{e}}\cdot\hat{\boldsymbol{e}} + (\hat{\boldsymbol{k}}\times\hat{\boldsymbol{e}})\cdot\boldsymbol{\gamma}_{\mathrm{m}}\cdot(\hat{\boldsymbol{k}}\times\hat{\boldsymbol{e}}). \tag{2.3}$$

The electric polarizability dyadic in (2.2) has only a $\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$-component, *i.e.*, $\boldsymbol{\gamma}_{\mathrm{e}} = \gamma_{\mathrm{e}}\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$, for flat perfectly magnetic conducting plates, *cf.*, [8]. This simplifies the sum rule (2.2) to an identity in the magnetic polarizability dyadic, $\boldsymbol{\gamma}_{\mathrm{m}}$, for polarizations such that $\hat{\boldsymbol{e}}\cdot\hat{\boldsymbol{z}} = 0$, *e.g.*, an incident wave propagating in the normal direction, $\hat{\boldsymbol{k}} = \pm\hat{\boldsymbol{z}}$, *viz.*,

$$\frac{2}{\pi^2}\int_0^\infty \sigma_{\mathrm{t}}(\lambda;\hat{\boldsymbol{z}},\hat{\boldsymbol{e}})\,\mathrm{d}\lambda = (\hat{\boldsymbol{z}}\times\hat{\boldsymbol{e}})\cdot\boldsymbol{\gamma}_{\mathrm{m}}\cdot(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{e}}). \tag{2.4}$$

Estimate the integral in (2.3) to obtain a bound on the transmission cross section times the bandwidth, *i.e.*,

$$|\Lambda|\min_{\lambda\in\Lambda}\sigma_{\mathrm{t}}(\lambda;\hat{\boldsymbol{z}},\hat{\boldsymbol{e}}) \leq \frac{\pi^2}{2}(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{e}})\cdot\boldsymbol{\gamma}_{\mathrm{m}}\cdot(\hat{\boldsymbol{z}}\times\hat{\boldsymbol{e}}), \tag{2.5}$$

where the bandwidth is $|\Lambda| = \lambda_2 - \lambda_1$ and $\Lambda = [\lambda_1, \lambda_2]$ denotes the wavelength interval. Note that the wavenumber, k, or the angular frequency can as well be used to derive sum rules and dispersion relations as $k = -1/\lambda$ maps the upper half plane into itself, *cf.*, [5, 15].

3 High contrast polarizability dyadic

The magnetic polarizability dyadic is related to the magnetic dipole moment, $\boldsymbol{m}$, as $\boldsymbol{m} = \boldsymbol{\gamma}_{\mathrm{m}}\cdot\boldsymbol{H}$, where $\boldsymbol{H}$ is the magnetic field strength. It equals the high contrast polarizability dyadic for perfectly magnetic conducting objects, *i.e.*, $\boldsymbol{\gamma}_{\mathrm{m}} = \boldsymbol{\gamma}_\infty$, *cf.*, the electric case in [15]. The high contrast polarizability dyadic is determined from the first moment of the normalized surface charge density ρ as

$$\boldsymbol{\gamma}_\infty\cdot\hat{\boldsymbol{e}} = \int_{S_{\mathrm{m}}} \boldsymbol{r}\rho(\boldsymbol{r})\,\mathrm{dS}, \tag{3.1}$$

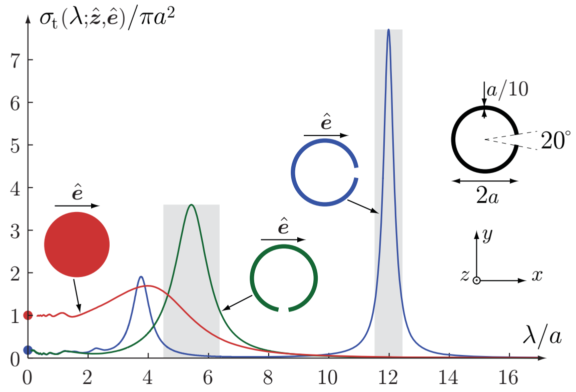

Figure 3: Transmission cross sections for circular and split ring apertures in an opaque screen for an incident plane wave propagating in the normal direction of the screen. The shaded boxes illustrate the bound (2.4), *cf.*, Fig. 2.

where ρ satisfies the integral equation

$$\boldsymbol{r}\cdot\hat{\boldsymbol{e}} + C = \int_{S_{\mathrm{m}}} \frac{\rho(\boldsymbol{r}')}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{dS}' \quad \text{for } \boldsymbol{r}\in S_{\mathrm{m}} \tag{3.2}$$

and the constant C is obtained from the requirement that the total normalized surface charge is zero, $\int_{S_{\mathrm{m}}}\rho(\boldsymbol{r})\,\mathrm{dS}=0$. The polarizability dyadic can also be determined by solving a partial differential equation and there are closed form expressions for the elliptic disc that reduce to $\boldsymbol{\gamma}_\infty = 16a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}})/3$ for a circular disc with radius a [8, 15].

Variational results show that $\boldsymbol{\gamma}_\infty$ is monotonic in the surface S_{m}, *i.e.*, $\boldsymbol{\gamma}_{\infty 1}\leq\boldsymbol{\gamma}_{\infty 2}$ if $S_{\mathrm{m}1}\subset S_{\mathrm{m}2}$, where $\boldsymbol{\gamma}_{\infty j}$, $j=1,2$ denote the high contrast polarizability dyadic of surface $S_{\mathrm{m}j}$, see [7, 15]. The high-contrast polarizability dyadics for a disc and a split ring are given in Fig. 2. It is observed that the polarizability dyadic of the disc is larger than the polarizability dyadic of the split ring. This is in agreement with the variational principles discussed above [7, 15]. However, it is also observed that the area can be substantially reduced with only a minor reduction of the polarizability dyadics, *e.g.*, the area of the circular split ring is 18% and the polarizability in the $\hat{\boldsymbol{y}}$-direction is 84% of the corresponding cases for the circular disc.

4 Numerical example

The sum rule (2.3) is illustrated for apertures in the form of a circular disc a split ring resonator, see Fig 3. The magnetic polarizability dyadics of the complementary perfectly magnetic conducting circular disc and split ring resonator are given in Fig. 2 and they determine the right-hand sides of the sum rule (2.3). Numerical integration of (2.3) over the considered wavelength interval gives 96%, 99% and 99% of the corresponding polarizabilities in the right hand side. The bound (2.4) is illustrated by the shaded boxes in Fig. 3. It is observed that the bound (2.4) offers

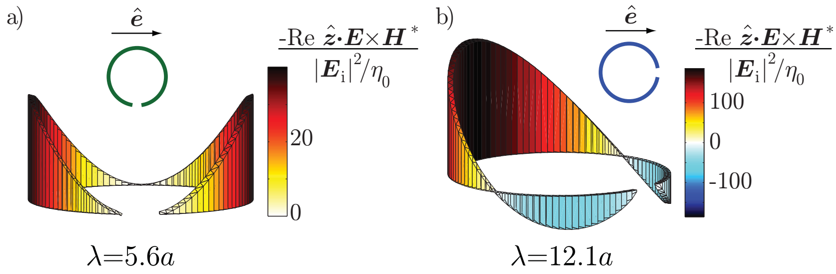

Figure 4: Illustrations of the normalized power-flux density though the split ring aperture at a) $\lambda = 5.6a$ and b) $\lambda = 12.1a$, *cf.*, Fig. 3.

a good estimate of the bandwidth amplitude product as σ_{t} is small outside the resonance wavelength for the split ring aperture.

The sum rule determines the area under the curves and it provides an estimate of the bandwidth times the amplitude of the corresponding resonances. It gives no information about the location of the resonances or if the resonance is wide with small amplitude or narrow with large amplitude. Although, the precise location and shape of a resonance require the solution of the Maxwell equations it is sometimes possible to estimate the resonance wavelength from simple physical arguments. Many objects are resonant when their length is approximately half a wavelength. This partly explains the increased transmission around $\lambda/a = 4$ for the circular disc. This resonance is also very broad due to the many length scales in the disc. However, as the area is limited by the sum rule (2.3), the broad resonance also limits the amplitude of the transmission. The resonances of the split rings are sharper and approximately located at $\lambda/a \approx 4\pi$ and $\lambda/a \approx 2\pi$. These resonances are also narrow and can hence provide a large transmission as seen in Fig. 3.

The enhanced transmission is also seen in Fig. 4, where the normalized power-flux density $-\operatorname{Re}\hat{\boldsymbol{z}}\cdot\boldsymbol{E}\times\boldsymbol{H}^*\eta_0/|\boldsymbol{E}_{\mathrm{i}}|^2$ is depicted for the split ring aperture at the resonances wavelengths $\lambda = 5.6a$ and $\lambda = 12.1a$. It is observed that the maximal power-flux density is increased more than 100 times at $\lambda = 12.1a$ compared to the short wavelength limit of 1. This ability to concentrate energy into sub-wavelength regions has potential applications in near field optics, enhancement of emission from molecules, non-linear optics, and optical switching [12].

5 Conclusions

The sum rule introduced in this letter shows that the transmission cross section integrated over all wavelengths is related to the polarizability of the aperture. Moreover, variational principles show that this total transmission is monotonic in the size of the aperture. This is, *e.g.*, illustrated by the split rings where extraordinary transmission is observed over a narrow bandwidth, however the total transmission is slightly lower than for the circular disc. These results are valid for a large class of apertures and they offer new understanding about the underlying principles of extraordinary

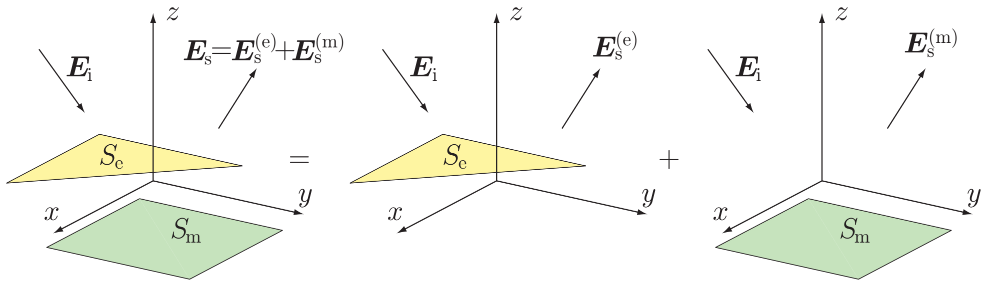

Figure 5: Illustration of the additive property of the scattered field from electric and magnetic surfaces in the xy-plane, *i.e.*, $\boldsymbol{E}_\mathrm{s} = \boldsymbol{E}_\mathrm{s}^{(\mathrm{e})} + \boldsymbol{E}_\mathrm{s}^{(\mathrm{m})}$.

transmission through sub-wavelength apertures in thin opaque screens.

Acknowledgment

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Appendix A Babinet's principle

The Babinet's principle states that the transmission pattern of apertures in a planar opaque screen is identical to the scattering pattern of the complementary structure [11, 17]. Here, an alternative derivation that is based on scattering by disjoint electric and magnetic surfaces in the same plane is presented. Consider two disjoint surfaces $S_\mathrm{e} \subset \mathbb{R}^2$ and $S_\mathrm{m} \subset \mathbb{R}^2$ located in the plane defined by $z = 0$, see Fig. 1. The surfaces are supposed to satisfy the boundary conditions $\hat{\boldsymbol{z}} \times \boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{0}$ for the electric surface $\boldsymbol{r} \in S_\mathrm{e}$ and $\hat{\boldsymbol{z}} \times \boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{0}$ for the magnetic surface $\boldsymbol{r} \in S_\mathrm{m}$. Now the magnetic field induced by arbitrary electric currents on S_e satisfy the related boundary condition on S_m and contrary for the electric field induced by magnetic currents on S_m. Hence, the scattered field by the surfaces is the sum of the scattered fields of S_e and S_m, where the electric and magnetic surfaces can be considered independently of each other, see Fig. 5.

The Babinet's principle follows by letting the union of S_e and S_m be the infinite plane defined by $z = 0$ and assuming $z < 0$ to be source free. As the incident field, $\boldsymbol{E}_\mathrm{i}$, cannot penetrate through the screen, the total field, $\boldsymbol{E} = \boldsymbol{E}_\mathrm{i} + \boldsymbol{E}_\mathrm{s}$, is zero in $z < 0$ and hence $\boldsymbol{E}_\mathrm{s} = -\boldsymbol{E}_\mathrm{i}$ for $z < 0$. Using that the scattered field is additive from the electric and magnetic surfaces, $\boldsymbol{E}_\mathrm{s} = \boldsymbol{E}_\mathrm{s}^{(\mathrm{e})} + \boldsymbol{E}_\mathrm{s}^{(\mathrm{m})}$, gives the scattered field from the electric surface, S_e, as $\boldsymbol{E}_\mathrm{s}^{(\mathrm{e})} = \boldsymbol{E}_\mathrm{s} - \boldsymbol{E}_\mathrm{s}^{(\mathrm{m})} = -\boldsymbol{E}_\mathrm{i} - \boldsymbol{E}_\mathrm{s}^{(\mathrm{m})}$. The total, *i.e.*, the transmitted, field in $z < 0$ with only S_e present is hence determined by the scattered field from the magnetic surface, *i.e.*, $\boldsymbol{E}_\mathrm{s}^{(\mathrm{e})} + \boldsymbol{E}_\mathrm{i} = -\boldsymbol{E}_\mathrm{s}^{(\mathrm{m})}$.

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