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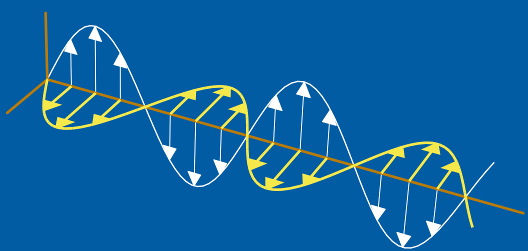

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Abstract

In this paper, physical limitations on scattering of acoustic waves over a frequency interval are discussed based on the holomorphic properties of the scattering amplitude in the forward direction. The result is given by a dispersion relation for the extinction cross section which yields an upper bound on the product of the extinction cross section and the associated bandwidth of any frequency interval. The upper bound is shown to depend only on the geometry and static material properties of the scatterer. The results are exemplified by permeable and impermeable scatterers with homogeneous and isotropic material properties.

1 Introduction

Linear acoustics with propagation and scattering of waves in air and water has been a subject of considerable interest for more than a century. Major contributions to the scattering theory of both acoustic and electromagnetic waves from bounded obstacles was provided by Rayleigh in a sequence of papers. From a theoretical point of view, scattering of acoustic waves share many features with electromagnetic and elastodynamic wave interaction. For a comprehensive introduction to linear acoustics, see, *e.g.*, Refs. 5 and 11.

The objective of this paper is to derive physical limitations on broadband scattering of acoustic waves. In more detail, the scattering problem discussed here involves how a scatterer of arbitrary shape perturbs some known incident field over a frequency interval. The analysis is based on a forward dispersion relation for the extinction cross section applied to a set of passive and linear constitutive relations. This forward dispersion relation, known as the integrated extinction, is a direct consequence of causality and energy conservation via the holomorphic properties of the scattering amplitude in the forward direction. As far as the authors knows, the integrated extinction was first introduced in Ref. 7 concerning absorption and emission of electromagnetic waves by interstellar dust. The analysis in Ref. 7, however, is restricted to homogeneous and isotropic spheroids. This narrow class of scatterers was generalized in Ref. 8 to include anisotropic and heterogenous obstacles of arbitrary shape.

The present paper is a direct application to linear acoustics of the physical limitations for scattering of electromagnetic waves introduced in Refs. 8 and 9. The broad usefulness of the integrated extinction is illustrated by its diversity of applications, see, *e.g.*, Ref. 9 for upper bounds on the bandwidth of metamaterials associated with electromagnetic interaction. The integrated extinction has also fruitfully been applied to antennas of arbitrary shape in Ref. 2 to establish physical limitations on directivity and bandwidth. The theory for broadband scattering of acoustic waves is motivated by the summation rules and the analogy with causality in the scattering theory for particles in Ref. 6.

In Sec. 2, the integrated extinction is derived based on the holomorphic properties of the scattering amplitude in the forward direction. The derivation is based on a

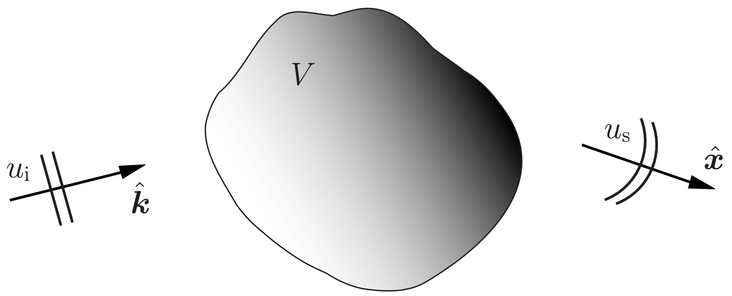

Figure 1: Illustration of the direct scattering problem: the scatterer V is subject to a plane wave $u_{\mathrm{i}} = \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}\cdot\boldsymbol{x}}$ impinging in the $\hat{\boldsymbol{k}}$-direction. The incident field is perturbed by V and a scattered field u_{s} is detected in the $\hat{\boldsymbol{x}}$-direction.

exterior problem, and is hence independent of the boundary conditions imposed on the scatterer. The effect of various boundary conditions are discussed in Sec. 3, and there applied to the results in Sec. 2. In the final section, Sec. 4, the main results of the paper are summarized and possible applications of the integrated extinction are discussed.

2 The integrated extinction

Consider a time-harmonic plane wave $u_{\mathrm{i}} = \mathrm{e}^{\mathrm{i}k\hat{\boldsymbol{k}}\cdot\boldsymbol{x}}$ (complex excess pressure) with time dependence $\mathrm{e}^{-\mathrm{i}\omega t}$ impinging on a bounded, but not necessary simply connected, scatterer $V \subset \mathbb{R}^3$ of arbitrary shape, see Figure 1. The plane wave is impinging in the $\hat{\boldsymbol{k}}$-direction, and $\boldsymbol{x}$ denotes the position vector with respect to some origin. The scatterer V is assumed to be linear and time-translational invariant with passive material properties modeled by general anisotropic and heterogeneous constitutive relations. The analysis includes the impermeable case as well as transmission problems with or without losses. The scatterer V is embedded in the exterior region $\mathbb{R}^3 \setminus \overline{V}$, which is assumed to be a compressible homogeneous and isotropic fluid characterized by the wave number $k = \omega/c$. The material properties of $\mathbb{R}^3 \setminus \overline{V}$ are assumed to be lossless and independent of time.

Let $u = u_{\mathrm{i}} + u_{\mathrm{s}}$ denote the total field in $\mathbb{R}^3 \setminus \overline{V}$, where the time-dependent physical excess pressure p is related to u via $p = \mathrm{Re}\{u\mathrm{e}^{-\mathrm{i}\omega t}\}$. The scattered field u_{s} represents the disturbance of the field in the presence of V. It satisfies the Helmholtz equation in the exterior of V, see Ref. 11, *i.e.*,

$$\nabla^2 u_{\mathrm{s}} + k^2 u_{\mathrm{s}} = 0, \qquad \boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{V}. \tag{2.1}$$

The boundary condition imposed on u_{s} at large distances $x = |\boldsymbol{x}|$ is the Sommerfeld radiation condition

$$\lim_{x\to\infty} x \left(\frac{\partial u_{\mathrm{s}}}{\partial x} - \mathrm{i}k u_{\mathrm{s}} \right) = 0, \tag{2.2}$$

which is assumed to hold uniformly in all directions $\hat{\boldsymbol{x}} = \boldsymbol{x}/x$. The condition (2.2) establishes the outgoing character of u_{s}, and provides a condition for a well-posed exterior boundary value problem. For a discussion of various boundary conditions imposed on V, see Sec. 3.

From the integral representations in Ref. 10 it is clear that every solution to (2.1) satisfying (2.2) has an asymptotic behavior of an outgoing spherical wave, *i.e.*,

$$u_{\mathrm{s}} = \frac{\mathrm{e}^{\mathrm{i}kx}}{x} S(k, \hat{\boldsymbol{x}}) + \mathcal{O}(x^{-2}) \quad \text{as } x \to \infty.$$

The scattering amplitude S is independent of x and describes the interaction of V with the incident field. From a time-domain description of the problem it follows that S is the Fourier transform of some temporal scattering amplitude S_{t}. Assume S_{t} is causal in the forward direction in the sense that $S_{\mathrm{t}}(\tau, \hat{\boldsymbol{k}}, \hat{\boldsymbol{k}}) = 0$ for $\tau < 0$, where $\tau = ct - \hat{\boldsymbol{k}} \cdot \boldsymbol{x}$. Based on this condition, the Fourier transform of S_{t} reduces to an integral over $\tau > 0$, *i.e.*,

$$S(k, \hat{\boldsymbol{k}}) = \int_0^\infty S_{\mathrm{t}}(\tau, \hat{\boldsymbol{k}}, \hat{\boldsymbol{k}}) \mathrm{e}^{\mathrm{i}k\tau} \, \mathrm{d}\tau. \tag{2.3}$$

The convergence of (2.3) is improved by extending its domain of definition to complex-valued k with $\operatorname{Im} k > 0$. Such an extension defines a holomorphic function S in the upper half plane $\operatorname{Im} k > 0$, see Sec. 1 in Ref. 6. Note that S in general is not a holomorphic function at infinity for $\operatorname{Im} k > 0$ in the absence of the causality condition.

The description of broadband scattering is simplified by introducing a weighted function ϱ of the scattering amplitude in the forward direction. For this purpose, let ϱ denote the holomorphic function

$$\varrho(k) = S(k, \hat{\boldsymbol{k}})/k^2, \qquad \operatorname{Im} k > 0.$$

Since S_{t} is real-valued it follows from (2.3) that ϱ is real-valued on the imaginary axis, and that it satisfies the cross symmetry $\varrho(-k^*) = \varrho^*(k)$ (the star denotes complex conjugation) for complex-valued k. Assume that ϱ vanishes uniformly as $|k| \to \infty$ for $\operatorname{Im} k \geq 0$. This assumption is justified by the argument that the high-frequency response of a material is non-unique from a modeling point of view. The assumption is also supported by the extinction paradox $\operatorname{Im} \varrho(k) = \mathcal{O}(k^{-1})$ as $k \to \infty$ for real-valued k, see Ref. 8 and references therein.

An important measure of the total energy that V extracts from the incident field in the form of radiation or absorption is given by the extinction cross section σ_{ext}. The extinction cross section is related to ϱ via the optical theorem, see Ref. 6,

$$\sigma_{\mathrm{ext}} = 4\pi k \operatorname{Im} \varrho, \tag{2.4}$$

where $k \in [0, \infty)$. The optical theorem is a direct consequence of energy conservation (or probability in the scattering theory of the Schrödinger equation) and states that the total energy removed from the incident field is solely determined by $\operatorname{Im} \varrho$. The extinction cross section is commonly decomposed into the scattering cross section σ_{s} and the absorption cross section σ_{a}, *i.e.*,

$$\sigma_{\mathrm{ext}} = \sigma_{\mathrm{s}} + \sigma_{\mathrm{a}}. \tag{2.5}$$

Here, σ_{s} and σ_{a} are defined as the scattered and absorbed power divided by the incident power flux. The scattering and absorption cross sections are related to u_{s} and u on the boundary ∂V via, see Ref. 1,

$$\sigma_{\mathrm{s}} = \frac{4\pi}{k} \operatorname{Im} \int_{\partial V} u_{\mathrm{s}}^* \frac{\partial u_{\mathrm{s}}}{\partial n} \, \mathrm{d}S, \qquad \sigma_{\mathrm{a}} = \frac{4\pi}{k} \operatorname{Im} \int_{\partial V} u \frac{\partial u^*}{\partial n} \, \mathrm{d}S,$$

where the normal derivative $\partial/\partial n$ is evaluated with respect to the outward pointing unit normal vector. In the permeable and lossy case, the absorption cross section σ_{a} represents the total energy absorbed by V. For a lossless scatterer, $\sigma_{\mathrm{a}} = 0$.

Under the assumption that ϱ vanishes uniformly as $|k| \to \infty$ for $\operatorname{Im} k \geq 0$, it follows from the analysis in Ref. 6 that ϱ satisfies the Hilbert transform or the Plemelj formulae

$$\operatorname{Re} \varrho(k') = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\operatorname{Im} \varrho(k)}{k - k'} \, \mathrm{d}k, \tag{2.6}$$

where k' is real-valued and $\mathcal{P}$ denotes Cauchy's principal value. It is particularly interesting to evaluate (2.6) in the static limit. For this purpose, assume that $\operatorname{Re} \varrho(k') = \mathcal{O}(1)$ and $\operatorname{Im} \varrho(k') = \mathcal{O}(k')$ as $k' \to 0$, and that ϱ is sufficiently regular to interchange the principal value and the static limit. Based on these assumptions, (2.4) yields

$$\lim_{k \to 0} \operatorname{Re} \varrho(k) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\operatorname{Im} \varrho(k)}{k} \, \mathrm{d}k, \tag{2.7}$$

where it has been used that $\operatorname{Im} \varrho(k) = -\operatorname{Im} \varrho(-k)$ for real-valued k. The optical theorem (2.4) inserted into (2.7) finally yields

$$\int_{0}^{\infty} \frac{\sigma_{\mathrm{ext}}(k)}{k^2} \, \mathrm{d}k = 2\pi^2 \lim_{k \to 0} \operatorname{Re} \varrho(k). \tag{2.8}$$

The left hand side of (2.8) is referred to as the integrated extinction. The identity provides a forward dispersion relation for the extinction cross section as a direct consequence of causality and energy conservation. In fact, due to the lack of any length scale in the static limit as $k \to 0$, the right hand side of (2.8) is proportional to the volume of V. Furthermore, the right hand side of (2.8) depends only on the static properties of V, and is presented in Sec. 3 for a large class of homogeneous and isotropic scatterers.

The weak assumptions imposed on ϱ in the derivation above is summarized as follows: $\varrho(k) \to 0$ uniformly as $|k| \to \infty$ for $\operatorname{Im} k \geq 0$, and $\operatorname{Re} \varrho(k) = \mathcal{O}(1)$ and $\operatorname{Im} \varrho(k) = \mathcal{O}(k)$ as $k \to 0$ for real-valued k. In general, the integrated extinction (2.8) is not valid if any of these assumptions are violated, as illustrated in Sec. 3.3. In fact, the requirements above can be relaxed by the introduction of the Plemelj formulae for distributions. The integrated extinction (2.8) can also be derived using Cauchy's integral theorem, see Ref. 8.

The integrated extinction (2.8) may be used to establish physical limitations on broadband scattering by acoustic waves. Since σ_{ext} is defined as the sum of the scattered and absorbed power divided by the incident power flux, it is by definition non-negative. Hence, the left hand side of (2.8) is estimated from below by

$$|K| \min_{k\in K} \frac{\sigma(k)}{k^2} \leq \int_K \frac{\sigma(k)}{k^2}\,\mathrm{d}k \leq \int_0^\infty \frac{\sigma_{\mathrm{ext}}(k)}{k^2}\,\mathrm{d}k, \tag{2.9}$$

where $|K|$ denotes the absolute bandwidth of any $K \subset [0, \infty)$, and σ represents either σ_{ext}, σ_{s} or σ_{a}. By combining the left hand side of (2.9) with the right hand side of (2.8), one obtain the fundamental inequality

$$|K| \min_{k\in K} \frac{\sigma(k)}{k^2} \leq 2\pi^2 \lim_{k\to 0} \mathrm{Re}\,\varrho(k). \tag{2.10}$$

The interpretation of (2.10) is that it yields an upper bound on the absolute bandwidth $|K|$ for a given scattering and/or absorption cross section $\min_{k\in K} \sigma(k)/k^2$. From (2.10), it is seen that the static limit of $\mathrm{Re}\,\varrho$ bounds the total amount of power extracted by V within K. The electromagnetic analogy to (2.10) is, *inter alia*, central for establishing upper bounds on the performance of antennas of arbitrary shape, see Ref. 2.

3 The effect of various boundary conditions

In this section, the static limit $\lim_{k\to 0} \mathrm{Re}\,\varrho$ is examined for various boundary conditions and applied to the integrated extinction (2.8). For this purpose, V is assumed to be homogeneous and isotropic with sufficiently smooth boundary ∂V to guarantee the existence of boundary values in the classical sense.

3.1 The Neumann or acoustically hard problem

The Neumann or acoustically hard problem corresponds to an impermeable scatterer with boundary condition $\partial u/\partial n = 0$ for $\boldsymbol{x} \in \partial V$. The physical interpretation of the Neumann boundary condition is that the velocity field on ∂V is zero since no local displacements are admitted. From the fact that u_{s} only exists in $\mathbb{R}^3 \setminus \overline{V}$, it follows that the corresponding scattered field in the time-domain cannot precede the incident field in the forward direction, *i.e.*, the causality condition imposed on S_{t} in Sec. 2 is valid for the Neumann problem. The static limit of S is derived in Refs. 1 and 3 from a power series expansion of u_{i} and u_{s}. The result in terms of $\mathrm{Re}\,\varrho$ reads

$$\lim_{k\to 0} \mathrm{Re}\,\varrho(k) = \frac{1}{4\pi}(\hat{\boldsymbol{k}} \cdot \boldsymbol{\gamma}_{\mathrm{m}} \cdot \hat{\boldsymbol{k}} - |V|), \tag{3.1}$$

where $|V|$ denotes the volume of V. Here, $\boldsymbol{\gamma}_{\mathrm{m}}$ models the scattering of acoustic waves in the low frequency limit. In analogy with the corresponding theory for electromagnetic waves in Ref. 8, $\boldsymbol{\gamma}_{\mathrm{m}}$ is termed the magnetic polarizability dyadic. The magnetic polarizability dyadic is proportional to $|V|$, and closed-form expressions of $\boldsymbol{\gamma}_{\mathrm{m}}$ exist for the ellipsoids.

An expression of the integrated extinction for the Neumann problem is obtained by inserting (3.1) into (2.8), *viz.*,

$$\int_0^\infty \frac{\sigma_{\text{ext}}(k)}{k^2}\,\mathrm{d}k = \frac{\pi}{2}(\hat{\boldsymbol{k}}\cdot\boldsymbol{\gamma}_{\text{m}}\cdot\hat{\boldsymbol{k}} - |V|). \tag{3.2}$$

Note that (3.2) is independent of $\hat{\boldsymbol{k}}$ when $\boldsymbol{\gamma}_{\text{m}}$ is isotropic, *i.e.*, $\boldsymbol{\gamma}_{\text{m}} = \gamma_{\text{m}}\mathbf{I}$ where $\mathbf{I}$ denotes the unit dyadic, corresponding to a scatterer which is invariant under certain point groups, see Ref. 8 and references therein. The product $\hat{\boldsymbol{k}}\cdot\boldsymbol{\gamma}_{\text{m}}\cdot\hat{\boldsymbol{k}}$ on the right hand side of (3.2) can be estimated from above by the largest eigenvalue of $\boldsymbol{\gamma}_{\text{m}}$, and associated upper bounds on these eigenvalues are extensively discussed in Ref. 8. The static limit of $\operatorname{Re}\varrho$ in (3.1) can also be inserted into the right hand side of (2.10) to yield an upper bound on the scattering and absorption properties of V within any finite interval K.

The integrated extinction (3.2) takes a particularly simple form for the sphere. In this case, $\boldsymbol{\gamma}_{\text{m}}$ is isotropic with $\gamma_{\text{m}} = 3|V|/2$, see Refs. 3 and 8, and the right hand side of (3.2) is reduced to $\pi|V|/4$. This result for the sphere has numerically been verified using the classical Mie-series expansion in Ref. 5.

3.2 The transmission or acoustically permeable problem

In addition to the exterior boundary value problem (2.1) and (2.2), the transmission or acoustically permeable problem is defined by the interior requirement that $\nabla^2 u_{\text{s}} + k_\star^2 u_{\text{s}} = 0$ for $\boldsymbol{x}\in V$ with the induced boundary conditions $u^+ = u^-$ and $\rho_\delta \partial u^+/\partial n = \partial u^-/\partial n$. Here, $k_\star = \omega/c_\star$ denotes the wave number in V, and u^+ and u^- represents the limits of u from $\mathbb{R}^3\setminus\overline{V}$ and V, respectively. The quantity ρ_δ is related to the relative mass density $\rho_{\text{rel}} = \rho_\star/\rho$ via $\rho_\delta = \rho_{\text{rel}}/(1-\mathrm{i}\omega\delta_\star\kappa_\star)$, where $\kappa_\star$ and $\rho_\star$ denotes the compressibility and the mass density of V, respectively. The compressibility represents the relative volume reduction per unit increase in surface pressure. The conversion of mechanical energy into thermal energy due to losses in V are modeled by the compressional viscosity $\delta_\star > 0$, which represents the rate of change of mass per unit length. In the lossless case, $\delta_\star = 0$, the phase velocity is $c_\star = 1/\sqrt{\kappa_\star\rho_\star}$ and $\rho_\delta = \rho_{\text{rel}}$.

The causality condition introduced in Sec. 2 is valid for the transmission problem provided $\operatorname{Re} c_\star < c$, *i.e.*, when the incident field precedes the scattered field in the forward direction. Unless V does not fulfill this requirement, ϱ is not holomorphic for $\operatorname{Im} k > 0$ and the analysis in Sec. 2 does not hold. Hence, the integrated extinction (2.8) is not valid if $\operatorname{Re} c_\star \geq c$. This defect can partially be justified by replacing the definition of ϱ by $\varrho = \mathrm{e}^{2\mathrm{i}ka} S(k,\hat{\boldsymbol{k}})/k^2$, where $a > 0$ is sufficiently large to guarantee the existence of causality in the forward direction. The compensating factor $\mathrm{e}^{2\mathrm{i}ka}$ corresponds to a time-delayed scattered field, and for homogenous and isotropic scatterers, a sufficient condition for a is $2a > \operatorname{diam} V$, where $\operatorname{diam} V$ denotes the diameter of V. A drawback of the introduction of the factor $\mathrm{e}^{2\mathrm{i}ka}$ in the definition of ϱ is that the optical theorem no longer can be identified in the derivation. Instead, the integrated extinction for scatterers which not obey the causality condition reduce to integral identities for $\operatorname{Re}\varrho$ and $\operatorname{Im}\varrho$. Unfortunately, in this case the integrands have not a definite sign and therefore the estimate (2.10) is not valid.

The static limit of the scattering amplitude S for the transmission problem is derived in Refs. 1 and 3. The result in terms of $\operatorname{Re}\varrho$ reads

$$\lim_{k\to 0}\operatorname{Re}\varrho(k)=\frac{1}{4\pi}((\kappa_{\mathrm{rel}}-1)|V|-\hat{\boldsymbol{k}}\cdot\boldsymbol{\gamma}(\rho_{\mathrm{rel}}^{-1})\cdot\hat{\boldsymbol{k}}), \tag{3.3}$$

where $\kappa_{\mathrm{rel}}=\kappa_\star/\kappa$ denotes the relative compressibility of V, and $\boldsymbol{\gamma}$ represents the general polarizability dyadic. In the derivation of (3.3), it has been used that possible losses $\delta_\star>0$ in V do not contribute in the static limit of $\operatorname{Re}\varrho$, which motivates that the argument in $\boldsymbol{\gamma}$ is ρ_{rel} rather than ρ_δ. Analogous to $\boldsymbol{\gamma}_{\mathrm{m}}$, the general polarizability dyadic is proportional to $|V|$, and closed-form expressions for $\boldsymbol{\gamma}$ exist for the ellipsoids, see Refs. 1, 3 and 8. From the properties of $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}_{\mathrm{m}}$ in the references above, it follows that $\boldsymbol{\gamma}(\rho_{\mathrm{rel}}^{-1})\to-\boldsymbol{\gamma}_{\mathrm{m}}$ as $\rho_{\mathrm{rel}}\to\infty$, and hence the static limit of $\operatorname{Re}\varrho$ reduces to (3.1) for the Neumann problem as $\kappa_{\mathrm{rel}}\to 0+$ and $\rho_{\mathrm{rel}}\to\infty$. Another interesting limit corresponding to vanishing mass density in V is given by $\boldsymbol{\gamma}(\rho_{\mathrm{rel}}^{-1})\to\boldsymbol{\gamma}_{\mathrm{e}}$ as $\rho_{\mathrm{rel}}\to 0+$, where $\boldsymbol{\gamma}_{\mathrm{e}}$ is termed the electric polarizability dyadic in analogy with the low frequency scattering of electromagnetic waves, see Refs. 1, 3 and 8.

The integrated extinction for the transmission problem is given by (3.3) inserted into (2.8). The result is

$$\int_0^\infty\frac{\sigma_{\mathrm{ext}}(k)}{k^2}\,\mathrm{d}k=\frac{\pi}{2}((\kappa_{\mathrm{rel}}-1)|V|-\hat{\boldsymbol{k}}\cdot\boldsymbol{\gamma}(\rho_{\mathrm{rel}}^{-1})\cdot\hat{\boldsymbol{k}}), \tag{3.4}$$

Note that (3.4) is independent of any losses $\delta_\star>0$, and that the directional character of the integrated extinction only depends on the relative mass density ρ_{rel}. For $\rho_{\mathrm{rel}}\to 1$, *i.e.*, identical mass densities in V and $\mathbb{R}^3\setminus\overline{V}$, the integrated extinction is independent of the incident direction $\hat{\boldsymbol{k}}$, depending only on the relative compressibility κ_{rel}. Furthermore, the integrated extinction (3.2) vanishes in the limit as $\kappa_{\mathrm{rel}}\to 1$ and $\rho_{\mathrm{rel}}\to 1$, corresponding to identical material properties in V and $\mathbb{R}^3\setminus\overline{V}$. Due to the non-negative character of the extinction cross section, this limit implies that $\sigma_{\mathrm{ext}}=0$ independent of the frequency as expected. Analogous to the Neumann problem, (3.4) is also independent of the incident direcion $\hat{\boldsymbol{k}}$ for scatterers with $\boldsymbol{\gamma}=\gamma\mathbf{I}$ for some real-valued γ. The product $\hat{\boldsymbol{k}}\cdot\boldsymbol{\gamma}\cdot\hat{\boldsymbol{k}}$ on the right hand side of (3.4) is estimated from above by the largest eigenvalue of $\boldsymbol{\gamma}$, and associated upper bounds on these eigenvalues are discussed in Ref. 8. The static limit of $\operatorname{Re}\varrho$ in (3.1) can also be inserted into the right hand side of (2.10) to yield an upper bound on the scattering and absorption properties of V over any finite interval K.

For the isotropic and homogenous sphere, $\gamma=3|V|(1-\rho_{\mathrm{rel}})/(2\rho_{\mathrm{rel}}+1)$, and the right hand side of (3.3) is independent of the incident direction as required by symmetry. Also this result for the sphere has been verified numerically to arbitrary precision using the classical Mie-series expansion.

3.3 Boundary conditions with contradictions

The integrated extinction (2.8) and the analysis in Sec. 2 are not applicable to the Dirichlet or acoustically soft problem with $u = 0$ for $\boldsymbol{x} \in \partial V$. The physical interpretation of the Dirichlet boundary condition is that the scatterer offers no resistance to pressure. The Dirichlet problem defines an impermeable scatterer for which u_{s} only exist in $\mathbb{R}^3 \setminus \overline{V}$. Hence, the causality condition introduced in Sec. 2 is valid. However, the assumption that $\operatorname{Re} \varrho(k) = \mathcal{O}(1)$ as $k \to 0$ for real-valued k is not valid in this case. Instead, Refs. 1 and 3 suggest that

$$\operatorname{Re} \varrho(k) = \mathcal{O}(k^{-2}) \quad \text{as } k \to 0$$

for real-valued k. The conclusion is therefore that the integrated extinction (2.8) is not valid for the Dirichlet problem.

The same conclusion also holds for the Robin problem with impedance boundary condition $\partial u/\partial n + \mathrm{i}k\nu u = 0$ for $\boldsymbol{x} \in \partial V$. The Robin problem models an intermediate behavior between the Dirichlet and Neumann problems, see Ref. 1. The real-valued constant ν is related to the exterior acoustic impedance η (defined by the ratio of the excess pressure and the normal velocity on ∂V) via $\eta\nu = \sqrt{\rho/\kappa}$, where κ and ρ denotes the compressibility and mass density of $\mathbb{R}^3 \setminus \overline{V}$, respectively. In the limits $\nu \to 0+$ and $\nu \to \infty$, the Robin problem reduces to the Neumann and Dirichlet problems, respectively. For the Robin problem, the static limit of $\operatorname{Re} \varrho$ for $\nu \neq 0$ reads, see Refs. 1 and 3,

$$\operatorname{Re} \varrho(k) = \mathcal{O}(k^{-1}) \quad \text{as } k \to 0$$

for real-valued k. Hence, the assumption in Sec. 2 that $\operatorname{Re} \varrho(k) = \mathcal{O}(1)$ as $k \to 0$ is not valid for the Robin problem either. The question whether a similar identity to the integrated extinction exists for the Dirichlet and Robin problems with other weight functions than $1/k^2$ in (2.8), is addressed in a forthcoming paper.

4 Conclusion

The static limits of $\operatorname{Re} \varrho$ in Sec. 3 can be used in (2.10) to establish physical limitations on the amount of energy a scatterer can extract from a known incident field in any frequency interval $K \subset [0, \infty)$. Both absorbed and radiated energy is taken into account. From the analysis of homogeneous and isotropic scatterers in Sec. 3, it is clear that the integrated extinction holds for both Neumann and transmission problems. However, the present formulation of the integrated extinction fails for the Dirichlet and Robin problems since the assumption in Sec. 2 that $\operatorname{Re} \varrho(k) = \mathcal{O}(1)$ as $k \to 0$ for real-valued k is violated for these boundary conditions. In fact, the eigenvalues of the polarizability dyadics $\boldsymbol{\gamma}$, $\boldsymbol{\gamma}_{\mathrm{e}}$ and $\boldsymbol{\gamma}_{\mathrm{m}}$ are easily calculated using the finite element method (FEM). Some numerical results of these eigenvalues are presented in Refs. 8 and 9 together with comprehensive illustrations of the integrated extinction for electromagnetic waves.

The integrated extinction (2.8) can also be used to establish additional information on the inverse scattering problem of linear acoustics. One advantage of the integrated extinction is that it only requires measurements of the scattering amplitude in the forward direction. The theory may also be used to obtain additional insights into the possibilities and limitations of manufactured materials such as acoustic metamaterials in Ref. 4. However, the main importance of the integrated extinction (2.8) is that it provides a fundamental knowledge of the physical processes involved in wave interaction with matter over any bandwidth. It is also crucial to the understanding of the physical effects imposed on a system by the first principles of causality and energy conservation.

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References

[1] G. Dassios and R. Kleinman. *Low frequency scattering.* Oxford University Press, Oxford, 2000.

[2] M. Gustafsson, C. Sohl, and G. Kristensson. Physical limitations on antennas of arbitrary shape. Submitted to Proc. Roy. Soc. London Ser. A, 2007.

[3] R. E. Kleinman and T. B. A. Senior. Rayleigh scattering. In V. V. Varadan and V. K. Varadan, editors, *Low and high frequency asymptotics*, volume 2 of *Acoustic, Electromagnetic and Elastic Wave Scattering*, chapter 1, pages 1–70. Elsevier Science Publishers, Amsterdam, 1986.

[4] J. Li and C. T. Chan. Double-negative acoustic metamaterial. *Phys. Rev. E*, **70**(5), 055602, 2004.

[5] P. M. Morse and K. U. Ingard. *Theoretical Acoustics.* McGraw-Hill, New York, 1968.

[6] H. M. Nussenzveig. *Causality and dispersion relations.* Academic Press, London, 1972.

[7] E. M. Purcell. On the absorption and emission of light by interstellar grains. *J. Astrophys.*, **158**, 433–440, 1969.

[8] C. Sohl, M. Gustafsson, and G. Kristensson. Physical limitations on broadband scattering by heterogeneous obstacles. Submitted to J. Phys. A: Math. Gen., 2006.

[9] C. Sohl, M. Gustafsson, and G. Kristensson. Physical limitations on metamaterials: Restrictions on scattering and absorption over a frequency interval. Technical Report LUTEDX/(TEAT-7154)/1–10/(2007), Lund University, Department of Electrical and Information Technology, P.O. Box 118, S-221 00 Lund, Sweden, 2007. `http://www.es.lth.se`.

[10] S. Ström. Introduction to integral representations and integral equations for time-harmonic acoustic, electromagnetic and elastodynamic wave fields. In V. V. Varadan, A. Lakhtakia, and V. K. Varadan, editors, *Field Representations and Introduction to Scattering*, Acoustic, Electromagnetic and Elastic Wave Scattering, chapter 2, pages 37–141. Elsevier Science Publishers, Amsterdam, 1991.

[11] V. V. Varadan and V. K. Varadan. Acoustic, electromagnetic and elastodynamics fields. In V. V. Varadan, A. Lakhtakia, and V. K. Varadan, editors, *Field Representations and Introduction to Scattering*, Acoustic, Electromagnetic and Elastic Wave Scattering, chapter 1, pages 1–35. Elsevier Science Publishers, Amsterdam, 1991.