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Wave propagators for transient waves in one-dimensional media

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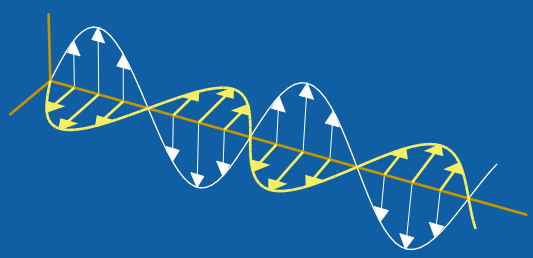

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Abstract

Wave propagators are introduced for transient wave propagation in a general one-dimensional medium. The definition of wave propagators is based upon a wave splitting technique. Several properties for the propagators are given and the relations between the method of propagators and the invariant imbedding method and the Green function approach are discussed.

1 Introduction

The method of propagators for one-dimensional transient wave-propagation is described and its connection to the invariant imbedding method and the Green function approach is discussed. The present method of propagators has been used in two earlier papers on wave propagation in periodic structures, see Refs 21, 26. In this paper the method is described in a more general setting that focus on the properties of the propagators. The propagator method, as well as the imbedding method and the Green function approach, is based upon a wave splitting technique for the wave equation. The field is thus split into generalized left and right moving waves. In all three methods scattering kernels, rather than the fields, are the significant quantities. The kernels are independent of the incident field and this is of importance for inverse scattering problems. The three methods can handle the same class of wave propagation problems. However, the choice of method for a particular problem is crucial since the solution to a specific problem can be complicated with one of the methods but elegant and simple with another. In most cases the Green function approach gives less complicated solutions, both analytically and numerically, than the imbedding method. However there are some important exceptions, such as wave propagation in homogeneous media, where the imbedding method is preferable. As will be seen in this paper the imbedding method and the Green function approach can be viewed as special cases of the propagator method. The strength of the propagator method is that it either provides the best solution obtained by the other two methods or, by utilizing properties that are not available in the other two methods, gives an even simpler solution. Two examples where the propagator method is superior to the other two methods are wave propagation in periodic structures, see Ref. 26, and internal source problems.

This introduction contains a review of the imbedding method and the Green function approach applied to direct and inverse scattering in the time domain. There are numerous other methods that have been applied to the same class of problems, but these methods are not touched upon here. The imbedding method and the Green function approach have been applied to one-dimensional problems in acoustics, elastodynamics and electromagnetics. Often these areas are mathematically equivalent. It is more appropriate to make a distinction between wave propagation in dispersive materials and non-dispersive materials. The former materials lead to wave equations that are non-local in time and the theoretical and numerical analysis of these equations differ from the analysis of the equations that are local in time.

The imbedding method was introduced to inverse scattering in the time domain

in Refs 4, 5, where non-dispersive, non-dissipative materials were considered. The theory has then been generalized to dissipative media [6] and dispersive materials [2, 3, 20]. A thorough description of the imbedding method is found in the series of papers [29–31]. In these papers the inverse problem of simultaneous reconstruction of two material parameters was considered. It was shown how both conductivity and permittivity of a non-dispersive slab can be reconstructed using both reflection and transmission data.

The Green function approach was introduced in Ref. 33. The method was then applied to a non-dispersive, inhomogeneous wave equation. Both the inverse and the direct problem were considered. Later the dispersive problem was also analyzed, see Ref. 27. In Refs 13 and 37 the inverse problem of simultaneous reconstruction of two material parameters of a slab was discussed. These papers are the Green function correspondence to the papers [29–31]. It was found that the Green function approach gives a somewhat simpler solution to the inverse problem than the imbedding method. However, the structure of the solution in the two cases is the same.

The direct source problem, i.e. to find the field from a slab with internal sources, and also the corresponding inverse source problem, i.e. to find the sources of a slab by measuring the field at the boundary of the slab, have been considered in Refs 8 and 36. Also the problem of finding the material parameters of a slab using the field from internal sources was considered in these papers. A related problem is to find the incident field on a slab by measuring the transmitted field. This is done by introducing what is referred to as the compact Green function, see Refs 13 and 35. The Green function approach has also been applied to gyrotropic media, [24, 25], anisotropic media [10, 11], biisotropic media [32, 44, 45] and to waveguides [28].

In most applications on wave propagation the materials are assumed to be invariant under time translation. However in some cases, non-stationary materials have to be considered. Recently the Green function approach was generalized to cover these type of problems [1].

An interesting class of inverse problems are so called design problems, where a medium is designed such that a prescribed type of wave propagation or reflection is achieved. Such problems have been studied in, e.g., Refs 19 and 34, using the Green function approach. A related problem is to design an incident wave on a given medium in order to obtain a prescribed power distribution or dissipation in the medium. This problem has been studied using the Green function approach together with optimization methods [40, 41].

There are some attempts to use the imbedding and Green function approaches in two and three-dimensional problems that due to geometrical symmetries can be reduced to direct and inverse problems in one spatial coordinate, see Refs 16,17,21–23. The fully three-dimensional problems are more challenging and much attention has been paid to these problems lately. A wave splitting for a three-dimensional geometry was introduced by Weston [46, 48, 49]. This splitting is the basis for the three-dimensional imbedding and Green function approaches, see Refs 47 and 48. There are several encouraging results concerning three-dimensional inverse scattering using these approaches, eg., Refs 50 and 18.

So far, the references given here have concerned time-domain problems. However, there are frequency domain counterparts of the imbedding method and the Green function approach. A frequency domain Green function approach has been applied to problems concerning transmission lines [14] and also to biisotropic media and bianisotropic media [15, 38, 39].

In this paper an operator notation is adopted in order to keep the formulas and equations as compact as possible. In the next section, the wave splitting is introduced and the wave equation is written in terms of the split fields. In Section 3 the wave propagators are introduced using the short hand operator notation. The explicit definition of the propagators are then introduced in Section 4. The propagators are expressed in terms of propagator kernels and the equations for these kernels are presented in Section 5. The connection between the method of propagators and the imbedding and Green function approaches are discussed in Section 6. In Section 7 the propagators of homogeneous media are analyzed. There is also and appendix where the equations for the propagator kernels are derived.

2 Wave equation and wave splitting

Consider a one-dimensional isotropic medium and let E be a transverse component of the electric field. The following wave equation for the electric field is considered in the half-space $z > 0$:

$$\partial_z^2 E(z,t) - \frac{1}{c(z)^2}\partial_t^2 E(z,t) - F[E] = 0$$

where $F[E]$ is the linear functional

$$F[E] = f(z)\partial_t E(z,t) + g(z)E(z,t) + \int_{-\infty}^{t} h(z,t-\tau)E(z,\tau)d\tau$$

The half-space $z < 0$ contains sources that generate the incident field, otherwise there are no restrictions on that half-space. The wave equation includes inhomogeneous wave speed, dissipation and material dispersion effects. Even though more general wave equations can be considered, this one is general enough in order to present the properties of the propagator method. In $z > 0$ it is assumed that $c(z)$, $\partial_z c(z)$ and $f(z)$ are continuous. The reason for these restrictions is to avoid delta pulses and jump discontinuities in the scattering and propagator kernels. By using a technique similar to the one described in Ref. 31 more general classes of $c(z)$ and $f(z)$ can be considered, but that technique will not be utilized in this paper. In a matrix form the wave equation reads

$$\partial_z \begin{pmatrix} E \\ \partial_z E \end{pmatrix} = \mathcal{A} \begin{pmatrix} E \\ \partial_z E \end{pmatrix} \tag{2.1}$$

where the calligraphic $\mathcal{A}$ denotes the matrix-valued operator

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ \frac{1}{c(z)^2}\partial_t^2 + F[\cdot] & 0 \end{pmatrix} \tag{2.2}$$

The calligraphic font is used to denote an operator in this paper.

The wave splitting is a transformation from the fields E and $\partial_z E$ to two new fields E^+ and E^- as

$$\begin{pmatrix} E^+ \\ E^- \end{pmatrix} = \mathcal{S} \begin{pmatrix} E \\ \partial_z E \end{pmatrix} \tag{2.3}$$

where

$$\mathcal{S} = \frac{1}{2} \begin{pmatrix} 1 & -\mathcal{K} \\ 1 & \mathcal{K} \end{pmatrix} \tag{2.4}$$

and $\mathcal{K}$ is a linear operator. A necessary condition for the splitting operator $\mathcal{K}$ is that the principal part of the system of wave equations for $E^{\pm}$ becomes diagonal. The simplest form of such a splitting operator is

$$\mathcal{K}\partial_z E(z,t) = c(z)\partial_t^{-1}\partial_z E(z,t) = c(z)\int_{-\infty}^{t} \partial_z E(z,t')dt' \tag{2.5}$$

From Eqs (2.1) and (2.3) it is seen that the split fields satisfy the equation

$$\partial_z \begin{pmatrix} E^+ \\ E^- \end{pmatrix} = \left((\partial_z \mathcal{S})\mathcal{S}^{-1} + \mathcal{S}\mathcal{A}\mathcal{S}^{-1}\right) \begin{pmatrix} E^+ \\ E^- \end{pmatrix}$$

This equation is referred to as the dynamic equation. By introducing four operators Δ_{11}, Δ_{12}, Δ_{21} and Δ_{22} the dynamic equation reads

$$\partial_z \begin{pmatrix} E^+ \\ E^- \end{pmatrix} = \frac{1}{c(z)} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \partial_t \begin{pmatrix} E^+ \\ E^- \end{pmatrix} + \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} \begin{pmatrix} E^+ \\ E^- \end{pmatrix} \tag{2.6}$$

The operators Δ_{ij} contain no time derivatives and the principal part of Eq. (2.6) is then diagonal.

3 Wave propagators

Consider a transient electromagnetic plane wave that has been generated in the region $z < 0$ and impinges at $z = 0$ at $t = 0$. The wave propagators are linear operators that map a field $E^+(z,t)$ at one position $z > 0$ to some other position $z' > 0$. The propagators are defined by

$$E^+(z', t + \tau(z', z)) = \mathcal{P}^+(z', z)E^+(z,t) \tag{3.1}$$

$$E^-(z', t + \tau(z', z)) = \mathcal{P}^-(z', z)E^+(z,t) \tag{3.2}$$

where

$$\tau(z', z) = \int_z^{z'} \frac{1}{c(z'')} dz'' \tag{3.3}$$

is the travel time from z to z'. Notice that there are no restrictions on the relative magnitudes of z and z' in the definition of the propagators. When $z' > z$ the propagators propagate the field E^+ forward in the z-direction and when $z' < z$ the propagation is backward or in the negative z-direction. The notations forward

propagators and backward propagators are adopted for $\mathcal{P}^{\pm}(z',z)$ when $z'>z$ and $z'<z$, respectively. As will be seen later, the propagators are generalized functions that are convolved with the field they operate on. Thus, if Eqs (3.1) and (3.2) are Laplace transformed, the propagators become functions of z and z' and they are multiplied with the Laplace transform of $E^{+}(z,t)$. From Eq. (2.6) it is seen that the propagators satisfy the equations

$$\begin{aligned}\partial_{z'}\begin{pmatrix}\mathcal{P}^{+}(z',z)\\ \mathcal{P}^{-}(z',z)\end{pmatrix}-\frac{2}{c(z')}\partial_t\begin{pmatrix}0\\ \mathcal{P}^{-}(z',z)\end{pmatrix}\\ =\begin{pmatrix}\Delta_{11}(z') & \Delta_{12}(z')\\ \Delta_{21}(z') & \Delta_{22}(z')\end{pmatrix}\begin{pmatrix}\mathcal{P}^{+}(z',z)\\ \mathcal{P}^{-}(z',z)\end{pmatrix}\end{aligned} \tag{3.4}$$

The definition of the propagators implies the following relations when z, z' and z'' are positive

$$\mathcal{P}^{+}(z,z)=\mathcal{I}=\text{identity operator} \tag{3.5}$$

$$\mathcal{P}^{+}(z'',z)=\mathcal{P}^{+}(z'',z')\mathcal{P}^{+}(z',z) \tag{3.6}$$

$$\mathcal{P}^{-}(z',z)=\mathcal{P}^{-}(z',z')\mathcal{P}^{+}(z',z) \tag{3.7}$$

A formal operator equation can be obtained for the propagator $\mathcal{P}^{+}$ using these properties. Thus

$$\mathcal{P}^{+}(z'+\Delta z,z)-\mathcal{P}^{+}(z',z)=\left(\mathcal{P}^{+}(z'+\Delta z,z')-\mathcal{P}^{+}(z',z')\right)\mathcal{P}^{+}(z',z)$$

When $\Delta z\to 0$ it follows that

$$\partial_{z'}\mathcal{P}^{+}(z',z)=\mathcal{M}(z')\mathcal{P}^{+}(z',z) \tag{3.8}$$

where, according to Eq. (3.4),

$$\mathcal{M}(z')=\partial_{z'}\mathcal{P}^{+}(z',z)|_{z=z'}=\Delta_{11}(z')+\Delta_{12}(z')\mathcal{P}^{-}(z',z')$$

The equation also follows from the system (3.4) and the relation (3.7) since

$$\begin{aligned}\partial_{z'}\mathcal{P}^{+}(z',z)&=\Delta_{11}(z')\mathcal{P}^{+}(z',z)+\Delta_{12}(z')\mathcal{P}^{-}(z',z)\\ &=\Delta_{11}(z')\mathcal{P}^{+}(z',z)+\Delta_{12}(z')\mathcal{P}^{-}(z',z')\mathcal{P}^{+}(z',z)\end{aligned}$$

The propagator $\mathcal{P}^{-}(z',z')$ is the reflection operator for the half space $z>z'$. The formal solution to Eq. (3.8) is

$$\mathcal{P}^{+}(z',z)=\exp\left(\int_{z}^{z'}\mathcal{M}(z'')dz''\right) \tag{3.9}$$

The corresponding expression for the propagator $\mathcal{P}^{-}(z',z)$ is

$$\mathcal{P}^{-}(z',z)=\mathcal{P}^{-}(z',z')\exp\left(\int_{z}^{z'}\mathcal{M}(z'')dz''\right)$$

The exponential function is defined by the expansion

$$\exp\left(\int_z^{z'} \mathcal{M}(z'')dz''\right) = \sum_{n=0}^{\infty} \frac{\left(\int_z^{z'} \mathcal{M}(z'')dz''\right)^n}{n!}$$

From Eqs (3.5) and (3.6) it is seen that

$$\mathcal{P}^+(z',z)\mathcal{P}^+(z,z') = \mathcal{I} \tag{3.10}$$

and thus the inverse of the propagator $\mathcal{P}^+(z',z)$ is

$$\left(\mathcal{P}^+(z',z)\right)^{-1} = \mathcal{P}^+(z,z') \tag{3.11}$$

The existence of this inverse is verified in the next section. In general, there exists no inverse of the propagator $\mathcal{P}^-(z',z)$.

4 Explicit representations of the propagators

The explicit representations of the propagators follow from invariance under time-translation and causality and read

$$\mathcal{P}^+(z',z)E^+(z,t) = a(z',z)E^+(z,t) + [P^+(z',z,\cdot) * E^+(z,\cdot)](t) \tag{4.1}$$

$$\mathcal{P}^-(z',z)E^+(z,t) = [P^-(z',z,\cdot) * E^+(z,\cdot)](t) \tag{4.2}$$

where the star denotes temporal convolution

$$[P^+(z',z,\cdot) * E^+(z,\cdot)](t) = \int_0^t P^+(z',z,t-t')E^+(z,t')dt'$$

The wave front factor $a(z',z)$ is given by

$$a(z',z) = \sqrt{\frac{c(z')}{c(z)}} e^{\left(-\frac{1}{2}\int_z^{z'} c(z'')f(z'')dz''\right)} \tag{4.3}$$

This expression is derived in the appendix. The form of the representations in Eqs (4.1) and (4.2) is independent of the splitting operator $\mathcal{K}$. Also the wave front factor $a(z',z)$ is independent of the splitting whereas the propagator kernels $P^\pm$ depend on $\mathcal{K}$.

From Eqs (3.10) and (4.1) it is seen that the kernel $P^+(z,z',t)$ for the inverse propagator in (3.11) is related to the propagator kernel $P^+(z',z,t)$ via the Volterra equation of the second kind

$$a(z',z)P^+(z,z',t) + a(z,z')P^+(z',z,t) + [P^+(z',z,\cdot) * P^+(z,z',\cdot)] = 0$$

Volterra equations of the second kind have unique solutions and thus the kernel $P(z,z',t)$ and the inverse propagator $(\mathcal{P}^+(z',z))^{-1}$ exist. The inverse propagator can be used to determine the field $E^+(z',t)$ from the field $E^+(z,t)$ where $z' < z$.

5 Equations for the propagator kernels

In this subsection two sets of equations for the propagator kernels are presented. The derivations of these equations are found in the appendix. The splitting operator used for these sets is the simple $\mathcal{K} = c(z)\partial_t^{-1}$, see Eq. (2.5). The dynamic equation corresponding to that splitting is given by Eq. (2.6) where

$$\Delta_{11} = \frac{\partial_z c(z)}{2c(z)} - \frac{c(z)}{2}\partial_t^{-1}F[\cdot] \tag{5.1}$$

$$\Delta_{12} = -\frac{\partial_z c(z)}{2c(z)} - \frac{c(z)}{2}\partial_t^{-1}F[\cdot] \tag{5.2}$$

$$\Delta_{21} = -\frac{\partial_z c(z)}{2c(z)} + \frac{c(z)}{2}\partial_t^{-1}F[\cdot] \tag{5.3}$$

$$\Delta_{22} = \frac{\partial_z c(z)}{2c(z)} + \frac{c(z)}{2}\partial_t^{-1}F[\cdot] \tag{5.4}$$

Both of the two sets presented in this section are appropriate for numerical calculations. The first set is directly related to Eq. (3.4). The two equations for the propagator kernels read

$$\begin{aligned}\partial_{z'}P^+(z',z,t) = &\frac{\partial_{z'}c(z')}{2c(z')}(P^+(z',z,t) - P^-(z',z,t)) \\ &- \frac{c(z')}{2}(g(z') + \partial_t^{-1}h(z',t))a(z',z) \\ &- \frac{c(z')}{2}\left(\partial_t^{-1}F[P^+(z',z,t) + P^-(z',z,t)]\right)\end{aligned} \tag{5.5}$$

$$\begin{aligned}\partial_{z'}P^-(z',z,t) - \frac{2}{c(z')}\partial_t P^-(z',z,t) = &\frac{\partial_{z'}c(z')}{2c(z')}(P^-(z',z,t) - P^+(z',z,t)) \\ &+ \frac{c(z')}{2}(g(z') + \partial_t^{-1}h(z',t))a(z',z) \\ &+ \frac{c(z')}{2}\left(\partial_t^{-1}F[P^+(z',z,t) + P^-(z',z,t)]\right)\end{aligned} \tag{5.6}$$

Notice again that these equations are valid for all positive values of z and z'. If the equations are used for $z' > z$ the forward propagator kernels are obtained and if they are used for $z' < z$ the backward propagator kernels are obtained. When $z = 0$ the above set of equations is the same as the one obtained for the kernels in the Green function approach, cf. Ref. 33 and the next section. The initial condition for the kernel P^- reads

$$P^-(z',z,0) = \frac{1}{4}\left(\partial_{z'}c(z') - c^2(z')f(z')\right)a(z',z) \tag{5.7}$$

The boundary condition $P^+(z,z,t) = 0$ is obvious and is needed in order to solve the equations uniquely. Only if the medium is of finite length, a boundary condition for the kernel $P^-(z',z,t)$ is also needed.

The other set of equations is obtained by varying the coordinate z. The equations are closely related to the equations obtained in the invariant imbedding method, cf. Ref. 29 and next section. The equation for the propagator kernel P^+ reads

$$\begin{aligned}\partial_z P^+(z',z,t) =& \frac{c_z(z)}{2c(z)}\big(a(z',z)R(z,t) - P^+(z',z,t) \\ &+ [P^+(z',z,\cdot) * R(z,\cdot)](t)\big) + \frac{c(z)}{2}\big(a(z',z)(g(z) + \partial_t^{-1}h(z,t)) \\ &+ \partial_t^{-1}F[a(z',z)R(z,t) + P^+(z',z,t) + [P^+(z',z,\cdot) * R(z,\cdot)](t)]\big)\end{aligned} \tag{5.8}$$

where $R(z,t) = P^-(z,z,t)$ is the reflection kernel for the half-space $[z,\infty)$. The equation for $R(z,t)$ is obtained from the relation (4.2)

$$\begin{aligned}2\partial_z R(z,t) - \frac{4}{c(z)}\partial_t R(z,t) = \frac{c_z(z)}{c(z)}[R(z,\cdot) * R(z,\cdot)](t) \\ + c(z)\big(g(z) + \partial_t^{-1}h(z,t)\big) \\ + c(z)\partial_t^{-1}F[2R(z,t) + [R(z,\cdot) * R(z,\cdot)](t)]\end{aligned} \tag{5.9}$$

The initial condition for $R(z,t)$ is

$$R(z,0) = \frac{1}{4}\big(\partial_z c(z) - c^2(z)f(z)\big) \tag{5.10}$$

The kernel $P^-(z',z,t)$ is obtained from the relation (3.7) and thus

$$P^-(z',z,t) = R(z',t)a(z',z) + [R(z',\cdot) * P^+(z',z,\cdot)](t)$$

There is also an equation for $P^-(z',z,t)$ with differentiation wrt z, but since P^- is obtained from (5.8) and (5.9) the equation is superfluous and is not presented here. In this paper, the numerical solution of the equations for the kernels $P^\pm$ and R are not discussed, it is merely emphasized that these equations are straightforward to solve numerically. Numerical algorithms for similar types of equations are found in, e.g., Refs 4, 33 and 35.

6 Relations to other techniques

The invariant imbedding technique, see Refs 4 and 29, and the Green function approach, see Ref. 33, are two methods that have been frequently applied to transient one-dimensional wave propagation problems. In this section it is shown how these methods are related to the method of propagators. It is also seen how the compact Green functions, see Refs 14 and 35, are related to the propagator kernels.

6.1 The invariant imbedding method

In Ref. 4 scattering from a finite slab $0 < z < L$, where the half-space $z > L$ is homogeneous and non-dispersive, is considered. The invariant imbedding technique

presented in that paper is based upon the reflection operator $\mathcal{R}(z)$ and transmission operator $\mathcal{T}(z)$ for the subslab $[z, L]$. The relation between the operators $\mathcal{R}$, $\mathcal{T}$ and the propagators are

$$\mathcal{R}(z) = \mathcal{P}^-(z, z)$$
$$\mathcal{T}(z) = \mathcal{P}^+(L, z)$$

The representations of $\mathcal{R}(z)$ and $\mathcal{T}(z)$ are identical with the representations of $\mathcal{P}^-(z, z)$ and $\mathcal{P}^+(L, z)$, see Eqs (4.1) and (4.2). The reflection and transmission kernels that appear in the representations are

$$R(z, t) = P^-(z, z, t)$$
$$T(z, t) = P^+(L, z, t)$$

The equations for the kernels $R(z, t)$ and $T(z, t)$ are the same as Eqs (5.9) and (5.8) (with $z' = L$), respectively.

In Ref. 29 a more general imbedding technique is considered where reflection and transmission operators for an imbedded subslab $[x, y]$, $0 < x < y < L$, are introduced. The slab is then imbedded between the two homogeneous half-spaces $z < x$ and $z > y$ where $F[E] = 0$ in both half-spaces and $c(z) = c(x)$ in the half-space $z < x$ and $c(z) = c(y)$ in the other half-space. The operators are functions of both endpoints of the subslab. In this case, relations between the reflection and transmission operators of the imbedding technique and the propagators can be found using a technique similar to the one described in Ref. 42. Since the relations are quite complicated and of limited interest they are not presented in this paper.

6.2 The Green function approach

In the Green function approach, cf. Ref. 33, operators $\mathcal{G}^\pm(z)$ are introduced that map an incident wave at $z = 0$ to the internal split fields as

$$E^+(z, t + \tau(0, z)) = \mathcal{G}^+(z)E^+(0, t)$$
$$E^-(z, t + \tau(0, z)) = \mathcal{G}^-(z)E^+(0, t)$$

The relations between these Green operators and the corresponding kernels and the propagators and propagator kernels are obviously

$$\mathcal{G}^+(z) = \mathcal{P}^+(z, 0)$$
$$\mathcal{G}^-(z) = \mathcal{P}^-(z, 0)$$

$$G^+(z, t) = P^+(z, 0, t)$$
$$G^-(z, t) = P^-(z, 0, t)$$

The equations for the kernels $G^\pm(z, t)$ are the same as Eqs (5.5) and (5.6) with $z = 0$. Thus the Green functions approach is a special case of the propagator method and some of the results presented in this paper for the propagators also hold for the Green functions.

6.3 Compact Green functions

In Refs 14 and 35 kernels referred to as the compact Green functions are used in inverse problems for a non-dispersive finite slab $0 < z < L$. The compact Green functions are the kernels for the operators that map the transmitted field to the internal split fields. It is seen that these operators are the backward propagators $\mathcal{P}^{\pm}(z, L)$ and hence the compact Green functions are the kernels $P^{\pm}(z, L, t)$. The attribute compact comes from the fact that for a non-dispersive slab, these kernels have compact support in the time interval $0 < t < 2\tau(z, L)$. To show this it is first noticed that for a non-dispersive non-dissipative medium the source terms in Eqs (5.5) and (5.6) vanish since $g(z) = 0$ and $h(z, t) = 0$. For a slab $0 < z < L$ with homogeneous half-space $z > L$ both $P^{+}(z, L, t)$ and $P^{-}(z, L, t)$ have homogeneous boundary conditions at $z = L$, i.e., $P^{+}(L, L, t) = P^{-}(L, L, t) = 0$. The directional derivatives of Eqs (5.5) and (5.6) and the boundary conditions at $z = L$ imply that $P^{\pm}(z, L, t)$ both are zero when $t > 2\tau(L, z)$ and hence compactly supported.

7 Homogeneous medium

In a homogeneous medium the wave front speed is constant, $c(z) = c$, and the functional F have no explicit z-dependence, i.e.,

$$F[E] = f\partial_t E(z, t) + gE(z, t) + \int_{-\infty}^{t} h(t - \tau)E(z, \tau)d\tau$$

where f and g are constants and $h(t)$ is a function of time. Due to the translational invariance in the spatial variable of the medium, the propagators are only dependent on the distance between z and z' and hence

$$\begin{aligned} E^{+}(z', t + \tau(z' - z)) &= \mathcal{P}^{+}(z' - z)E^{+}(z, t) \\ &= a(z' - z)E^{+}(z, t) + [P^{+}(z' - z, \cdot) * E^{+}(z, \cdot)](t) \end{aligned}$$

$$E^{-}(z', t + \tau(z' - z)) = \mathcal{P}^{-}(z' - z)E^{+}(z, t) = [P^{-}(z' - z, \cdot) * E^{+}(z, \cdot)](t)$$

where $\tau(z' - z)$, $P^{\pm}(z' - z, t)$ and $a(z' - z)$ correspond to $\tau(z', z)$, $P^{\pm}(z', z, t)$ and $a(z', z)$ in Eqs (4.1) and (4.2). From Eqs (3.5), (3.6) and (3.7) it follows that the propagator $\mathcal{P}^{+}$ in a homogeneous medium obeys the rules

$$\mathcal{P}^{+}(x)\mathcal{P}^{+}(y) = \mathcal{P}^{+}(x + y), \qquad \mathcal{P}^{+}(0) = \mathcal{I}, \qquad x, y \geq 0$$

These rules are sufficient for the collection of propagators $\mathcal{P}^{+}(z)$ to form a semi-group of operators. Since the inverse operator $\mathcal{P}^{-1}(z) = \mathcal{P}(-z)$ exists, cf. Eq. (3.11), the collection of propagators $\mathcal{P}^{+}(z)$ is even a one-parameter group of linear operators, see Ref. 43. The theory for semi-groups is found in, e.g., Refs 43 and 9. In this paper the theory of semi-groups is not further utilized.

Since the reflection operator of a homogeneous medium $\mathcal{P}^{-}(0)$ is independent of z, the formal solution to Eq. (3.8) takes the simple form

$$\mathcal{P}^{+}(z) = \exp(\mathcal{M}z) = \exp((\Delta_{11} + \Delta_{12}\mathcal{P}^{-}(0))z) \tag{7.1}$$

The equations for the propagator kernels simplify in the case of a homogeneous medium. The major simplification is that the reflection kernel becomes independent of z and hence from Eq. (5.9) the following Volterra equation is obtained

$$R(t) + \frac{c^2}{4}\left(f + gt + \partial_t^{-2}h(t) + \partial_t^{-2}F[2R(t) + [R(\cdot) * R(\cdot)](t)]\right) = 0 \tag{7.2}$$

It is also possible to obtain a Volterra equation for the propagator kernel $P^+(z,t)$. One way to do this is to take the Laplace transform of the formal solution, Eq. (7.1), giving

$$\tilde{P}^+(z,s) = \exp((\tilde{\Delta}_{11}(s) + \tilde{\Delta}_{12}(s)\tilde{R}(s))z) - a(z)$$

where s is the transformed time variable and $\tilde{f}(s)$ denotes the Laplace transform of $f(t)$. By differentiating this equation wrt s it is seen that

$$\partial_s\tilde{P}^+(z,s) = z(\partial_s((\tilde{\Delta}_{11}(s) + \tilde{\Delta}_{12}(s)\tilde{R}(s)))\tilde{(}P^+(z,s) + 1)$$

Since the inverse Laplace transform of $\partial_s\tilde{f}(s)$ is $-tf(t)$ the following equation is obtained

$$\frac{2ct}{z}P^+(t) + \beta(t) + [\beta(\cdot) * P^+(\cdot)](t) = 0$$

where

$$\beta(t) = gt + t(\partial^{-1}h(t)) + ftR(t) + gt(\partial^{-1}R(t)) + t[(\partial_t^{-1}h(\cdot)) * R(\cdot)](t)$$

The equation for $P^+(t)$ is a Volterra equation of the second kind and is straightforward to solve numerically. It has been used in earlier papers on transmission problems, see Refs 20 and 12

7.1 Exact splitting operator

In the case of a homogeneous medium it is always possible to find a splitting operator $\mathcal{K}$ for which the dynamic equation, Eq. (2.6), is diagonal. Hence, also the system in Eqs (3.4) is diagonal with that splitting operator. The derivation of the expressions for this exact splitting operator is done in three steps. The first step is to prove that the propagator $\mathcal{P}^+(z)$ for a homogeneous medium is the propagator also for the field E^- as well as for the total field $E = E^+ + E^-$. This follows from the relations

$$\begin{aligned} E^+(z', t + \tau(z'-z)) &= \mathcal{P}^+(z'-z)E^+(z,t) \\ E^-(z', t + \tau(z'-z)) &= \mathcal{P}^+(z'-z)\mathcal{P}^-(0)E^+(z,t) \\ E^-(z,t) &= \mathcal{P}^-(0)E^+(z,t) \end{aligned}$$

Using $E(z,t) = E^+(z,t) + E^-(z,t)$ it is seen that $\mathcal{P}^+(z)$ propagates E as well as E^+ and E^- since

$$\begin{aligned} E^-(z', t + \tau(z'-z)) = \mathcal{P}^+(z'-z)\mathcal{P}^-(0)E^+(z,t) &= \mathcal{P}^+(z'-z)E^-(z,t) \\ E(z', t + \tau(z'-z)) &= \mathcal{P}^+(z'-z)E(z,t) \end{aligned}$$

It has been used that $\mathcal{P}^+(z)$ and $\mathcal{P}^-(0)$ commute. Since the propagator $\mathcal{P}^+(z'-z)$ is a propagator for the entire field it must be independent of the splitting. In the second step, it is first noticed that $\Delta_{11}+\Delta_{12}\mathcal{P}^-(0)$ is independent of the splitting. This is a direct consequence of the fact that $\mathcal{P}^+(z'-z)$ is independent of the splitting and of Eq. (7.1). In the third step, the expression for $\Delta_{11}+\Delta_{12}\mathcal{P}^-(0)$ using the simplest splitting operator $\mathcal{K}_0 = c\partial_t^{-1}$ is put equal to the expression for $\Delta_{11}+\Delta_{12}\mathcal{P}^-(0)$ using the exact splitting operator. From the obtained relation the expressions for the exact splitting operator are obtained. For the homogeneous medium

$$\begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix} = \mathcal{S}\mathcal{A}\mathcal{S}^{-1} - \frac{1}{c}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\partial_t$$

Using the expressions for $\mathcal{S}$ and $\mathcal{A}$, see Eqs (2.4) and (2.2), it follows that

$$\Delta_{11} = -\Delta_{22} = \frac{1}{c}\partial_t - \frac{1}{2}(\mathcal{K}^{-1} + \mathcal{K}(c^{-2}\partial_t^2 + F[\cdot])) \tag{7.3}$$

$$\Delta_{12} = -\Delta_{21} = \frac{1}{2}(\mathcal{K}^{-1} - \mathcal{K}(c^{-2}\partial_t^2 + F[\cdot])) \tag{7.4}$$

The simple splitting operator $\mathcal{K}_0 = c\partial_t^{-1}$ has an inverse $\mathcal{K}_0^{-1} = c^{-1}\partial_t$ and by inserting these operators into Eqs (7.3) and (7.4) the corresponding expressions for Δ_{ij} follow

$$\Delta_{11}^0 = \Delta_{12}^0 = -\Delta_{21}^0 = -\Delta_{22}^0 = -\frac{c}{2}\partial_t^{-1}F[\cdot]$$

These expressions were also given in Eqs (5.1)–(5.4). Next consider a splitting operator $\mathcal{K}$ that diagonalizes Eq. (2.6) and also (3.4). In that case $\Delta_{12} = \Delta_{21} = 0$ and according to Eq. (7.4) and the relation $\mathcal{K}^{-2} = c^{-2}\partial_t^2 + F[\cdot]$

$$\Delta_{11} = -\Delta_{22} = \frac{1}{c}\partial_t - \mathcal{K}^{-1} \tag{7.5}$$

Since $\Delta_{11} = \Delta_{11}^0 + \Delta_{12}^0\mathcal{P}_0^-(0)$ it follows from Eq. (7.5) that

$$\begin{aligned} \mathcal{K}^{-1} &= \frac{1}{c}\partial_t - \Delta_{11}^0 - \Delta_{12}^0\mathcal{P}_0^-(0) \\ \mathcal{K} &= (\frac{1}{c}\partial_t - \Delta_{11}^0 - \Delta_{12}^0\mathcal{P}_0^-(0))^{-1} \end{aligned} \tag{7.6}$$

These are the relation needed in order to construct the exact splitting operator.

It is illustrative to see explicitly how the exact splitting operator is constructed in the case of a dispersive medium where the linear functional F is given by

$$\begin{aligned} F[E] =& \frac{1}{c_0^2}\partial_t^2[\chi(\cdot) * E(z,\cdot)](t) = \\ & \frac{1}{c_0^2}\left(\chi(0)\partial_t E(z,t) + \partial_t\chi(t)|_{t=0}E(z,t) + [(\partial_{tt}\chi(\cdot)) * E(z,\cdot)](t)\right) \end{aligned}$$

and where $c(z) = c_0$. The kernel $\chi(t)$ is the susceptibility kernel that relates the displacement field to the electric field by

$$D(z,t) = \varepsilon_0 E(z,t) + \varepsilon_0\int_{-\infty}^{t}\chi(t-\tau)E(z,\tau)d\tau$$

First the reflection kernel $R(t)$ is obtained from the Volterra equation Eq. (7.2) that in this case reads

$$R(t) + \frac{1}{4}\left(\chi(t) + [\chi(\cdot) * (2R(\cdot) + [R * R](\cdot))]\right) = 0 \tag{7.7}$$

The initial condition for $R(t)$, see Eq. (5.10), is

$$R(0) = -\frac{1}{4}\chi(0)$$

The operator $\mathcal{K}^{-1}$ is then given by Eq. (7.6) which now reads

$$\mathcal{K}^{-1}E = \frac{1}{2c_0}\partial_t\left(2E + [\chi * E] + [\chi * R * E]\right)$$

The operator $\mathcal{K}$ satisfies $\mathcal{K}\mathcal{K}^{-1} = \mathcal{K}^{-1}\mathcal{K} = \mathcal{I}$ and thus

$$\mathcal{K}E = c_0\partial_t^{-1}\left(E + [k * E]\right)$$

where $k(t)$ satisfies the Volterra equation of the second kind

$$\chi(t) + [\chi * R](t) + 2k(t) + [k * \chi](t) + [k * \chi * R](t) = 0$$

It is seen that two Volterra equations of the second kind have to be solved in order to obtain the exact splitting operators $\mathcal{K}$ and $\mathcal{K}^{-1}$ numerically.

8 Conclusions

There were three main purposes with this paper. The first one was to present the method of propagators and to illuminate the useful properties that this method provides. The second purpose was to connect the method to the invariant imbedding method and the Green function approach and the third purpose was to give a review of the imbedding method and the Green function approach. It was seen that both the invariant imbedding method and the Green function approach are special cases of the method of propagators. Features of both of these two methods can then be utilized by the method of propagators which make it a powerful tool for analyzing transient wave propagation in one-dimensional media.

An interesting question is how the method of propagators is translated to the three-dimensional case. Most of the theory can probably be generalized to three dimensions using a technique similar to the one presented in Ref. 48. The three-dimensional case is currently under consideration.

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Appendix A Appendix

In this appendix the equations for the propagator kernels, Eqs (5.5), (5.6), (5.8) and (5.9) are derived. The splitting operator used for these equations is $\mathcal{K} = c(z)\partial_t^{-1}$. The derivation follows a standard procedure that has been used in a number of papers, see Refs 3 and 33. A somewhat different technique to derive imbedding equations was presented in Ref. 7. The initial step in the derivation is to differentiate the representations of the propagators

$$E^+(z', t + \tau(z', z)) = a(z', z)E^+(z, t) + [P^+(z', z, \cdot) * E^+(z, \cdot)](t) \quad \text{(A.1)}$$
$$E^-(z', t + \tau(z', z)) = [P^-(z', z, \cdot) * E^+(z, \cdot)](t), \quad \text{(A.2)}$$

cf. Eqs (3.1), (3.2), (4.1) and (4.2), wrt either z or z'. Differentiation wrt z' leads to Eqs (5.5) and (5.6) and differentiation wrt z leads to Eqs (5.8) and (5.9).

Differentiation wrt z' of Eq. (A.1) gives

$$(\partial_{z'} - (\partial_{z'}\tau(z', z))\partial_t)E^+(z', t + \tau(z', z)) = \\ \partial_{z'}a(z', z)E^+(z, t) + [\partial_{z'}P^+(z', z, \cdot) * E^+(z, \cdot)](t)$$

This equation is now expressed in terms of the field $E^+(z, t)$. The z' derivative of $E^+(z', t + \tau(z', z))$ is eliminated using the dynamic equations (2.6). Thus

$$\big(-c(z')^{-1}\partial_t + \Delta_{11} - (\partial_{z'}\tau(z', z))\partial_t\big)\, E^+(z', t + \tau(z', z)) \\ + \Delta_{12}E^-(z', t + \tau(z', z)) = \\ \partial_{z'}a(z', z)E^+(z, t) + [\partial_{z'}P^+(z', z, \cdot) * E^+(z, \cdot)](t)$$

where $E^\pm(z', t + \tau(z', z))$ in the left hand side are expressed in terms of $E^+(z, t)$ by Eqs (A.1) and (A.2). The explicit expressions of Δ_{ij} are found in Eqs (5.1)–(5.4). By identifying terms proportional to $\partial_t E^+(z, t)$, $E^+(z, t)$ and terms containing convolutions of $E^+(z, t)$ in the left and right hand sides the following three equations are obtained

$$\partial'_z\tau(z', z) = c(z')^{-1}$$

$$\partial_{z'}a(z', z) = \left(\frac{\partial_{z'}c(z')}{2c(z')} - \frac{c(z')}{2}f(z')\right)a(z', z)$$

$$\partial_{z'}P^+(z', z, t) = \frac{\partial_{z'}c(z')}{2c(z')}(P^+(z', z, t) - P^-(z', z, t)) \\ - \frac{c(z')}{2}a(z', z)(g(z') + \partial_t^{-1}h(z', t)) \\ - \frac{c(z')}{2}\partial_t^{-1}F[P^+(z', z, t) + P^-(z', z, t)]$$

The first equation gives the expression for the travel time in Eq. (3.3), the second gives the expression for the wave front factor in Eq. (4.3) and the third one is

Eq. (5.5). The equation (5.6) is derived in the same manner. First, Eq. (A.2) is differentiated wrt z' where the z' derivative of $E^-(z', t+\tau(z', z))$ is eliminated by Eq. (2.6). By expressing $E^\pm(z', t+\tau(z', z))$ in $E^+(z, t)$ and identifying terms in the left and right hand sides, two equations are obtained

$$P^-(z', z, 0) = \frac{1}{4} a(z', z) \left(\partial_{z'} c(z') - c^2(z') f(z')\right)$$

$$\begin{aligned}\partial_{z'} P^-(z', z, t) - \frac{2}{c(z')} \partial_t P^-(z', z, t) = \frac{\partial_{z'} c(z')}{2c(z')} (P^-(z', z, t) - P^+(z', z, t)) \\ + \frac{c(z')}{2} \left(a(z', z)(g(z') + \partial_t^{-1} h(z', t))\right) \\ + \frac{c(z')}{2} \left(\partial_t^{-1} F[P^+(z', z, t) + P^-(z', z, t)]\right)\end{aligned}$$

The first equation is the initial condition for $P^-(z', z, t)$, see Eq. (5.7), and the second one is Eq. (5.6).

The derivation of Eq. (5.8) starts with a differentiation of Eq. (A.1) wrt z giving

$$\begin{aligned}\partial_z \tau(z', z) \partial_t E^+(z', t+\tau(z', z)) = (\partial_z a(z', z)) E^+(z, t) + a(z', z) \partial_z E^+(z, t) \\ + [(\partial_z P^+(z', z, \cdot)) * E^+(z, \cdot)](t) + [P^+(z', z, \cdot) * \partial_z E^+(z, \cdot)]\end{aligned}$$

The z-derivatives of $E^+(z, t)$ are expressed in $E^\pm(z, t)$ using the dynamic equations (2.6). In the obtained equation $E^-(z, t)$ is expressed in terms of $E^+(z, t)$ using

$$E^-(z, t) = [P^-(z, z, \cdot) * E^+(z, \cdot)](t) = [R(z, \cdot) * E^+(z, \cdot)](t)$$

When terms proportional to $\partial_t E^+(z, t)$, $E(z, t)$ and terms containing convolutions of $E^+(z, t)$ are identified in the right and left hand sides the following three equations are obtained

$$\partial_z \tau(z', z) = -c(z)^{-1}$$

$$\partial_z a(z', z) = -\left(\frac{\partial_z c(z)}{2c(z)} - \frac{c(z)}{2} f(z)\right) a(z', z)$$

$$\begin{aligned}\partial_z P^+(z', z, t) = \frac{c_z(z)}{2c(z)} \left(a(z', z) R(z, t) - P^+(z', z, t)\right. \\ \left. + [P^+(z', z, \cdot) * R(z, \cdot)](t)\right) + \frac{c(z)}{2} \left(a(z', z)(g(z) + \partial_t^{-1} h(z, t))\right. \\ \left. + \partial_t^{-1} F[a(z', z) R(z, t) + P^+(z', z, t) + [P^+(z', z, \cdot) * R(z, \cdot)](t)]\right)\end{aligned}$$

The first two equations give the travel time and the wave front factor and the third equation is Eq. (5.8). Finally Eq. (5.9) is derived by first differentiating the equation

$$E^-(z, t) = [R(z, \cdot) * E^+(z, \cdot)](t)$$

wrt z and then proceed in the same manner as in the derivation of Eq. (5.8). Two equations follow from the identification of terms in the last step of the derivation, namely

$$R(z,0) = \frac{1}{4}\left(\partial_z c(z) - c^2(z) f(z)\right)$$

$$\begin{aligned} 2\partial_z R(z,t) - \frac{4}{c(z)}\partial_t R(z,t) = & \frac{c_z(z)}{c(z)}[R(z,\cdot) * R(z,\cdot)](t) \\ & + c(z)\left(g(z) + \partial_t^{-1} h(z,t)\right) \\ & + c(z)\partial_t^{-1} F[2R(z,t) + [R(z,\cdot) * R(z,\cdot)](t)] \end{aligned}$$

The first equation is the initial condition for the reflection kernel, cf. Eq. (5.10), and the second one is Eq. (5.9).

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