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Some Results on Convolutional Codes over Rings

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Abstract — **Convolutional codes over rings were motivated from phase-modulated signals. Some structural properties of generator matrices of convolutional codes over rings have been studied. Here, a condition for a convolutional code over a ring to be systematic is given and shown to be equivalent to the condition given by Massey and Mittelholzer. Furthermore, the conditions of generator matrices over $\mathbb{Z}_{p^e}$ being catastrophic, basic, and minimal are considered, and the predictable degree property of polynomial generator matrices is considered.**

I. Summary

Massey and Mittelholzer [1] introduced convolutional codes over rings together with their motivation from phase-modulated signals. They also showed that convolutional codes over rings behave much differently than convolutional codes over fields. Structural properties of convolutional codes over rings were discussed in [2] [3]. We have here studied some structural properties of convolutional codes over rings in more detail.

Let R be a commutative ring with identity, $R[D]$ be the polynomial ring over R, and $R(D)$ be the *ring of rational functions* over R in the indeterminate D, such that the trailing coefficient of the denominator polynomials are units in the ring R. Let $R_r(D)$ be the subring of $R(D)$ consisting of those elements (equivalence classes) which contain a representative $\frac{f(D)}{q(D)}$ with $q(0)$ being a unit in R. We call this the *ring of realizable functions* and the elements in $R_r(D)$ *realizable functions*. Obviously, we have the relation $R_r(D) \subset R(D)$ between these two rings.

Definition 1 *A realizable transfer function matrix $G(D)$ with entries in $R(D)$ is called a* generator matrix *if its rows are free over $R(D)$.*

For convolutional codes over fields, all codes have both systematic and nonsystematic generator matrices. Thus, in the field case, being systematic is an encoder property. However, in the ring case, being systematic is a code property [2]. A convolutional code $\mathcal{C}$ over a ring R is defined to be *systematic* if it has a systematic generator matrix.

Theorem 1 *A convolutional code $\mathcal{C}$ over a ring R is systematic if and only if it has a generator matrix $G(D)$ that has a $b \times b$ subdeterminant which is a unit in $R_r(D)$.*

It is worth noting, that it is not required that *every* generator matrix of a systematic code $\mathcal{C}$ has a $b \times b$ subdeterminant that is a unit in $R_r(D)$. But a generator matrix $G(D)$ that does not have $b \times b$ subdeterminant which is a unit in $R(D)$, cannot generate a systematic code!

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Let $\mathcal{C}_0$ be the *start module* of a rate-b/c convolutional code $\mathcal{C}$ over a ring R, i.e., $\mathcal{C}_0$ consists of all c-tuples $\mathbf{v}(0)$ for which $\mathbf{v}(D)$ is a causal codeword in $\mathcal{C}$. In [2], a convolutional code $\mathcal{C}$ over a ring R is defined to be *proper* if one can select b components so that the c-tuples in $\mathcal{C}_0$, when restricted to these components, form the free module R^b. Then they proved

Proposition 1 *[2] A convolutional code is systematic if and only if it is proper.*

We proved

Theorem 2 *Proposition 1 is equivalent to Theorem 1.*

Consider a generator matrix $G(D)$ over $\mathbb{Z}_{p^e}$. We can write $G(D) = G_0(D)+G_1(D)p+\dots G_{e-1}(D)p^{e-1}$, where the entries of all $G_i(D)$ are in $\mathbb{Z}_p(D)$. Then
$G(D) \bmod p = G_0(D)$. Furthermore,

(i) $G(D)$ is catastrophic if $G(D) \bmod p$ is catastrophic, and for polynomial generator matrices only if $G(D) \bmod p$ is catastrophic,

(ii) $G(D)$ is basic if and only if $G(D) \bmod p$ is basic.

From (i) it follows that for a polynomial generator matrix $G(D)$ over $\mathbb{Z}_{p^e}$, $G(D)$ is catastrophic if and only if $G(D) \bmod p$ is catastrophic, which was announced by Massey and Mittelholzer in [1]. However, for nonpolynomial generator matrices over $\mathbb{Z}_{p^e}$, the catastrophicity of $G(D)$ does not imply catastrophicity of $G(D) \bmod p$. For minimality of generator matrices, such an easy relation between the generator matrices $G(D)$ and $G(D) \bmod p$ does not exist. For example, the generator matrix $G(D) = (1 + pD)$ over $\mathbb{Z}_{p^2}$ is not minimal, but $G(D) \bmod p = (1)$ is minimal over $\mathbb{Z}_p$.

Forney defined the predictable degree property of polynomial generator matrices [4]. For convolutional codes over rings we have,

Theorem 3 *A polynomial generator matrix $G(D)$ whose row-wise highest degree coefficients are units in R has the predictable degree property if and only if the rows of $[G(D)]_h$ are free over R, where $[G(D)]_h$ consists of the row-wise highest degree coefficients of $G(D)$.*

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