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2010

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CODEN:LUTEDX/(TEAT-7202)/1-25/(2010)

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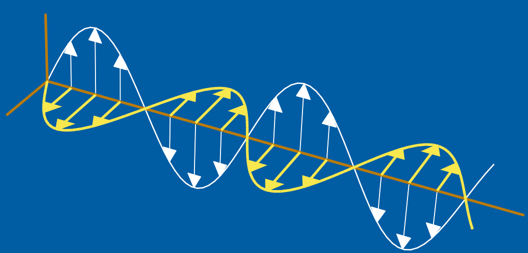

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Editor: Gerhard Kristensson

Abstract

The time dependent Maxwell system is supplemented with the constitutive relations of linear bianisotropc media and is treated as a neutral integro-differential equation in a Hilbert space. By using the theory of abstract Volterra equations and strongly continuous semigroups we obtain general well-posedness results for the corresponding mathematical problem.

1 Introduction and motivation

Our aim is to apply the general functional–analytic methods of evolution equations to prove well–posedness for Maxwell equations. The constitutive relations are considered to be the more general ones describing linear materials in electromagnetics, namely those of linear bianisotropic media exhibiting memory. The results in the present work constitute a generalization of those found in [10, 17, 30] and are strongly motivated by [20] (a shorter version of which is [19]) and [28]. See also [23].

An optically active medium is an example of a linear bianisotropic medium. Such a medium displays *both* electric and magnetic polarization phenomena by *either* electric *or* magnetic excitation. In the electromagnetic community there is a vast literature in the last twenty five years on composite media; see the books [22, 25, 29] and their references lists. In the last fifteen years there has been active mathematical research on different aspects of the theory of electromagnetics of composite media. In particular, the rigorous mathematical analysis of propagation and scattering problems for time–harmonic electromagnetic fields in *chiral* media is fairly well developed. On the other hand, the study of related problems in the time domain is more recent and not equally extensive; in this direction we refer to [3, 4, 6, 10, 17, 19, 30].

As it is well known [18], every electromagnetic phenomenon is specified by four vector quantities: the *electric field* $\boldsymbol{E}$, the *magnetic field* $\boldsymbol{H}$, the *electric flux density* $\boldsymbol{D}$ and the *magnetic flux density* $\boldsymbol{B}$. These quantities are considered as time–dependent vector fields on a domain $\Omega \subset \mathbb{R}^3$, so they are functions of the spatial variable $\boldsymbol{r} \in \Omega$ and the time variable $t \in \mathbb{R}$. The inter–dependence among these quantities is given by the celebrated *Maxwell system*

$$\begin{aligned} \frac{\partial \boldsymbol{D}}{\partial t} &= \operatorname{curl} \boldsymbol{H} - \boldsymbol{J} \\ \frac{\partial \boldsymbol{B}}{\partial t} &= -\operatorname{curl} \boldsymbol{E} \end{aligned} \tag{1.1}$$

where $\boldsymbol{J}$ is the density of the electric current. The first equation in (1.1) is the Ampère Law and the second one is the Faraday Law. Subject to (1.1) are the two Gauss laws

$$\begin{aligned} \operatorname{div} \boldsymbol{D} &= \rho \\ \operatorname{div} \boldsymbol{B} &= 0 \end{aligned} \tag{1.2}$$

ρ being the density of the electric charge. If we differentiate (1.2) with respect to t then, by using (1.1), we derive the equation of continuity

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \boldsymbol{J} = 0 \tag{1.3}$$

Conversely, let us suppose that (1.3) holds. Then, by differentiating (1.1) with respect to t, we find

$$\frac{\partial}{\partial t}(\operatorname{div}\boldsymbol{D}-\rho)=0 \quad , \quad \frac{\partial}{\partial t}\operatorname{div}\boldsymbol{B}=0 \tag{1.4}$$

which, in turn, implies

$$\operatorname{div}\boldsymbol{D}-\rho=f(\boldsymbol{r}) \quad , \quad \operatorname{div}\boldsymbol{B}=g(\boldsymbol{r}) \tag{1.5}$$

Now, without loss of generality, we can assume that f, g are zero fields. This is done by using a standard argument: add appropriate time independent fields to $\boldsymbol{D}$ and $\boldsymbol{B}$ (having divergence equal to f and g, respectively). The new fields will obey the Gauss equations and their time derivatives will coincide with those of the original fields. Note that the operators *curl*, *div* are applied to vectors as functions of the spatial variable $\boldsymbol{r}\in\Omega$.

The equations (1.1) and (1.2) form a differential system from which we wish to calculate the quadruplet $(\boldsymbol{B},\boldsymbol{D},\boldsymbol{E},\boldsymbol{H})$ assuming that the vector $\boldsymbol{J}$ and the scalar ρ are known. The equation (1.3) can be considered then as a consistency condition between the known quantities. Thus, one has to calculate twelve scalar functions from a system of eight scalar equations. This means that the system is under–determined. To overcome this deficiency, constitutive relations are introduced

$$\begin{aligned}\boldsymbol{D}&=\boldsymbol{D}(\boldsymbol{E},\boldsymbol{H})\\ \boldsymbol{B}&=\boldsymbol{B}(\boldsymbol{E},\boldsymbol{H})\end{aligned} \tag{1.6}$$

As a matter of fact, we assume both $\boldsymbol{D}$ and $\boldsymbol{B}$ to be functions (of a known form) of $\boldsymbol{E}$ and $\boldsymbol{H}$. This is a rather rational hypothesis which is experimentally established. Roughly speaking, equations (1.6) provide a mathematical model of the material, that occupies the domain Ω where the phenomenon takes place. Now, the system of equations (1.1), (1.2) and (1.6) is over–determined, since there are more equations than the unknowns. But, as we will see later, the Gauss laws are in some sense redundant if we accept the equations of continuity as axioms during the modeling (this seems plausible since they refer to the known quantities). This fact has lead to the usual practice of considering as "the Maxwell system" the equations (1.1) plus the constitutive relations of the form (1.6), that describe various materials.

For conducting media, the currents are not entirely freely chosen, but they obey a generalized Ohm law instead

$$\boldsymbol{J}=\boldsymbol{J}(\boldsymbol{E},\boldsymbol{H})$$

More precisely it is assumed that the currents are expressed as the sum of one "constitutive" part and one "forced" part [9, p. 15], [27]

$$\boldsymbol{J}=\boldsymbol{F}(\boldsymbol{E},\boldsymbol{H})+\boldsymbol{J}_{\mathrm{f}} \tag{1.7}$$

Such a consideration does not change the essence in the treatment of the problem but inserts some extra complications in notation and in the study of energy. To avoid these complications, we will not assume a "constitutive" part in the current, that is it will be completely "forced". Moreover, $\boldsymbol{J}$ will serve as the "inhomogeneous data" subject to the mathematical problem.

Additionally, one has to impose initial and boundary conditions. The former will be of the form

$$\begin{array}{l} \boldsymbol{E}(0,\boldsymbol{r}) = \boldsymbol{E}_0(\boldsymbol{r}) \\ \boldsymbol{H}(0,\boldsymbol{r}) = \boldsymbol{H}_0(\boldsymbol{r}) \end{array} \quad , \qquad \text{for } \ \boldsymbol{r} \in \Omega \tag{1.8}$$

and the latter can have a variety of types; for simplicity we consider in this work the "perfect conductor" boundary condition

$$\boldsymbol{E}(t,\boldsymbol{r}) \times \boldsymbol{n}(\boldsymbol{r}) = 0 \quad , \qquad \text{for } \ \boldsymbol{r} \in \partial\Omega \ , \ t \in I \tag{1.9}$$

where $\boldsymbol{n}(\boldsymbol{r})$ denotes the outward normal applied to $\boldsymbol{r} \in \partial\Omega$, which – throughout this paper – will be assumed to be Lipschitzian. Under this assumption, the outward normal is defined almost everywhere in $\partial\Omega$ (with respect to some surface measure).

Until now we have not referred to the time interval $I \subset \mathbb{R}$, *i.e.*, the connected set where the variable t may take values. Actually this is a part of the problem; to find the maximum interval of definition for all the above mentioned fields. Of course, the ideal case would be for the fields to be defined in the whole line, that is $I = \mathbb{R}$. Then we would be aware of the whole history of the phenomenon. But this is not the case, in general, here: the materials which are involved in the problem via the constitutive relations are usually only *causal* ones. This means that the beginning of the observation of the fields $\boldsymbol{E}$, $\boldsymbol{H}$ at the time instant $t = 0$ (the "present") cannot reveal the past of the fields $\boldsymbol{D}$, $\boldsymbol{B}$. More precisely, the functional equation (1.6) determines the values of $\boldsymbol{D}$, $\boldsymbol{B}$ at the time instant $t > 0$, only when the values of $\boldsymbol{E}$, $\boldsymbol{H}$ are known in the interval $[0, t]$. Furthermore, it does not make sense to consider the values for $t < 0$ and we assume that the fields are identical to zero there. Thus, in the rest of this work we will assume that $I = \mathbb{R}_+ := [0, \infty)$.

To summarize, for a rigorous mathematical treatment of a problem in Electromagnetics, one has to:

1. introduce a compact notation,
2. find (by experiment), prove (by reasoning), or just guess the constitutive relations,
3. formulate the problem as an abstract problem, and
4. investigate the solvability.

This is the route that we will follow.

2 Formulation of the problem

In this section we will formulate the Maxwell system as an abstract evolution problem. This is necessary in order to apply the abstract methods of functional analysis.

2.1 The six–vector notation

The vector nature of the Maxwell system forces us to work on product spaces and follow a matrix–oriented treatment. This is exactly what one does by using the so called six–vector notation [24], see for example [11]. As we have mentioned in the previous section, we mainly deal with vector fields in the three dimensional space which are functions of the time variable $t \in \mathbb{R}_+$ and the spatial variable $\boldsymbol{r} = (x, y, z) \in \Omega$. The image of this field is then realized as a time–dependent vector applied at the point $\boldsymbol{r}$. We shall denote such a generic field by $\boldsymbol{F} = (F_x, F_y, F_z)^T$, where the superscript "T" denotes transposition; we consider $\boldsymbol{F}$ as a vector-valued function $t \mapsto \boldsymbol{F}(t, \cdot)$. The operator *curl* is defined formally as the (antisymmetric) matrix operator

$$\mathrm{curl} := \begin{bmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{bmatrix}$$

We furthermore define

- the EM flux density $\boldsymbol{d} := (\boldsymbol{D}, \boldsymbol{B})^T$,
- the EM field $\boldsymbol{e} := (\boldsymbol{E}, \boldsymbol{H})^T$,
- the current $\boldsymbol{j} := (-\boldsymbol{J}, \boldsymbol{0})^T$,
- the initial state $\boldsymbol{e}_0 := (\boldsymbol{E}_0, \boldsymbol{H}_0)^T$.

A linear operator acting on $\boldsymbol{e}$ is written as a 2×2 (block) matrix with linear operators as its entries. Such an example is the *Maxwell operator*, which we employ in this work

$$\mathcal{M} := \begin{bmatrix} 0 & \mathrm{curl} \\ -\,\mathrm{curl} & 0 \end{bmatrix}$$

The constitutive relations are now modeled in terms of an operator $\mathbb{V}$ and can be understood as a functional equation

$$\boldsymbol{d} = \mathbb{V}\boldsymbol{e}$$

2.2 The general abstract problem

After that, the Maxwell system is written as the initial–value problem for an abstract evolution equation

$$\begin{cases} (\mathbb{V}\boldsymbol{e})'(t) = \mathcal{M}\boldsymbol{e}(t) + \boldsymbol{j}(t) \quad , \quad \text{for } t \geqslant 0 \\ \qquad \boldsymbol{e}(0) = \boldsymbol{e}_0 \end{cases} \tag{2.1}$$

The prime stands for the time derivative. Actually, by using standard terminology (e.g. see [7] or [13]), in (2.1) we have an inhomogeneous neutral functional differential equation, where $\boldsymbol{e}$ is the unknown. Note that it is the derivative of the functional argument, $\mathbb{V}\boldsymbol{e}$, that appears in (2.1). This argument is very crucial and affects the problem in various senses. For example, a nonlinear operator $\mathbb{V}$ gives rise to a nonlinear problem, while a very general $\mathbb{V}$ may result to an under–determined problem: the initial value $\boldsymbol{e}_0$ may not be sufficient for the uniqueness of the solution and one may need to know additional past values of the field $\boldsymbol{e}$ (it is like to have "more equations than the unknowns").

Thus it becomes clear that a careful study of the operator $\mathbb{V}$ is a very important part of the treatment of the abstract problem (2.1). Especially in our case, the properties of this operator reflect the properties of the medium in question, hence this study presents both physical and mathematical interest. This will be done in detail in the sequel. Another crucial point in our investigation is to decide where does the unknown $\boldsymbol{e}$ "live", that is to specify the state space of the problem (2.1). This choice will be again directed by physical properties.

2.3 Energy considerations

Following [21, Ch. 1.3] we define the Poynting vector as $\boldsymbol{S} := \boldsymbol{E} \times \boldsymbol{H}$. The vector $\boldsymbol{S}$ models the power flux density passing across the boundary $\partial\Omega$. By using the identity

$$\operatorname{div} \boldsymbol{S} = \boldsymbol{H} \cdot \operatorname{curl} \boldsymbol{E} - \boldsymbol{E} \cdot \operatorname{curl} \boldsymbol{H}$$

and the Maxwell system (1.1), we obtain the "Poynting Theorem"

$$\operatorname{div} \boldsymbol{S} + \boldsymbol{e} \cdot \frac{\partial \boldsymbol{d}}{\partial t} = \boldsymbol{e} \cdot \boldsymbol{j} \tag{2.2}$$

(2.2) is essentially an equation of continuity and from this the conservation of EM energy is deduced. Actually, the part $\boldsymbol{e} \cdot \frac{\partial \boldsymbol{d}}{\partial t}$ models the time rate change of the stored EM energy, while the part $\boldsymbol{e} \cdot \boldsymbol{j}$ represents the power supplied by the current $\boldsymbol{j}$. Indeed, the integrated quantity of the former inner product

$$\mathcal{E}(t) := \int_0^t \int_\Omega \boldsymbol{e}(\tau; \boldsymbol{r}) \cdot \frac{\partial \boldsymbol{d}}{\partial t}(\tau; \boldsymbol{r}) \, d\boldsymbol{r} \, d\tau \tag{2.3}$$

is what usually measures the variation in EM energy for $t \geqslant 0$.

Definition 2.1. *A material is called lossless if $\mathcal{E}(t) = 0$ for all fields $\boldsymbol{e}$.*

By using (2.2) and the Gauss Divergence Theorem, we find that

$$\mathcal{E}(t) = -\int_0^t \int_{\partial\Omega} \boldsymbol{S}(\tau; \boldsymbol{r}) \cdot \boldsymbol{n}(\boldsymbol{r}) \, dS(\boldsymbol{r}) \, d\tau + \int_0^t \int_\Omega \boldsymbol{e}(\tau; \boldsymbol{r}) \cdot \boldsymbol{j}(\tau; \boldsymbol{r}) \, d\boldsymbol{r} \, d\tau \tag{2.4}$$

Equations (2.3) and (2.4) suggest that we have to consider fields

$$\boldsymbol{e}(t; \cdot) \ , \ \boldsymbol{d}(t; \cdot) \ , \ \boldsymbol{j}(t; \cdot) \in L^2(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3) \tag{2.5}$$

In (2.4) we see that the energy consists in two parts: one with negative sign, expressing the energy flux through the boundary $\partial\Omega$, and one with positive sign, expressing the energy supplied by the external force. The former expresses the part of the energy produced inside the medium that occupies Ω and the latter the part of the energy that is supplied by the external environment. In a lossless material these two contributions are equal.

Remark 1. *Equation* (2.4) *is valid for boundary conditions more general than* (1.9). *Actually,* (1.9) *implies that the energy flux through the boundary vanishes and, in the absence of currents, the medium is lossless.*

2.4 State space

In view of (2.5) we define the state space as

$$\mathcal{X} := L^2(\Omega;\mathbb{R}^3) \times L^2(\Omega;\mathbb{R}^3)$$

This is real a Hilbert space with the natural inner product

$$\left\langle \begin{pmatrix} \boldsymbol{E}_1 \\ \boldsymbol{H}_1 \end{pmatrix}, \begin{pmatrix} \boldsymbol{E}_2 \\ \boldsymbol{H}_2 \end{pmatrix} \right\rangle_{\mathcal{X}} := \int_\Omega \boldsymbol{E}_1 \cdot \boldsymbol{E}_2 \, d\boldsymbol{r} + \int_\Omega \boldsymbol{H}_1 \cdot \boldsymbol{H}_2 \, d\boldsymbol{r}$$

Consequently, all the relevant fields are considered as $\mathcal{X}$-valued functions of the real variable t. More precisely, as we have already stated and will clarify later, the fields are actually functions $\mathbb{R} \to \mathcal{X}$ vanishing for $t < 0$. For the purposes of our analysis, and as we can see by a double application of Cauchy–Schwarz inequality, it is sufficient to consider $\boldsymbol{e}$, $\boldsymbol{d}$ and $\boldsymbol{j}$ as elements of the space $L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{X})$.

At this point we recall the definitions of some, appropriate for electromagnetics, Sobolev spaces, [5, 8]:

$$\begin{aligned} H(\mathrm{curl}\,;\Omega) &:= \left\{ \boldsymbol{F} \in L^2(\Omega;\mathbb{R}^3) \ : \ \mathrm{curl}\,\boldsymbol{F} \in L^2(\Omega;\mathbb{R}^3) \right\} \\ H_0(\mathrm{curl}\,;\Omega) &:= \left\{ \boldsymbol{F} \in H(\mathrm{curl}\,;\Omega) \ : \ \boldsymbol{n} \times \boldsymbol{F} = \boldsymbol{0} \text{ on } \partial\Omega \right\} \\ H(\mathrm{div}\,;\Omega) &:= \left\{ \boldsymbol{F} \in L^2(\Omega;\mathbb{R}^3) \ : \ \mathrm{div}\,\boldsymbol{F} \in L^2(\Omega) \right\} \\ H(\mathrm{div}\,0;\Omega) &:= \left\{ \boldsymbol{F} \in H(\mathrm{div}\,;\Omega) \ : \ \mathrm{div}\,\boldsymbol{F} = 0 \right\} \end{aligned}$$

Now we assume that

$$\boldsymbol{e} \ : \ \mathbb{R}_+ \to H_0(\mathrm{curl}\,;\Omega) \times H(\mathrm{curl}\,;\Omega)$$

and

$$\boldsymbol{J} \ : \ \mathbb{R}_+ \to H(\mathrm{div}\,;\Omega)$$

Then automatically (1.1) and (1.3) give

$$\boldsymbol{d} \ : \ \mathbb{R}_+ \to H(\mathrm{div}\,;\Omega) \times H(\mathrm{div}\,;\Omega)$$

and

$$\rho \ : \ \mathbb{R}_+ \to L^2(\Omega)$$

respectively.

As a further requirement of the modeling, one should take into account equations (1.2). As far as they are concerned, equations (1.5) show that, given the continuity (and only the weak differentiability) of fields for $t \geqslant 0$ and (1.3), (1.2) hold at each time instant provided that they are satisfied at one time instant only, and particularly at $t = 0$. This reads as follows

$$\operatorname{div} \boldsymbol{d}(0) = \begin{pmatrix} \rho(0) \\ 0 \end{pmatrix}$$

Observe that the *div* operator acts formally as a scalar to a six-vector

$$\operatorname{div} \begin{pmatrix} \boldsymbol{D} \\ \boldsymbol{B} \end{pmatrix} = \begin{pmatrix} \operatorname{div} \boldsymbol{D} \\ \operatorname{div} \boldsymbol{B} \end{pmatrix}$$

That is to say, one has to be able to calculate the value $(\mathbb{V}\boldsymbol{e})(0)$ in advance. So, in our setting, it should be

$$(\mathbb{V}\boldsymbol{e})(0) = V_\infty(\boldsymbol{e}_0) \tag{2.6}$$

The operator V_∞ is defined in the space $\mathcal{X}$ and models the *optical response* of the medium, *i.e.*, the instantaneous reaction to an excitation.
Define the closed hyperplane in $L^2(\Omega; \mathbb{R}^3)$

$$H(\operatorname{div} \rho(0); \Omega) := \{ \boldsymbol{D} \in H(\operatorname{div}; \Omega) \ : \ \operatorname{div} \boldsymbol{D} = \rho(0) \} = \boldsymbol{R} + H(\operatorname{div} 0; \Omega)$$

where $\boldsymbol{R} \in L^2(\Omega; \mathbb{R}^3)$ is a field satisfying $\operatorname{div} \boldsymbol{R} = \rho(0)$. Define also the subset of $\mathcal{X}$

$$\mathcal{X}_0 := V_\infty^{-1}[H(\operatorname{div} \rho(0); \Omega) \times H(\operatorname{div} 0; \Omega)]$$

In view of the above analysis, we have the following remarkable consequence: assume that the modeling of the EM propagation problem in a cavity takes as its starting point

1. the Maxwell system (1.1), which refers mainly to the unknown quantities, and
2. the equation of continuity (1.3), which refers exclusively to the known quantities.

Then the Gauss laws (1.2) hold true as far as the initial datum is chosen from $\mathcal{X}_0$.

2.5 The Maxwell operator

For the realization of the Maxwell operator in the Hilbert space $\mathcal{X}$, we understand the *curl* operator in its distributional version; it applies on six-vectors by following the usual matrix rules. So

$$\mathcal{M} \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} = \begin{pmatrix} \operatorname{curl} \boldsymbol{H} \\ -\operatorname{curl} \boldsymbol{E} \end{pmatrix}$$

The domain of definition is taken to be

$$D(\mathcal{M}) := H_0(\operatorname{curl}; \Omega) \times H(\operatorname{curl}; \Omega)$$

Observe that the boundary condition (1.9) is absorbed in the first component of the above cartesian product. $\mathcal{M}$ defines then a skew-adjoint operator in the Hilbert space $\mathcal{X}$ [8, p. 266, Th. 1]. Consequently, Stone's Theorem [12, Theorem 3.24] implies that $\mathcal{M}$ is the generator of a strongly continuous group of isometries $(U(t))_{t\in\mathbb{R}}$. Equivalently, this means that both $\mathcal{M}$ and $-\mathcal{M}$ are generators of strongly continuous semigroups. More precisely, $\mathcal{M}$ generates $(U(t))_{t\geqslant 0}$ whereas $-\mathcal{M}$ generates $(U(t)^*)_{t\geqslant 0}$. Observe, incidentally, that $U(t)^* = U(t)^{-1}$.

It is crucial to observe that $-\mathcal{M}$ appears actually in (2.1) after the change of variable $t \mapsto -t$, which corresponds to an inversion of time. Stated differently, this gives an opportunity to calculate the past of $\boldsymbol{e}$ also. This is possible, for example, if the operator $\mathbb{V}$ is autonomous, *i.e.*, independent of time. A causal model, however, as the one we propose, requires fields that vanish for $t < 0$ and then (2.1) is trivially satisfied. Thereby, we focus on, and we only use the semigroup $(U(t))_{t\geqslant 0}$.

3 Assumptions on the medium and the constitutive operator

In this section we state the axioms which govern the evolution of the electromagnetic field. Since we have taken the Maxwell equations as granted, these axioms concern the properties of the material inside the domain Ω. We follow essentially [11, §2.2] and we give both physical and mathematical interpretations. The approach is system–theoretic in the sense that we consider the EM field $\boldsymbol{e}$ as the the cause and the EM flux density $\boldsymbol{d}$ as the effect. Our goal is to specify the form of the operator $\mathbb{V}$ which, as we have shown above, can be realized as an operator in the Fréchet space $L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{X})$. A detailed mathematical study of the constitutive operator can be found in [16].

Hypothesis 1 (`Determinism`)**.** *For every cause there exists exactly one effect.*

This postulate is to ensure that $\mathbb{V}$ exists and is not the zero operator.

Hypothesis 2 (`Linearity`)**.** *The effect is produced linearly by its cause.*

That is to say $\mathbb{V}$ is a linear operator. Actually, it can be realized in the matrix form

$$\mathbb{V} := \begin{bmatrix} \boldsymbol{\varepsilon} & \boldsymbol{\xi} \\ \boldsymbol{\zeta} & \boldsymbol{\mu} \end{bmatrix} \tag{3.1}$$

where the entries $\boldsymbol{\varepsilon}$, $\boldsymbol{\mu}$, $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$ are all linear operators. We now can classify the media in accordance with the generally accepted terminology. More precisely, we consider the following cases.

- If $\boldsymbol{\xi} = \boldsymbol{\zeta} = 0$ and
  - both $\boldsymbol{\varepsilon}$, $\boldsymbol{\mu}$ are multiples of the identity operator, the medium is called isotropic.
  - at least one of $\boldsymbol{\varepsilon}$, $\boldsymbol{\mu}$ is not a multiple of the identity operator, the medium is called anisotropic.

- If all the $\boldsymbol{\varepsilon}$, $\boldsymbol{\mu}$, $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$ are multiples of the identity operator, the medium is called biisotropic.
- In any other case, the medium is called bianisotropic.

Hypothesis 3 (`Locality in space`). *For every $\boldsymbol{r} \in \Omega$, the value $\boldsymbol{d}(\cdot, \boldsymbol{r}) = (\mathbb{V}\boldsymbol{e})(\cdot)(\boldsymbol{r})$ is calculated by using only the value $\boldsymbol{e}(\cdot, \boldsymbol{r})$.*

The most important postulate for our theory is given now.

Hypothesis 4 (`Causality`[1]). *The effect cannot precede its cause.*

The following one is a technical assumption which simplifies considerably the mathematical treatment.

Hypothesis 5 (`Non-aging medium`). *The properties of the medium remain invariant in time.*

This time–translation invariance has many equivalent formulations. E.g. one can say that the time instant at which the observation starts does not play any significant role. Thereby, the "present" can be chosen arbitrarily. We then choose as the beginning of observation the time instant $t = 0$. Consequently, the cause $\boldsymbol{e}(t)$ is defined for $t \geqslant 0$ with $\boldsymbol{e}(0) = \boldsymbol{e}_0$ and vanishes for $t < 0$.

As a consequence of these hypotheses, the constitutive operator is continuous and should have the following convolutive form

$$\boldsymbol{d}(t, \boldsymbol{r}) = (\mathbb{V}\boldsymbol{e})(t)(\boldsymbol{r}) = V_\infty(\boldsymbol{r})\boldsymbol{e}(t, \boldsymbol{r}) + \int_0^t V_{\mathrm{d}}(t-\tau, \boldsymbol{r})\boldsymbol{e}(\tau, \boldsymbol{r})\, d\tau \tag{3.2}$$

where

$$V_\infty := \begin{bmatrix} \boldsymbol{\varepsilon}_\infty & \boldsymbol{\xi}_\infty \\ \boldsymbol{\zeta}_\infty & \boldsymbol{\mu}_\infty \end{bmatrix} \ , \ V_{\mathrm{d}}(t) := \begin{bmatrix} \boldsymbol{\varepsilon}_{\mathrm{d}}(t) & \boldsymbol{\xi}_{\mathrm{d}}(t) \\ \boldsymbol{\zeta}_{\mathrm{d}}(t) & \boldsymbol{\mu}_{\mathrm{d}}(t) \end{bmatrix}$$

This fact is intuitively correct, it has been already proposed in [19, 20], and it is proved in detail in [16]. Due to linearity, the optical response operator V_∞ becomes a block matrix not depending on time. Let us note that, in an abstract distributional setting, a necessary and sufficient condition has been established for a linear operator to be a convolution operator ([32, Theorem 5.8-2]).

The entries of the above matrices are 3×3 matrices whose elements are measurable, essentially bounded functions on Ω. A suggestive notation to express this fact is to write $V_\infty \in M_2[M_3(L^\infty(\Omega))]$, $V_{\mathrm{d}} : \mathbb{R}_+ \to M_2[M_3(L^\infty(\Omega))]$), where by $M_n(A)$ we denote the linear space of $n \times n$ matrices with entries from the set A.

Note that each V_∞, $V_{\mathrm{d}}(t)$ defines a bounded multiplication operator in $\mathcal{X}$; the formal definition follows.

[1] This notion of "primitive" causality is compatible with a notion of causality according to which the effect depends on its cause and the cause produces its effect. Nevertheless, the philosophical accounts of causation vary over a heterogeneous range of positions.

Definition 3.1. *Let* $m : \Omega \to M_2(M_3(\mathbb{R}))$. *The operator* T_m *with domain of definition*

$$D(T_m) := \{ \boldsymbol{u} \in \mathcal{X} \; : \; m\boldsymbol{u} \in \mathcal{X} \}$$

and formula

$$T_m \boldsymbol{u} := m\boldsymbol{u}$$

is called the multiplication operator corresponding to m.

It turns out that

(i) T_m is characterized by m, *i.e.*, $T_{m_1} = T_{m_2}$ if and only if $m_1 = m_2$, thereby we often identify the operator T_m with the function m.

(ii) T_m is always closed and densely defined.

(iii) T_m is bounded if and only if the entries of m are $L^\infty(\Omega)$ functions.

We have already stated that V_∞ models the optical response of the medium whereas V_{d}, called the *susceptibility kernel*, models the dispersion phenomena.

Concerning the "time regularity" of V_d, we make the following

Assumption 1. $V_d \in L^\infty_{loc}(\mathbb{R}; M_2[M_3(L^\infty(\Omega))]))$ *and vanishes for* $t < 0$.

Note that with Assumption 1, the convolution in (3.2) is well defined if we assume continuous fields.

4 The abstract problem

The analysis in the above sections leads us to the following abstract evolution initial value problem: find $\boldsymbol{e} : \mathbb{R}_+ \to \mathcal{X}$ which satisfies

$$\begin{cases} \left(V_\infty \boldsymbol{e}(t) + \int_0^t V_{\mathrm{d}}(t-\tau)\boldsymbol{e}(\tau)\, d\tau \right)' = \mathcal{M}\boldsymbol{e}(t) + \boldsymbol{j}(t)\,,\; t \geqslant 0 \\ \boldsymbol{e}(0) = \boldsymbol{e}_0 \end{cases} \tag{4.1}$$

Technically speaking, (4.1) is a neutral integro-differntial equation of convolution type. We consider first the homogeneous version of (4.1)

$$\begin{cases} \left(V_\infty \boldsymbol{e}(t) + \int_0^t V_{\mathrm{d}}(t-\tau)\boldsymbol{e}(\tau)\, d\tau \right)' = \mathcal{M}\boldsymbol{e}(t)\,,\; t \geqslant 0 \\ \boldsymbol{e}(0) = \boldsymbol{e}_0 \end{cases} \tag{4.2}$$

Definition 4.1. *A function* $\boldsymbol{e} : \mathbb{R}_+ \to \mathcal{X}$ *is called a classical solution of* (4.2) *if*

C1 $\boldsymbol{e} \in C^1(\mathbb{R}_+; \mathcal{X})$.

C2 $\boldsymbol{e}(t) \in D(\mathcal{M})$ *for* $t \in \mathbb{R}_+$.

C3 $\boldsymbol{e}$ *satisfies* (4.2) *pointwise in* $\mathbb{R}_+$.

Following the established theory for the Abstract Cauchy Problem [12, pp. 146-147], we introduce a weaker notion of solution.

Definition 4.2. *A function* $\boldsymbol{e} : \mathbb{R}_+ \to \mathcal{X}$ *is called a mild solution of* (4.2) *if*

M1 $\boldsymbol{e} \in C(\mathbb{R}_+; \mathcal{X})$.

M2 $\int_0^t \boldsymbol{e}(\tau)\, d\tau \in D(\mathcal{M})$ *for* $t \in \mathbb{R}_+$.

M3 $\boldsymbol{e}$ *satisfies pointwise in* $\mathbb{R}_+$ *the equation*

$$V_\infty \boldsymbol{e}(t) = V_\infty \boldsymbol{e}_0 + \mathcal{M} \int_0^t \boldsymbol{e}(\tau)\, d\tau - \int_0^t V_d(t-\tau)\boldsymbol{e}(\tau)\, d\tau \tag{4.3}$$

Note that (4.3) is, in some sense, the integrated version of (4.2). Moreover, a classical solution is defined only for $\boldsymbol{e}_0 \in D(\mathcal{M})$ whereas a mild one can be defined for every $\boldsymbol{e}_0 \in \mathcal{X}$. Clearly, a classical solution is also a mild one. For the inverse implication, we have the following result.

Proposition 1. *Let* $\boldsymbol{e}$ *be a mild solution of* (4.2) *satisfying* (C1). *Then it is a classical solution.*

Proof. Put for convenience

$$\boldsymbol{g}(t) = (V_d * \boldsymbol{e})(t) = \int_0^t V_{\mathrm{d}}(t-\tau)\boldsymbol{e}(\tau)\, d\tau$$

Since one of the "components" of the above convolution is continuously differentiable, the same holds for the convolution itself. Equation (4.3) gives

$$V_\infty \left(\frac{\boldsymbol{e}(t+h) - \boldsymbol{e}(t)}{h} \right) + \frac{\boldsymbol{g}(t+h) - \boldsymbol{g}(t)}{h} = \mathcal{M} \left(\frac{1}{h} \int_t^{t+h} \boldsymbol{e}(\tau)\, d\tau \right) \tag{4.4}$$

The left hand side has a limit as $h \to 0$ (as $h \to 0^+$ if $t = 0$). Moreover,

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} \boldsymbol{e}(\tau)\, d\tau = \boldsymbol{e}(t)$$

The closedness of $\mathcal{M}$ and (4.4) imply (C2) and (C3) for $\boldsymbol{e}$. □

The above proposition suggests the following approach: to prove first mild well-posedness for (4.2) and then search for which initial data the solution becomes continuously differentiable and thus a classical one.

5 On the solution method

We calculate the mild solutions of (4.2) by transforming (4.3) to a Volterra integral equation. The relevant theory and the general machinery are given in the appendices. The idea is to apply a variation-of-constants procedure in the equation (4.3). The fundamental prerequisite for that is given in the following

Assumption 2. *V_∞^{-1} exists as a bounded operator in $\mathcal{X}$ and $\mathcal{Q} := V_\infty^{-1}\mathcal{M}$ is the generator of a strongly continuous semigroup $(T(t))_{t\geqslant 0}$.*

Observe that Assumption 2 consists in two conditions:

S1 V_∞ is boundedly invertible.

S2 $\mathcal{Q}$ remains a generator.

(S2) ensures that $\mathcal{Q}$ is closed and densely defined with $D(\mathcal{Q}) = D(\mathcal{M})$. Equipped with the graph norm

$$\|\boldsymbol{u}\|_q := \sqrt{\|\boldsymbol{u}\|^2 + \|\mathcal{Q}\boldsymbol{u}\|^2}$$

$D(\mathcal{Q})$ becomes a Hilbert space, denoted by $[D(\mathcal{Q})]$. Now (4.3) is written as

$$\boldsymbol{e}(t) = \boldsymbol{e}_0 + \mathcal{Q}\int_0^t \boldsymbol{e}(\tau)\,d\tau + \int_0^t K(t-\tau)\boldsymbol{e}(\tau)\,d\tau \tag{5.1}$$

where

$$K(t) := -V_\infty^{-1}V_{\mathrm{d}}(t)$$

Let now $\boldsymbol{e}$ be a solution of (5.1), *i.e.*, a mild solution of (4.2). Fix a $t > 0$ and define the function

$$\boldsymbol{u}(s) := T(t-s)\int_0^s \boldsymbol{e}(\tau)\,d\tau\ ,\ 0 \leqslant s \leqslant t \tag{5.2}$$

The function $\boldsymbol{u}$ is continuously differentiable and

$$\boldsymbol{u}'(s) = -T(t-s)\mathcal{Q}\int_0^s \boldsymbol{e}(\tau)\,d\tau + T(t-s)e(s) \tag{5.3}$$

By using (5.1), equation (5.3) is written as

$$\boldsymbol{u}'(s) = T(t-s)\boldsymbol{e}_0 + T(t-s)\int_0^s K(s-\tau)\boldsymbol{e}(\tau)\,d\tau \tag{5.4}$$

We integrate now (5.4) in $[0,t]$ to obtain (the Fubini Theorem is used)

$$\int_0^t \boldsymbol{e}(\tau)\,d\tau = \int_0^t T(\tau)\boldsymbol{e}_0\,d\tau + \int_0^t (T*K)(t-\tau)\boldsymbol{e}(\tau)\,d\tau \tag{5.5}$$

The algebra of bounded operators in X is denoted by $\mathcal{B}(\mathcal{X})$. When endowed with the strong topology, we write $\mathcal{B}_{\mathrm{s}}(\mathcal{X})$.

Assumption 3. $T * K \in W^1_{loc}(\mathbb{R}_+; \mathcal{B}_s(\mathcal{X}))$.

With this at hand, we can differentiate (5.5) and thus obtain the Volterra equation of convolution type

$$\boldsymbol{e}(t) = (F * \boldsymbol{e})(t) + T(t)\boldsymbol{e}_0 \ , \ t \geqslant 0, \tag{5.6}$$

where

$$F(t) := \frac{d}{dt}(T * K)(t) = \frac{d}{dt}\int_0^t T(t-\tau)K(\tau)\, d\tau$$

6 Mild solvability

The solvability of (5.6) is established by the standard theory presented in Appendix B. Namely, for a fixed but arbitrarily large $b > 0$, we consider the Volterra operator

$$W \ : \ C([0,b];\mathcal{X}) \to C([0,b];\mathcal{X})$$

given by

$$(W\boldsymbol{g})(t) = (F * \boldsymbol{g})(t)$$

The Banach space $C([0,b];\mathcal{X})$ is equipped with the supremum norm, and (5.6) can be realized as an equation in $C([0,b];\mathcal{X})$

$$\boldsymbol{e} := W\boldsymbol{e} + T(\cdot)\boldsymbol{e}_0 \tag{6.1}$$

which admits the unique solution

$$\boldsymbol{e} = (I - W)^{-1}(T(\cdot)\boldsymbol{e}_0) \tag{6.2}$$

Now, for $n = 0, 1, ...$ and $t \geqslant 0$, define the operators $S_n(t) \in \mathcal{B}(\mathcal{X})$ by the recursive scheme

$$S_0(t) := T(t) \ \ , \ S_n(t) := (F * S_n)(t) = \int_0^t F(t-s)S_n(s)\, ds \tag{6.3}$$

Then estimates (B.1) (Appendix B) give the following

Proposition 2. *The (Volterra) series*

$$S(t) := \sum_{n=0}^{\infty} S_n(t)$$

converges in norm and uniformly with respect to t on bounded intervals. Moreover, letting

$$\omega_t := \sup_{0 \leqslant s \leqslant t} \| F(s) \| \,,$$

the following estimate holds

$$\| S(t) \| \leqslant e^{\omega_t t} \tag{6.4}$$

and we have for every $\boldsymbol{x} \in \mathcal{X}$,

$$S(\cdot)\boldsymbol{x} = (I - W)^{-1}(T(\cdot)\boldsymbol{x}) \tag{6.5}$$

Thereby, we obtain the main result.

Proposition 3. *The problem (4.2) is mildly uniquely solvable for each $\boldsymbol{e}_0 \in \mathcal{X}$ and the solution is given by $\boldsymbol{e}(t) = S(t)\boldsymbol{e}_0$. More precisely, (4.2) is mildly well-posed.*

7 Classical solvability

We have to check for which initial data $\boldsymbol{e}_0 \in \mathcal{X}$ the mild solution $\boldsymbol{e}(t) = S(t)\boldsymbol{e}_0$ is continuously differentiable. By employing induction over n one can prove the following:

Lemma 7.1. *Let $\boldsymbol{e}_0 \in D(\mathcal{Q})$. Then, for each $n = 0, 1, ...,$ the function $S_n(\cdot)\boldsymbol{e}_0$ is continuously differentiable. More precisely,*

$$\begin{cases} (S_0(t)\boldsymbol{e}_0)' = \mathcal{Q}T(t)\boldsymbol{e}_0 = T(t)\mathcal{Q}\boldsymbol{e}_0 \\ (S_{n+1}(t)\boldsymbol{e}_0)' = F(t)S_n(0)\boldsymbol{e}_0 + (F * (S_n(\cdot)\boldsymbol{e}_0)')(t) \end{cases} \tag{7.1}$$

At this point, we make a further assumption of regularity. Actually, we slightly strengthen Assumption 3.

Assumption 4. *$T * K(\cdot)\boldsymbol{e}_0 \in C^1(\mathbb{R}_+; \mathcal{X})$, equivalently $F(\cdot)\boldsymbol{e}_0 \in C(\mathbb{R}_+; \mathcal{X})$, for $\boldsymbol{e}_0 \in D(A)$.*

It is then easily seen that the sequence defined in (7.1) is summable and the corresponding series converges to the unique solution of the Volterra equation

$$\boldsymbol{e}'(t) = (F * \boldsymbol{e}')(t) + (F(t) + \mathcal{Q}T(t))\boldsymbol{e}_0 \tag{7.2}$$

Observe that $\boldsymbol{e}'$ denotes just the unknown in (7.2). But this notation is justified by the fact that (7.2) is obtained by the formal differentiation of (5.6). This analysis has as a result that, whenever $\boldsymbol{e}_0 \in D(\mathcal{Q})$, the series $\sum S_n(t)\boldsymbol{e}_0$ is continuously differentiable and thus we have

Proposition 4. *Let Assumption 4 hold and $\boldsymbol{e}_0 \in D(A)$. Then $\boldsymbol{e}(t) := S(t)\boldsymbol{e}_0$ is the unique classical solution of (4.2).*

8 The inhomogeneous problem

We turn now our attention to the general problem (4.1) and we are going to apply again a variation-of-constants procedure in the spirit of Section 6, see also [2] and [33]. We first define the analogous notions of solution for the inhomogeneous case.

Definition 8.1. *Let $\boldsymbol{j} \in C(\mathbb{R}_+; \mathcal{X})$. A function $\boldsymbol{e} : \mathbb{R}_+ \to \mathcal{X}$ is called a classical solution of (4.1) if*

iC1 *$\boldsymbol{e} \in C^1(\mathbb{R}_+; \mathcal{X})$.*

iC2 *$\boldsymbol{e}(t) \in D(\mathcal{M})$ for $t \in \mathbb{R}_+$.*

iC3 *$\boldsymbol{e}$ satisfies (4.2) pointwise in $\mathbb{R}_+$.*

Definition 8.2. *Let $\boldsymbol{j} \in L^1_{loc}(\mathbb{R}_+; \mathcal{X})$. A function $\boldsymbol{e} : \mathbb{R}_+ \to \mathcal{X}$ is called a mild solution of (4.1) if*

iM1 $\boldsymbol{e} \in C(\mathbb{R}_+; \mathcal{X})$.

iM2 $\int_0^t \boldsymbol{e}(\tau)\, d\tau \in D(\mathcal{M})$ *for* $t \in \mathbb{R}_+$.

iM3 $\boldsymbol{e}$ *satisfies pointwise in* $\mathbb{R}_+$ *the equation*

$$V_\infty \boldsymbol{e}(t) = V_\infty \boldsymbol{e}_0 + \mathcal{M} \int_0^t \boldsymbol{e}(\tau)\, d\tau - \int_0^t V_d(t-\tau)\boldsymbol{e}(\tau)\, d\tau + \int_0^t \boldsymbol{j}(\tau)\, d\tau$$

We also have a result analogous to Proposition 1.

Proposition 5. *Let* $\boldsymbol{j} \in C(\mathbb{R}_+; \mathcal{X})$ *and* $\boldsymbol{e}$ *be a mild solution of* (4.1) *satisfying* (iC1). *Then it is a classical solution.*

With our notation, if $\boldsymbol{e}$ is a mild solution of (4.1), then it satisfies

$$\boldsymbol{e}(t) = \boldsymbol{e}_0 + \mathcal{Q} \int_0^t \boldsymbol{e}(\tau)\, d\tau + \int_0^t K(t-\tau)\boldsymbol{e}(\tau)\, d\tau + \int_0^t \boldsymbol{f}(\tau)\, d\tau \tag{8.1}$$

where $\boldsymbol{f} := V_\infty^{-1}\boldsymbol{j}$. Fix now a $t > 0$. Define again the function

$$\boldsymbol{u}(s) := T(t-s) \int_0^s \boldsymbol{e}(\tau)\, d\tau \ \ , 0 \leqslant s \leqslant t$$

and repeat the calculations made in section 6; one sees immediately that $\boldsymbol{e}$ satisfies the Volterra equation

$$\boldsymbol{e} = F * \boldsymbol{e} + (T(\cdot)\boldsymbol{e}_0 + T * \boldsymbol{f}) \tag{8.2}$$

By Lemma B.1 (Appendix B) we obtain a representation formula for the solution of (4.1).

Corollary 8.1 (Variation-of-constants formula)**.** *Let* $\boldsymbol{j} \in L^1_{loc}(\mathbb{R}_+; \mathcal{X})$. *The unique mild solution of* (4.1) *is given by*

$$\boldsymbol{e}(t) = S(t)\boldsymbol{e}_0 + \int_0^t S(t-\tau)\boldsymbol{f}(\tau)\, d\tau \tag{8.3}$$

More precisely, the problem (4.1) *is mildly well posed.*

We also give a criterion for classical solvability.

Corollary 8.2. *Let* $\boldsymbol{e}_0 \in D(\mathcal{Q})$ *and* $\boldsymbol{j} \in C(\mathbb{R}_+; \mathcal{X})$. *Then* (8.3) *is the classical solution of* (4.1) *if (and only if)* $S * \boldsymbol{f} \in C^1(\mathbb{R}_+; \mathcal{X})$.

We see that we face again a problem of differentiating the convolution.

9 Checking Assumption 2

We have already stated that the validity of Assumption 2 boils down to the investigation of two sub-problems. We check each one separately. The results presented in this section are treated in detail in Sections 3 and 4 of [15].

9.1 (S1) Bounded invertibility for V_∞

We have to check whether V_∞^{-1} defines a bounded multiplication operator on $\mathcal{X}$. First of all, note that the inverse operator is defined, again as a multiplication operator, when $V_\infty(\boldsymbol{r})^{-1}$ is defined for almost all $\boldsymbol{r} \in \Omega$. Moreover, we have the following pointwise characterization.

Proposition 6. *The following are equivalent:*
a) V_∞ is boundedly invertible.
b) $V_\infty(\boldsymbol{r})$ is almost uniforlmy bounded below, i.e., there exists a positive constant c such that

$$|\, V_\infty(\boldsymbol{r})\mathsf{y}\,| \geqslant c\,|\,\mathsf{y}\,|$$

for almost all $\boldsymbol{r} \in \Omega$ and $\mathsf{y} \in \mathbb{R}^3 \times \mathbb{R}^3$.

9.2 (S2) Generation property for $\mathcal{Q}$

The general philosophy of our treatment, at least in its mathematical part, is to consider the dispersive material ($V_{\mathrm{d}} \neq 0$) as a perturbation of the non-dispersive material ($V_{\mathrm{d}} = 0$). The latter is described by the constitutive relation

$$\boldsymbol{d}(t, \boldsymbol{r}) = V_\infty(\boldsymbol{r})\boldsymbol{e}(t, \boldsymbol{r}) \tag{9.1}$$

The following ensures that the relevant unperturbed problem is well posed.

Assumption 5. *The bilinear form*

$$\langle\, \boldsymbol{u}, \boldsymbol{v}\,\rangle_V := \langle\, V_\infty \boldsymbol{u}, \boldsymbol{v}\,\rangle_{\mathcal{X}}$$

defines an inner product in $\mathcal{X}$.

After this, and as a direct corollary of Stone's theorem, we have the following

Proposition 7. *Let Assumption 5 hold and assume that $(\mathcal{X}, \langle\,\cdot,\cdot\,\rangle_V)$ is a Hilbert space. Then $\mathcal{Q}$ generates a C_0-group of isometries with respect to the new norm $\|\cdot\|_V$.*

Following our notation, let $T(t)$ be the aforementioned group. According to (2.3), the variation of EM energy is given by

$$\mathcal{E}(t) = \int_0^t \langle\, V_\infty \boldsymbol{e}'(t), \boldsymbol{e}(t)\,\rangle_{\mathcal{X}}\, dt \tag{9.2}$$

Then by performing a typical calculation in (9.2), we confirm that the considered material is indeed lossless:

$$\mathcal{E}(t) = \int_0^t \langle\, \boldsymbol{e}'(t), \boldsymbol{e}(t)\,\rangle_V \, d\tau = \frac{1}{2}\left(\|\, T(t)\boldsymbol{e}_0\,\|_V^2 - \|\,\boldsymbol{e}_0\,\|_V^2\right) = 0$$

9.3 The main result

Taking into account the above, we can formulate sufficient conditions which cover both (S1) and (S2) and thus imply Assumption 2. We would like to underline the pointwise character of the result, with reference to the optical response matrix V_∞.

Proposition 8. *Let V_∞ be*
a) almost uniforlmy bounded below,
b) almost everywhere a symmetric matrix, i.e.,

$$V_\infty(\boldsymbol{r}) = V_\infty(\boldsymbol{r})^T$$

for almost all $\boldsymbol{r} \in \Omega$, and
c) almost everywhere a positive definite matrix, i.e.,

$$V_\infty(\boldsymbol{r})\mathsf{y} \cdot \mathsf{y} > 0$$

for almost all $\boldsymbol{r} \in \Omega$ and all non-zero $\mathsf{y} \in \mathbb{R}^3 \times \mathbb{R}^3$.
Then Assumption 2 holds.

Remark 2. *Let us note that the assumptions (a) and (b) of Proposition 8 are equivalent to the fact that V_∞ is almost uniformly coercive, i.e., there is a positive constant a such that*

$$V_\infty(\boldsymbol{r})\mathsf{y} \cdot \mathsf{y} \geqslant a\,|\,y\,|^2 \tag{9.3}$$

for almost all $\boldsymbol{r} \in \Omega$ and all $\mathsf{y} \in \mathbb{R}^3 \times \mathbb{R}^3$. This compels $\langle\,\cdot,\cdot\,\rangle_\infty$ to define an inner product equivalent to $\langle\,\cdot,\cdot\,\rangle_{\mathcal{X}}$ and turns $\mathcal{X}$ into a Hilbert space.

10 Checking Assumption 3

This is to find conditions such that the convolution $T * K$ is a weakly differentiable function in the strong topology of $\mathcal{B}(\mathcal{X})$. These conditions are essentially smoothness conditions for $\mathcal{X}$-valued functions $K(\cdot)\boldsymbol{x}$, $\boldsymbol{x} \in \mathcal{X}$. Such a (minimal) condition is:

Assumption 6. *$K(\cdot)\boldsymbol{x}$ is continuous, that is $K \in C(\mathbb{R}_+; \mathcal{B}_s(\mathcal{X}))$.*

We should note here that any particular condition of smoothness on K implies a "corresponding" condition of smoothness on V_d.

Remark 3. *The assumption of smoothness of the susceptibility kernel is not physically controversial.*

Travis [31] has given a necessary and sufficient condition on T for the continuous differentiability of the convolution $T * \boldsymbol{f}$ for every $\boldsymbol{f} \in C(\mathbb{R}_+; \mathcal{X})$. The Lemma 3.5 of that reference provides us with the following negative result: Assumption 3 cannot be expected to hold for an arbitrary choice of K. Thereby, we have to impose further restrictions on K. We consider two such cases in the sequel.

10.1 Time Regularity: $K \in C^1(\mathbb{R}_+; \mathcal{B}_{\mathbf{s}}(\mathcal{X}))$

Then (4.2) becomes an initial value problem for a usual integro-differential equation of Volterra type

$$\begin{cases} \boldsymbol{e}'(t) = [\mathcal{Q} + K(0)]\boldsymbol{e}(t) + \int_0^t K'(t-\tau)\boldsymbol{e}(\tau)\, d\tau\,,\ t \geqslant 0 \\ \boldsymbol{e}(0) = \boldsymbol{e}_0 \end{cases} \tag{10.1}$$

where K' denotes the strong derivative of K. The problem (10.1) has been considered by Marti [26] already in the 1960s. Note that, since $K(0) \in \mathcal{B}(\mathcal{X})$, the operator

$$\mathcal{M}_p := \mathcal{M} + K(0),$$

with domain of definition $D(\mathcal{M}_p) = D(\mathcal{M})$ is the generator of a C_0-semigroup $(T_p(t))_{t\geqslant 0}$ which satisfies again a Volterra integral equation [12, Corollary III.1.7]

$$T_p(t)\boldsymbol{x} = T(t)\boldsymbol{x} + \int_0^t T(t-\tau)K(0)T_p(\tau)\boldsymbol{x}\, d\tau \tag{10.2}$$

for every $t \geqslant 0$ and $\boldsymbol{x} \in \mathcal{X}$. Equation (5.6) now reads

$$\boldsymbol{e} = W_1\boldsymbol{e} + W_2\boldsymbol{e} + T(\cdot)\boldsymbol{e}_0 \tag{10.3}$$

where

$$W_1\boldsymbol{g} := T(\cdot)K(0) * \boldsymbol{g} \quad , \quad W_2\boldsymbol{g} := (T * K') * \boldsymbol{g}$$

We now rewrite (10.3) as

$$\boldsymbol{e} = (I - W_1)^{-1}W_2\boldsymbol{e} + (I - W_1)^{-1}(T(\cdot)\boldsymbol{e}_0) \tag{10.4}$$

Using Lemma B.1 (Appendix B) and (10.2), (10.4) becomes again a Volterra equation

$$\boldsymbol{e} = (T_p * K')\boldsymbol{e} + T_p(\cdot)\boldsymbol{e}_0, \tag{10.5}$$

and the solution of (10.3), which is equivalent to (10.1), agrees with Marti's solution.

Remark 4. *Time regularity involves only the susceptibility kernel V_d.*

10.2 Space Regularity: $K(t)[\mathcal{X}] \subset D(\mathcal{Q})$ for every $t \geqslant 0$

In this case, and due to the fact that $T(\cdot)\boldsymbol{x} \in C^1(\mathbb{R}_+; \mathcal{X})$ for every $\boldsymbol{x} \in D(\mathcal{Q})$ with $(T(t)\boldsymbol{x})' = \mathcal{Q}T(t)\boldsymbol{x} = T(t)\mathcal{Q}\boldsymbol{x}$, we have

$$F(t) = K(t) + (\mathcal{Q}T(\cdot) * K)(t) = K(t) + (T * \mathcal{Q}K(\cdot))(t) \tag{10.6}$$

Let $\mathcal{K}(t) := \mathcal{Q}K(t)$. It is easy to see that $\mathcal{K} \in C(\mathbb{R}_+; \mathcal{B}_{\mathrm{s}}(\mathcal{X}))$. The integral equation (5.6) now reads

$$\boldsymbol{e} = K * \boldsymbol{e} + (T * \mathcal{K}) * e + T(\cdot)\boldsymbol{e} \tag{10.7}$$

Another way to attack (4.2) in this case is to make the change of variable

$$\boldsymbol{u}(t) := \boldsymbol{e}(t) - (K * \boldsymbol{e})(t) \tag{10.8}$$

in view of which the problem is written as

$$\begin{cases} \boldsymbol{u}'(t) = \mathcal{Q}\boldsymbol{u}(t) + (\mathcal{K} * \boldsymbol{e})(t)\,,\ t \geqslant 0 \\ \boldsymbol{e}(0) = \boldsymbol{e}_0 \end{cases} \tag{10.9}$$

Equation (10.9) is an inhomogeneous abstract Cauchy problem for the operator $\mathcal{Q}$, having as inhomogeneous data $\mathcal{K} * \boldsymbol{e}$. The solution of (10.9) is given by

$$\boldsymbol{u}(t) = T(t)\boldsymbol{e}_0 + ((T * \mathcal{K}) * \boldsymbol{e})(t) \tag{10.10}$$

By substituting (10.8) in the left hand side of (10.10), we obtain again (10.7).

Remark 5. *Space regularity involves both the optical response V_∞ and the susceptibility kernel V_d.*

11 A comment on the Laplace Transform method

Another method to obtain mild solutions of (4.2) is to apply the Laplace Transform. The relevant theory, notation and the general "arsenal" of the Laplace Transform is presented in detail in [1]. Here we deal only with the formal considerations of the matter. Namely, we apply formally the Laplace Transform on both sides of (4.3) to obtain

$$\widehat{\boldsymbol{e}}(\lambda) = \frac{1}{\lambda}\boldsymbol{e}_0 + \frac{1}{\lambda}\mathcal{Q}\widehat{\boldsymbol{e}}(\lambda) + \widehat{K}(\lambda)\widehat{\boldsymbol{e}}(\lambda) \tag{11.1}$$

We now define the resolvent operator of $\mathcal{Q}$, $R(\lambda) := (\lambda I - \mathcal{Q})^{-1}$. It is known that $R(\lambda) = \widehat{T}(\lambda)$, so (11.1) becomes

$$\widehat{\boldsymbol{e}}(\lambda) = \widehat{T}(\lambda)\boldsymbol{e}_0 + \lambda\widehat{T}(\lambda)\widehat{K}(\lambda)\widehat{\boldsymbol{e}}(\lambda) \tag{11.2}$$

Observe that (11.2) is exactly the Laplace Transform of (5.6). In addition, by assuming that the operator $I - \lambda\widehat{T}(\lambda)\widehat{K}(\lambda)$ is invertible, we obtain

$$\widehat{\boldsymbol{e}}(\lambda) = [I - \lambda\widehat{T}(\lambda)\widehat{K}(\lambda)]^{-1}\widehat{T}(\lambda)\boldsymbol{e}_0 \tag{11.3}$$

Thereby, one has to prove that the (operator-valued) function

$$\Phi(\lambda) := [I - \lambda\widehat{T}(\lambda)\widehat{K}(\lambda)]^{-1}\widehat{T}(\lambda)$$

is a Laplace Transform, *i.e.*, there is a function $S : \mathbb{R}_+ \to \mathcal{B}(\mathfrak{X})$ such that $\Phi(\lambda) = \widehat{S}(\lambda)$. Then by inverting the Laplace transform in (11.3) we obtain again the mild solution as $\boldsymbol{e}(t) = S(t)\boldsymbol{e}_0$.

Acknowledgments

This work is based on results from the doctoral thesis [14] of ADI under the supervision of IGS. The authors would like to thank Professor A. N. Yannacopoulos (Athens University of Economics and Business, Greece) for interesting discussions.

Appendix A The integral for vector and operator valued functions

Let X a real separable Hilbert space where the inner product is denoted by $\langle \cdot, \cdot \rangle$ and the norm by $|\cdot|$. We fix an orthonormal basis (e_n). We consider the real line $\mathbb{R}$ with the Lebesgue measure. Let $A \subset \mathbb{R}^N$ be measurable and consider a function $f : A \to X$.

Definition A.1. *f is said to be measurable if every real function $\langle f(\cdot), e_n \rangle$ is measurable.*

Evidently, f is measurable if and only if $\langle f(\cdot), y \rangle$ is measurable for every $y \in X$. There is a norming sequence $(w_n) \subset X$, $|w_n| \leqslant 1$, *i.e.*,

$$|x| = \sup_n |\langle x, w_n \rangle|$$

Thereby, $|f(\cdot)|$ is also measurable.

Lemma A.1. *Let f be measurable. The following are equivalent:*
a) $\langle f(\cdot), y \rangle \in L^1(A)$ for every $y \in X$.
b) $|f(\cdot)| \in L^1(A)$.

Definition A.2. *Let f be measurable. We say that f is integrable if it satisfies one of the equivalent conditions of Lemma A.1.*

Let f be integrable. Consider the mapping

$$I_f : X \in y \mapsto \int_A \langle f(t), y \rangle \, dt \in \mathbb{R}$$

Then $I_f \in X^*$ and, by the Riesz Representation Theorem, there is a unique $x_f \in X$ such that $I_f(y) = \langle x_f, y \rangle$.

Definition A.3. *x_f is the integral of f on A and is denoted by $\int_A f(t)\, dt$.*

Note that the sequence of Fourier coefficients of $\int_A f(t)\, dt$ is $\left(\int_A \langle f(t), e_n \rangle \, dt\right)$. For $1 \leqslant p \leqslant \infty$ we define the space $L^p(A; X)$; it contains all the measurable functions $f : A \to X$ for which $|f(\cdot)|^p \in L^1(A)$. With the usual norm

$$\| f \|_{L^p} := \left(\int_A |f(t)|^p \, dt \right)^{1/p}$$

$L^p(A;X)$ becomes a Banach space (a Hilbert space for the special case $p = 2$). Observe that $L^1(A;X)$ is exactly the space of integrable functions. The space $L^p_{\mathrm{loc}}(A;X)$ is defined to contain all the measurable functions $f : A \to X$ for which $|\, f(\cdot)\,|^p \in L^1(K)$ for all $K \subset\subset A$. This space is endowed with the natural Fréchet topology.

We now consider functions $F : A \to \mathcal{B}(X)$.

Definition A.4. *We say that F is measurable (integrable) if the X-valued function $F(\cdot)x$ is measurable (integrable) for every $x \in X$. The space of integrable $\mathcal{B}(X)$-valued functions is written as $L^1(A;\mathcal{B}_s(X))$.*

If $F \in L^1(A;\mathcal{B}_{\mathrm{s}}(X))$, the integral of F is the bounded operator given by the formula

$$\Big(\int_A F(t)\,dt\Big)\,x = \int_A F(t)x\,dt$$

Next we need a notion of weak derivative. Let $f : [\alpha,\beta] \to X$ be a measurable function.

Definition A.5. *f is said to be weakly differentiable if there is a function $g \in L^1([\alpha,\beta];X)$ such that, for $\alpha \leqslant t \leqslant \beta$,*

$$f(t) - f(\alpha) = \int_\alpha^t g(\tau)\,d\tau$$

g is then called the weak derivative of f and is denoted by f'.

It is evident that the weak derivative generalizes the classical one and that each weak differentiable function is continuous. It is also directly seen from the definition that f is weakly differentiable if and only if each scalar function $\langle\, f(\cdot), y\,\rangle$, $y \in X$ is weakly differentiable. For $1 \leqslant p \leqslant \infty$ we consider the space

$$W^{1,p}([\alpha,\beta];X) := \{\, f \in L^p([\alpha,\beta];X) \;:\; f' \in L^p([\alpha,\beta];X)\,\}$$

Endowed with the norm $\|\, f\,\|_{W^{1,p}} := (\|\, f\,\|^p_{L^p} + \|\, f'\,\|^p_{L^p})^{1/p}$ it becomes a Banach space (a Hilbert space for $p = 2$).

Lemma A.2. *Let $I \subset \mathbb{R}$, $F : I \to X$ be strongly continuously differentiable in D (a subspace of X) with $(F(t)x)' = G(t)x$ and $f : I \to X$ a continuously differentiable function taking values in D. Then $F(t)f(t)$ is continuously differentiable and*

$$(F(t)f(t))' = G(t)f(t) + F(t)f'(t)$$

Appendix B Convolution

Let now F, $G \in L^1_{\mathrm{loc}}(\mathbb{R}_+;\mathcal{B}_{\mathrm{s}}(X))$, $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+;X)$. We then can define the convolutions $F * f \in C(\mathbb{R}_+;X)$, $F * G \in C(\mathbb{R}_+;\mathcal{B}_{\mathrm{s}}(X))$ as

$$(F * f)(t) := \int_0^t F(t-\tau)f(\tau)\,d\tau \quad\text{and}\quad (F * G)(t) := \int_0^t F(t-\tau)G(\tau)\,d\tau$$

The Fubini Theorem implies that convolution is an associative operation. Furthermore, the formal convolution operator $W_F f := F * f$ is linear. When $F \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; X)$ and $f \in C(\mathbb{R}_+; X)$, we have the estimates

$$| (W_F^n f)(t) | \leqslant \frac{(\omega_t t)^n}{n!} \sup_{0 \leqslant s \leqslant t} | f(s) | \tag{B.1}$$

where ω_t is defined to be the (essential) supremum of $\| F(\cdot) \|$ in $[0, t]$. Estimates (B.1) imply especially two well known facts:

1. W_F defines a bounded operator on $C([0, b]; X)$ for arbitrary large but fixed $b > 0$.

2. $(I - W_F)^{-1}$ exists as a bounded operator on $C([0, b]; X)$ and

$$(I - W_F)^{-1} = \sum_{n=0}^{\infty} W_F^n = I + W_R \tag{B.2}$$

that is the inverse is written as the identity plus a convolution operator.

Lemma B.1. *Let F, $G \in C(\mathbb{R}_+; \mathcal{B}_s(X))$ and consider the corresponding convolution operators W_F, W_G in $C([0, b]; X)$. Then*

$$(I - W_F)^{-1} W_G f = W_H f = H * f$$

where $H \in C(\mathbb{R}_+; \mathcal{B}_s(X))$ is defined by

$$H(\cdot)x := (I - W_F)^{-1}(G(\cdot)x)$$

The proof follows directly from the relation

$$(I - W_F)^{-1} W_G = W_G + W_R W_G = W_G + W_{R*G} = W_{G+R*G}$$

The differentiability of the convolution is a crucial problem for our study. When one of the components is differentiable, this is essentially a problem of differentiation under the integral sign. From the Leibnitz rule we have

Lemma B.2. *Let either $F \in W^1_{loc}(\mathbb{R}_+; \mathcal{B}_s(X)$, or $f \in W^1_{loc}(\mathbb{R}_+; X)$, Then the convolution $F * f$ is differentiable and either*

$$(F * f)'(t) = F(0) f(t) + (F' * f)(t)$$

or

$$(F * f)'(t) = F(t) f(0) + (F * f')(t)$$

respectively.

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