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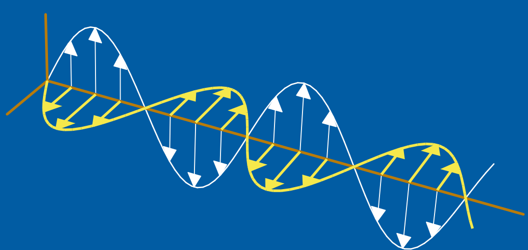

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Abstract

This paper discusses the restrictions that dissipation forces on the material parameters and responses of bi-anisotropic media. The treatment is a general time domain analysis where six-vector formalism is applied. Energy conditions set limitations on the optical response dyadics and dyadic susceptibility kernels of the material that have to be satisfied regardless the time dependence of the fields. This paper will treat in detail the resulting restrictions for the optical response of bi-anisotropic media. Examples are given for some special cases. Also uniqueness and equivalence aspects of constitutive relations in the time domain are discussed.

1 Introduction

When material effects are discussed in the electromagnetics literature, it often happens that the analysis does not set restrictions for the freedom of medium parameters and response functions. Particularly in time-harmonic treatments where the problem is considered from a fixed-frequency point of view, it is tempting to ignore the physical limitations of the material constitutive relations because the medium is described by the set of complex numbers. The basic material characterization rule for a microwave engineer is that a lossless medium possesses a real permittivity, and for a lossy medium, the imaginary part of permittivity has to be a negative number [2].[1] Obviously, this approach fails to pay explicit attention to the dispersive characteristics of the electromagnetic response of materials.

There exists a growing need to study the limitations of the constitutive response functions of materials because of the recent increased emphasis on complex and exotic materials in electromagnetics and microwave technology. The response of a dielectric medium can be characterized by the dielectric constant and the scalar susceptibility kernel function, but for an anisotropic medium these are 3×3 tensors. Bi-anisotropic media are characterized by four tensor constants and four tensor-valued susceptibility functions. For isotropic chiral media, the fixed-frequency rules of thumb have been spelled out [8, 11]: in the lossless case one has to use real values for the permittivity, permeability, and chirality parameters. In addition, the chirality parameter cannot exceed the refractive index of the medium. And for a lossy material, the imaginary part of the chirality parameter is limited above by the imaginary parts of permittivity and permeability.

Lossless medium is an idealization. Some dispersion or memory effects always exist in the response of the material to electromagnetic excitation, and part of the incoming energy will be dissipated in the associated losses. In a frequency-domain description, this means that the material responds with different permittivity amplitude at different frequencies. Causality arguments lead to the existence of the

[1] The sign of the imaginary part depends on the functional time dependence, and $\exp(j\omega t)$ is dominant in electrical engineering textbooks. But in the case of the convention $\exp(-i\omega t)$ [7], the imaginary part of the permittivity should be positive for a lossy medium.

imaginary part of the permittivity function, which is related to the real part through the Kramers–Kronig relations [5].

However, a lossless material may be a useful idealization. One example could be an application where the electromagnetic signal is applied to a material and the frequencies of the signal are restricted to a narrow band which lies far outside all resonances of the material. Then the material parameters can be considered to be constant throughout the band and losses can be neglected to a first approximation, and the analysis is certainly easier than a full treatment involving dispersive material functions.

The present paper studies bi-anisotropic materials and the restrictions that general physical principles imply on the constitutive relations. The analysis is a general time domain treatment which means that the temporal dependence of the fields is limited neither to sinusoidal variation nor to any other predetermined function. Moreover, all quantities in this description are real-valued. In a time domain description, the bi-anisotropic constitutive relations contain the optical responses (four dyadics) and the susceptibility kernels (four dyadic-valued functions). The constitutive relations can be written in several different forms. The equivalence between various representations is discussed with special emphasis on the electric and magnetic conductivity terms.

Because of the coupled nature of the electric and magnetic fields and responses, six-vector notation turns out to be suited for the analysis. Six-vectors contain two dual electric and magnetic vector quantities, each with three components, and the result is a single vector with six components. The restrictions for a dissipative material are derived from the condition that the electromagnetic energy contained within any volume has to be always non-negative, regardless of the field excitation. The present paper focuses on the restrictions of the optical responses. The detailed limitations of the susceptibility kernels are left to a forthcoming study.

In the following, all three-vectors are typed in italic bold-face (for example, $\boldsymbol{E}$), all three-dyadics in Roman boldface or Greek boldface ($\mathbf{I}, \boldsymbol{\varepsilon}$). All six-vectors are typed in lowercase sans serif (e) and six-dyadics in uppercase sans serif (M).

2 Definitions and basic equations

To begin the electromagnetic analysis of bi-anisotropic materials, the following definitions for field relations, constitutive parameters, power, dissipation, and energy are needed.

2.1 Maxwell equations

The Maxwell equations between the space- and time-dependent electric and magnetic fields $\boldsymbol{E}$, $\boldsymbol{H}$, and electric and magnetic flux densities $\boldsymbol{D}$, $\boldsymbol{B}$ are

$$\begin{cases} \nabla \times \boldsymbol{E} = -\boldsymbol{J}_m - \partial_t \boldsymbol{B} \\ \nabla \times \boldsymbol{H} = \boldsymbol{J}_e + \partial_t \boldsymbol{D} \end{cases} \tag{2.1}$$

In these equations, both electric $\boldsymbol{J}_e$ and magnetic $\boldsymbol{J}_m$ current densities are included.

These equations have to be complemented with six more scalar equations, independent of Maxwell equations, that relate the fields and flux densities. These equations model the dynamics of the charges in the material.

2.2 Constitutive relations of bi-anisotropic media

For the general class of bi-anisotropic materials that is considered in the present paper, the constitutive relations couple the electric and the magnetic fields. Electric field excitation causes both electric and magnetic response, and so does also magnetic field excitation. The anisotropic character of the medium leads to dyadic, or tensor-valued, relation between the cause and response vector functions.

The constitutive relations have a general form for a bi-anisotropic medium, see Ref. [6]:

$$\begin{cases} c_0\eta_0\boldsymbol{D}(t) = \boldsymbol{\varepsilon}\cdot\boldsymbol{E}(t) + (\boldsymbol{\chi}_{ee} * \boldsymbol{E})(t) + \eta_0\boldsymbol{\xi}\cdot\boldsymbol{H}(t) + \eta_0(\boldsymbol{\chi}_{em} * \boldsymbol{H})(t) \\ c_0\boldsymbol{B}(t) = \boldsymbol{\zeta}\cdot\boldsymbol{E}(t) + (\boldsymbol{\chi}_{me} * \boldsymbol{E})(t) + \eta_0\boldsymbol{\mu}\cdot\boldsymbol{H}(t) + \eta_0(\boldsymbol{\chi}_{mm} * \boldsymbol{H})(t) \end{cases} \tag{2.2}$$

where $c_0 = 1/\sqrt{\epsilon_0\mu_0}$, $\eta_0 = \sqrt{\mu_0/\epsilon_0}$, and μ_0 and ϵ_0 are the free-space constants. Here, the spatial dependence, of all vector fields and dyadics, is suppressed. Moreover, the star symbol $*$ denotes temporal convolution with a scalar product included, *i.e.*,

$$(\boldsymbol{\alpha} * \boldsymbol{B})(t) = \int_{-\infty}^{t} \boldsymbol{\alpha}(t-t')\cdot\boldsymbol{B}(t')dt'$$

The susceptibility functions $\boldsymbol{\chi}_{\kappa\lambda}$ and the fields appearing in (2.2) have explicit time dependence and are real-valued functions. In equations (2.2), each of the four responses is separated into two parts: optical response and dispersive part. The real-valued dyadics $\boldsymbol{\varepsilon}, \boldsymbol{\mu}, \boldsymbol{\xi}, \boldsymbol{\zeta}$ describe the instantaneous (optical) response of the material. The dispersive part is described by the dyadic susceptibility kernels $\boldsymbol{\chi}_{\kappa\lambda}(t)$ as functions of time. The components of the susceptibility kernels are assumed to be in $C^1(\mathbb{R}_+)$.[2] Due to causality, the susceptibility kernels $\boldsymbol{\chi}_{\kappa\lambda}$ vanish for $t < 0$ but can have a discontinuity in the origin, *i.e.*, $\boldsymbol{\chi}_{\kappa\lambda}(t = 0^+) \neq \boldsymbol{0}$. Although this possibility sometimes is doubted in the literature (see, for example [5, p. 310]), it is not in conflict with the analysis presented in this paper. Moreover, from an engineering point of view it is interesting to approximate fast processes or raise-times with a discontinuous function. This amounts to working on a time scale that is much coarser than the fastest phenomena in the response of the medium (the introduction of optical responses is equivalent to including delta functions in the susceptibility kernels, which is an even stronger use of this principle of different time scales).

[2] Throughout this paper $C(A)$ denotes the set of continuous functions on the set A. The set $C^1(A)$ denotes the set of functions in $C(A)$ with time derivatives in $C(A)$. Furthermore, $\mathbb{R}_+$ denotes the set of non-negative real numbers.

2.2.1 Uniqueness of the constitutive relations

A more general set of constitutive relations than in (2.2) that includes also dispersive effects of the conductivity is

$$\begin{cases} c_0\eta_0\boldsymbol{D} = \boldsymbol{\varepsilon}\cdot\boldsymbol{E} + \boldsymbol{\chi}_{ee} * \boldsymbol{E} + \eta_0\boldsymbol{\xi}\cdot\boldsymbol{H} + \eta_0\boldsymbol{\chi}_{em} * \boldsymbol{H} \\ \boldsymbol{J}_e = \boldsymbol{\sigma}_{ee}\cdot\boldsymbol{E} + \epsilon_0\boldsymbol{\Sigma}_{ee} * \boldsymbol{E} + \eta_0\boldsymbol{\sigma}_{em}\cdot\boldsymbol{H} + \epsilon_0\eta_0\boldsymbol{\Sigma}_{em} * \boldsymbol{H} \end{cases}$$

and

$$\begin{cases} c_0\boldsymbol{B} = \boldsymbol{\zeta}\cdot\boldsymbol{E} + \boldsymbol{\chi}_{me} * \boldsymbol{E} + \eta_0\boldsymbol{\mu}\cdot\boldsymbol{H} + \eta_0\boldsymbol{\chi}_{mm} * \boldsymbol{H} \\ \boldsymbol{J}_m = \boldsymbol{\sigma}_{me}\cdot\boldsymbol{E} + \epsilon_0\boldsymbol{\Sigma}_{me} * \boldsymbol{E} + \eta_0\boldsymbol{\sigma}_{mm}\cdot\boldsymbol{H} + \epsilon_0\eta_0\boldsymbol{\Sigma}_{mm} * \boldsymbol{H} \end{cases}$$

where the time-dependence has been suppressed. Again, due to causality, the real-valued conductivity kernels $\boldsymbol{\Sigma}_{\kappa\lambda}$ have to vanish for $t<0$. The electric and magnetic conductivity dyadics $\boldsymbol{\sigma}_{\kappa\lambda}$, which are real, represent the instantaneous electric and magnetic current responses.

These general constitutive relations are, however, not unique. Consider a gauge transformation of the form, see also [14]

$$\begin{cases} \boldsymbol{\chi}_{\kappa\lambda} \to \boldsymbol{\chi}_{\kappa\lambda} + \mathbf{f}_{\kappa\lambda} \\ \boldsymbol{\Sigma}_{\kappa\lambda} \to \boldsymbol{\Sigma}_{\kappa\lambda} - \partial_t\mathbf{f}_{\kappa\lambda}, \qquad t>0 \\ \boldsymbol{\sigma}_{e\lambda} \to \boldsymbol{\sigma}_{e\lambda} - \epsilon_0\mathbf{f}_{e\lambda}(0^+) \\ \boldsymbol{\sigma}_{m\lambda} \to \boldsymbol{\sigma}_{m\lambda} - \dfrac{1}{c_0}\mathbf{f}_{m\lambda}(0^+) \end{cases}$$

for κ, $\lambda = e$, m. This transformation does not affect the right-hand side of the Maxwell equations (2.1). Here, the dyadics $\mathbf{f}_{\kappa\lambda}$ are arbitrary real-valued dyadics satisfying $\mathbf{f}_{\kappa\lambda} = \mathbf{0}$ for $t<0$, and $\mathbf{f}_{\kappa\lambda}(0^+) = \mathbf{f}_{\kappa\lambda}(t=0^+)$. The effect of this transformation is to reorganize the amount of bound charges, which is included in the field $\boldsymbol{D}$, and the amount of free charges, which is represented by $\boldsymbol{J}_e$. A similar interpretation can be given for the fields $\boldsymbol{B}$ and $\boldsymbol{J}_m$.

Two special cases are easily obtained.

1. Define $\mathbf{f}_{\kappa\lambda}$ to be

$$\begin{cases} \epsilon_0\mathbf{f}_{e\kappa}(t) = H(t)\left[\boldsymbol{\sigma}_{e\kappa} + \epsilon_0\displaystyle\int_0^t \boldsymbol{\Sigma}_{e\kappa}(t')\,dt'\right] \\ \dfrac{1}{c_0}\mathbf{f}_{m\kappa}(t) = H(t)\left[\boldsymbol{\sigma}_{m\kappa} + \dfrac{1}{c_0}\displaystyle\int_0^t \boldsymbol{\Sigma}_{m\kappa}(t')\,dt'\right] \end{cases}$$

where $H(t)$ is the Heaviside step function. This implies $\boldsymbol{J}_e \equiv \boldsymbol{J}_m \equiv \mathbf{0}$, *i.e.*, a pure displacement representation

$$\begin{cases} c_0\eta_0\boldsymbol{D} = \boldsymbol{\varepsilon}\cdot\boldsymbol{E} + (\boldsymbol{\chi}_{ee} + \mathbf{f}_{ee}) * \boldsymbol{E} + \eta_0\boldsymbol{\xi}\cdot\boldsymbol{H} + \eta_0(\boldsymbol{\chi}_{em} + \mathbf{f}_{em}) * \boldsymbol{H} \\ c_0\boldsymbol{B} = \boldsymbol{\zeta}\cdot\boldsymbol{E} + (\boldsymbol{\chi}_{me} + \mathbf{f}_{me}) * \boldsymbol{E} + \eta_0\boldsymbol{\mu}\cdot\boldsymbol{H} + \eta_0(\boldsymbol{\chi}_{mm} + \mathbf{f}_{mm}) * \boldsymbol{H} \end{cases}$$

and all effects of the current density are included in the dispersion terms.

2. Define $\mathbf{f}_{\kappa\lambda}$ to be $\mathbf{f}_{\kappa\lambda} = -\boldsymbol{\chi}_{\kappa\lambda}$ This implies a permittivity-conductivity representation

$$\begin{cases} c_0\eta_0\boldsymbol{D} = \boldsymbol{\varepsilon}\cdot\boldsymbol{E} + \eta_0\boldsymbol{\xi}\cdot\boldsymbol{H} \\ \boldsymbol{J}_e = (\boldsymbol{\sigma}_{ee} + \epsilon_0\boldsymbol{\chi}_{ee}(0^+))\cdot\boldsymbol{E} + \epsilon_0(\boldsymbol{\Sigma}_{ee} + \partial_t\boldsymbol{\chi}_{ee}) * \boldsymbol{E} \\ \qquad + \eta_0(\boldsymbol{\sigma}_{em} + \epsilon_0\boldsymbol{\chi}_{em}(0^+))\boldsymbol{H} + \epsilon_0\eta_0(\boldsymbol{\Sigma}_{em} + \partial_t\boldsymbol{\chi}_{em}) * \boldsymbol{H} \end{cases}$$

and

$$\begin{cases} c_0\boldsymbol{B} = \boldsymbol{\zeta}\cdot\boldsymbol{E} + \eta_0\boldsymbol{\mu}\cdot\boldsymbol{H} \\ \boldsymbol{J}_m = (\boldsymbol{\sigma}_{me} + \dfrac{1}{c_0}\boldsymbol{\chi}_{me}(0^+))\boldsymbol{E} + \dfrac{1}{c_0}(\boldsymbol{\Sigma}_{me} + \partial_t\boldsymbol{\chi}_{me}) * \boldsymbol{E} \\ \qquad + \eta_0(\boldsymbol{\sigma}_{mm} + \dfrac{1}{c_0}\boldsymbol{\chi}_{mm}(0^+))\cdot\boldsymbol{H} + \dfrac{\eta_0}{c_0}(\boldsymbol{\Sigma}_{mm} + \partial_t\boldsymbol{\chi}_{mm}) * \boldsymbol{H} \end{cases}$$

Therefore, without loss of generality the constitutive relations in (2.2) can be assumed, and these will be used in the analysis of the present paper.

2.2.2 Transformations between equivalent constitutive relations

The constitutive relations in (2.2) are equivalent to the following:

$$\begin{cases} c_0\eta_0\boldsymbol{D}(t) = \tilde{\boldsymbol{\varepsilon}}\cdot\boldsymbol{E}(t) + (\tilde{\boldsymbol{\chi}}_{ee} * \boldsymbol{E})(t) + c_0\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{B}(t) + c_0(\tilde{\boldsymbol{\chi}}_{em} * \boldsymbol{B})(t) \\ \eta_0\boldsymbol{H}(t) = \tilde{\boldsymbol{\zeta}}\cdot\boldsymbol{E}(t) + (\tilde{\boldsymbol{\chi}}_{me} * \boldsymbol{E})(t) + c_0\tilde{\boldsymbol{\mu}}^{-1}\cdot\boldsymbol{B}(t) + c_0(\tilde{\boldsymbol{\chi}}_{mm} * \boldsymbol{B})(t) \end{cases} \tag{2.3}$$

which is a form that is quite often used in the literature [12]. The relations between the optical response dyadics of (2.2) and (2.3) are

$$\begin{cases} \tilde{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon} - \boldsymbol{\xi}\cdot\boldsymbol{\mu}^{-1}\cdot\boldsymbol{\zeta} = \boldsymbol{\varepsilon} + \boldsymbol{\xi}\cdot\tilde{\boldsymbol{\zeta}} = \boldsymbol{\varepsilon} - \tilde{\boldsymbol{\xi}}\cdot\boldsymbol{\zeta} \\ \tilde{\boldsymbol{\xi}} = \boldsymbol{\xi}\cdot\boldsymbol{\mu}^{-1} \\ \tilde{\boldsymbol{\zeta}} = -\boldsymbol{\mu}^{-1}\cdot\boldsymbol{\zeta} \\ \tilde{\boldsymbol{\mu}} = \boldsymbol{\mu} \end{cases}$$

and the susceptibility dyadics of (2.3) are obtained via the equations

$$\begin{cases} \tilde{\boldsymbol{\chi}}_{ee} = \boldsymbol{\chi}_{ee} + \boldsymbol{\xi}\cdot\tilde{\boldsymbol{\chi}}_{me} + \boldsymbol{\chi}_{em}\cdot\tilde{\boldsymbol{\zeta}} + \boldsymbol{\chi}_{em} * \tilde{\boldsymbol{\chi}}_{me} \\ \tilde{\boldsymbol{\chi}}_{em} = \boldsymbol{\chi}_{em}\cdot\boldsymbol{\mu}^{-1} + \boldsymbol{\xi}\cdot\tilde{\boldsymbol{\chi}}_{mm} + \boldsymbol{\chi}_{em} * \tilde{\boldsymbol{\chi}}_{mm} \\ \tilde{\boldsymbol{\chi}}_{me} = -\tilde{\boldsymbol{\chi}}_{mm}\cdot\boldsymbol{\zeta} - \boldsymbol{\mu}^{-1}\cdot\boldsymbol{\chi}_{me} - \tilde{\boldsymbol{\chi}}_{mm} * \boldsymbol{\chi}_{me} \\ \boldsymbol{\mu}^{-1}\cdot\boldsymbol{\chi}_{mm} + \tilde{\boldsymbol{\chi}}_{mm}\cdot\boldsymbol{\mu} + \tilde{\boldsymbol{\chi}}_{mm} * \boldsymbol{\chi}_{mm} = \mathbf{0} \end{cases} \tag{2.4}$$

It has been implicitly assumed that $\boldsymbol{\mu}$ is a full dyadic, *i.e.*, it has an inverse. The last relation of (2.4) is a Volterra equation of the second kind for $\tilde{\boldsymbol{\chi}}_{mm}$, *i.e.*, it possesses a unique solution[3]. After solving for $\tilde{\boldsymbol{\chi}}_{mm}$, the remaining kernels can be easily

[3]This equation is also a Volterra equation of the second kind in $\boldsymbol{\chi}_{mm}$ if $\tilde{\boldsymbol{\chi}}_{mm}$ is known. This observation is used in the transformation in the opposite direction.

obtained by insertion in the other three equations. This means that the relations (2.2) and (2.3) are equivalent. There exists a one-to-one correspondence between all constitutive parameters.

It can be shown that if the following relations hold

$$\boldsymbol{\varepsilon}^T = \boldsymbol{\varepsilon}, \quad \boldsymbol{\mu}^T = \boldsymbol{\mu}, \quad \boldsymbol{\xi}^T = \pm\boldsymbol{\zeta}$$

then

$$\tilde{\boldsymbol{\varepsilon}}^T = \tilde{\boldsymbol{\varepsilon}}, \quad \tilde{\boldsymbol{\mu}}^T = \tilde{\boldsymbol{\mu}}, \quad \tilde{\boldsymbol{\xi}}^T = \mp\tilde{\boldsymbol{\zeta}}$$

Furthermore, if in addition the susceptibility kernels satisfy

$$\boldsymbol{\chi}_{ee}^T = \boldsymbol{\chi}_{ee}, \quad \boldsymbol{\chi}_{mm}^T = \boldsymbol{\chi}_{mm}, \quad \boldsymbol{\chi}_{em}^T = \pm\boldsymbol{\chi}_{me}$$

then we have also

$$\tilde{\boldsymbol{\chi}}_{ee}^T = \tilde{\boldsymbol{\chi}}_{ee}, \quad \tilde{\boldsymbol{\chi}}_{mm}^T = \tilde{\boldsymbol{\chi}}_{mm}, \quad \tilde{\boldsymbol{\chi}}_{em}^T = \mp\tilde{\boldsymbol{\chi}}_{me}$$

These conditions are relevant for reciprocity aspects. In this paper the concept of reciprocity is adopted according to the definition [7, Sect. 5.5, Equation (17)], *i.e.*

$$\boldsymbol{\chi}_{ee}^T = \boldsymbol{\chi}_{ee}, \quad \boldsymbol{\chi}_{mm}^T = \boldsymbol{\chi}_{mm}, \quad \boldsymbol{\chi}_{em}^T = -\boldsymbol{\chi}_{me} \tag{2.5}$$

2.3 Six-vector formulation

In electromagnetic studies of bi-anisotropic materials, the six-vector notation [10] is helpful. A six-vector is composed of an ordinary electric vector and a magnetic vector, both having three components. Correspondingly, a six-dyadic is an operation between two six-vectors. The electromagnetic six-vector field e and six-vector flux density d are defined as

$$\mathsf{e} = \begin{pmatrix} \boldsymbol{E} \\ \eta_0 \boldsymbol{H} \end{pmatrix}, \qquad \mathsf{d} = \begin{pmatrix} c_0\eta_0 \boldsymbol{D} \\ c_0 \boldsymbol{B} \end{pmatrix} \tag{2.6}$$

Hence, all components of the six-vector fields have the same (electrical) dimension, *i.e.*, volts per meter. Now the constitutive relations (2.2) can be written as a single six-vector equation:

$$\mathsf{d} = \mathsf{M}\cdot\mathsf{e} + \mathsf{K} * \mathsf{e} \tag{2.7}$$

where

$$\mathsf{M} = \begin{pmatrix} \boldsymbol{\varepsilon} & \boldsymbol{\xi} \\ \boldsymbol{\zeta} & \boldsymbol{\mu} \end{pmatrix} \tag{2.8}$$

is the six-dyadic of the optical response of the material parameters. It has a 6×6-element matrix representation and therefore its full description requires 36 parameters. Correspondingly,

$$\mathsf{K} = \begin{pmatrix} \boldsymbol{\chi}_{ee} & \boldsymbol{\chi}_{em} \\ \boldsymbol{\chi}_{me} & \boldsymbol{\chi}_{mm} \end{pmatrix} \tag{2.9}$$

is the susceptibility kernel six-dyadic of the bi-anisotropic material which models the dispersive effects.

2.4 Poynting theorem

To describe power flow carried by the electromagnetic field in materials, the Poynting vector

$$\boldsymbol{S}(t) = \boldsymbol{E}(t) \times \boldsymbol{H}(t)$$

is used. This vector gives the instantaneous power density and direction of the power flow at a point. The power balance equation can be written with the use of the Maxwell equations in a source-free medium ($\boldsymbol{J}_e \equiv \boldsymbol{0},\ \boldsymbol{J}_m \equiv \boldsymbol{0}$):

$$\nabla \cdot \boldsymbol{S} + \boldsymbol{E} \cdot \partial_t \boldsymbol{D} + \boldsymbol{H} \cdot \partial_t \boldsymbol{B} = 0 \tag{2.10}$$

In six-vector notation the Poynting theorem reads

$$\eta_0 c_0 \nabla \cdot \boldsymbol{S} + \mathsf{e} \cdot \partial_t \mathsf{d} = 0$$

The quantities in this equation are scalars. The first term on the left-hand side is a three-vector divergence. The second one is a dot product between two six-vectors, understood as a sum of two parts: the electric components of both six-vectors (the upper three components) dot-multiplied, and so also the lower, magnetic components. The divergence of the Poynting vector represents the power per volume created at a certain point.

2.5 Dissipation and energy

Integration of equation (2.10) over a simply connected volume $V(\boldsymbol{r})$ centered at $\boldsymbol{r}$ with boundary $\partial V(\boldsymbol{r})$ ($\hat{n}$ outward directed normal) and integration over time from $-\infty$ to τ and use of (2.7) imply

$$\mathcal{E}(\tau) = \iiint\limits_{V(\boldsymbol{r})} \{w_{opt}(\tau) + w_d(\tau)\}\, dv \tag{2.11}$$

where $\mathcal{E}$ is the normalized electromagnetic energy stored in the volume V, the volume measure is dv, and the energy density subscripts refer to the optical response and to the dispersive terms:

$$\mathcal{E}(\tau) = -\iint\limits_{\partial V(\boldsymbol{r})} \int\limits_{-\infty}^{\tau} \eta_0 c_0 \boldsymbol{S}(t) \cdot \hat{n}\, dt\, dS$$

$$w_{opt}(\tau) = \int\limits_{-\infty}^{\tau} \mathsf{e}(t) \cdot \mathsf{M} \cdot \partial_t \mathsf{e}(t) dt \tag{2.12}$$

$$w_d(\tau) = \int\limits_{-\infty}^{\tau} \mathsf{e}(t) \cdot \partial_t \left[\mathsf{K} * \mathsf{e}\right](t) dt \tag{2.13}$$

In these formulas, the spatial dependence of the functions is suppressed.

Now we are ready to formulate a definition for the important concept of dissipation.

Definition 2.1. *The medium is dissipative (passive) at a point $\boldsymbol{r}$ in the region V if and only if for all τ and for every electromagnetic field e in $C^1(V \times \mathbb{R})$, and every volume $V(\boldsymbol{r})$, such that $V(\boldsymbol{r}) \subset V$, the total energy $\mathcal{E}(\tau) \geq 0$.*

Definition 2.2. *The medium is dissipative in V if and only if it is dissipative at all points in V.*

Physically, this definition states that the total electromagnetic energy entering into a sample of a dissipative material is always non-negative at all times. Or stated differently, no net production of electromagnetic energy inside ∂V is possible. The medium is passive. Furthermore, dissipation is a local property in space. Thus, in principle, parts of the medium can be dissipative, while others not.

Since the integrand in (2.11) is continuous in the spatial variables this definition is equivalent to

$$w_{opt}(\tau) + w_d(\tau) \geq 0$$

for all τ and all fields $\in C^1$.

One of the aims of the present paper is to study in detail the consequences of dissipation regarding the optical response dyadics of the medium. Therefore, let us focus in the following on the term w_{opt}. The dispersive term w_d will be studied in a subsequent paper.

3 Dissipation due to optical response

If a material is dissipative, the sum of the energy density terms has to be positive for all fields. From this fact one can derive conditions for the optical response six-dyadic M.

3.1 Symmetry and positive semi-definiteness

Perform a partial integration in (2.12). This yields the alternative form

$$w_{opt}(\tau) = \frac{1}{2}\mathsf{e}(\tau) \cdot \mathsf{M} \cdot \mathsf{e}(\tau) + \int_{-\infty}^{\tau} \mathsf{e}(t) \cdot \mathsf{M}_{as} \cdot \partial_t \mathsf{e}(t) dt \tag{3.1}$$

The subscript as denotes the anti-symmetric part of the six-dyadic: $\mathsf{M}_{as} = \frac{1}{2}(\mathsf{M} - \mathsf{M}^T)$. In a dissipative medium $w_{opt}(\tau) + w_d(\tau)$ must be non-negative. This implies the following theorem.

Theorem 3.1. *In a dissipative medium the six-dyadic M is symmetric and positive semi-definite.*

Proof. opt.response Let $\{\hat{\mathsf{u}}_i\}_{i=1}^6$ be a base in six-space. Let $\mathsf{e} = f(t_0, a, b; t)\hat{\mathsf{u}}_i + f(t_0, c, d; t)\hat{\mathsf{u}}_j$, where the function f refers to the roof-top pulse depicted in Figure 1.[4]

Choose constants $a > c > 0$, $b > d > 0$ and $t_0 + b < \tau$. The first term of (3.1) then

[4]Although the function f is not in C^1, this shortcoming can be remedied by a mollifier technique [13].

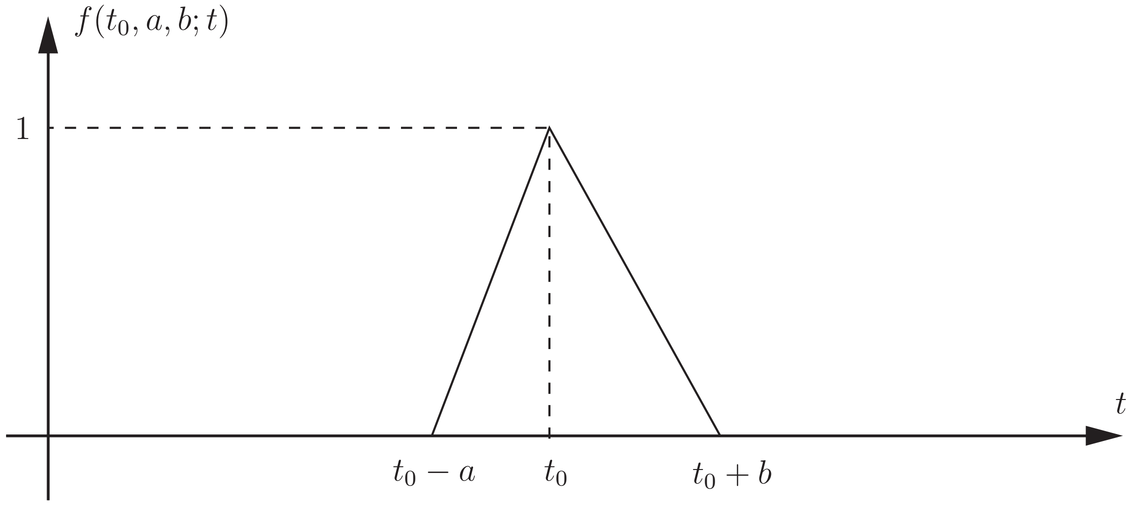

Figure 1: A roof-top pulse.

vanishes by construction. Moreover, the remaining part yields

$$w_{opt} = \hat{\mathsf{u}}_i \cdot \mathsf{M}_{as} \cdot \hat{\mathsf{u}}_j \left[\frac{ad - bc}{ab} \right] \tag{3.2}$$

The following estimate holds for the dispersive term (2.13)

$$w_d \leq 2\|\mathsf{K}\|(a + b + c + d)$$

Here, $\| \cdot \|$ denotes the norm

$$\|\mathsf{K}\| = \max_{ij} \|\hat{\mathsf{u}}_i \cdot \mathsf{K} \cdot \hat{\mathsf{u}}_j\|_\infty$$

wherein $\| \cdot \|_\infty$ denotes the temporal maximum norm. Now, let all coefficients a, b, c and d be proportional to ϵ. Hence, $\lim_{\epsilon \to 0} w_d = 0$. The only non-vanishing term in the total energy density is thus the expression (3.2). This is a quantity of indefinite sign unless $\mathsf{M}_{as} = \mathsf{0}$. Therefore, in a dissipative medium the optical response is modeled by a symmetric six-dyadic M. Moreover, the optical response can be written on the form

$$w_{opt} = \frac{1}{2} \mathsf{e}(\tau) \cdot \mathsf{M} \cdot \mathsf{e}(\tau)$$

This completes the first part of the proof. Further, to show that M is positive semi-definite, let $\mathsf{e} = f(\tau, \epsilon, \epsilon; t)\mathsf{g}$ where g is an arbitrary constant six-vector. Consequently, $\lim_{\epsilon \to 0} w_d = 0$ and

$$w_{opt} = \frac{1}{2} \mathsf{g} \cdot \mathsf{M} \cdot \mathsf{g}$$

This implies that in a dissipative medium M is positive semi-definite. □

3.2 Interpretation

Several corollaries to Theorem 3.1 follow concerning the properties of the four constitutive three-dyadics of the optical response.

3.2.1 Symmetry of the material six-dyadic

First, because of the symmetry of M, the permittivity part $\boldsymbol{\varepsilon}$ and the permeability part $\boldsymbol{\mu}$ are both symmetric three-dyadics. On the other hand, the magnetoelectric dyadics that are located off-diagonally in M may contain both symmetric and anti-symmetric parts. For these dyadics, the symmetry of M means $\boldsymbol{\xi} = \boldsymbol{\zeta}^T$, which leads to the following connection concerning the symmetric (s) and antisymmetric (as) parts of the dyadics:

$$\mathsf{M} = \begin{pmatrix} \boldsymbol{\varepsilon}_s & \boldsymbol{\xi}_s + \boldsymbol{\xi}_{as} \\ \boldsymbol{\xi}_s - \boldsymbol{\xi}_{as} & \boldsymbol{\mu}_s \end{pmatrix} \tag{3.3}$$

It is interesting to compare the result with the requirements of reciprocity, see (2.5). For reciprocal materials, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are symmetric but the magnetoelectric components are connected by $\boldsymbol{\xi} = -\boldsymbol{\zeta}^T$ [7, Sect. 5.5]. Hence dissipation means that the permittivity and permeability dyadics have to be reciprocal, but not so for the magnetoelectric dyadics. In fact, dissipation implies that the magnetoelectric parts of the optical response are non-reciprocal. This result can also be interpreted such that there is no chirality component in the optical response in a medium with dissipation [15].

Dissipation limits the number of parameters of the material description. Altogether there are 21 free parameters in the optical response dyadic.

3.2.2 Positive semi-definiteness of the material six-dyadic

Moreover, from the property of positive semi-definiteness of M one can extract several corollaries. This leads to conditions between the 21 matrix elements. The conditions can be expressed either with eigenvalues or with the principal minors of the six-dyadic [4]. In general a dyadic is positive semi-definite if all its eigenvalues are non-negative. Alternative conditions can then be derived using principal minors or other dyadic invariants such as the trace.

The permittivity and permeability dyadics, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$, are both positive semi-definite, as can be easily seen by letting the electric field or the magnetic field vanish in the energy expression and requiring the energy to be non-negative for all possible fields. This implies that the eigenvalues of the dyadics cannot be negative, or equivalently:

$$\begin{cases} \mathrm{det}\{\boldsymbol{\varepsilon}\} \geq 0, \mathrm{det}\{\boldsymbol{\mu}\} \geq 0 \\ \mathrm{spm}\{\boldsymbol{\varepsilon}\} \geq 0, \mathrm{spm}\{\boldsymbol{\mu}\} \geq 0 \\ \mathrm{tr}\{\boldsymbol{\varepsilon}\} \geq 0, \mathrm{tr}\{\boldsymbol{\mu}\} \geq 0 \end{cases} \tag{3.4}$$

where det, spm, and tr denote the determinant, sum of principal minors, and the trace of a dyadic [8, Chap. 2].

Furthermore, by treating combinations of the electric and magnetic fields it is possible to deduce that the dyadics $\boldsymbol{\varepsilon} - \boldsymbol{\xi} \cdot \boldsymbol{\mu}^{-1} \cdot \boldsymbol{\xi}^T$ and $\boldsymbol{\mu} - \boldsymbol{\xi}^T \cdot \boldsymbol{\varepsilon}^{-1} \cdot \boldsymbol{\xi}$ have to be positive semi-definite.[5] The magnetoelectric dyadics are therefore limited, *e.g.* by the conditions

$$\det\left(\boldsymbol{\varepsilon} - \boldsymbol{\xi} \cdot \boldsymbol{\mu}^{-1} \cdot \boldsymbol{\xi}^T\right) \geq 0 \tag{3.5}$$

$$\det\left(\boldsymbol{\mu} - \boldsymbol{\xi}^T \cdot \boldsymbol{\varepsilon}^{-1} \cdot \boldsymbol{\xi}\right) \geq 0 \tag{3.6}$$

where it has been implicitly assumed that the inverses exist for the permittivity and permeability dyadics. These results are in agreement with the property that the determinant of M can be written as the following products [10]

$$\det\{\mathsf{M}\} = \det\{\boldsymbol{\varepsilon}\} \det\left(\boldsymbol{\mu} - \boldsymbol{\zeta} \cdot \boldsymbol{\varepsilon}^{-1} \cdot \boldsymbol{\xi}\right) = \det\{\boldsymbol{\mu}\} \det\left(\boldsymbol{\varepsilon} - \boldsymbol{\xi} \cdot \boldsymbol{\mu}^{-1} \cdot \boldsymbol{\zeta}\right)$$

An interpretation of the inequalities in (3.5)–(3.6) is that the magnetoelectric parameters have to be "weaker" than the permittivity and permeability.[6]

3.3 Examples

Let us next discuss how these general results for the optical response six-dyadic of a dissipative medium affect some special cases of complex materials.

3.3.1 Bi-isotropic medium

For a bi-isotropic medium, the four three-dyadics in the optical response six-dyadic are multiples of the identity three-dyadic $\mathbf{I}$. The isotropy decreases the 36-dimensional material six-dyadic into a subspace of four degrees of freedom, and furthermore, the symmetry of M restricts this number to three and we can write

$$\mathsf{M} = \begin{pmatrix} \varepsilon & \xi \\ \xi & \mu \end{pmatrix} \mathbf{I} \tag{3.7}$$

where ε, μ, and ξ are scalars. Here ξ is the measure of the non-reciprocity of the optical response. The reciprocal magnetoelectric quantity, chirality, is not compatible with the symmetry requirement. The conditions for the scalar parameters are

$$\varepsilon \geq 0, \qquad \mu \geq 0, \qquad \xi \leq \sqrt{\varepsilon\mu}$$

which results agree with the often assumed limitations for the material parameters in frequency-domain analyses [11, Section 2.6.1].

[5] Note that the first of these dyadics appears in the transformation formulas between the constitutive relations (2.2) and (2.3).

[6] If a dyadic $\mathbf{A}-\mathbf{B}$ is positive semi-definite, $\mathbf{A}$ and $\mathbf{B}$ real symmetric dyadics, then it can be shown that the largest eigenvalue of $\mathbf{A}$ is larger than all eigenvalues of $\mathbf{B}$, and the smallest eigenvalue of $\mathbf{B}$ is smaller than all eigenvalues of $\mathbf{A}$.

3.3.2 Co-axial magnetoelectric medium

For a co-axial magnetoelectric material, all four constitutive dyadics are symmetric and have the same eigenvectors:

$$\boldsymbol{\alpha} = \sum_{i=1}^{3} \alpha_i \boldsymbol{u}_i \boldsymbol{u}_i$$

where $\boldsymbol{\alpha} = \boldsymbol{\varepsilon}, \boldsymbol{\mu}, \boldsymbol{\xi}, \boldsymbol{\zeta}$. Then the requirement of positive semi-definiteness as expressed in (3.5) leads to the following conditions:

$$\xi_i^2 \leq \varepsilon_i \mu_i, \qquad i = 1, 2, 3$$

3.3.3 Uniaxial medium with antisymmetric magnetoelectric component

As another example, consider a medium with uniaxial permittivity and permeability three-dyadics, and with the magnetoelectric dyadic $\boldsymbol{\xi}$ such that its symmetric part is also uniaxial, and all optical axes parallel with z. Furthermore, let $g\boldsymbol{u}_z$ be the vector of the antisymmetric part. Although this is not the most general optical response dyadic of a material, it includes many special cases of media. The parameters look like

$$\begin{cases} \boldsymbol{\varepsilon} = \varepsilon_t \mathbf{I}_t + \varepsilon_z \boldsymbol{u}_z \boldsymbol{u}_z \\ \boldsymbol{\mu} = \mu_t \mathbf{I}_t + \mu_z \boldsymbol{u}_z \boldsymbol{u}_z \\ \boldsymbol{\xi} = \xi_t \mathbf{I}_t + \xi_z \boldsymbol{u}_z \boldsymbol{u}_z + g \boldsymbol{u}_z \times \mathbf{I} \\ \boldsymbol{\zeta} = \xi_t \mathbf{I}_t + \xi_z \boldsymbol{u}_z \boldsymbol{u}_z - g \boldsymbol{u}_z \times \mathbf{I} \end{cases}$$

Here the unit dyadic in the plane transversal to the optical axis is $\mathbf{I}_t = \mathbf{I} - \boldsymbol{u}_z \boldsymbol{u}_z$. The antisymmetric dyadic with respect to the same axis is $\boldsymbol{u}_z \times \mathbf{I} = \boldsymbol{u}_y \boldsymbol{u}_x - \boldsymbol{u}_x \boldsymbol{u}_y$.

The conditions for the parameters read:

$$\begin{cases} \varepsilon_t \geq 0 \\ \varepsilon_z \geq 0 \\ \mu_t \geq 0 \\ \mu_z \geq 0 \end{cases} \qquad \begin{cases} \xi_z^2 \leq \varepsilon_z \mu_z \\ g^2 + \xi_t^2 \leq \varepsilon_t \mu_t \end{cases}$$

These conditions restrict the magnitude of the magnetoelectric parameter in z-direction, ξ_z, by the axial refractive index. Similarly, the transversal magnetoelectric part, ξ_t, and its antisymmetric component, g, are limited by the transversal refractive index.

4 Conclusions and discussion

The aim of the present paper has been to find what restrictions dissipation implies on the optical response of the medium. Time domain analysis used in the paper requires that the constitutive relations of the medium are modeled by four convolution operators and four instantaneous responses which all are dyadics in general.

The major result is that the optical response six-dyadic has to be symmetric and positive semi-definite. This implies that the four optical response dyadics cannot be arbitrary real numbers but instead of 36 components they only can contain 21. Also some of the components are limited in magnitude.

The permittivity and permeability parts of the optical response have to be symmetric dyadics, in other words they have to be reciprocal. Therefore, electric and magnetic gyrotropic effects are forbidden in the optical response. The magnetoelectric dyadics contain both symmetric and antisymmetric parts but the symmetric and antisymmetric parts of $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ are closely related. One consequence of this paper is that no reciprocal magnetoelectric effects of the optical response can exist but non-reciprocal ones are allowed. In other words, the symmetry of the six-dyadic is not a reciprocity condition, although in frequency-domain studies, the reciprocity requirement is known to limit the number of bi-anisotropic material parameters from 36 to 21, too.

A well-known reciprocal magnetoelectric effect is optical activity. Chiral materials display optical activity which means that the polarization plane of a propagating electromagnetic wave rotates. Although "gyrotropy" is not so much used in the Western literature for chiral materials, this term very often comprises chiral effects in Russian physics textbooks, well evidenced by the title "Theory of Gyrotropy" of the main authority of chiral electromagnetics in the Former Soviet Union [3]. Therefore with certain rationality one might call also the magnetoelectric requirements resulting from dissipation as a non-gyrotropy condition, both for $\boldsymbol{\varepsilon}, \boldsymbol{\mu}$ and $\boldsymbol{\xi}, \boldsymbol{\zeta}$.

It is interesting to compare the present result excluding gyrotropic effects in the optical response with the results in frequency-domain. There, gyrotropic effects are accepted for fields oscillating with microwave frequencies. As a matter of fact, the frequency-domain treatment looks more reluctantly at gyrotropy for low frequencies. It is pointed out [1] that chiral effects do not exist for static fields, and the frequency-domain chirality parameter is an odd function of frequency which makes this parameter zero in statics. The importance of dispersion in real media is here obvious. Without any dispersion the frequency-domain susceptibility dyadics are independent of frequency and identical to the optical response. Any lack of optical activity for static fields imply that this effect also vanishes in the optical response.

Regarding the limitations of the magnitudes of material parameters, some results exist in the frequency-domain literature. For the magnetoelectric dyadics of bi-anisotropic materials, conditions regarding the positive definiteness have been given in Ref. 9. These look similar to the relations (3.4)–(3.6) although in Ref. 9 the conditions are written for the frequency-domain constitutive dyadics, not only for the optical response dyadics. Consequently, in this reference no results appear that would forbid gyrotropic effects in the optical response, which is a major difference to the results of the present paper.

Finally it is worth emphasizing the importance of the equivalence between various representations of constitutive relations. The relations of the form (2.2) are general enough to represent electric and magnetic conductivity effects, as has been shown above. Furthermore, although these relations give the flux densities $\boldsymbol{D}$ and $\boldsymbol{B}$ as functions of the field vectors $\boldsymbol{E}$ and $\boldsymbol{H}$, they are completely equivalent with the constitutive relations where the "primary fields" $\boldsymbol{E}$ and $\boldsymbol{B}$ are considered as causes for the mathematical induction fields $\boldsymbol{D}$ and $\boldsymbol{H}$. A one-to-one correspondence exists between the material dyadic functions that appear in these relations.

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