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Flexibility in knowing school mathematics

In the contexts of a Swedish and an Indian school class

Dash, Ingrid

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LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

Flexibility in knowing school mathematics
in the contexts of a Swedish and an Indian school class

Ingrid Dash

AKADEMISK AVHANDLING

som för avläggande av filosofie doktorsexamen vid
samhällsvetenskapliga fakulteten, Lunds universitet,
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Abstract

A central question in mathematics education research concerns understanding. The main objective of the present thesis has been to obtain insights into flexible modes of knowing in school mathematics in two school class contexts, and how these relate to modes of being a learner in these contexts, with specific focus on learners' flexible ways of discerning parts and delimiting wholes, and how they understand part- and whole-relationships while doing mathematics. The theoretical exploration of knowing school mathematics was informed by perspectives from phenomenography and variation theory, as well as constructionist theoretical standpoints. Empirical material was collected from a school class in Southern Sweden and a school class in Orissa, Central-Eastern India. The meaning the learners expressed during interviews and observations, verbally or with the help of mathematics, was analysed using contextual analysis. In line with methods in phenomenographic research, the main results of the thesis are different categories of description. Three modes of knowing emerged from the empirical material. These were: associative flexible experiencing; compositional flexible experiencing and contextual flexible experiencing.

These modes of knowing feature distinct differences: in the depth of understanding mathematics, in how learners use variation when dealing with an object of knowledge, and in learner identity. The associative mode of knowing involved the learner in arbitrary ways of making sense of the material s/he was working with, with a focus on arbitrarily discerned aspects in chains of associations. The compositional mode of knowing meant that the learner made an effort to understand, keeping a focus on compositions, such as number-relations or formulas. Finally, the contextual mode of knowing engaged the learners in ways of understanding the context from which critical aspects were to be discerned. The contexts gave meanings to the content. The knowledge about the context, mathematical and also reality-based, gave meaning to the theoretical constructs. The logic of the mathematical context and content was understood in a more differentiated way than within the two other modes of knowing. In all parts of empirical material, the compositional flexible mode of knowing predominated. The dominant mode of being a learner in the Swedish school class context was simultaneously independent and collaborative, as well as creative and productive. In the Indian school class context, the dominating mode of being a learner was autonomous and committed.

A major finding is that in mathematics education there is a need to give pupils tasks containing possibilities both for experiencing variation and for

authorship. This also demands of the teacher to observe and evaluate the individual pupil's understanding and use of the possibilities offered.

Keywords: Mathematics education, compulsory school, Sweden, Orissa, India, flexibility in knowing, modes of knowing, authorship, agency, phenomenography, contextual analysis, intercultural perspective.

To Ipsa

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Lund, December 2008
Ingrid Dash

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DOKTORSAVHANDLINGAR FRÅN PEDAGOGISKA INSTITUTIONEN, LUNDS UNIVERSITET FR.O.M
2000 222

Prelude

I nod my head, I smile. I am wearing a sari. I have come to this remote village in Orissa, a state situated in the central eastern part of India, as a daughter-in-law from far away Sweden. Many, many people come to 'see' me. We meet in smiles, laughter as well as in facial expressions of shared bad news, suffering, illness, I don't know what. I appreciate their coming, I feel compassion for them, I feel theirs. Some have come barefoot over the rough roads, walking up to fifteen kilometres, just to see me? My ego bows down to their feet. They have a good talk with me, they say, though I'm not even uttering a word. I greet them when they come and also when they finally say: *I am coming*. Why do they all say that they are coming, when they leave?

This was nearly twenty years ago. My interactions with people in the village and nearby town have led me to a better understanding of life. No school could have given me better language tutoring than my mother-in-law, when she used to sit on the veranda with me, and talk about insects, birds, animals, trees and flowers. Nowadays I have a more complex picture of what, for instance, concepts like privacy, work or hospitality and poverty mean in different cultures, and in various contexts within the cultures. I have revised my own heartfelt understanding of different concepts.

I listened to many people's talk. I interacted, with the use of more and more words. I learnt the cultural use of language. I later was told that *I am going* rarely is used. We only 'go' once in this life. I learned how to read and write Oriya through listening to people's conversations and asking my husband for spelling, pronunciation and meanings, again and again.

My husband and I once went to a village which is situated in an isolated part of Orissa. Walking down the road, people came to us and everybody invited us to their home. You cannot say no. I especially remember one family. As soon as we came into their yard, many of them got busy arranging sitting mats and sweeping the veranda. After my husband had talked for some minutes, they said: *See, we are poor, we don't have anything to give you. But we can give you some food*. We said that we had just eaten. Then they all started discussing, and those who had come to see me from the neighbouring homes also took part. *What can we give her?*

In the meantime, a man climbed the nearest coconut tree and came back to the yard with a coconut, stripping and cutting it open. He handed it to me so that I

could drink. Everybody was happy and some were laughing. *See, we had something she could have from us!*

I have many times asked myself the same questions. What happens when we don't share the language, when we cannot express a meaning in words? What do we communicate, what do we know and what do we learn? These questions constitute a background to my research interest and inform my desire to explore knowing, breaking the boundaries of absolutist views of knowledge and learning as independent of who we are, where we are and how we constitute ourselves as learners.

The thesis is about knowing school mathematics. Since 1994, when I started to work as a mathematics and science teacher at schools in Sweden and in India, I have wondered what the pupils *really* understand of what they are working with, and how I can improve the way I deal with the subject matter to enable deep understanding and appreciation of content and meaning. I developed different ways to assess understanding and to encourage the pupils to talk about how they understand mathematical problems, and express what they think about mathematics in general. I was interested about finding out the content of their thinking and how they were doing mathematics.

My experiences of living and being in two cultures have schooled me into taking other people's perspectives, a sensitivity I have carried with me during the work with this thesis. Intercultural interaction has led me to adopt a more critical approach to research methods used in social science, and specifically concerning claims to 'objectivity' and 'neutrality'. It is my hope that by this thesis I have contributed to widening perspectives on knowing in school mathematics, and thus ultimately made a contribution to the development of teaching practice, enriched by intercultural perspectives.

I Introduction

1. Outline of the thesis

The thesis begins with an introduction, Part I, *Introduction*, describing the background to how the research questions came to be formulated, the aim of the study, as well as a brief description of the research context and of mathematics education in the compulsory school systems in Sweden and India.

Part II, *Theoretical Explorations*, describes the focus of the study, flexibility in knowing, as well as presenting certain interpretations of the theoretical assumptions informing the thesis. It deals with implications of taking a learner perspective and an intercultural research approach for studying knowing within research in mathematics education. A learner perspective is pursued in order to gain insight into different modes of knowing. For a long time, the aim of educational research was above all to attempt to predict learning results, and to understand the general nature of learning by arranging test situations and experiments (Svensson 2005). The specific research focus in the present work is not on causal relations between Presage, Process and Product, what Biggs (1996) calls the 3P-model, but instead on how the learner expresses h/er/is understanding. One basic theoretical assumption is that what the learner expresses, reflects h/er/is relation to some part of the reality he/she experiences. Understanding of mathematical content is in other words assumed to be dependent on qualitatively different ways of experiencing part and part-whole relationships.

The design of the empirical investigation is described in Part III, *Empirical Explorations*. Results are then presented in Part IV, *Empirical Results*, which is divided into three sections. The first is entitled *Three modes of knowing* and describes the outcome space for flexibility in knowing. The second is entitled *The Swedish school class context*, while the third is *The Indian school class context*. In these sections, the themes described are Educational discourses, Classroom discourses and Modes of being a learner. Educational discourses and classroom discourses were themes which emerged from analysis of the whole empirical material. As part of conclusions from the two studies, these discourses are related to how the learner constitutes h/er/itself as a learner.

Finally, in Part V *Conclusions and discussion*, results are summarised, and certain important implications for theory, method and practice are discussed.

1.1 Background to the thesis

The work started with an understanding that the theory of conceptual change, which underlies much of the research in mathematics education today, is not sufficient as a theoretical base, and that there is a need to look for new models on learning in mathematics classrooms. Although the research area of mathematics education has long been rooted in mathematics and psychology, writes Burton (1999), new perspectives are pervading it from disciplines and fields as diverse as philosophy, logic, sociology, anthropology, history, women's studies, cognitive science, linguistics, semiotics, hermeneutics, post-structuralism and post-modernism.

As Ernest (1985; 1989) has pointed out, a philosophy of mathematics deals with views on the nature of mathematics, which can strongly influence the mathematics curriculum taught to pupils. Views on the nature of mathematics, in terms of a mathematical belief (Schoenfeld 1989), also influences the way pupils try to understand mathematics. As part of my own learning process as a researcher, I entertained the idea of including *uncertainty* into educational theories of knowing. My aim was to explore, through conversations with the knower, how the relation between knower and the known can be understood as an internal relation. My main assumption is that, when we come to know mathematics, we also learn how to be a learner of mathematics. This reasoning is in line with how Svensson (1984) describes the relation between the person and the world as an internal relation:

If the relation is seen as internal, then it must be understood as simultaneously depending on subject and object (the person and the world). The internal nature of the person's relation to the world can be understood in terms of intentionality (p. 9, *My translation*)

In Part II, *Theoretical explorations*, intentionality and the learner's internal relation to the object of knowledge are explored more deeply. The theoretical explorations of knowing in school mathematics in the present thesis ran parallel to posing specific research questions to be investigated empirically. The general question was how pupils understand school mathematics.

The phenomenon *flexible experiencing* became the research focus. This phenomenon was delimited in two different school class contexts, and parallel to the empirical delimitation and exploration, the theoretical exploration enriched the analysis. This process of gaining a fuller picture of flexible experiencing is in line with the use of phenomenography as a research tool

(Svensson 1997), and is consistent with the perspectives and ideas of the research design adopted.

When the *participatory aspect* is included as a fundamental element of knowing the question of objective reality and subjective reality becomes crucial. Knowing mathematics becomes an open concept (Skovsmose 1994). But Skovsmose points out the difficulties of ‘opening’ the concept - how it even becomes an *explosive* concept, when it is no longer limited to being a part of a cognitive theoretical framework:

But, as soon as negotiation and dialogue enter epistemology, knowing gets out of control. To think in terms of a dialogical epistemology means that knowing can no longer be thought of as an imprecise concept with an open texture. Even this would be to assume too much about the precision of the concept. Because negotiation implies an interpersonal relationship, knowing becomes related not only to uncertainty and to prejudice but also to power (pp. 206-207).

One of the problems that has to be dealt with in opening knowing to dialogue and negotiation is that there follows a closure. As a consequence, Kuhn (1979) has observed, both opening and closure have to be continuously discussed in research on mathematics education.

Knowing school mathematics is in the present thesis also considered in relation to a philosophy of mathematics (Ernest 1989). When mathematics is no longer considered a universal map of reality, a space for reflection is opened on objectivity, subjectivity, as well as object and subject in relation to mathematics learning. The essence of mathematics is questioned, destabilising the centre of communication and reflection in mathematics education.

The empirical investigation was designed on the basis of theoretical assumptions that mathematics is a *language*, like our verbal language. Mathematics is considered as a language, in the sense that within the language there are different narratives with socio-cultural roots, which through a range of human interactions in history have been formulated into varying models of reality. Some of the mathematical models that have been formulated are ‘purely’ mathematical, and do not have any clear significance in relation to reality. In other words, mathematics both includes representations of reality, reflecting the various ways we know it, and a specifically mathematical universe, which only exists in the imagination of mathematicians.

In the present thesis, school mathematics is the research focus. Although my personal assumptions on mathematics must have influenced the analysis of the empirical material to some extent, my aim was to understand the learner's own assumptions regarding the ontology and epistemology of mathematics. During the interviews, I have shifted the centre of communication as much as possible to the learner, taking a learner-perspective. My assumption is that during the dialogue between the learner and the teacher/researcher, meanings are constituted, depending on *the learner's understanding* of content, on the one hand, and informed by his/her *mode of being a learner*, on the other. However, it is important to point out that it is not within the scope of the thesis to include factors such as ethnicity, social class, gender, and cultural or family background in the analysis, all of which clearly influence teachers' and pupils' interpretations of and positions in the Indian and Swedish school system.

Instead, the question I choose to explore further in Part II, *Theoretical Explorations*, is the relation between flexibility in knowing and contexts. An underlying consideration is that experiences and discourses influence learning approaches in any field of learning. Content and process of knowing are here understood as a relation, and are seen as each other's figure and background. I further assume that knowing is in a constant state of flux, and that there is a proximal zone of development (Vygotskij 1975, 1978). To some degree I share the socio-cultural view that mathematics are socio-cultural artefacts, which are learned in a *context of interaction*. I assume, however, that there is an *agentive participation* in the process of learning, and that the types of learning that can take place are affected by *what the learner experiences to be the object of knowledge* (Marton, Runesson & Tsui 2004).

The most easily observed form of communication in mathematics classrooms is through language, either spoken or written. Alvesson & Kärreman (2004) distinguish three levels at which language use can be studied, namely: the level of expression, the level of meaning and the level of practice. In the present thesis, I have chosen to explore all three levels in the empirical material. The aspect of uncertainty which I would like to include in theories on knowing, and which is explored here, is above all related to the *flexibility* in how pupils experience parts and part-whole relationships in knowledge content, and how they delimit parts and wholes. This flexibility is a quality in learning which is important for the way learners understand mathematics at school. Flexibility in knowing school mathematics is seen in the light of learners' experiences in contexts of a Swedish and an Indian school class.

1.2 The meaning of comparisons in the thesis

While conducting the studies at two different locations of our world, one in India and one in Sweden, I got involved in certain questions of principal concerning comparisons across cultures and contexts. These were: *What comparisons are usually made in comparative education?* and *What comparisons are possible in the present study?*

Implicit issues exist in comparative education research, concerning *what* comparisons are possible, *how* these comparisons can be made, and even *why* one should make comparisons. Different answers can be given, depending on, among other things, how close or how distant to the context of the participant the studies are expected to be conducted. Comparative education research has increasingly gained a socio-cultural focus, including both smaller in-context studies, and international studies conducted by the IEA (International Association for the Evaluation of Educational Achievement). In this perspective, the challenge of comparative education research is, according to Broadfoot (Alexander, Broadfoot & Phillips 1999), to understand how variables are influenced by certain cultural features. He writes:

Comparative education is definitely not travelers' tales nor the basis for unsystematic policy-borrowing. Neither is it descriptive accounts of what is however carefully done. It is not de-contextualized comparisons of particular dimensions such as educational achievements. These lack the theoretical framework that is essential to justifying the drawing of any conclusions from the data gathered (p. 29).

The problems involved in making comparisons cross-culturally becomes evident when we turn our attention to assumptions on cultures that are taken for granted, as well as when we look at the concept 'culture', which in itself may have dubious meanings within various theoretical frameworks. We have to consider how views on different cultures often are linked together in 'mirror-images', or what Davies (2000) calls dualistic binary hierarchies, based on dualistic values and propositional logic, which determine the hierarchy of the 'mirror images'. Said (1978) tried to understand the way West had depicted the East in a mystified and mythical way, when he wrote his famous account of Orientalism. But he did not go beyond the modern discourse of culture as singular entities that can be understood by a rational mind, as Sen (2005) has remarked. In the same vein, Mohanty (2000, p. 99) stresses that 'A culture is not a self-enclosed world. It is not an identity, but a system of differences held together by history'.

In the context of what during colonial times in India usually characterized comparative research methods used in cross-cultural studies, Sen has described three different Western research interests. The *magisterial* interest was the interest of the colonial British with respect to the culture of India, the main contribution stemming from authors who had never been to India. The purpose of the literature deriving from this interest was to depict Indian culture as inferior and 'backward' compared to European culture. The *curatorial* interest was the missionaries' view on India, who with their literature contributed to a picture of Indian culture closer to the natives' views, but still had a Eurocentric focus. The *exoticist* approach to India has propagated the view that India mainly represents a 'spiritual' interest, a view which has influenced both India's own image and the West's image of India. It therefore seems to me that it is important to ask oneself as a researcher: What research interest has led me to make comparisons in the way I do?

Harding (1998) argues that Eurocentric perspectives represented in mathematics education research share what is known as the 'internalistic' view on science, where 'objectivity' is ensured by scientific processes, and not viewed as a product of history, economy, and socio-cultural exchanges. It is assumed, for instance, that a specific range of scientific processes are especially suited to discovering nature's order, using mathematics to describe it (Davis & Hersch 1988). As the present study progressed, an attempt was made to go beyond a number of assumptions on the nature of school mathematics, on learning school mathematics and on teaching the subject, which ultimately rest on such absolutist and Eurocentric ideas.

To address the question *What comparisons are usually made in comparative education?* a point of departure has been the discussion by Biggs (1996), in the book *The Chinese Learner*. In mathematics education, research assumptions on what characterises 'good learning environments' are often taken for granted. The researchers point at challenges to Western research with regard to student learning, and among other examples discuss the assumptions that 'good learning' is an outcome of a 'good learning environment', presenting the following characteristics (Biggs 1996):

- Teaching methods are varied
- Content is presented in a meaningful context
- Warm classroom climate
- High cognitive level outcomes are expected and addressed in assessment
- Assessment is classroom-based and conducted in a non-threatening environment

(p. 45)

Biggs specifically discusses the 3P- model (p. 51), where the three components Presage, Process and Product often are the focus of research into learning. This type of research interest is often based on assumptions that there are stable relationships between teaching context, student learning processes and learning outcomes, and that through establishing these relationships it is possible to predict the outcome of learning, based on knowledge about the learning environment.

Biggs points out that when Western researchers observe Chinese learning environments, none of the characteristics of a 'good' learning environment seems to be present, although Chinese students have significantly higher levels of achievement than those of Western students. Chinese students regard themselves as deep-learners, in the sense of dealing with tasks meaningfully. The author concludes that the Western misperceptions of, or problems with understanding Asian learning environments, demand a culturally modified research model.

Other assumptions on learning relate to particular theoretical frameworks and specific views on the nature of knowledge and human development. These include assumptions on children's incompetence and competence in mentalist theories on knowing, that still dominate much of mathematics education research. Walkerdine (1988) observed that correct accomplishment in mathematics learning is by most teachers considered to be the result of cognitive development. The child's behaviour is to be observed by the teacher. Moreover, as the discourse on 'child-centred' education inscribes the child as 'the bedrock of practice' (p. 205), certain distinct features of the child's behaviour tend to be taken for granted in formal learning. Correct behaviour implies that the child has grasped the subject matter. It is further assumed that the object of knowledge is to be treated in a formal setting, and the child should understand the difference between play and work. The implications of such discourses are for mathematics learners that 'mathematics becomes cognitive development and cognitive development becomes a description of a child' (p. 205).

Consequently, the child whose behaviour does not correspond to the norm is considered not to be thinking rationally. Another conclusion that is frequently taken for granted is that teacher-dominated blackboard teaching is less learning-oriented, and to a lesser degree enhances understanding of content than so-called 'progressive teaching'. Alexander (2001) provides an example of

how this kind of assumption can miss out on cultural aspects in classroom discourse. In a case study on primary education in India, Alexander draws some conclusions concerning the discourse, based on empirical material from Indian primary schools:

Yet although in her demonstration lesson this teacher indeed does not harangue or lecture her pupils but seeks to engage their attention and involve some of them in the demonstrations, and although the two-move exchange of the previous sequences is partly replaced by exchanges including feedback on responses, the traditional catechistic discourse retains its hold, just as the English variant did throughout the Plowdenite ascendancy. Questioning remains largely closed, key words are chanted, and at exactly those points which offer greatest potential for questioning of a more exploratory kind, it is replaced by exposition. In these utterances the teacher does not direct the questions to her pupils but answers them herself ('How can I get the sand out of the liquid? I can get the sand out by...') (pp. 449-450).

The interpretation is based on observation material from a sequence where the teacher has demonstrated, together with a pupil, the fact that some substances change or resist change in size and/or shape. A pupil dissolved sugar in water, but could not dissolve sand in water. The teacher's demonstration is followed by giving the pupils the assignment to do the same thing at home using simple materials. They come back next day and tell the teacher about it.

We can see that, on an organisational level, as indicated by verbal expressions and actions in the sequence, if we choose to use Alexander's terminology, the learning discourse can certainly be seen as an example of 'catechistic' discourse. However, Alexander's analysis is based on data which has not been considered in its context from where it was derived, which is very common in comparative education research. Taking an intercultural perspective, we can try to reach beyond and beneath the intentions of participation in learning that appear to be *encouraged by teachers*. We can assume that pupils have to take an active stand in their reflections on their own learning, and that learning means not just 'receiving information' but also includes reflections. Consequently, learning has to be *performed by the learner* (Skovsmose 1994).

Performance, in turn, raises questions related to how the learners understand themselves as learners. What intentions do they bring into the classroom? Do

these correspond with the teacher's intentions and in what way? A number of studies have indicated different ways the learners' modes of being a learner can impact on the learning situation. Davies (1983) has shown us how she found that learners to a high degree contribute to the sense of order in the classroom. She saw that pupils act and talk in a number of ways to maintain the work in classrooms, so that the intentions of the teacher can be maintained and/or negotiated upon.

To move beyond a description of teaching practices interpreted through my own structures of relevance, to include the learners' perceptions and intentions, I used phenomenography as a research tool (Svensson 1997). This involves a second-order perspective aiming at describing different conceptions of aspects of reality (Marton 1981). The second-order perspective draws our attention to *the meaning the learner invests into content*, which does not necessarily correspond to the theoretical meaning of the content. In the present study, the meanings emerging from data collection (interviews, observations and secondary materials) were found to depend on both the learners' and the teachers' experiences of the content, as well as on their respective positioning in relation to educational and classroom discourses.

The overall aim of the research was to explore flexibility in knowing and the nature of this *flexibility* in its relation to context. Since I do not view mathematics, knowledge and learning as independent of the participants or the context, it is not possible here to make generalised universal comparisons, based on quantitative comparisons of performance using standardised variables. Not only would such comparisons be difficult, but even qualitative characteristics need to be understood in their own structures of relevance. Since the meanings of content, intentions and participation in the field of formal learning represent an *experienced* reality, comparisons need to go beyond a simple observation of behaviour, to explore the meaning experienced by the individuals involved. This does not mean that the results lack relevance for other contexts, but only that any conclusions based on these results need to be tested, re-negotiated and re-interpreted within the particular context they are applied to.

As a concluding introductory remark, I have to point out, however, that I have neither used theory as a deductive system, nor do my studies in the two school classes constitute representative cases. The classes were selected to provide examples of committed and didactically conscious teaching practice, rather than more 'typical' conditions.

1.3 A general description of mathematics education in Sweden and India

Sweden, a highly technological society, and India, an emerging technological society, share the interest to increase the mathematical skills and knowledge of the coming generation. The countries meet different challenges at a societal level, which can be translated into pedagogic challenges in the field of education. Mathematics is considered to implicitly have a democratic value, and mathematical skills and competences are often attributed with power (Skovsmose 1994). But the possession of mathematical skills and competences does not, as many researchers in mathematics education research have recognised, imply that the pupils actually get the tools required for practicing democracy or gain empowerment. In both countries, important pedagogical questions can be posed in relation to how to deal with mathematics, and how to stage the learning situations so that pupils can democratically engage in a critique concerning the role of mathematics and technology in society, and explore narratives of mathematics which enable the pupils to be owners of their knowledge (Skovsmose 1994).

In Sweden, the public school system is based on compulsory schooling, which includes regular compulsory school, Sami school, special school, and programmes for pupils with learning disabilities. All education throughout the public school system is free. The curriculum, national objectives and guidelines are laid down by the Swedish Parliament and government. Within the established objectives and framework, the individual municipality may determine how its schools are run. The municipalities are obliged to provide a place in a preschool class for all children beginning the autumn term of the year the child turns 6. The preschool class curriculum is included in the 9-year compulsory school curriculum (LpO 94). The 9-year compulsory school programme is for all children ages 7-16. In compulsory school grades 1-9, the pupils are normally guaranteed at least 900 hours of mathematics education (NCM 2004, p. 18).

The NCM, National Centre of Mathematics education in Sweden (2001), concludes that in Sweden, one of the most important didactical questions concerns how to make pupils understand basic mathematics:

How can we make mathematics education so interesting and challenging that these groups of youths achieve the goals and learn basic mathematics for citizenship and working life? (p.23).

As many as 22.8% of the class 9 pupils who finished compulsory school in year 2004/5 (National Agency of Education in Sweden 2006) failed in one or many subjects. 90.7% of the girls and 87.2% of the boys passed the core subjects mathematics, Swedish and English (National Agency of Education in Sweden 2005). A recent international evaluation study performed by PISA (2003) showed that Swedish pupils are better than the average pupil in the investigated countries in solving routine tasks, which demand application of basic knowledge, and can interpret and use mathematics in known contexts. However, the Swedish pupils achieve less than average on mathematical tasks which demand analysis, reflection, communication and argumentation. In the national evaluation (NU03) of pupils' mathematical skills, the National Agency of Education in Sweden observed similar trends. The proportion of low performing pupils has increased, whilst there has been a decrease in the proportion of high performing pupils. Via data from the national test results in mathematics in year 9 for the period 2001-2003, between 80 and 90 percent of the pupils were assessed as attaining each of the stipulated goals. Based on teachers' estimates of goal attainment in year 5, about 70 percent of the pupils were assessed as having fulfilled all the goals, and at least 80 percent have attained at least one of the goals (NU 03,p. 35). At the same time, there is a steady increase, compared to evaluations 1992 and 1995, in individualised teaching, where pupils work in isolation from the teacher and their class friends.

From January 2003 to September 2004, the delegation for mathematics (*Matematikdelegationen*), funded by the Swedish Government, investigated the situation in mathematics education. In its final report (SOU 2004:97) it was concluded that pupils generally develop poor mathematics skills in learning contexts where the teachers let the pupils sit and work with word problems in silence, which is a trend in mathematics education in Sweden. So called 'silent problem solving' (*tyst räkning*), is a response to the increased individualisation of learning at school. Teachers give additional tasks of the same kind to the better achievers, but not necessarily with increasing levels of complexity. In general, mathematics teachers tend to teach in the way they believe that mathematics is taught traditionally, with the foremost aim of preparing the pupils for examination. Teachers often have responded to the goals concerning the *pupil's own search for knowledge* stipulated in the national curriculum of 1994 (LpO 94), by letting the pupils do their tasks individually and in silence (SOU 2004:97, p. 142). This has led to an even greater loss of interest in mathematics. *Matematikdelegationen* concludes that teachers to a greater extent *supervise* pupils and to a lesser extent *give instructions*, which has affected mathematics education adversely, as it is tradition-bound and teachers lack relevant education (SOU 2004:97, p.57). Other important conclusions are that mathematics need to be

popularised through different community activities and media presentations (p. 109). Teachers should be given possibilities to study mathematics didactics.

As pointed out in the report *Lusten att lära: med fokus på matematik* (Wanting to Learn: focus on mathematics) written by the Swedish National Agency of Education (Skolverket 2003), teachers in general do not take the opportunity to interpret the goals of education into varied ways of teaching. The report also shows that teachers in general tend to follow the textbook examples and relate to mathematics as it is presented in textbooks (Skolverket 2003, p. 44). Mathematics at school often becomes a dry subject. Pupils spend hours studying meaningless examples that can be solved with simple arithmetic.

One important issue in mathematics education is concerned with the communication of mathematical objects of knowledge. Wistedt (1994) has explored how the pupils' intentions and the teachers' intentions concerning communication and reflection in mathematics give rise to different responses in relation to given tasks. She found in a case study, where four pupils and their teacher were observed during a session of practical exercises, that the reflections which the pupils made were not always mathematically significant, and not always relevant to the didactical aims. The pupils were given the task to use wooden cubes to 'build small numbers'. The communication included five rules which the pupils were supposed to understand in order to understand the task. These rules implied that the blocks were seen as:

1. Illustration of numbers
2. That the number should have the form of a rectangle
3. Representing two dimensions, without consideration of the volume
4. Representing factors by the number of rows and the number of blocks in each row
5. An exploration of small numbers, both as factors and products

Wistedt argues that concrete material should be understood in relation to some implicit learning goal. In this case, the assumption that the blocks would represent numbers is often taken for granted in mathematics classrooms and in other learning situations at school. The learning goals are not explicit and therefore they become difficult to discover for some pupils. Wistedt saw that the primary difficulty lies in that:

The didactical rules are temporary, introduced for the sole purpose of helping the pupils to bridge up the gap between a familiar context and a context not yet constructed (p.135).

Wistedt concluded that in the sessions she observed, three children, Alex, Ellie and John, were making productive use of the teacher's rules. Given the task and the rules, the pupils creatively directed themselves towards the intended content, and modified and refuted the rules. Ellie suggested that the blocks can be divided, and made a drawing of number four as three cubes and 'a little more'. She not only introduced a new rule, the pupils also shifted their attention from aspects of multiplication to numbers and infinity. Ellie tried to explain her ideas to the two boys both from a personal and a common reference frame. She made a drawing illustrating thirds, then she described thirds in decimal form, and finally as an invented algorithmic solution. The rule 'decimal fractions can be used to describe parts of a block' is introduced by the three children. Even though Alex and John did not understand Ellie's detailed explanations, they understood the significance of the ideas and cooperated with Ellie. Perhaps they did not understand the decimal system in the same way as Ellie:

Ellie is using the decimal system to structure the problem, and this doesn't seem to work for Alex, nor for John, maybe because they view the decimal system as a way of describing findings, a language tool rather than a system, the principles of which can be used as cognitive instruments. There are limits to communication (p. 137).

The fourth pupil, Tom, had problems to understand the rules given by the teacher, while the teacher saw the rules as obvious, and not requiring further comments. The communication between the teacher and Tom failed. When Tom asked 'Shall I put the blocks on top of each other?', he saw the blocks in an everyday context. Tom referred to the use of blocks during play. In response to Tom's question, the teacher asked questions which reflect rule number 1, that the blocks illustrate numbers, e.g., 'What number do you see?'. At other times, the teacher asked questions which demanded of Tom to understand all the implicit rules of the exercise, e.g., 'How do you build numbers?'. The one-way communication of telling Tom what to do directed the pupil's attention towards formal matters, 'such as how to build or how and where to write things' (p. 137). The communication in the classroom involved the pupils in the interpretation of rules, which were not explicit. Three pupils understood the rules, and Ellie used the understanding to modify the rules. Tom, on the other hand did not understand the rules, and the teacher failed to understand what problems Tom saw with the rules.

Bunar (2001) found that schools in Sweden are becoming increasingly multicultural, in the sense of that pupils come from different ethnic backgrounds. But, at the same time, since children from non-academic backgrounds tend to go to the schools in the vicinity of their homes, social segregation has increased. Especially children from non-academic homes tend not to continue further study in courses where mathematics knowledge is required, and this applies to an even greater extent to girls. Although 98 % of all pupils apply to upper secondary school, as much as one third of these enter an upper secondary school programme which does not demand in-depth knowledge in mathematics and which also does not give further depth to the knowledge in mathematics (NCM 2001, p. 23). Currently, a national investigation has resulted in a 1000-pages long proposal for a reformed upper secondary school.

India is also facing many pedagogical challenges in mathematics education. In India, the public school system is broadly structured the same across the entire country, which consists of 25 states and seven union territories. The education budget is not national, but instead a state government responsibility, with the result that there is considerable variation in expenditure on education across different states. Free and compulsory education for children aged 6-14 is divided into primary education (usually classes 1-5, or ages 5-10+), and upper primary or middle stages (usually classes 6-8 or ages 10+-13+). Secondary education is class 9-11 (or ages 13+-16+).

The empirical study for this thesis was conducted at a school in a provincial town, Balasore in the state of Orissa, where people during the past decades increasingly have migrated from surrounding rural areas, and by doing so taken with them some of the elements of rural non-formal education. A limited range of pre-school and nursery provision is available, as well as private schools in the provincial town (Human Development Report 2004).

National educational policies in India emphasize that children who attend ten years' schooling should be equipped with knowledge to function in the society, and vocational training should be offered parallel to theoretical studies. Dyer (2000) writes:

India is now home to the largest non-literate population in the world, and while socio-economic circumstances of non-literate groups have traditionally been blamed for this, it is no longer possible to claim that the quality of India's elementary schools is not a key contributory factor: a recently published report based on

comprehensive field research (PROBE 1999, p. 8) claims that 'the state of elementary education is in dismal condition' (p. 11).

By the 1980s, a major planning objective had been met, for the country had achieved one part of the goal for achieving Universal Elementary Education (UEE), which was first stated in the new nations' Constitution of 1948. Every child now had access to an elementary school within a 'reasonable' distance from their home, writes Dyer. Alexander (2001) describes the situation:

In rural areas, however, the distance to the nearest school still causes frustration and is one of the main factors behind a high rate of dropout after class 5. Considering that 73 percent of the Indian population live in rural areas, the availability of schooling still must be counted as a problem. Especially the enrolment and dropout rates of girls, members of scheduled castes, as well as tribes, are much higher than the overall figures. A total of one-third of the children aged 6-14 are out of school (p. 85).

Operation Blackboard emanated from the 1986 National Policy on Education, and was implemented in the 1990s. The aim was to upgrade physical facilities in elementary schools across the entire country. There would be 'at least two reasonably large rooms that are usable in all weather, and necessary toys, blackboards, maps, charts and other learning material. At least two teachers, one of whom a woman, should work in every school, the number increasing as early as possible to one teacher per class' (NPE 1986, p. 11).

Primary teachers' education was formulated as a priority in the NPE, and a child-centred approach was to be promoted, with activity-based learning, no repetition at the primary stage, and exclusion of physical punishment, writes Dyer. The policy logic underlying *Operation Blackboard* was that improvements in educational quality could not take place without better physical facilities (Dyer 2000, p. 152). However, it turned out that full use was not made of the teaching aids provided to schools, due to the absence of a process of consultation with practicing teachers. The equipment was mostly used for mathematics and science, while Dyer's study shows that there was a general underuse of library books. The teachers were not familiar with the use of interactive games for language learning, and did not exploit fiction for developing listening skills, vocabulary building or pleasure reading in most schools, since they prioritised the prescribed texts. Dyer saw that an extra room

provided a possibility for the teacher to move around in the classrooms so that s/he could follow up on the children's work progress, rather than just presenting the text from the textbook. In the single teacher schools that Dyer studied, she saw that *Operation Blackboard* had not sufficiently acknowledged the significance of the quality of teaching. The additional teacher was not selected on basis of professional competence. Nevertheless, this additional teacher was seen by the class teacher as an essential input for quality, although the additional teacher mainly was engaged in trying to maintain the attention of a larger number of pupils. The majority of the teachers saw their work in terms of their *own teaching*, rather than the children's *learning*, and commonly looked for external explanations when they had difficulties in the classroom.

Operation Blackboard may well have marked the end of an era of centralised and top-down modes of implementation of minimum norms for the Indian education system, states Dyer. The decentralisation of education in the late 1990s, which was one of M K Gandhi's visions for an independent India, has had the effect of making the education system more diversified and far from equivalent, considering issues of educational outcomes and of social justice. There is still a considerable disparity between different schools' standard of infrastructure and teachers' pedagogic skills and awareness. In the field of mathematics education, NCERT (National Centre for Education, Research and Training) suggests improvements of in-service and pre-service teachers' abilities to implement curriculum texts into practice. Issues of making education available for all, and subject matters understandable by all, have been dealt with in the curriculum texts since the National Education Plan of 1986. Sarangapani (2000) puts forward the point that between the 1950s and 1990 the curriculum development in India approached both knowledge and learning from an essentially behaviourist paradigm:

This approach is marked by a minimalist model of the child - essentially in terms of 'previous knowledge' and the ability to respond and show 'responsible behaviour' when motivated, which the teachers then can select and shape' (p. 5).

Sarangapani traces back the concepts and vocabulary to Benjamin Bloom and other American curriculum experts of the 1950s. By the early 1990s, the phrase 'child-centred' reflected the discussions of the debate initiated in the 1970s on 'whether to model the child's learning in behaviourist or constructivist terms' (p. 5). The behaviourist approach is still dominating and implicit to the nature of the MLLs (Minimum Levels of Learning) for classes 1 to 5. For

mathematics, the author states the examples for class 1 : ‘ Count 1-20 using objects and pictures’, ‘Identify zero as the number representing nothing or the absence of objects in a collection’ and ‘Arrange numbers 1-100 in ascending and descending order’ (MLL, pp. 20-21 quoted in Sarangapani 2000). The highly specific and fragmentary learning objectives in terms of ‘observable behaviour’ are to be developed through teacher-led and predefined activities over a period of time.

The high level of de-contextualization in the textbook examples and tasks has also been criticised by Sarangapani, who refers to the National Curriculum Framework 2000:

‘Contextualisation’ is not simply a matter of detail that can be taken care of in textbooks or through pedagogic exercises such as story telling, dramatics and puppetry (p. 5).

Sarangapani argues that the content introduced by textbooks should be possible to comprehend, regardless of student background. The author also means that memorisation contributes to lack of meaning of what children learn in school and is a central problem in the Indian school curriculum.

A report with the title *Learning Without Burden*, was published in 1993 as a result of an investigation, initiated by the Indian Government. One conclusion of the report was that textbook writers’ understanding of children’s world views reflected in textbooks does not correspond to an understanding of learning processes. Learning becomes a burden when mathematics presented at school seems to belong to a culture that has very little to do with the children’s own experiences and context, especially the rural context.

1.4 Research context

At the core of the thesis, stands the problem of re-centring the theoretical research focus on ‘knowing’ in the field of mathematics education. In the Cartesian thought tradition, there are specific points of departure, ‘centres’, from which knowledge, learning and communication are defined. Specifically, most research in mathematics education has been inspired by cognitive psychology. Knowledge has been described as structured in the individual’s schemata, personal theories or mental models. The focus has been on the misfit of the children’s own everyday concepts compared to scientific concepts, mostly focusing and identifying the learners’ weaknesses (Confrey 1990), rather than their competences. Within cognitivist theory, the learner is viewed as an

epistemological being. Von Glaserfeld stresses the importance of examining epistemology (the study of knowledge) rather than ontology (the study of the nature of being) (Confrey 1990, p. 14). Confrey concludes that we have to accept that we are trapped in our own human ways of knowing, and that we seek a 'fit' rather than a 'match' in our conceptual structures. As I see it, Confrey means that the cognitive units should 'fit' into schemata, in order to establish concepts. Whenever cognitive units do not 'fit', they have to be modified, or else another cognitive unit has to be distinguished. This can sometimes give rise to 'trial and error' strategies, or what has also been discussed in terms of *computational* skills, e.g. in addition and subtraction, a unit is added to or subtracted from another unit. *Flexible thinking*, as discussed later in the present thesis, corresponds to being able to shift one's attention between different cognitive units in order to understand and be able to use a *concept* (Crowley 2000).

In phenomenography, experiences of wholes are seen as *conceptions*. The experiences are based on thinking about the world as it appears to the person. The flexibility in knowing lies in how well the parts discerned by the learner are experienced to be a 'match', what in-coherences can be observed, and how the variation in a material is experienced and used. Booth, Wistedt, Halldén, Martinsson and Marton (1999) draw the interesting conclusion that:

Encountering variation of one sort or another can bring a person to see new dimensions of potential in a phenomenon that were previously taken for granted, and this spying new aspects of a phenomenon is fundamental to learning (p. 73).

In research on preconceptions and misconceptions (Confrey 1990), a distinction is made between *meaningful* learning and rote learning. Rejecting the verbatim repetition of definitions and the algorithmic reproduction of procedures, Piaget specified the conditions for meaningful learning, including appropriate materials and the disposition of the student to relate old and new ideas, as well as preconceptions which allow the learner to act on this disposition. However, Piaget's theory on children's conceptions of mathematics has received critique, arguing that the heterogeneity of children's thinking structures is not considered, and that we cannot state that cross-cultural studies are comparisons of different conceptions of the same reality (cf Matusov & Hayes 2000 and Walkerdine & Sinha 1978).

The deep and surface approach to learning (Marton & Säljö 1976) is discussed by Biggs (1996) in relation to Chinese learners. His main concern is that the approaches are depending on context, the task and the way the learner understands both (p. 53).

Within both phenomenography and variation theory, a central and general research interest is concerned with how learners understand knowledge objects. In the theoretical foundations of phenomenography, there is a specific research interest in *conceptions*, which are empirically explored, while in variation theory the specific research interest is in *variation* as an aspect of teaching and learning. Although originating from the same general perspective, the specific research interests have developed into different fields of research.

As we have seen, one of the centres of interest has been conceptions. Conceptions are understandings of objects of knowledge as wholes. In other words, the content of conceptions are objects of knowledge as experienced by the knower. Descriptions of conceptions are mostly obtained from interview transcripts, and are categorised on a collective level. Anderberg (1999) makes the following precisions:

The concern for development of knowledge in education is not always a concern for the development of understanding of objects of knowledge. However, in most educational contexts there is an agreement that better understanding of objects of knowledge dealt with is aimed at. The development aimed at is a development of the understanding of students in a direction described in terms such as from more superficial to deeper understanding or from more fragmentary or piecemeal understanding to deeper more integrated, holistic and comprehensive understanding. (article I, p. 8)

Svensson (1997) underlines that the variation in understanding is related to language and culture. With respect to the issues we are concerned with here, the question is how the process and content of understanding can be described. The theoretical assumption I make is that there is a *flexibility*, both in how the knower discerns parts and delimits wholes, and in how the knower understands the relation between parts and wholes. Flexibility in knowing depends on how the knower flexibly experiences the content of the knowledge object. It is

expressed in the use of language, and in the present thesis an attempt is made to understand this flexibility in two school class contexts, and in terms of how the pupils who participated in the study constitute themselves as learners. The specific aspects of learner identity observed are *agency* and *authorship* (Davies 2000; Burton 2004a), which will be more extensively discussed and described in Part II, *Theoretical Explorations*.

1.4.1 Flexible thinking

The thesis is, thus, concerned with an exploration of a particular quality in knowing and learning, involving two essential aspects: the *flexibility* in how one understands parts and whole relationships while solving mathematical problems at school, and flexibility in learner identity. Flexibility in knowing and flexibility in learner identity are both described, based on empirical results in Part IV, and is also explored more in depth in Part II, *Theoretical Explorations*.

Flexibility in thinking has in earlier research been considered mainly from a cognitive perspective. According to this perspective, flexibility depends on shifts in mental images. My intention here is to present the concept *flexible thinking*, defined and used within a cognitive theoretical framework, as a background to the focus of the thesis on *flexible experiencing*. In cognitive theory, a distinction is made between knowing *of* something, the knowledge content, and knowing *with*, the process of coming to know, including learning styles and skills, motivational attitudes, beliefs and affect, which have been widely researched in research in mathematics education (Schoenfeld 1989). The process of coming to know is assumed to be a purely mental process, where a theoretical concept starts its life in the mind of the learner as a metaphor which has to undergo a process of transformation (Bruner 1986).

The cognitive perspective on flexible thinking is frequently connected to mental images. Gray and Tall (1991) derive their theoretical framework from Piaget's works, and identify flexible thinking as a feature of mathematical thinking. *Procepts*, which Gray and Tall define as 'mental objects', are combinations of process and concept, produced by the process. According to Gray and Tall, the initial learning process is not concerned with learning definitions or concepts through visual perceptions, but a matter of dealing with mathematical symbolism as a process and the underlying mathematical concept. Here we need to observe that the process mentioned is a 'mental' process. Gray and Tall (1991) give a number of examples of procepts:

- 1.The process of counting and the concept of number. Number 7 is for instance both a process of counting up to the number 7 and the number produced by that counting.
- 2.The process of ‘counting on’ and ‘counting all’ and the concept of addition. For instance in $5 + 4$, where the process of adding units and the result 9 is part of the same procept.
- 3.The process of repeated addition and its product in multiplication.
- 4.The process of division of whole numbers and the concept of fraction (p. 2)

David and da Penha Lopez (2005) argue that pupils who are able to think flexibly with the mathematical symbols are also successful in mathematics. Their conclusions are drawn against a background of studies conducted by David & Macahado (1996), and David, Machado, & Moren (1992), where it was found from the analysis of students’ errors that approaches adopted by teachers can contribute to errors and failures. For example, excessive training in routine procedures sometimes leads the students to practice false rules. David and da Penha Lopez use Grey and Tall’s concept of flexible thinking. The authors suggest that flexible thinking can be included in what they term *modes of cognition*, which also includes modelling, optimisation, symbolism, inference, logical analysis, and abstraction. Further, they consider modes of cognition and thinking, related to beliefs, attitudes and affect, to be important aspects of *mathematical thinking*. Referring to Schoenfeld (1992, p. 362) they mention these aspects as contributing to the development of mathematical problem solving, and the inclination or disposition to ‘act like mathematicians’.

Crowley and Tall (1999) have also developed the notion of *flexible thinking*. They used the web metaphor of ‘cognitive units’, pieces of cognitive structure that can be held in the focus of attention all at one time, as forming the nodes of a cognitive structure linked to other units. Crowley and Tall write:

The notion of whether a link is ‘internal’ within a unit, or ‘external’ between units, is largely a matter of personal choice. The actual connections within the brain are not topologically divided into an inside and an outside (p. 2).

Crowley and Tall (1999, p. 2) advance that the metaphor of internal links can be useful in seeing separate ideas of, for instance an equation and its

corresponding graphical representation as a line, as different aspects of a single 'entity', that is itself a node in a larger network. They take the example of the equation $y = 3x + 5$ to illustrate how units are linked. The following units are discussed:

The equation $y = 3x + 5$

The equation $3x - y = -5$

The equation $y - 8 = 3(x-1)$

The graph of $y = 3x + 5$ as a line

The line through (0.5) with slope 3

The line through the points (1.8), (0.5)

There is a possibility to transform the equation $y = 3x + 5$ into the $3x - y = -5$ and $y-8 = 3(x-1)$. The graphical representation of the equation $y = 3x + 5$ is a line. Points through which the line goes are for instance (1.8), (0.5). All this information can be obtained from the given equation.

Crowley (2000) describes flexibility in thinking as the way students 'go between' cognitive units in their thinking, when solving algebraic problems. She discovered that those students who can use different cognitive units flexibly also become more successful. However, the theoretical assumptions underlying concepts like *procepts* and *cognitive units* are also connected to assumptions that learning can be universally generalised as mental structural changes.

1.4.2 Flexibility in knowing

My interpretations of theoretical foundations within the phenomenographic research tradition, variation theory and constructionist theoretical standpoints, led me to a definition on knowing as 'acquiring knowledge when experiencing variation' and 'making sense of the knowledge content in acts of participation'. This is discussed in more detail in Part II, *Theoretical Explorations*. One overarching aim of the thesis is to try to understand what happens if we introduce uncertainty to theories on learning, in terms of flexibility in how learners choose to focus and find meaning in part and whole relationships of mathematical problems. In speaking of *knowing*, instead of 'thinking', the use of the term 'experiencing', gives from a phenomenological perspective a focus on how we experience our reality. The dividing line between 'inner' mental acts and 'outer' behaviour disappears (Marton & Booth 1997):

There is not a real world "out there" and a subjective world "in here". The world is not constructed by the learner, nor is it imposed upon her; it is constituted as an internal relation

between them. There is only one world, but it is a world that we experience, a world in which we live, a world that is ours (p. 13).

In the present thesis, knowing is explored as understanding knowledge content, on the one hand, and as a process of constituting oneself as a learner, on the other. Knowing is in other words seen as both a *process of reflection on content*, in order to reach insight and meanings, and a *process of Self*, where intentionality is expressed during participation in the classroom.

According to variation theory (Marton, Runesson & Tsui 2004), the invariant aspects of a mathematical problem are focused simultaneously to the varied aspects. The discernment of the invariant aspects is made against a background of what is varied. The content of the mathematical problem can be varied in ways to encourage pupils to experience variation in the content. A *relevance structure*, the learner's experience of what the given situation or task demands, can be experienced, based on the assumption that that the learner focuses on invariant critical aspects among varied aspects, which make up a background (Marton & Booth 1997). The flexibility in knowing is concerned with how learners flexibly experience part and whole relationships, based on how they experience and use variation, and how that influences their understanding of content.

1.5 Aims of the thesis

The thesis deals with a problem central to educational research on learning within the research field of mathematics education. At first it was broadly formulated as: *How do learners learn school mathematics with understanding?* This general question developed into a more specific aim:

To explore meanings of flexibility in knowing school mathematics within the contexts of educational discourses and classroom discourses in two classrooms

To address this research question, the Indian and the Swedish school class contexts were selected in order to gain a strongly varied contextual meaning of flexibility in knowing school mathematics. The differences between the contexts enriched the empirical material and provided valuable information, which might have been missed if two very similar contexts had been chosen.

II Theoretical explorations

2. Subject and object, subjectivity and objectivity

The division between the ‘Self’ and the ‘world’ has been one of the foundations of the natural and social sciences for the last two centuries. Dogmatic realism, where it is considered possible to ‘objectify’ all statements concerning the object of study (Heisenberg 1958), preferably with the help of mathematics, is in educational research represented in some of Piaget’s later theoretical developments, for instance, where he used algebraic expressions to draw generalised conclusions concerning the evolution of thinking in children. Reflected abstractions on concepts such as ‘Time’, represented in cognitive units or schemata, are seen as vehicles for the development of mathematical knowledge. Children are seen as observing and using physical objects to make reflective abstractions that can be described by the researcher independently of the situation or context (Skovsmose 1994). Piaget attempted to find universal and generalised theories on how children acquire knowledge, and how they progress from everyday concepts to scientific ones. Piaget (1971) contended that ‘structures’ – by which he means ‘systems of transformations’ - can be revealed from reality, and that it is possible to formalise them into theories. He claimed to have revealed both empirical evidence for the transformation of children’s conceptions of the world, and to have formalised these in algebraic terms. Piaget described how concepts like Time could be acquired, and also generalised the structures of development with the help of mathematics. He has been strongly criticized from various quarters, especially for his standpoint that a child’s conceptual development can be theorised as an epistemology - in this case the epistemology of mathematics. Burton (1999) makes a clarifying point that there is no correspondence between the theory of algebraic groups and how learners come to know mathematics:

The theory of algebraic groups is an example of a particular element of mathematics culture rather than a universal model applicable across disciplines, cultures, times (p. 26).

Skovsmose (1994) draws upon the consequences in mathematics education, when sources of mathematics and logic are considered in some sense ‘deeper’

and different from the sources of language, as in the Piagetian theoretical framework:

A main perspective on mathematics education, the piagetian monologism, has put the focus on the intellectual development of the child and reduced communication to a pedagogical and methodological implement. Communication makes it difficult to see interaction in education as a precondition for reflection upon the content (p. 204).

When interaction in mathematics classrooms becomes based on the evaluation of how 'able' the learner is in grasping given theoretical constructs, interaction loses much of its communicative and reciprocal dimension. Skovsmose puts forward the powerful idea that the interaction between the teacher and pupil, or between pupil and pupil, includes more than a purely performative dimension of communication. It includes intentions, actions, language use and meaning constitution.

The insistence on going beyond the dichotomy of *either/ or* in the unresolved scientific debate on subjectivity/objectivity in social science, has been criticized for its sometimes extreme subjectivity, represented in the 'relativist' view. It has been observed that there is a risk that relativism develops into another single 'objective' truth, just like the absolutist view (Bernstein 1983).

Both theoretical and empirical explorations of the present thesis clearly involve subjectivity at a variety of levels. However, the thesis does not represent a relativist standpoint. Rather, the research process, like the pupils' learning, is seen as an act of participation, as well as a process of intentionally approaching an object of knowledge.

Firstly, the theoretical exploration is based on assumptions on mathematics and mathematics education. These assumptions are subjective in the sense that they are based on my personal experience and knowledge, while a different experience and knowledge base may well have produced a different set of assumptions. Secondly, the use of different theoretical standpoints engaged me in a dialogue of interpretative exercise, where I used my personal judgement, rather than reproducing the collective judgements of a single research tradition.

When it comes to the empirical explorations, there is always a degree of subjectivity in the interview situation itself. It is difficult to estimate how the interview was experienced by the respondents. The dialogue involved both the

interviewer and respondent in a reflective process on mathematical experience and on the experiences of learning mathematics. There was an attempt made from the interviewer's point of view to come close to how the learners understood the context in which the questions were asked.

Another degree of subjectivity is intertwined with the assumptions underlying the analysis of the material. The methodological thinking behind these assumptions can be discussed in relation to three fundamental positions in human science epistemology (Kvale 1989). Among these, one position states that there is objective, certain knowledge which can be obtained through following particular methods. A second position is that there is an underlying logic to reality, but not to human behaviour, from which follows that methods cannot be predefined but have to be designed according to the aim of the study. The third position holds that no research can produce objective and context-free knowledge. All knowledge is construction. By contrast, the main methodological assumption in the present thesis is that subjectivity is a part of how we arrive at objective knowledge.

Mathematics holds a particular status in the debate on subjectivity and objectivity, since some see mathematics as the underlying deep structure in the 'laws of Nature' that we need to discover. Mathematics thus become more than an aspect of reality that we explore: mathematics define what we consider 'scientific', since only findings exhibiting mathematical regularity are considered to express 'truth' and the 'laws of Nature'. In a more utilitarian perspective, mathematics are seen as a convenient tool for summarising regularities and calculating probabilities. Finally, mathematics can be seen as a language, to express particular views or models of reality. Heisenberg (1958) has reflected upon the meaning of subjectivity and the implications of such positions in relation to physics and mathematics:

Quantum theory does not allow a completely objective description of nature (p. 107).

But quantum theory is in itself an example for the possibility of explaining nature by means of simple mathematical laws without the basis of dogmatic realism (p. 82).

Mathematics can, according to Heisenberg, help to 'explain' reality, but are not part of an objective reality. Mathematics is, in other words, a socio-cultural product. Heisenberg makes a distinction between *dogmatic* and *practical* realism. Practical realism, contrary to dogmatic realism, assumes that there are

statements that can be objectivated. He argues that in fact the largest part of our experience in daily life consists of such statements. Heisenberg takes examples like Time, Space and Existence.

Practical realism has always been and will always be an essential part of natural science. Dogmatic realism, however, is, as we see it now, not a necessary condition for natural science (p. 82).

Heisenberg's thoughts bring us closer to an appreciation of subjectivity as an essential part of objectivity. Husserl (quoted by Dahlin 2001) holds that the ontological higher status of science, compared to our sense experiences, represents a fallacy, since science can be verified only by sense experiences. The primary concern with epistemology hides this 'ontological reversal'. In the same vein, Bloor (1973) makes a point that the realist position entails an ontology of mathematics, where mathematics is seen as existing independently of human beings and where mathematicians are seen as discoverers of Truth. This view that mathematical truths are independent and structured, as a consequence makes activity within mathematics structured according to certain logical lines of reasoning.

The causes of knowledge, someone's seeing certain truths, are those that put them into the position to be able to make the appropriate form of intuitive contact (p. 177).

The mathematicians remain the interpreters of mathematics and their work has to be validated in interaction with other mathematicians.

The epistemology of mathematics is based on the assumption that mathematical beliefs are 'true', implying that they require no explanation. The ontological assumption is that it is possible to obtain abstract models for a hidden reality behind concretely experienced phenomena. Harvey quoted by Dahlin (2001) contends that there is a need to put more emphasis on the aesthetic dimension of knowledge formation:

By aesthetic I mean a point of view which cultivates a careful and exact attention to all the qualities inherent in sense experience. The objective of such an approach to natural phenomena would not be merely to appreciate their beauty, but also to understand them (p. 454).

Dahlin makes the argumentation that the ‘mathematization of nature’ (referring to Husserl) has made us listen to one of nature’s hundred languages. We are deaf to ninety-nine.

2.1 Dualism and non-dualism

In the present work, a creative research approach is applied, involving several important characteristics (Giri 2008). Ultimately, the research approach employed takes a critical stand to the aims of earlier educational research and the problems of dualisms of dichotomies. Most educational research today can be related either to mentalist or social theories. Lerman (1996) has analysed the gap between the mental and the social in later developments of Piaget’s theories, termed *social* and *radical constructivism*. He feels that there is a theoretical gap between the socially constructed meanings, which are assumed to be ‘appropriated’ in different ways, compared to how an individual ‘comes to know’ something cognitively. He argues that this gap continues to be problematic, since a theoretical distinction is made between scientific knowledge and everyday knowledge. Adding a social dimension to cognition, as in social constructivism, does not solve the problem. Lerman concludes that mental processes and the processes of appropriating cultural knowledge are too different to integrate into *one* understanding.

Subject and object are in Western psychology seen as *separated* in acts of knowing, subjects having separate consciousness from their surroundings and an ‘ego’ at their centre. Berman (1981, p. 72) makes the point that the underlying view has been that the subject is separate from object, and could therefore inspect and evaluate the object. Seiffert (2008) takes the example of the so-called S-R-model. The model was controversial from the start, and not even stringently developed as a theory. Nevertheless, according to Seiffert, it has had profound effects on the social sciences, preparing the ground for the continued prevalence of dualistic thinking and language in social science. It has also contributed to cultivating the corollary dualisms of Sender and Receiver and of cause and effect.

Berman (1981) uses the term *modern consciousness*, to describe how in everyday life, we have lost a sense of surplus of meaning and our consciousness has become partly fragmented. He argues that modern consciousness is disenchanting with the world, due to too much emphasis on objectivity dominating and controlling knowledge. He contrasts ‘modern consciousness’ with that of a *participatory consciousness*, which includes a holistic world view, and a notion that subject and object, Self and Other, human and environment, are ultimately related and sometimes even experienced as identical. In contrasting

these two forms of consciousness, he ultimately claims that through our 'scientific' outlook we have reached a state where we can hold that people of the times before the Scientific Revolution in Europe were 'childish' and had an anthropocentric world view that was not yet mature. Berman insists that modern consciousness influences our 'Western' culture still today. He makes a strong point that it is impossible for the individual today to abandon our 'scientific' attitude, especially the division between mind, body and soul, and if we do so we become insane.

It seems to me that Berman's argumentation is also trapped in a set of dualisms - such as modern/pre-modern, Western/non-Western, fragmented/holistic - with the consequence that analytic borders can be drawn, separating world views. Nevertheless, his reasoning is important in the context of understanding some of the implications of the Cartesian world view on science and mathematics. He also describes how the collapse of the feudal economy and the emergence of capitalism on a broad scale alternated social relations and provided the context for a scientific revolution in Western Europe.

In much the same line, Harding (1998) describes how this social context carries part of the explanation to why scientific 'truth' became connected to notions of *utility*, as well as to cognition and technology. Experiments, quantifications, predictions and control formed the parameters of a world view (Berman 1981, p. 51). A world view aiming at the *control* of the observable by the observer entailed an ontology that saw things which do not possess purpose, but only behaviour, which can and must be described in atomistic mechanical and quantitative ways. The consequences of such a world view are particularly extreme when the 'observed' are other human beings, who are thus methodologically divested of their own intentions.

Seiffert (2008) argues that since dichotomies are inherent to the language used in scientific discussions, it also influences the scientific thinking to become dualistic in its nature. To think non-dually, he advances, would require from the researcher to do research in a new way, and to recognise that dualism is a part of non-dualism, since it is not possible to reach beyond what we can think and express. We cannot reach a better 'non-dual' state of thinking and being. But we can open up for new ways of being, which embrace complexity (Seiffert 2008; Vattimo 1994). One consequence of a creative social research which puts emphasis on the researcher's ontological cultivation, as outlined by Giri (2004), is that we can consider ontology parallel to epistemology, and take interest in ontology, not as being dominated by the concern of epistemology, but as emerging from conversation and learning.

Giri proposes an *ontological epistemology of participation*. He believes that ontology is an act of involvement, where the researcher in conversation with h/er/imsself, the Other and the world understands the world. In the present thesis, the non-dualism of Self and the world is expressed in a *relational view* on knowing. Ontological cultivation, the understanding of the world, is interpreted as learning from *experience*. The internal relation between the knower and the known is understood in terms of *intentionality*. In section 2.1 *Experienced learning context and participation in learning*, intentionality is dealt with in more detail.

I have focused on the *participatory* part of an ‘ontological epistemology of participation’ (Giri, 2004), to study flexibility in knowing school mathematics. For this purpose, I used the so-called *intentional-expressive dialogue model* developed by Anderberg et al. (2005; 2006) which I have found helpful to promote conversation and to explore various modes of knowing and modes of being a learner. A point of departure of the conversations was to look at language used and meanings of concepts used, from a participant perspective, to gain a picture of how the knowledge content and the meanings attributed with it showed how the participant understood the content. My focus was therefore not on whether the pupil gave correct or incorrect solutions, but instead to capture the way the pupil understood parts and whole-relationships of the problem, and how s/he delimited parts and wholes. In that way, I could also gain understanding of different ways the pupils conceived the parts and wholes. The dialogue model and methodological implications of an ontological epistemology of participation will be further discussed in Part III, *Empirical Explorations*.

I have attempted to make justice to the complexity of the material. The aim has not been to replace the presuppositions from one theoretical model with another. It has not been my intention to replace generalising statements represented in cognitive theories, which still dominate mathematics education research, by equally generalising statements of the same order. Instead, attempting to board the task of keeping the complexity in a dialogical ontology and epistemology of mathematics, a *relational view* on knowledge is embraced. Giri (2004) has suggested that the interaction in the dialogue between researcher and respondent could rest on attempts to deal with dualisms in a non-dualist manner. The dualism which I focus on here is the one which distinguishes ontology from epistemology, and the subject from the object in mathematics education.

In her research on mathematicians' ways of knowing mathematics, Burton (1995) found that five categories, based on philosophical, pedagogical and feminist literature, could define what it means to 'know' mathematics:

1. its person- and cultural/social relatedness
the aesthetics of mathematical thinking it invokes
2. its nurturing of intuition and insight
3. its recognition and celebration of different approaches, particularly in styles of thinking
4. the globality of its applications(p. 287).

We can conclude from Burton's definitions, that the epistemology and ontology of mathematics belong to a whole, and that knowing mathematics includes mathematical ontology and epistemology, as well as the relation between the knower and the mathematical content. Relating ontology to epistemology makes it possible to understand knowing as related to learner agency and authorship Giri (2004) writes:

We have to realize that ontology emerges as much from contestation, conversation and learning as it is an initial part in self and science (p. 29).

Making an attempt to re-centre the ontology and epistemology of school mathematics, we find that mathematics become 'objectively true' only through the *processes of working* with them, in relation to *who* is stating the known, and *in what circumstances* the known is stated. Absolutist certainty is replaced by an element of uncertainty, in both epistemology and ontology. This uncertainty has been expressed by Heisenberg (1958) in terms of how objectivity is related to experience:

Any concepts or words which have been formed in the past through interplay between the world and ourselves are not really sharply defined with respect to their meaning; that is to say, we do not know exactly how far they will help us in finding our way in the world. We often know that they can be applied to a wide range of inner or outer experience, but we practically never know precisely the limits of their applicability. This is true even of the simplest and most general concepts like "existence" and "space" and "time". Therefore, it will never be possible by pure reason to arrive at some absolute truth (p. 86).

Heisenberg explains how mathematical concepts are related to mathematical systems of logic, and at the same time how there is an ongoing reflective argumentation in the relation between mathematics and reality:

The concepts may, however, be sharply defined with regard to their connections. This is actually the fact when the concepts become a part of a system of axioms and definitions which can be expressed consistently by a mathematical scheme. Such a group of connected concepts may be applicable to a wide field of experience and will help us to find our way in this field. But the limits of the applicability will in general not be known, at least not completely. At the same time, mathematical logic itself is a mathematical scheme which we can not know how well it describes the reality (p. 9).

When we accept uncertainty and subjectivity, mathematics become a creative and historically evolving part of our reality, just as art is (Taguchi 2004):

Mathematical descriptions of our reality are in general seen as closer to the Truth than if the same phenomenon was expressed as a piece of art. It is all about how we have chosen to value these different ways of expression. It is not a matter of mathematics being truer than art. They are expressions of the same but in different ways, for different contexts and most often, but not necessarily, with different purposes or concrete consequences (p. 55, *my translation*).

Modes of being a learner in terms of language user and knowledge producer can be one centre of research and practice in mathematics education, where we might shift our scientific questions from *What is Time?*, turning our attention to questions like *What is human experience of Time?*

I was inspired by the phenomenographic research orientation. During the late 1970s in Sweden, this empirical research orientation made a break with the Piagetian internalistic way of viewing human thinking. Research questions instead came to concern people's qualitatively different ways of seeing the world. Conceptions were studied and categorised, while methodological issues were a central part of the research conducted. In the phenomenographic

tradition, assumptions on learning are made going beyond the dualism of inner and outer, or the person and the world. Learning is seen as a *change in the relation* between the person and the world (Marton & Booth 1997). Understanding knowledge content includes both content and process. A person relates to the world at any given moment, depending on context and situation, and simultaneously the world appears to the person in this relation. The totality of an individual's experiences are manifested in that individual's awareness (Marton & Booth 1997).

Svensson (1997) argues that studies which focus conceptions and discursive and contextual features can use phenomenography as a research tool. In this respect, Säljö (1996) makes an interesting point, from a socio-cultural perspective, arguing that studying conceptions reduces the human being to a Thinker and that the relational view on learning misses out on studying the subject as an Actor, participating in a socio-cultural context. One of Säljö's main points is that a theoretical relationship is lacking between *experience* and *discourse*. Let us therefore look closer at this relationship, and see which potential theoretical meeting points there may be.

2.2 Experienced learning context and participation in learning

Every field of learning is immersed in culture. Stigler & Hiebert (1999) analysed the video material from TIMMS (Third International Mathematics and Science Study 1993). When the video record of representative samples from USA, Germany and Japan was analysed, it was found that the differences in ways of instruction in mathematics classrooms between countries were more evident than the differences within a country. They concluded that the way many US teachers seemed to believe that mathematics should be taught is mostly a set of procedures, meaning that mathematics is learned piece by piece. They found that the differences between the US teaching and Japanese beliefs practised in their mathematical classrooms were most illustrative. The level of difficulty is in the US classroom led by a belief that practice should be error-free and that confusion and frustration should be minimised. They mention the example of working with fractions with like denominator, then to proceed with simple unlike denominators, and later practice more difficult problems. Japanese teachers seem to believe that 'constructing connections between methods and problems requires time to explore and invent, to make mistakes, to reflect, and to receive the needed information at an appropriate time' (Stigler & Hiebert (1999) p. 91). For an example $1/2$ and $1/4$ would be given first, and then comparisons of different methods for solution that pupils develop would be made. Stigler and Hiebert conclude that:

Obviously, struggling and making mistakes and then seeing why they are mistakes are believed to be essential parts of the learning process in Japan (p. 91)

The aspects of culture I have focused on in the present thesis concern localised epistemologies, discourses on education and the classroom discourse. The learning context of the learner is seen as an ‘internal relation’ between the learner’s own intentions, approaches to learning, understanding of content, modes of being a learner, and modes of knowing, on the one hand, and the external learning situation, on the other, including the intentions and teaching approaches of the teacher. The learner performs learning in the classroom, with a set of expectations based, among other things, on prior experiences of learning contexts and of knowledge objects.

It is my assumption that discourses on being a learner, in the practice of learning mathematics, are experienced as *discursive possibilities* for the constitution of learner identities. These possibilities are *used* by learners in constituting their learner identity, not simply ‘appropriating’ culturally determined behaviour. In social constructionism it is also generally admitted that a person can have multiple identities, depending on the storylines available to constitute the identity (Holland et al. 1998).

It is further my assumption that ‘knowing mathematics’ means to be absorbed in tasks with the whole person, and that *intentionality* is connected to *what forms of participation pupils engage in*, in the learning practice. A ‘practical intentionality’ is related to participation and language use, and is described by J.N Mohanty (2002, p. 128) in the following terms: “If every intentionality intends an object as having certain significance or meaning, we can speak of ‘practical meaning’.” He underlines that a person is an intentional entity. Mind and body exist as a whole, a person, which is in itself a source of intentionality:

A person thinks, believes, loves, hopes, desires. Its entire being and nature consist in such intentional relatedness to the world and other persons (p. 74).

Although Mohanty draws his assumptions from phenomenological theory on intentionality, from Husserl and Merleau-Ponty, I believe that practical intentionality can be seen as a potential theoretical meeting point with the poststructuralist notion of Self, considered as a process (Davies 2000). I also take inspiration from Burton in the way she combines the *what* and *who* of

mathematical learning, i.e. *authorship* and *agency*. The concept of *authorship* Burton (2005) borrows from Holland et al. (1998), stating:

By choosing authorship I want to make clear that I understand the ‘what’ as well as the ‘who’ of mathematics learning as deriving from the interpersonal, and as a consequence, being entirely a socio-cultural artefact (p. 22).

Burton further contends that the epistemological perspective on mathematics tends to influence the pedagogical approach as well. Either the approach to mathematics is ‘objective’ or ‘negotiable’. When mathematics is codified and *authored* by authoritative others and transmitted to learners, the *agency* remains external to the learner. Mathematical narrative, on the other hand, may be told and re-told in the style and with emphasis chosen by the agents (ibid, p. 24).

With the concept *agency*, she sees a link between learners’ understandings of mathematics and the learners’ responsibility for and role in its construction. Knowing and knowledge are therefore not separate as process and/or product, Burton (2004b) concludes:

Coming to know and what you come to know are interdependent, not individual or socio-cultural pure (p. 23)

In other words, seeing individuals as actors or empowered agents participating in social activities is not in contradiction to considering their ‘intentionality’ when engaging in such activities, nor does it exclude investigating their experience or their conceptions, in a phenomenological perspective.

2.3 Culture of mathematics and mathematical culture

Bloor (1973) argues that the domination of epistemology over the ontology in mathematics can be overcome in research on sociology of knowledge. Referring to Wittgenstein’s *Remarks on the Foundations of Mathematics* from 1936, Bloor states that ‘Wittgenstein’s example of production of number sequences like 2,4,6,8...is a representative example of a mathematical inference’ (p.181). Wittgenstein argues, according to Bloor, that even in the mentioned number sequence there is an origin of thought which has been created by people in a certain context. The realist conception of rule following does not provide answers to the problems that it was designed to solve. Instead we can ask ourselves, in line with Bloor:

How can we make the same steps again and again; what makes 'the same' the same; what guarantees the identical steps at the different stages of rule's application? (p. 181).

Wittgenstein, quoted by Bloor, considers mathematical formulas in relation to how they are consistently applied in a social practice. Moreover, the application of a formula is a social process, since every communication involving a formula is embedded in terms of 'the way we always use it' and 'the way we are taught to use it'. That meaning and use are interconnected in communication was formulated by Wittgenstein (1968) in his theory on language game. Bloor continues his discussion on Wittgenstein, stating the example of the use of 'zero' in Babylonian mathematics. The concept of zero was understood in context. There was a sign for distinguishing 204 from 24, but not for distinguishing 240 from 24. The structure the concept 'zero' was a part of at that time, was not logical, but social. Wittgenstein argues that the logical structuring follows the social. The truth is relocated to utility, and mathematics can be seen as invention rather than discovery.

Mathematics is an institution, and institutions, though human products, are not subject to individual whim. There is a sense in which institutions exist in their own right over and above the specific acts of the people who play roles within them. This is because institutions involve ways of behaving which have become settled and routinized. Certain ways of behaving have become ingrained in the dispositions of a group of actors and expectations have crystallized (p. 188).

One main conclusion Bloor draws from Wittgenstein's social theory of mathematics, is that mathematical logic should be *explained*, rather than be treated as a revelation of a truth to be justified. Another is that Wittgenstein opens for a sociology of mathematics, where it is possible to understand the development and acceptance of mathematical knowledge, not only error or confusion.

Based on interviews with mathematicians, Burton (2004a) makes a distinction between *culture of mathematics* and *mathematical culture*. The 'culture of mathematics' is, for instance, aesthetics in terms of structure, compactness and

connections, according to mathematicians. Mathematicians, mostly women mention use of power, hierarchies, isolation and competition within what Burton terms as ‘mathematical culture’. One of Burton’s conclusions is that we should address the mathematical culture more than the culture of mathematics in mathematics education research. My personal standpoint, with respect to the issues discussed by Burton, is that it is important to look to the mathematical culture as well as the culture of mathematics, so that we can make visible the ethnocentric and mono-cultural ideas and evaluations that tend to prevail in school and education (for a discussion, see also Lahdenperä 2004, p. 206). My concern is that we can re-think the content of school mathematics, as well as work with the mathematical culture reflected in classroom practice.

Needham’s historical descriptions of Chinese science and technology that began appearing in the 1950s are an example of an important change of position from Eurocentric to postcolonial science histories (Harding 1998, p. 147). In Needham’s work, the emphasis lies on multiple origins of science. Ethnomathematics (cf D’Ambrosio 1994; Joseph 1991; Ascher 1991) has a somewhat different focus, compared to Needham, and instead explores the universality of mathematical ideas and different developments of mathematical models. Sen (2005) writes about the West and East as contributing to each other’s image, and how such analytic partitioning simultaneously excludes from the history of science and mathematics many important aspects of history, in terms of cultural flows and intercultural communication.

Not only can the content of mathematics be seen as ‘objectively’ given in an absolutist sense. Similar absolutist assumptions can also influence the view of how learners should be. Burton (2004a) argues that both mathematics itself and those learning it tend to be treated in schools as homogeneous:

Within mathematics teaching, homogeneity may be experienced as single method and/or single solution type approaches; in classrooms, attempts are made to homogenize students often through ability grouping. The first does a disservice to mathematics, the second to the learners(p. 268).

From the learner’s perspective, this homogeneous view leads to the exclusion of many intentions. Learners who do not correspond to the norm of the group are seen as deviating, and sometimes as a disturbance. Walkerdine (1988) observes that correct accomplishment in mathematics learning is by many teachers considered as an indication that pupils have understood the content.

Failure, on the other hand, is regarded as symptomatic of not trying hard enough, or of that the level of maturity is not reached. This is also considered in relation to appropriate behaviour. If a pupil fails to live up to the ideals of what it means to be a learner, then that is interpreted as an indication of not being able to reason rationally and logically.

Nardi and Steward (2003) observed and interviewed seventy pupils, aged 13-14, in three year 9 classrooms at a school in Norfolk, UK. The researchers found that so-called 'quiet disaffection', which is seldom focused in mathematics education research, is the non-disruptive behaviour expressed as disengagement and invisibility in a tacit, non-disruptive manner. Nardi and Steward found that several pupils seem to experience mathematics as a set of rules that suggest that one should remember them by rote. Many also do not like a large amount of teacher exposition or long-winded/pedantic approaches to mathematical explanation. They find their teacher's explanations difficult to understand or make sense of. The pupils in the study did not seem opposed to textbook activities as such, but objected to the exclusive and invariable use these were put to. A style of working that emphasizes negotiation and explanation to others is not only emotionally more satisfying or more efficient in terms of task completion, but is also prone to generate a better understanding of the mathematics. Further, the pupils in Nardi and Steward's study seemed to resent mathematical learning as a *rote-learning* (emphasis given in the text), an activity that involves the manipulation of unquestionable rules, and yields unique methods and answers to problems. Despite a perceived efficiency of memorisation and mimicking of correct procedures as cued by the teacher (for example in tests and examinations), the intellectual appeal of these approaches to the pupils was limited, especially because their use implied having to tolerate extensive exposition by the teacher. There was strong evidence from Nardi and Steward's data that pupils at secondary level perceived enjoyment (relevance, excitement and variety) to be central to learning. The pupils in the study seemed to appreciate a teacher who uses tasks, including games, that are useful, enjoyable and that one can relate to, combined with a reasonable degree of challenge and initiative, provided in an open-ended mathematical task (e.g. investigations, project work, coursework). The pupils in the study seemed to appreciate a teacher who explains clearly and in different ways, uses concrete examples and builds on previous knowledge; a teacher who is friendly and invites student questions without picking on them; a positive teacher who gives feedback and praise, and who makes you work or think.

The homogeneous view of mathematics does not engage pupils actively in learning activities that are based in their own experience and interests. It tends

to make mathematical language seem omnipotent, purely abstract and far from everyday experiences. The homogeneous view of mathematics also denies that it has developed during human history at multiple places and different times, as part of socially situated practices (Ernest 2004). From a ‘multi-logic’ point of view, mathematics becomes contingent to *who* practices it, their purposes, the social context, and the assumptions underlying the activity. The mathematical culture at school is often linked to philosophies of mathematics (Ernest 1991). A range of perspectives may be termed *absolutist*:

These view mathematics as an objective, absolute, certain and incorrigible body of knowledge, which rest on the firm foundations of deductive logic. Thus according to absolutism, mathematical knowledge is timeless, although we may discover new theories and truths to add; it is superhuman and ahistorical, for the history of mathematics is irrelevant to the nature and justification of mathematical knowledge; it is pure isolated knowledge, which happens to be useful because of its universal validity; it is value-free and culture-free, for the same reason (pp. 8-9).

Ernest argues that this view has contributed to a negative image of mathematics, due to the epistemological foundationalism. At school, the absolutist view can be communicated in routine mathematical tasks, which involve the application of learnt procedures and where achievement in mathematics is measured by producing correct answers.

However, over the past few decades, *fallibilist* perspectives on mathematics have gained ground. These propose a different and opposing image of mathematics as human, corrigible, historical and changing (Davis & Hersh 1988; Ernest 1989). Burton (1999) urges us to understand the impact of a purely absolutist view on mathematics on how we understand, not only its epistemology, but also its ontology, as ‘objective’ and external to our experiences of the world:

This is despite accumulating evidence that such external authorship protects and encourages an ineffective transmissive mode of teaching of what is mistakenly viewed as ‘objective’ knowledge. With the rapid technologization of the academe, such socio-cultural differences as previously existed (Joseph 1990) are more and more likely to be over-written by a homogeneity of publicly recognised mathematics

that conforms to an international common practice within a discourse controlled and validated by the most powerful voices (p. 29).

The development of assessment methods to be used on an international basis implicitly holds that mathematics is culture-free, and that there is a homogeneity in how it is possible to understand mathematics. International evaluations of mathematics skills and knowledge are regularly conducted by PISA (Programme for International Student Assessment) and IEA (International Association for the Evaluation of Educational Achievement). PISA produces internationally standardised test items in mathematics for 15 year old students in order to measure how pupils solve *authentic* problems. One mathematics example given in the PISA 2003 Assessment Framework deals with a saving account and an interest rate. The situation is classified by PISA as ‘public’, since it relates to the local community and society, and it is also assumed to be authentic:

Note that this kind of problem is one that could be part of the actual experience or practice of the participant. It provides an *authentic* context for the use of mathematics, since the application of mathematics in this context would be genuinely directed to solving the problem. This can be contrasted with problems frequently seen in school mathematics texts, where the main purpose is to practise the mathematics involved rather than to use mathematics to solve a real problem (p. 28).

PISA refers to what it calls ‘mathematisation cycle’ as a process where the pupil organises the information given in a problem situated in reality according to mathematical concepts, and transforms the real-world problem to a mathematical problem. The completion of the process entails solving the problem, and ‘making sense of the mathematical solution in terms of the real situation, including identifying the limitations of the solution’ (p. 38). It seems to me, that there is an underlying theoretical assumption that authentic problems do not necessarily have to have a context which needs to be experienced and understood as authentic, although the word problems provided by PISA have been divided into scientific, practical and personal situations. The mathematical concepts should, according to PISA, be the pupil’s primary focus and these should be discovered in a word problem context, and at the same time the reality-based problem is to be transformed into a mathematical problem. PISA discusses the increasing level of difficulty in

the mathematical tasks and its relation to mathematical proficiency. Concerning how pupils at different levels of proficiency relate to contexts of the tasks, PISA states that (OECD 2003):

At the lowest described proficiency level, students typically carry out single-step processes that involve *recognition of familiar contexts and mathematically well-formulated problems*, reproducing well-known mathematical facts or processes, and applying simple computational skills.

At higher proficiency levels, students typically carry out more complex tasks involving more than a single processing step. They also combine different pieces of information or interpret different representations of mathematical concepts or information, *recognising which elements are relevant and important and how they relate to one another*.

At the highest proficiency level, students take a more creative and active role in their approach to mathematical problems. They typically interpret more complex information and negotiate a number of processing steps. They produce a formulation of a problem and often develop a suitable model that facilitates its solution. Students at this level typically *identify and apply relevant tools and knowledge in an unfamiliar problem context* (p. 51, *my italics*)

The theoretical framework which PISA applies to mathematical proficiency reflects an absolutist view on mathematics (Ernest 1994). Mathematics is seen as disconnected to the world of the students, which is contradictory to PISA's own definition of '*mathematical literacy*' (OECD 2003):

Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen (p. 24).

The degree to which tasks are experienced as authentic is an important issue, both for teaching and evaluation. When mathematicians work with mathematics, the feelings of being involved in mathematical work, feelings of frustration, excitement, satisfaction and sometimes euphoria, are connected to their process of 'coming to know'. The *involvement* makes them motivated to find a pattern, make a connection, eradicate a difficulty (Burton 2004). It is possible for a mathematician to both hold an 'absolutist' philosophical position concerning the epistemological foundation and justification of mathematical

knowledge, combined with a fallibilist, looser, more descriptive account. It is perhaps the case among teachers at school as well, that they can hold the absolutist position as an ideal, while working with ideas that are more fallibilist, or vice versa, Ernest (1994) supposes. However, he concludes that the implications of the absolutist view dominate.

Skovsmose (1994) suggests that in educational practice, pupils should not only know the aims of education through textbooks or discussions in the classroom, but also through a process of critical *engagement* with mathematics, to become *owners* of the aims of the educational process. He argues that it is important to note that mathematics education ‘entails an introduction to a mode of speech and thought and provides an introduction to a certain culture’(p. 6).

It is the final product rather than the emergence of mathematicians’ process of coming to know that is generally dealt with at school, thereby making understanding difficult, as many pupils fail to make sense of the piecemeal presented mathematics they are supposed to deal with. In his recent contribution to the philosophy of mathematics, Ernest (2004) writes about the need to look at conversation, not only as a means to stimulate cognition, but as a way to make learners participants in mathematical practice, as language users and knowledge producers:

Taking conversation as an epistemological starting point has the effect of re-grounding mathematical knowledge in physically-embodied, socially-situated acts of human knowing and communication (p. 26).

Ernest (2004) also contends that even though mathematics is conversational, its dialogical nature is hidden under a monological appearance, and ‘has hidden the traces of multiple voices of human authorship behind a rhetoric of objectivity and impersonality’ (p. 27). That is one of the reasons why in mathematics education, ‘mathematics seems very clear and reasonable, yet when the reasoning is not understood it becomes the most irrational and authoritarian of subjects’ (p. 27).

‘Knowing mathematics’ therefore means more than just being temporarily able to give answers to already practiced textbook problems, which can be compared to how one gains lexical knowledge of a language. It does not help us much to only have this kind of passively reproduced knowledge when we need to interact in life. Burton (1995) stresses:

Knowing is not about uncovering truths, but is part of a greater project of demystifying mathematics and a greater inclusivity. Mathematics is both contributory of and defined by the context within which it is derived (p. 285).

She makes assumptions on knowing mathematics as a constitutive process of both subject and object:

Knowing in mathematics cannot be differentiated from the knower even though the knowns ultimately become public property (p. 286).

Burton believes that it is necessary to debate the epistemology of mathematics, as it is crucial in order to arrive at definitions on knowing mathematics. She states that:

Knowing mathematics would, under this definition, be a function of who is claiming to know, related to which community, how that knowing is presented, what explanations are given for how that knowing was achieved, and the connections demonstrated between it and other knowings (applications) (p. 287).

There is thus a theoretical opening toward thinking about an educational theory on knowing school mathematics, which involves the subject and the object, epistemology and ontology.

2.4 Flexible experiencing and knowing

In cognitivist theory *flexible thinking* is considered to be a mental process of shifting between cognitive units. In recent years, a 'computational' view of mind has increasingly come to dominate the study of learning and cognition (Marton & Neuman 1990):

According to this view, the mind can be described as an information processing device analogous to a computer. Cognition can thus be described in terms of computational procedures, operating on the information received and stored in the form of an internal representation (p. 51).

One of the specific problems with the computational view of thinking is the assumption that if an individual can learn how to draw his/her attention to

structural aspects of a mathematical problem, for instance, and apply it to any similar occasion or test questions, then the individual can learn to 'understand' the problem. Fennema & Romberg (1999) contend that structured knowledge is easier to access and that it contains more than one way to understand information:

Developing understanding means more than just connecting new knowledge to prior knowledge, it also involves the creation of rich integrated knowledge structures. When students see a number of critical relationships among concepts and processes they are more likely to recognize how their existing networks of knowledge might be related to new situations. Structured knowledge is less susceptible to forgetting. When knowledge is highly structured, there are multiple paths for retrieving it, whereas isolated bits of information are more difficult to remember (p. 20).

This view of structured knowledge is reminiscent of Gray & Tall's (1991) procepts. Structured knowledge is by Gray and Tall defined as *procepts*, discussed earlier on page 32 in the present thesis. The flexibility in the process of counting and its product, the concept itself, is termed *flexible thinking*. A computational view of thinking in relation to school mathematics often deals with the final product of a long chain in the process of coming to know.

In Gestalt psychology, the focus on structural aspects of wholes is seen as crucial for productive thinking and for understanding (Wertheimer 1971):

Envisaging, realizing structural features and structural requirements; proceeding in accordance with, and determined by, these requirements; thereby changing the situation in the direction of structural improvements, which involves: that gaps, trouble-regions, disturbances, superficialities, etc. be viewed and dealt with structurally;

That inner structural relations - fitting or not fitting - be sought among such disturbances and the given situation as a whole and among its various parts;

That there be operations of structural grouping and segregation, of centering, etc. (pp. 235-36).

Marton and Neuman (1990) found that, rather than doing additions or subtractions in one's mind one unit at a time, according to 'algorithmic rules', 'sudden restructuring' is crucial for how children understand number relations. The immediate grasp of numbers found in these studies can be called intuitive 'in so far it is certainly based on an implicit perception of the whole problem' (Marton & Neuman discussed in Marton, Fensham & Chaiklin 1994, p. 470). Their conclusions were based on studies of how children deal with tasks posed as questions in an everyday context, for example:

If you have only 2 kronor in your purse and you want to buy a comic that costs 9 kronor, how many kronor do you need? ("2+ _=9") (p. 15).

Based on their results of interviewing eighty-two 7-year old children, all new school starters in four different first grade classes in Sweden, about how they solve different tasks, which were a variation of parts- and whole relations, Marton and Neuman (1990) provide evidence suggesting that the basis for arithmetic skills are non-computational. The outcome space consisted of 12 qualitatively different ways of experiencing numbers. Numbers were experienced as:

Movements
Fair shares
Names
Estimated numbers
Estimated fingers
Counted numbers
Heard numbers
Multiple patterns
Counted fingers
Finger numbers
Abstract numbers
(p. 18).

There were only 5 cases of purely computational counting in 815 responses to 11 problems. One counting procedure was based on *counting fingers*, understanding the ordinal meaning only, counting 2+7, beginning with two fingers and 7 fingers thereafter, after which they could count all the fingers together (p. 52). This procedure was seen in 4 cases. A procedure based on *counting numbers* was seen in one case only of double counting.

A more frequent strategy was *semi-computational*, where the children counted numbers both as unit-by unit, incrementing or decrementing, and with a grasp of the whole as a visual pattern. In two other strategies, named *estimated numbers* and *estimated fingers*, number words are uttered; there is an attempt to hear or see the cardinality of numbers, which is partly achieved in heard numbers, and totally in finger numbers (p. 53). Marton and Neuman conclude that the origin of arithmetic skills can be seen in the use of finger numbers and the understanding of the number relations, the oneness, twoness, threeness or possibly the fourness. They write:

When using finger numbers, all numbers up to ten are made immediately perceptible and so are the relations between the numbers; they are not only seen *in* each other but they are also felt in a tactile and body-anchored way. This capability is uniquely human and it is in our understanding the essential element in the development of arithmetic skills. Although counting is a necessary prerequisite to the acquisition of arithmetic skills, the skills themselves are fundamentally non-computational, and difficulties with mathematics seem to evolve when children are led to believe the reverse is true. (p. 53)

Some of the children gave an explanation to that the answer is 7 to the question $2 + _ = 9$: “Because I know that $9 - 2 = 7$ ”. The interpretation of this and other statements about similar problems, was that ‘the children understand numbers in terms of other numbers, of which they are composite parts, or which are composite parts of them’ (p. 35). Number 9 is composed of two parts of which one is given. The solution procedure is guided by the actual numbers, and *not by the type of arithmetic operation* (Marton & Neuman 1990, p. 47, *my italics*).

In dealing with a mathematical problem, the variation represented within the mathematical task is crucial to what possibility the learner has to experience variation. The varied aspects focused by the learner and the constancy of the unfocused aspects are simultaneously experienced. Marton, Runesson & Tsui (2004) discusses Voigt’s example of how two students, Jack and Jamie, went about solving a group of arithmetic tasks (Voigt 1995, pp. 173-174, quoted in Marton, Runesson & Tsui 2004):

1. $50-9=41$

2. $60-9=51$
3. $60-19=41$
4. $41+19=60$
5. $31+29=60$
6. $31+19=50$
7. $32+18=$ _
8. (p. 36).

Two students were working with seven tasks. They saw a pattern of how the numbers in the given tasks were related to each other through, what Voigt referred to as *negotiation of mathematical meaning*. Voigt concluded that a new strategy was developed through interaction between the students. Marton, Runesson and Tsui agree that interaction undoubtedly influences the strategy in the case described by Voigt, but argue that other elements are also at hand. Marton, Runesson and Tsui suggest that within the given tasks, it is possible to experience a certain structural variation.

Voigt's study has been explored in more detail from the point of view of variation theory by Runesson (1999), who points out that in the series of task something is kept invariant and something is varied. The three first tasks are all concerned with subtraction. The second number in the task ends with 9, but the number of tens are varied. Between task number 1 and 2, only the ten is changed (50 and 60) and between 2 and 3, the second term is changed (from 9 to 19). Runesson also observed that in the following three tasks, the arithmetical operation is addition, and the numbers end with 1 and 9. Between tasks 4 and 5 the number of tens in the first and second figure changes (from 41 to 31, and 19 to 29, respectively). The sum of these two additions, however, is the same: 60. Task number 6 contains and combines numbers from tasks 4 and 5. The first number is the same as the first number in task 5, while the second number is the same as the second number in task 4. In task number 7, the number 31 increases with 1 and makes 32, while 19 decreases with 1 and makes 18. Runesson concludes that there is a variation in the arithmetical operations presented in the tasks (subtraction and addition), as well as in the numbers involved in the operations. She looks more carefully at how Jack and Jamie solved the last task ($32 + 18$), and notices a difference in how the students solved task 5, where they added units and tens separately and how they understood 32 and 18 in relation to other numbers. Jamie saw that the sum in task 6 and task 7 are the same: 50. He can understand the relation between the parts and wholes within and between the tasks, and notices that one number increases with 1 and the other number decreases with 1, which means that the difference is zero (pp. 74-76). Runesson draws two important conclusions (p. 76). First, the change in strategy between tasks 5 and 7 can be described as a

result of a change in the students' ways of experiencing the variation. Secondly, the change in the ways of experiencing was related to the character of the group of tasks the students were working with.

Marton, Runesson & Tsui (2004) describe four different patterns of variation in how we experience something in relation to something else:

- Contrast
- Generalisation
- Separation
- Fusion

The aspects we take for granted and the aspects which we focus upon make up the relation to the whole experienced phenomenon. Contrast is the simplest experience of variation. As an example, when contrast is experienced, the individual can understand what 'three' means in relation to 'two' and 'four'. Generalisation is when we understand that 'three' has different appearances, for instance three apples or three honey bees. Separation means separating invariable aspects from variable, as in the example above, when Jamie saw that there were some aspects which were invariable, like the arithmetical operation, while some aspects were variable; a number increased. Fusion means that many different aspects are simultaneously focused. The fusion allows different cases to be separated within the same understanding. When Jamie understood, in a simultaneous way, that between tasks 6 and 7 one number was increased by 1 and the other decreased by 1, he understood that this implied that the sums would be 50 in both tasks. The forms of variation mentioned by Marton et al. can also be seen as examples of different ways to organise experiencing of certain phenomena in mathematics education.

Svensson (1997) has described *cognitive approaches* to problems, such as a mathematical problem, as atomistic or holistic. An *atomistic* approach means that focus lies on unrelated units within a problem, while a *holistic* approach has its focus on parts of the whole, as well as the relations between the parts and the whole. Cognitive approaches are related to previous experiences and to how the problem is described or presented. In variation theory, both the *what*- and the *how*-aspects of learning are considered. The content that is being learned is the *what*-aspect. The *how*-aspect includes the process of learning, where there is an intended act of learning related to certain capabilities for understanding or memorisation (Marton & Booth 1997).

Marton and Booth (1997) argue that an experience has a structural and a referential meaning aspect:

The structural aspect of a way of experiencing something is thus twofold: discernment of the whole from the context, on the one hand, and discernment of the parts and their relationships within the whole, on the other. Moreover, intimately intertwined with the structural aspect of the experience is the referential aspect, the meaning (p. 87).

An intentional-expressive perspective on learning was developed by Svensson (1978, 1997) and Anderberg (1999). The perspective is based on an assumption that there is a difference between an individual's intention, the thought content and h/er/is intentions expressed in social language. Thus, there is no identity between collective meanings of semantic-linguistic form and expressed meanings of subjective thought. Svensson (1997) clarifies the relation between conceptions, expressions and different levels of understanding in the following way:

One variation is from an implicitly expressed not articulated conception, to an explicitly focused on and formulated conception. The question of how to view this variation brings in the question of the relation of conceptions to language and to cultural and social context.

The character of a conception will vary from being the meaning of an immediately experienced part of reality to being a more general thought about what is common to several parts of reality which are more or less vaguely identified (p. 166).

In fact, since any categorisation of experience is grounded in culturally informed structures of relevance, particular care needs to be taken in research methodology to avoid pre-categorisation. Variation of aspects and the focus on the relation between them, in mathematical content, can give rise to different experiences of variation. The learner can gain more differentiated relations between parts and wholes through experiencing variation. The understanding of the content therefore increases.

Taking the assumption that there is a limited number of *ways people experience the world* as a point of departure, it is possible to explore variation in understanding. The ontology of mathematics should be considered parallel to the epistemology of mathematics. Acknowledging the learner's own experiences of mathematics, in terms of *conceptions of the world*, rather than focusing on *conceptions possessed by the knower* (Marton, Runesson & Tsui 2004) changes the ontological status of mathematics. From being omnipotent language, it becomes a language with which one can express understanding of an experienced reality, and the nature of mathematics becomes re-connected to people's experiences.

Intentionality is in the present thesis also seen as connected with *participation* and *empowerment*. In this connection, intentionality becomes interesting in relation to discourse and context (Skovsmose 1994) and practical intentionality (Mohanty 2002, pp. 31-32 and 35). Pupils' agency in mathematics classrooms is often visible in their choices of being participants or non-participants. The teacher brings into the learning practice intentions concerning mathematical experiences, and pupils are more or less aligned with these intentions. Although the pupils mostly play the 'game of participation', not all understand or accept the intentions of the teacher (Davies 1983). A small portion understand the teacher's intentions, and these pupils are often considered 'bright' and well-behaved. There are others whose intentions diverge from the teacher's. They decide to participate in other ways: keeping silent, bullying, or resisting (Valero 2004, p. 48). These pupils would from a traditional mathematics education viewpoint be considered as 'deviant' or problematic pupils who need to be 'normalised'. By contrast, negotiations on intentions in the learning practice would give real empowerment, Valero suggests. Then there would be potentialities for pupils to participate in the mathematics experience, in ways based on intentions to participate in development of the practice (Valero 2004, p. 49).

The theoretical interpretation I have made of phenomenography and variation theory, combined with an emphasis on agency, intentionality, participation and empowerment, have led to the focus of the present thesis on two aspects of knowing:

- Learners' performance of learning depending on agency and authorship
- Variation in learners' ways of understanding school mathematics, related to individual meaning-making processes and discursive possibilities in the learning context

The way that variation is experienced thus not only depends on the mathematical content acted upon, but also on the *experienced learning context* and *how the learner constitutes her/himself as a learner*. These theoretical conclusions are in agreement with what I found in the empirical material. ‘Flexible experiencing’ as defined in the present thesis is, in other words, related to content of learning, modes of being a learner, and modes of knowing.

III Empirical explorations

3. Research approach and perspectives

The field studies were conducted at two locations of our world; in Lund, Sweden and in Balasore, Orissa, India. I was born and raised in Sweden. For the last 19 years, India has also been my home, in the sense that I feel and experience belongingness. From 1994, I worked in both countries as a mathematics and science teacher, until I started my thesis work in 2002. In India, specifically in Orissa, where my husband comes from, I have lived at regular intervals, the periods of stay ranging from one year (in 1989) to two months.

After innumerable encounters with what was before only vaguely understood and not part of my personal consciousness, I have come to know *interculturality* as something that involves the whole being and widens one's horizon in a compelling way. For me, *interculturality* has had effects on how I live, think, act and understand my experiences and other people. Vattimo (1994) contends that a *weak ontology* opens possibilities for reflecting and acting on the construction of the human place in the world. For him, the sense of *being in the world* has changed at the end of modernity. He shows us that any passage from modernity is possible only because of, and after weakening the formerly strong modern 'truths' that have served as grounding in science and informed our world view for centuries. It becomes increasingly important to understand Self, Other and the world in new ways.

My reflections on Self led me to an *intercultural* research approach in the explorations of knowing school mathematics. During interviews, opportunities were given for reflections on learner identities, which would encourage the learner to understand h/er/is agency and authorship. It seems to me that the more I interact in different cultural contexts, the more culturally sensitive I become.

When people learn about different perspectives on reality, the horizon from which they are looking widens. It becomes easier and more motivating to put themselves in a 'third position' (Lahdenperä 2004). Lahdenperä maintains that research is *intercultural*, if phenomena are viewed from different cultural viewpoints or perspectives, and that it is an advantage when research groups

consist of different ethnicities. During my thesis studies, I have been in continuous dialogue with Ananta Kumar Giri, who comes from a village near the provincial town where the Indian part of the study was conducted. He is an associate professor at MIDS, Madras Institute of Development Studies, Chennai, India, and has worked as guest lecturer in many parts of the world. The dialogue has been very fruitful and - taken together with my background - I recognise what Lahdenperä (2000) calls a process of 'intercultural learning' in the manner work has progressed in this thesis. She describes it in the following terms:

This learning process makes it possible to deal with different culture-bound conceptions, the reconstruction of one's old belief systems and practices, as well as the creation of something new (p. 206).

There seems to be a complex relation between being present in time and space, and being conscious of the Other in a partly shared experience. Personally, I have, sometimes painfully, become aware of the cultural character of discourse, and its effects on how we position ourselves in relation to each other. Something that has made me feel great joy is the thrilling experience that what we emphasize as 'cultural identity', most of the time is re-interpreted and re-created by people in interaction. Perhaps this is what J.N. Mohanty (2002, p. 19) has in mind when he writes:

There is a need of a philosophy of a subjectivity open to others, which has windows to the world, is responsible for and sensitive to others. This begins already with the body; my body responds to the other's presence as though an invisible tie links the two. There are shared thoughts, feelings and desires: however each may be with its unique perspective.

As part of the preliminary analysis, dualisms were focused, in order to find new aspects derived from context and discourse, with an aim to bring about a change in perspective. Perceived differences were shifted, and new differences and similarities have emerged, which are not interlocked or produced in contrast to each other, but which are valid in relation to the empirical material grasped on its own premises. In Nussbaum's *Cultivating Humanity* (1998), several examples are given of how 'Non-Western' concepts can be critically analysed. She takes the example of a lecturer discussing different terms which could be translated as 'compassion', and how these are compared to Indian Buddhist views. The lecturer concludes:

The Buddhist view is not easy to compare with the views of compassion or sympathy of thinkers such as Rousseau and Adam Smith. Unlike its Western counterparts (*my italics*), the Buddhist view rests on a radical attack on the concept of the individual self, asking us to respond to suffering in a way that denies the distinctness of each person's individual life course (p. 114).

But instead of focusing elements which are perceived as fundamentally 'different', we could look at 'compassion', in its own value-context, and see that there is a sense of responsibility attached to neglecting Self, with an ideal of reaching Buddha consciousness, where every action or thought is directed towards doing good for others. The contextual meaning is therefore different from the meaning which was found when the lecturer simply *contrasted* Western individualism to Buddhist compassion. 'Western compassion' is also a complex concept in its own right, with its own contextual meanings that are lost in simplistic comparisons of dichotomies.

My research approach in this thesis has been to find *contextual meanings* within *localised epistemologies* in my empirical material. In other words, my aim has not been to find differences (or similarities) for the sake of contrast, but rather to try to understand what different practices mean in their localised and culturally enriched contexts. The analysis process, which started with the beginning of the field studies, can be described as a process of continuous change in perspectives, where I consulted the material and reflected upon what was said, what was not said, and what I could see if I reflected upon the whole material, repeatedly asking myself and the material questions. Dahlin & Regmi (1997) has captured the underlying thoughts in this analysis process:

Actually, to hold that 'they' are just as rational as 'we' are, implies that we can understand them. Actually, we can even come to the realization that the way we first understood their ideas was less in accord with their own understanding. That is, the sense 'we' make of 'their' ideas can correspond better and better to that made by 'them' themselves if we engage in serious study and dialogue'(p. 18).

If we attempt to approach each other in a partly non-dual mode of thinking and being, we can learn from each other in a continued manner, where Self and

Other are a process of experiencing our shifting boundaries of Self. This means, of course, a change in thinking about social science research, and a new kind of language perhaps, as well (Seiffert 2008). The non-dual mode of being, thinking and expressing would remind us all that we are related to each other (Mohanty 2002):

Then one finds the other in oneself - I do not understand all my motives, choices and desires. The stranger and the foreigner are right in my neighbourhood. My culture and the other culture are not separated as the known, the familiar and the unknown and the unfamiliar but rather by degrees of familiarity, foreignness, strangeness. Sometimes only through the Other, I come to understand myself. At other times, the reverse happens. The boundaries are shifting (p. 99)

The Swedish empirical material had to stand for itself, just like the Indian material, not being interlocked with the findings in the other study, or produced by a focus on the most extreme differences and divergence. Although it was interesting to play with the thought of how 'contradictory' the results were, I struggled to avoid the idea of drawing 'tables of differences' (Giri 2004). I realised how imprisoned we are in dualisms and difference.

The binary logic which follows with a dualistic world view has been scrutinised by Chandra Mohanty (2003). She examines the discursive power of Western discourse on third world women. Her analysis shows how discourses can be oppressive under the cover of good intentions, as in feminist analysis of third world women as being predominantly passive, submissive and illiterate. Sen (2005) has argued that different cultures are generally interpreted in ways that reinforce the underlying political conviction that Western civilisation is somehow the main, perhaps the only, source of rationalistic and liberal ideas - among them analytical scrutiny, open debate, political tolerance and the agreement to differ. The West is seen, in effect, as having exclusive access to the values that lie at the foundation of rationality and reasoning, science and evidence, liberty and tolerance and, of course, rights and justice.

'Once established, seen in confrontation with the rest, this view of the West tends to vindicate itself, Sen maintains (2005, p. 285). Since each civilisation contains diverse elements, a non-Western civilisation can be characterised by referring to those tendencies that are most distant from the identified Western traditions and values. These selected elements are then taken to be more 'authentic', or more indigenous than the elements that are relatively similar to

what can also be found in the West. For example, Indian religious literature, such as the Bhagavad Gita or the Tantric texts - which are identified as differing from secular writings seen as 'Western'- elicits much greater interest in the West than other kinds of Indian writing, including India's long history of heterodoxy. Sanskrit and Pali have a larger atheistic and agnostic literature than in any other classical tradition. There is a similar neglect, Sen points out, of Indian writing on non-religious subjects, ranging from mathematics, epistemology and natural science to economics and linguistics. In this way, through selective emphasis that stresses differences with the West, other civilisations can be redefined in 'alien' terms, which may be exotic and charming, or else bizarre and terrifying, or simply strange and engaging. When identity is thus defined by contrast, divergence with the West becomes central (Sen 2005, p. 286).

Sen also draws the picture of how the West's image of India has influenced India's own views on culture and history. There are still images from West that continue to influence the way India is trying to 'catch up' with the technological societies or counter with discourses on 'spiritual preference' facing a materialistic or rationalistic discourse. From this argumentation, Sen concludes that one consequence of Western dominance of the world today is that other cultures and traditions are often identified and defined by their contrasts with contemporary Western culture (2005, p. 285). A deconstruction of Western culture would further contribute to showing the dynamics of this dominance, argues Derrida (1997).

During the interviews, I tried to understand how the learners understand the mathematical content and their views of mathematics. My intention was to shift the centre of communication from myself as an interviewer to the interviewee, and to shift the centre of mathematics from epistemology, to an ontological epistemology, opening possibilities for reflections on the nature of mathematics and on being a learner. Neither the knowledge content, nor the learner, the researcher and the interviewee, are taken for granted.

Sinha (1999) discusses the presupposition in most theories on learning, that there is a goal to characterise 'the capacities, processes and mechanisms' of cognition (p. 34). When taking into consideration that ontology and epistemology can be understood in participation in a learning practice, a *relational* view of knowledge is helpful - in other words, a view inclusive of ontology and epistemology, as well as the subject and object. In line with the purpose in the present thesis, the epistemology of mathematics the learner experiences is nurtured by an experienced ontology of mathematics, and of

ontological cultivation of Self. The learners are seen as performers of learning. The epistemology of mathematics, as experienced by the learner in interactions in the classroom, enriches the learner's understandings of mathematics and modes of being a learner.

Finally, the work on this thesis has been characterised by a wish to be open to learn and expand my own horizons. The following short and beautiful line from Giri (in press) has helped me to silence my ego, to be humble and share thoughts, not allowing myself to be trapped in my research in dichotomies of researcher/respondent, teacher/pupil or adult/child: *What can we learn holding the hand of the Other and looking up to the face of the Other?*

3.1 Methodology

The choice of a particular methodology, in preference of another, responds to the questions 'why' and 'why not?' Why did I conduct the conversations and explorations in the way I did, and what influenced me to choose to do the research in the manner described here?

My original idea was to explore how pupils solved mathematical problems when given different tasks under circumstances and conditions that I controlled. I also wanted to understand *in what qualitatively different ways the pupils varied their ways of thinking* in relation to what they were working with in different situations. This idea I soon abandoned, above all because I realised that the type of problems I wanted to study in classrooms would interfere with what the pupils were actually working with at that time. Also, the mathematical problems were not formulated in a way that corresponded to how the pupils were used to be instructed. Although this had not been the purpose of our conversation at that time, teacher M, the teacher in the Swedish study, helped me to understand several points where the research design needed to be altered. When she told me about how she looked upon my 'problem bank', I additionally gained valuable insights into how she understands what a mathematical problem is. The problem bank which I had initially constructed and the situations I had designed in order to observe the pupils' work were modified after the first visit to M:s classroom. In the course of this visit, I quickly understood that M:s teaching and the interactions in the classroom were in themselves valuable to explore. My idea of how to conduct the field studies shifted from having an implicitly experimental character, to striving for a naturalistic character (Merriam 2006).

An effect of the naturalistic approach adopted in the thesis was that the process of analysis started with the beginning of the field studies. Rather than

observing the pupils' problem solving in an 'experimental' manner, on premises that I as a researcher had defined in advance, I wanted to approach the situation from the learners' perspective, focusing the learner's own experience. I did not want to impose my prejudices on the object of my study, neither on the manner I collected material, nor in the way it was analysed. I also did not want to simply describe certain behaviour, but instead try to understand the significance such behaviour could have for the participants involved. These are some of the reasons why I subsequently chose to use a dialogue structure based in an intentional-expressive perspective to collect data (Anderberg 1999; 2003) Contextual analysis (Svensson 2005) was chosen as analysis procedure, to base analysis in the learner's experience and take into consideration the situational, contextual and discursive aspects of the teachers' and pupils' action and speech. Discursive possibilities to constitute learner identities - what I here call modes of learning and modes of being a learner outside the classroom - have also been considered.

During and after my first field study in Sweden, I wrote down preliminary reflections on what I had seen. I also wrote reflections during the Indian study. A major concern was that I had to 'go beyond' my own preconceptions on what constitutes 'good' teaching and a 'good' learning environment, since the comparisons I first made were both simplistic and dualistic.

In the first part of the research process, I gained a general idea of the phenomenon flexible experiencing I wanted to study. The specific focus was the *flexibility* in how pupils discern and delimit parts and wholes, and how they experience the relationships between the parts and wholes. The theoretical and empirical explorations ran parallel. The dialogue structure which was used as a guideline for my conversations with the informants was as much as possible based on a generosity of understanding. I tried to relate to a multi-valued logic (Mohanty 2002), which seeks to encourage interest to understand the Self and Other. My interpretation of multi-valued logic as an interview approach is that meanings and intentions should be explored and kept open. The process of understanding the informant's statements about aspects of a discerned whole in a mathematical instance was concerned with an assumption that the meanings and intentions were to some extent partly differentiated, partly undifferentiated and partly undetermined. This leaves a space for possible reflection on what aspects might be more critical than others.

3.2 Design of the empirical investigation

For the purpose of the study, it was necessary to use classroom observation to learn about actions of pupils and teachers in their context, explore how

language was used in a classroom, and see how mathematics was presented, worked with and talked about. The interviews with pupils and teachers were conducted to gain a picture from the learners' and teachers' perspectives on educational/classroom discourses, which gave meaning to the teaching and learning processes.

The selection of cases and the method used to gain empirical material aimed to reflect a variation and to get knowledge of the teaching, as experienced by the learner and approached in qualitatively different ways.

The two school class contexts were chosen mainly on the basis of recommendations concerning teachers who were known to have thought about and worked with children's reflections in mathematics. The case in Sweden consists of a class six, with 24 pupils aged 12 to 13. The case in India consists of a class nine, with 43 enrolled pupils aged 12, 13 and 14 (around 25 pupils, on average, were present at each lesson). The selected participants were ages 12 and 13. Both the teachers were class-teachers, teaching mathematics in the class for which they were class-teachers. These teachers were asked by me to select ten pupils of different levels of achievement and differences in how well they communicated on their reflections in mathematics.

I first did eight observations, followed by two interview occasions with the ten selected pupils from a class six at a school in Lund, Sweden. The medium of instruction was Swedish and the extracts from the observation material given in the present thesis were translated into English. This resulted in a total of 19 interviews, since one of the ten pupils went for a holiday abroad before I had the opportunity to talk with him a second time. The interviews with the pupils lasted between 40 minutes to one hour. Lastly, teacher M was interviewed at two different occasions. Each interview lasted one to one and a half hour. All the interviews were conducted in Swedish and transcribed in Swedish. Those extracts which are described in the part IV *Empirical Results*, have been translated into English. As part of the data collected, I also included a report, written by teacher M as part of a course for teachers at the university college of Kristianstad. Another report, which she wrote as a report for a project work, which she had received a scholarship for, is also a part of the data.

After collecting material in the Swedish context, I did five observations in the classroom of the selected class nine at the school in Balasore, Orissa, India. The medium of instruction was Oriya and the extracts from the observation material given in the present thesis were translated into English. The observations were followed by two interview occasions with the selected ten

pupils. A total of 19 interviews were conducted here too, since one of the selected pupils was absent from class during the last part of my stay in Orissa. Lastly, the teacher A was interviewed at two different occasions. All the interviews were conducted in Oriya, and the extracts given in the present thesis were translated into English. Included in the data material were also teachers' reflections on problem-solving. They were asked to write about three questions: *How do children learn? What does it mean to work with problem solving at school?* and *What does problem solving in mathematics mean in your classroom?*

3.2.1 Introduction to the field: the Swedish study

After a conversation with honorary doctor in educational science, Gudrun Malmer, in spring 2002, I proceeded to contact a teacher by her recommendation. Gudrun told me about teacher M:s teaching, which she found exemplary. She also said that teacher M had very strong opinions about her teaching and a commitment to what she was doing.

I met teacher M at the local teacher college, where she was working part time at that time. I gave details about the planned study and also about what questions I would like to ask the pupils. M suggested that I should come to see how she worked and to know about the pupils. She also discussed the proposed mathematical tasks that I wanted to give the pupils. She firmly stated that: *These are not mathematical problems. They lack context and don't carry meaning in themselves as problems. It is just tasks to do.* I had spent a lot of time on what I called a 'problem bank', consisting of a set of problems, to be tested in different learning situations. After the first visit to the class, which also was my first observation, I dropped the whole idea of doing controlled task-situations with pre-determined tasks. From then on, I assumed an open and explorative research approach. The observations became important for preparing conversations with pupils and with teachers.

Information to the parents was sent through the pupils' weekly report book. The parents were asked for permission to interview their children. The information stated that I was a PhD student who was going to study how the pupils were thinking in mathematics. The pupils were used to visitors, and carried on as usual, with little or no attention paid to me during the observations.

3.2.2 Introduction to the field: the Indian study

Prior to going to India, I wrote down a few reflections on the Swedish study. The aim was to make it possible to go back and compare aspects. The school I visited in the provincial town of Balasore, Orissa, central-eastern India, works

under the Orissa state government policy of education. Introduction to the field was done with the help of a family acquaintance, a mathematics teacher, we can call her S, who I had earlier observed and interviewed, during my bachelors studies.

S had now become the vice-headmistress, and did not teach mathematics any more. Vice-headmistress S suggested that we (my husband and I), should first talk with the headmistress of the school about my study. A proper introduction of how I am connected to the locality would be a good start to gain access. It was important to state my place of belongingness. Every new acquaintance begins with an exploration of the point in the grid of the local social life one belongs to. There is in many cases a hierarchical positioning of self in India (Roland 1988). People 'always perceive in any relationship a hierarchical order, wherein the superior and the subordinate will be connected emotionally, and in terms of reciprocal responsibilities and expectation' (p. 101). There is a high level of 'attentiveness to the other in the hierarchical relationship. The feeling of connectedness means that one's opinions, abilities and characteristics are assigned secondary roles and must be constantly controlled and regulated Markus & Kitayama (1991):

Thus, persons are only parts that when separated from the larger social whole cannot be fully understood (Phillips 1976; Schweder 1984 quoted in (p. 227).

The day we met the headmistress, my husband introduced himself and me in terms of information about common acquaintances, and about ancestral belongingness to certain villages around the provincial town. The fact that I am married to someone from the locality, that I lived with my mother-in-law in a small village when I came every year, that I spoke the language, wore sari and that I understood the culture was important information in order to make a social estimation of the hierarchal positioning and for the interaction with teachers and pupils at the school. I very quickly gained the status of 'one of the staff'.

After the personal introduction, vice-headmistress S entered the room. After my husband had explained briefly that I would like to see a few lessons and talk to a few pupils, S asked what my study was about. I said that it is about children's reflections in mathematics, and also mentioned that it is very important that the teacher encourages reflection and dialogue. S recommended that we talked to teacher A. Teacher A was called for. My husband gave teacher A a similar, but short-versioned introduction to who I am in relation to him,

and how I belong to the place through our ancestral property, not far from the town. Then he explained briefly that I would like to see a few lessons and talk to a few pupils. Finally, the focus of the study was stated as ‘a study of children’s reflections in mathematics’, with an additional focus on dialogue between pupils and teacher. Teacher A repeated the focus loudly to herself as if to confirm to herself and everybody that this idea was fine for her. Then we settled a day for the first observation. Introduction to the field was done in this manner, in order to state in which way I belonged to the cultural context. It is important to know each others’ belonging, to see how we relate to each other through this belongingness and to understand our points of reference.

Vice-headmistress S received me in the staff room and we walked together to the classroom. As we entered, the whole class stood up, as is usual when any teacher, headmistress or guest comes into the classroom. S introduced me very briefly as a guest from Sweden who had come to see them and who was interested to see how they worked. The introduction was kept brief so that I would be able to conduct the observations and avoid that I would be observed myself. The statement that I ‘was interested to see how they worked’, had the almost immediate effect that the pupils directed their attention towards the teacher and their work. I went straight down to an empty place. After Teacher A had resumed her lesson, the pupils’ attention was almost fully on the teacher. All throughout the observations, I had to take care to keep a low profile. It was too interesting with a guest from faraway Sweden to keep on as usual. As I spent most of the time inside the staff room, the contact with the pupils was mainly when I came into the school yard and walked across to the staff room, or when I walked down the corridor between classroom and staff room.

3.2.3 Observations

I first did observations, a total of eight in the Swedish school class context, and five in the Indian school class context. In the Swedish study, the observations were spread over a period of five months. This was a good idea, since I could study two themes, one on Fractions and one on Time. I chose to interact with the pupils if they asked for some information which didn’t need a longer response, since I mainly saw the observations as a background to my conversations with the pupils and the teachers. As the observations progressed in the Swedish school class, I also focused on introductions of a theme, opening for questions, half class activities, patterns of discussions, and work with the concrete material. In the Indian study, the observations were conducted within a period of two months. I could observe lessons on both Mensuration and Euclid’s theorem. Specifically, as the study progressed and I did more observations, I focused on patterns of teaching during a lesson, and

on progression during a lesson and over a longer period of time. I looked at when and how the pupils were invited to talk, how they were asked to begin their work, how they got further instructions, the ways the lessons ended, what interruptions occurred, and of what kind. I observed how examples were presented, the dialogue in the classroom when working with examples outside of the text book, introduction of new examples or themes, working with tasks which required lengthy solutions.

3.2.4 A qualitative and explorative interview approach

In social science, empirical research design is frequently formulated as if the language, strictly controlled by the researcher, is a neutral and static tool through which s/he can mirror and control the reality, 'out there'. Language and language use are seen as something 'representing' meaning external to the interview situation. The assumptions made on language and language use in the present thesis follow the intentional-expressive perspective (Svensson 1978; Anderberg 1999). The meaning of expressions and content of conceptions are seen as an ongoing activity in relation to the learning context. There is an emphasis that all language use can be seen as having a productive, functional, interactive and context-dependent character.

A dialogue model (Anderberg 2003; Anderberg et al. 2005; 2006) based on the intentional-expressive perspective on knowing has been adapted, and used as a guideline for my interviews. The model was developed in relation to theoretical and empirical investigations, with the purpose of understanding the relation between language use and learning (Anderberg et al. 2005; 2006). A theoretical assumption underlying the model is that the meaning a verbal expression used in a particular context takes is a question of which understanding is developed by the learner, and how this evolves. In other words, expressions are seen as based on the learner's agency.

As in the phenomenographic interview, this dialogue has two parts. The first part introduces a problem which the interviewee is asked to handle in an open and concrete form. In the second part, the interviewee h/er/himself has to discern the phenomenon and distinguish it from the situation as a whole (Marton & Booth 1997, p. 130). The structure of the interview makes it possible to encourage the learner to reflect over h/er/his understanding of content, how s/he understands the nature of mathematics and h/er/himself as a learner. At the same time, a first analysis is made during the interview (Anderberg 1999; Theman 1985). The interview follows the following structure (Anderberg et al. 2006, pp. 2-3):

1. The original question
2. Analysis of the function of expressions and meanings used
3. The original question

The first phase of the interview is where the (in the present thesis mathematical) content is presented by the researcher in the form of a question. Here, time is spent exploring how the interviewee understands the content. During the second phase, the focus is on the expressions (in the present studies both verbal and mathematical), that have been used by the pupil to describe the understanding of the content. The pupil is invited to reflect upon these key-formulations. The interview finishes by returning to the question addressed at the beginning. Here the researcher takes time to verify that there is a correspondence between the pupil's statements, and the researcher's own interpretations of the pupil's understanding. The use of the dialogue-model made it possible to explore modes of knowing and to make a preliminary analysis. In asking questions about a mathematical problem or a theme, I wanted to explore:

- Which aspects the pupils were focusing.
- What meaning they gave these aspects.
- What they understood of these aspects in relation to how they understood the problem/theme/mathematics.

The aim of the interviews was to make a naturalistic study (Merriam 2006) of how the pupils in the two school classes experienced mathematical content and their views on mathematics. The specific research focus was to understand the nature of shifts in their ways of knowing from an interpretation of the pupil's talk or writing. The mathematical content dealt with during the interviews differed between the studies, but was the same for the different interviews within the respective studies.

In the Swedish study, I first gave the selected ten pupils a hand-out with a number of mathematical problems, similar to those I had seen them working with (Appendix 3). During classroom work, they had had fraction-sticks (paper cut out into rectangles with different length, and equal width, a concrete material the pupils themselves had produced according to teacher M:s instruction). The interviews were conducted in a room adjacent to the classroom, during lesson-time. The first interview began with asking questions to understand how the pupil experienced mathematics. Then followed questions concerning the tasks on the hand-out. The second interview focused the documentation the pupils had done on the theme 'Time'. The pupils were

asked to talk about a mathematical content of their own choice within the thematic project work. This became the topic, and then the interview followed the structure of the dialogue model, as presented in section 3.2.6. below. The topics 'My 24 hours', 'Vasaloppet' (Vasaloppet is a famous cross-country skiing event that takes place every year in Dalacarla, in central Sweden) and 'Maths in Town' (See Appendices 4 and 5), were three thematic projects the pupils had worked with in the classroom, and which they talked about during the second interview.

In the Indian study, during the first interview the pupils were given a matchstick problem. My idea was to see how the pupils responded to area-problems set with matchsticks, to give them a mathematical problem that related to what they were doing in class, but which had a non-textual character. The purpose was to shift their attention from text to talk. I wanted to find out how they responded to a different design, and a different way of posing questions concerning the mathematical concept 'area'. During the second interview, the pupils were asked to talk in general about mathematics, and to state a problem that was difficult to understand. The pupils were asked to explain how they had understood it, and were encouraged to also write the solution.

3.2.5 Interview settings and structure

During the interviews with the pupils, my intention was to make the atmosphere comfortable, so that they would feel that what they said was important and that I was not interested in testing their skills in mathematics, or to get correct answers. In the same way, I explained to the teachers who participated that it was not my purpose to evaluate their teaching. I particularly made an effort to make the situation seem as a continuation of the classroom talk, referring to classroom activities and talk which I had observed. The focus was first on classroom activities, and during the interviews the focus changed to be concerned with the mathematical content. During the first interview, general questions concerning the pupil's view of mathematics were asked. The second interview followed the structure of the intentional-expressive dialogue model. This dialogue model encourages the individual pupil to reflect over their language use and what meanings they use in relation to the content (Anderberg et al. 2005, 2006). During the interviews, incoherencies which appeared in the pupil's own description of h/er/is solution of a mathematical problem were further discussed, using follow-up questions. These incoherencies were interesting, since they shed light on flexibility in knowing. The structure of the dialogue model (Anderberg et al. 2006) helped me to design the interviews in two steps:

1. The conversations started with a question on knowledge content. Here I began by asking the pupil to solve a problem (Indian study); or to describe h/er/is theme work (Swedish study). Then I listened and asked questions so that the pupil would h/er/im-self describe the solution or some aspects of the problem or the theme.
2. The next step was designed so that the pupil could reflect over certain key-formulations and express what these meant. I let the pupils use pen and paper if they liked. Everyone produced some kind of writing. Not only were the pupils encouraged to reflect upon their use of language meaning; the reflective processes encouraged during interviews also gave them the possibility of understanding their intentions and their agency and authorship.

Listening to the pupil talk, I focused on aspects that seemed to be crucial to the pupil's description, especially incoherencies in their descriptions, and we talked around these. If some incoherencies were found in what the pupil said, I asked questions making it possible for the pupil to reflect: did you mean like this and like this...? The pupil was encouraged to talk. As the interview progressed, I gave new problems, or asked the pupil to give me another example of a task/theme. Every conversation was unique, but the focus was on aspects which the pupils themselves gave meaning to.

The interviews with the teachers were conducted at the end of each study, so that all information from classroom observations and interviews with pupils could be used as a background to the teacher interviews. During these conversations, I had interview guides with a number of question areas, which I had focused on during observations (Appendices 1, 2, 9 and 10). During the interviews, I also encouraged the teachers to develop further those aspects and themes that they themselves had discerned while talking about teaching and learning, using a similar structure to explore intentional-expressive meaning as during the interviews with the pupils.

3.2.6 Analysis

All interviews and most of the observations were recorded on audiotapes and transcribed. During some of the observations in the Swedish study, only notes were taken, to make the teacher more comfortable with the observation situations. During all the observations, reflections were written in notes.

The analysis of the material consisted of two parts:

- Analysis of flexibility in knowing and its relation to context
- Analysis of the educational and classroom discourses and their relation to modes of being a learner and approaches to learning

The analysis of the empirical material was a process parallel to the theoretical exploration. The statements were interpreted from two different reference points (Theman 1985). There was an *outer reference point*, where the statements were considered in relation to the whole material, and an *inner reference point* within the individual interview. Every statement has a content, and expresses a way of thinking. The meaning the researcher ascribes to the statement has to be documented in terms of the inner reference (Theman 1985). The inner reference is concerned with the relation between a specific statement and other statements, and the context in which the statement was made during one conversation. The outer reference is represented by the meaning the researcher finds that content and approach to content express in different statements, within one conversation, or in other conversations. In the present thesis, it includes the issue of how these statements represent a mode of knowing.

From the preliminary analysis of the empirical material, I observed that there were contextual, discursive and experiential aspects which need to be considered in order to understand flexibility in knowing. At the same time, these aspects were also an outcome of the theoretical exploration that ran parallel to the empirical investigation. The learners' performance of learning was seen as depending on agency and authorship. The variation in learners' ways of understanding school mathematics was understood in relation to individual meaning-making processes and discursive possibilities afforded in the learning context.

From the beginning, 'flexibility in knowing' was the research focus. However, the picture of the object became clearer as the conversations with pupils and teachers progressed. It was helpful to make a number of observations before the plan for the interviews was made. The observations gave me a preliminary insight into the conditions for flexibility in knowing. The phenomenographic analysis method has been described by Marton and Booth in the following terms (1994, p. 133):

The analysis starts by searching for extracts from the data that might be pertinent to the perspective, and inspecting them against the two contexts: now in the context of other extracts drawn from all the

interviews that touch upon the same and related themes; now in the context of the individual interview.

In the meantime, as the analysis of the empirical material progressed, the theoretical explorations made me gain a better picture of flexibility in knowing school mathematics. The process of delimitation was theoretical as well as empirical.

In the phenomenographic interview, the researcher's intuitive way of understanding others' intentions is explored (Theman 1985). Differences in language use are read as possible variations in how something is experienced. The interview transcripts are referred to as a pool of meaning, from which the researcher can try to make sense of the material (Marton & Booth 1997). But, as Runesson (1999) has pointed out, it might be labelled a *pool for meaning*, to include the researcher's own process of understanding the meaning that the material stands for.

Contextual analysis aims at understanding a phenomenon - in the present thesis flexibility in knowing school mathematics - from how it appears from the empirical material (Svensson 1978). Although there was from the beginning only a vague idea of how the phenomenon looked, it became necessary in the process of collecting data to increasingly discern the phenomenon from the empirical material. The categories of modes of knowing represent groups of cases of flexible experiencing, based on observed similarities and differences.

The procedure of contextual analysis is based on the reduction of empirical material into a description of 'internal relations' emerging from the analysis in relation to the context in which the conversations and explorations took place. These internal relations, experienced by the learner, are described in sections 2 and 3, Part IV, *Empirical Results*. Svensson (1989) describes the analytical and contextual qualities of contextual analysis in the following terms:

It is interpretative when searching for the meaning of specific data in relation to other specific data and in relation to the whole material, and analytic when trying to sort out what are significant main aspects and parts of the data and the phenomena studied(p. 530).

The analysis process also involved making a continuous shift in understanding, to allow other ways of understanding, a situation of 'both/and' instead of the Cartesian 'either/or' (Taguchi 2004, p.177).

The analysis process started with the formulation of the aim of the study. The interview structure allowed for the analysis process to begin already at the interview occasion. The themes of questions were prepared against a background of what had been noticed to be important features of the learning context during observations. The analysis process continued through the process of gathering data from the field, and in reading relevant literature. As I read through the transcriptions from the interviews with the teachers, I looked for *topics* and *aspects of topics*. From the beginning, I had an idea that the teacher's supply of possibilities for learning would be foreground and background to the pupils' use of and experienced possibilities to learn. The content in the empirical material was then analysed, exploring the meanings the individuals intended to express about parts of the world. On this subject, Svensson (1997, p. 170) writes:

The basis for differentiating thought and language, and conceptions and formulations in the interview, is to be found in the content of what is said in the interview. Content is, then, not primarily considered in terms of meanings of linguistic units, but from the point of view of expressing a relation to parts of the world.

The topics which the teachers and the pupils discerned during interviews and observations were the first I looked for. These topics, as well as different aspects of them, were either implicitly or explicitly expressed. Thereafter I turned my attention to all the material I had gathered: hand-outs, observations of discussions in classroom and from interviews. I selected some extracts where I found that there were incoherencies in what the pupils expressed using verbal or mathematical language. Then I tried to understand these extracts in relation to the whole material. Some questions became important to answer as I attempted to understand the extracts in the contexts. In the analysis of the empirical material, it appeared that the discursive possibilities for being a learner were *negotiated*, in both the Swedish and the Indian school class context. Most of the pupils experienced themselves as knowledge producers and language users. The pupils constituted themselves as learners in qualitatively different ways. The analysis took its point of departure in the following questions:

- In what ways did the pupils in the Swedish study understand the mathematical ideas they were working with?

- Why did they mostly speak about what they did and give a step-by-step account of their strategies, considering that they did now work with mathematics in that way in the classroom?
- Why did all pupils in the Indian study say that mathematics is about understanding when, as it seemed to me first, they were learning by rote and were practically ‘served’ the content right out of the textbooks?

I tried to see if I could find any recurring topics which had to do with how the pupils in the two cases *constituted themselves as learners* and which *discourses* were offered by the classroom discourse and educational discourses. In sections 2 and 3 of Part IV, *Empirical Results*, the analysis of the material is presented, based on these and similar ways that the theoretical and empirical explorations have enriched each other. The classroom discourse is immersed in localised epistemologies, teaching traditions, policies and structures, which make teaching altogether a cultural activity, where language plays a dominant role. The themes that emerged from the teachers’ and pupils’ talk during the lessons I observed were analysed in relation to the observed teaching practice, as well as themes the teachers chose to elaborate upon during interviews. *Classroom discourses* are described in relation to ways the teacher and the pupils constitute mathematics. Forms of organising learning situations are focused, and how language was used to underline the teachers’ intentions, which were found to be partly contradictory to the pupils’ intentions. The classroom discourse includes:

- The approaches to the nature of mathematics (its ontology)
- The teacher’s ways of dealing with mathematical content - Jaworski, Wood & Dawson (1999, p. 180) proposes that the term pedagogical power be defined as ‘the ability to draw in whatever pedagogical knowledge is needed to solve mathematical problems’
- Engagement in forms of knowledge

1. Interview transcripts
2. Notes and transcripts from observations
3. Classroom material and policy documents

A. The following data material was the basis for the analysis of flexibility in knowing:

1. Number of transcripts of interviews with pupils: 18 in the Swedish study and 18 in the Indian study. A total of 36 interview transcripts.
2. Number of observations: 8 in the Swedish study and 5 in the Indian study. A total of 13 observations.
3. In the Swedish study: 9 hand-outs I had made (Appendix 3) .1 sample of the thematical project work report on 'Time, containing worksheets, assembled by teacher M from different pupils' reports. Notes from pupils produced during lessons and interviews.
In the Indian study: One textbook in mathematics for class 9. Notes from pupils produced during interviews.

B. The following data material was used for analysis of what discursive possibilities were offered for learner identity, in terms of agency and authorship, and for an analysis of teachers' intentions concerning their teaching practice:

1. Teacher M in the Swedish study was interviewed three times. Teacher A in the Indian study was interviewed two times.
2. Number of observations: 8 in the Swedish study and 5 in the Indian study. A total of 13 observations
3. Educational plans for compulsory school in Sweden (LpO 94) and National Plan of Educational (1986), as well as other relevant reports and policy documents.
Two reports written by M in the Swedish study.
Written reflections on problem-solving by 9 teachers, including by teacher A in the Indian study.

Figure 1. An overview of the collected data material.

I carried out the analysis so that I could see how categories emerged from the interviews, the observation material and task solutions, and focused on statements made by the selected pupils and teachers. The delimitation of flexible experiencing was made in the context of the two studies, and in relation to the theoretical perspectives I used to interpret the material.

In a second step in my analysis, the incoherencies initially identified during the interviews became my focus of interest. After many readings of the whole interview material, three modes of knowing could be described. The *categories of description* represent the qualitative variation of flexible experiencing. These findings are described in detail in section 1 in Part IV, *Empirical Results*. The first section of Part IV, *Empirical results*, is a presentation of three modes of knowing, as they emerged from the contextual analysis. A few exemplars illustrate each of these modes. The second and third sections are descriptions of the educational and classroom discourses, which were found to frame the teachers' approaches to teaching and their pedagogical power, defined by Jaworski et al. (1999) as 'the pedagogical knowledge a teacher has to have access to' in order to solve mathematical problems. The educational and classroom discourses also are expected to frame the learners' modes of knowing and modes of being learners. The empirical results are presented below, in sections 1-3 of this part, while in Part V, *Discussion and conclusions*, relations between flexibility in knowing school mathematics and the contexts are discussed.

Taking the learner's perspective, I attempted to understand, within the experienced learning context: the learner's intentions, approaches to learning, understanding of content, and modes of being a learner. Finally, in Part V, *Conclusions and discussion*, the differences in meanings related to the contexts of study are described, and the usefulness of the categories is discussed in relation to the conclusions.

IV Empirical results

1: Three modes of knowing

1. Modes of knowing

One of the main purposes of the present study was to gain insight in *flexibility in knowing school mathematics*, and to understand flexibility in knowing in two different school class contexts, one in Sweden and one in India.

Flexibility in knowing school mathematics is in the theoretical section of this thesis dealt with in relation to theoretical foundations in phenomenography, variation theory and feminist epistemology of mathematics outlined by Burton (1995), using in particular the concepts *agency* and *authorship* (Burton, 2004; Holland et al. 1998; Davies 2000). This part of the result section deals with the modes of knowing which emerged from analysis of the empirical material. The exemplars presented are those which I found to be most illustrative. I have chosen to give exemplars from both the school classes, in order to provide an overall view of the outcome space. The qualitative differences identified *within* each of the three modes of knowing are discussed, in relation to which school class context the exemplars are derived from.

This part starts with a general description of the three modes of knowing. Then follows a description of a few exemplars for each of the categories. Finally, conclusions are drawn concerning how these exemplars can be understood in their context.

1.1 Three categories of knowing

As my field studies progressed, the following categories of description were discerned:

- Associative flexible experiencing
- Compositional flexible experiencing
- Contextual flexible experiencing

When a learner experiences a difference between arbitrarily discerned aspects in a mathematical problem, and uses the difference to make a problem-solving

strategy, it can be described as *associative flexible experiencing*. The strategy is based on seeing contrast (Marton, Runesson & Tsui 2004). *Contrast* can be experienced between two numbers; e.g. 2 and 3. There is a narrowing of focus to cues that can be found in the immediate task, or from the teacher's presentation/discussion. The way of only seeing contrast makes understanding of content undifferentiated. Since no invariant aspects are distinguished from variant aspects, all variation is perceived as equally arbitrary.

When a learner experiences simultaneously structural and referential aspects of variation (Marton & Booth 1997), it leads to an increasingly differentiated understanding. This is what characterises both *compositional* and *contextual flexible experiencing*. When the learner uses this experienced variation, it can be used on the basis of an experienced relevance within the mathematical logical content, as in compositional flexible experiencing, or of an experienced contextual relevance, as in contextual flexible experiencing.

In *compositional flexible experiencing*, there are generalised experiences of variation, and separation of invariant aspects from variant aspects. For example, in one of the exemplars given below, there is a generalised understanding that dm^3 and m^3 have different appearances. There is also an understanding that there is a decrease in numbers of zeroes when one determines how many m^3 are equivalent to 1 dm^3 .

The *compositions*, the combining of different parts to make a whole, are sometimes 'view-turned'. Here, I use Ahlberg's (2004) definition of view-turns (*synvändor*): 'what people experience constitutes changes in ways of experiencing something'. In compositional flexible experiencing, the change in how the pupils experience a whole is based on which *compositions* they discern within the mathematical problem. 'Compositions' are here taken to be any relatively stable set of relationships between aspects that the pupils discern, and which are used as an argument in their reasoning. By 'stable' I here mean that there is a certain measure of consistency in the pupil's explanations, but this does not preclude changes in the relationships that are discerned in the course of an interview, or that the pupils may shift between different arguments or points of focus.

Contextual flexible experiencing is based on comparing and contrasting with other problems, an understanding of what the particular problem-situation requires, and what the mathematical principle or relation means, so that different aspects can be fused (Marton, Runesson & Tsui 2004) in relation to the specific context of the problem. For an example, in one of the exemplars given, the number 1 in the task $1 - 1/5$ is understood as a whole and corresponding to

one unit of 50 candies. After eating $\frac{1}{5}$ of the whole, 40 candies were left. The understanding that a 'whole' can correspond to many sub-items - in this case, candies - and be represented by the number, is an example of fusion.

In compositional flexible experiencing, mathematical principles or relations are seen as compositions. This implies a deduction of the parts of the compositions from their mathematical relevance structure. It also means a reduction of the compositions from the context of the task. The strategies can give good results, but in order to know in what way the compositions are understood mathematically, the teacher and the pupil have to meet in communication. By contrast, in contextual flexible experiencing, mathematics is connected to meanings in different contexts, and there is an understanding that the meaning varies with the context in which mathematics is used.

1.1.1 Two exemplars of associative flexible experiencing

Two exemplars of *associative flexible experiencing* are illustrated in the following two extracts, one from an interview with V and the other with D, both from the Indian school class. These were the only two cases of associative flexible experiencing I found in the entire material.

-∞-Exemplar from an interview with V-∞-

V is a low-attaining pupil who seems to be very unsure of what she knows in mathematics. Teacher A confirmed that she is a very low-attaining and silent pupil. I tried to make the interview situation as comfortable as possible. I showed non-verbally and also expressed verbally that I was interested in what she was thinking, and that I was not interested in whether it was correct or not.

The following extract is taken from the last part of the first interview with V. I could only meet V once, since the other planned occasion was cancelled, due to the fact that V was absent from school. I did not get so much information on how V understood the questions and tasks I gave her. I started with the same question as I posed to all the selected pupils in the Indian school class. The question was to calculate the area of the square given below, given that the length of one matchstick is 2 cm (the length of the side in a small square in the figure is 2 cm):



V drew two squares. I asked her what area is and then I asked if she could tell me what a square is. When V remained silent, I finally asked her (ID stands for the interviewer):

ID: Can you give me an example of a mathematical problem you have done that you have understood?

ID: Is this what you have recently understood?

V: Yes.

V writes the solution to an algebraic expression (see below).

ID: What was the question to this?

V: Question and answer.

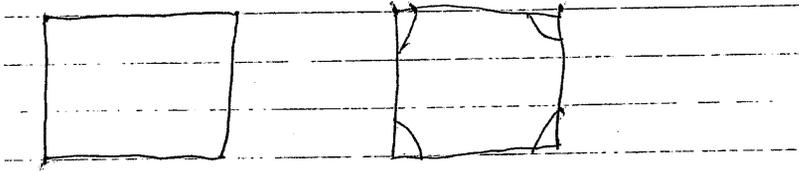
ID: What do you understand from the question?

V is silent.

ID: What does the question mean?

V is silent.

Here is what V drew. She first drew two squares, when I asked her what area is. When I asked her to tell about what she had recently understood, she wrote the whole solution to a problem from set theory:



$$(0) \quad (2) \quad \left(\frac{5}{4}\right) \quad (10)$$

$$= +\sqrt{0^2+4}$$

$$a^2$$

$$= +\sqrt{(2)^2+4} = 4+4 =$$

$$= +\sqrt{0^2+4}$$

$$= +\sqrt{0^2+4} = 0+4 = 4$$

$$= +\sqrt{(2)^2+4} = 4+4 = 8$$

$$= +\sqrt{\left(\frac{5}{4}\right)^2+4} = \frac{25}{4}+4 =$$

It seems to me, that V wrote the values for an algebraic expression first: 0, 5/4 and 10. Thereafter she proceeded to find out the value of the expression, by including the given values. She has forgotten to state the algebraic expression. For me that implies that the calculations are seen by V as more important than the question itself. The algebraic expression she wants to calculate the value for seems to be the square-root of the sum of a squared unknown and 4. She seems to have a vague idea of how 'the solution should look'. She consistently writes the plus-symbol in front of the square-root.

V mainly attempts to give an answer to a question, without necessarily understanding the question in the first place. She cannot tell me anything about the question, and from her silence I can only guess that she has focused on the whole sequence of question and answer. She has in fact not discerned a single aspect from the content of the task. Hence, the task is not understood as a

mathematical problem at all, only some task to do. As the question and the answer is understood as whole, there is no point, from V:s point of view, in making sense of any aspect of the content. This is an example of an extreme surface approach, which is not atomistic.

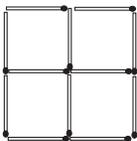
The answer is given as a response to my request to her for an example of what she has recently learnt. Her response indicates that she is used to demands to give correct answers to questions. V has given up her authorship to the extent that she seems to feel that every question has a specified answer. If she can reproduce the specified answer, she is able to answer, but otherwise her only option is to remain silent. Wistedt (1994) found in her case study, described on page 9 in the present thesis, that the rules communicated during the instruction of a task may be interpreted in different ways by pupils. Pupil V has probably experienced school mathematics as being communicated in a direction from teacher/textbook to her. She has interpreted the intentions of the teacher as providing correctly reproduced answers to questions, as there is no communication on alternative ways of understanding the task. The result is that V engaged in reflection over formal matters only.

Here, the *flexibility* has a character we do not ordinarily notice. The flexibility is concerned with an arbitrary way of discerning as much of the formalised mathematical expressions as possible, to 'fit' to the question asked.

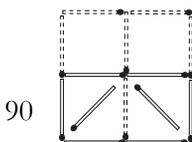
-∞-Exemplar from an interview with D-∞-

Next, we look at D:s arbitrary shifts in how she combines numbers in arbitrary ways to produce self-invented formulas.

Here I have posed the same question as mentioned above, namely to calculate the area of the square, given that the length of one matchstick is 2 cm (the length of the side in a small square is 2 cm):



Then I placed the following figure with matchsticks:



The length of one match-stick is still 2 cm. Note that two of the matchsticks that were placed to make two sides of the triangle were a piece of a broken matchstick. There was a possibility for the pupil to see that the inscribed triangle's area is equal to the area of the small square in the first example, or half the area of the rectangle.

ID: How large is the area of the inscribed right-angled triangle?

She writes and speaks aloud while writing:

D: The area of the quadrangle is 3 into 2 is equal to 6 centimetres ('into' here means 'multiplied by')

My reflection: Does she mean that the length is 3 cm and the width 2 cm? The length is actually 4 cm. Two match-sticks long.

D: The area of the triangle is 4 cube, is equal to 4 into 4 into 4 is equal to 64. ('into' also here means 'multiplied by')

D writes:

$$4^3=4 \times 4 \times 4=64 \text{ (Ans)}$$

My reflection: What does she mean by 4 to the power of 3, or as D says, 4 cube? Does she experience the relation between the 4 and the cube as a relationship between side and area? In this case, the length, which is 4 (cm) is cubed. Or does she merely manipulate with the numbers to get 4 and 3? In that case 4 and 3 can mean anything or maybe nothing at all.

I place the following figure in front of us and ask how big the areas of the two triangles are (the length of one match-stick is still 2 cm).



D writes and at the same time she says:

D: The triangle's area is: $64/2 = 32$

D: The triangle's area is 32

D speaks with certitude. She must believe that she is doing mathematics, but these ‘self-invented’ strategies do not have a consistent inner logic. She did not see first that the right-angled triangle’s area and the area of the quadrangle were related to each other and could have been calculated with ease, if D had understood the concept ‘area’. This lack of understanding she compensated with manipulation of numbers and formalisation into algebra. The consequence of her first attempt to formalise the calculation of the area of the triangle is that she starts with a numerically wrong area, although the strategy of dividing the area of the right-angled triangle into 2 to get the area of one small triangle is correct. The problem is that she most arbitrarily chooses the answer she got from her calculation of the area of the right-angled triangle and divides it into two. We cannot know how she reasoned when she did these calculations, or what her conception of area is. The flexibility is here characterised by a high degree of discontinuity in focus.

We can see that although D writes and speaks with a trust in her ability, she uses tools which are not useful in order to solve the area-problems. Her concern is mostly with manipulation with numbers and denotations. These tools seem to be experienced as external to her own authoring. She understands her knowing as dependent of others’ authoring. Nevertheless, her agency is independent, and she is active in her work.

1.1.2 Four exemplars of compositional flexible experiencing

Four exemplars of *compositional flexible experiencing* are illustrated in the following extracts. The first two extracts are taken from interviews with C and S from the Swedish study. The last two extracts come from interviews with D and N from the Indian study.

-∞-Exemplar from interview with pupil C -∞-

C talked about how he understands volume units. The extract begins after C has explained that it is like ‘going backwards’ when one figures out how many m^3 there are in 1 dm^3 .

ID: You mentioned that there are 1000 dm^3 in 1 m^3 . How do you get this transition when you ‘go backwards’?

C: Well, there are 1000 dm^3 in 1 m^3 , that you know. 1 dm^3 is 0.001 m^3 . Then it’s centimetres, then you have to add four more zeroes. Thereafter you divide with another ‘notch’ (*snäpp*).

My reflection: Here, C seems to mean that the next transition after 0.001 m^3 is to $0.000\ 001 \text{ m}^3$. He says that you ‘add four more zeroes’. He explained earlier in the interview that he counts the first digit zero before the decimals.

ID: One ‘notch’, means that you ‘go down’, then... So how many times do you ‘go down’ per ‘notch’?

C: Four zeroes. But *there* you don’t go down four zeroes, *there* you go a little differently. C indicates that he has written $1 \text{ m}^3 = 0.000\ 000\ 1 \text{ km}^3$.

ID: So what about *there*? (ID points to where C has written $1 \text{ m}^3 = 0.000\ 000\ 1 \text{ km}^3$).

C: Eh, two ‘notches’. That’s 1000, because you have 999 so that you add a 1 there so that it becomes 1 there. That’s something you know. (C points at the last digit 1 in $0.001 \text{ m} = 1 \text{ km}$ and adds 999 to the 1 and gets 1.000).

My reflection: C says two things in this statement. He says that there are two ‘notches’ (snäpp) between 1 m^3 and 1 km^3 . Two notches would correspond to 8 zeroes, including the first digit zero, following his earlier reasoning. Then he would arrive at $1 \text{ m}^3 = 0.00\ 000\ 001 \text{ km}^3$. But, as we shall see, he takes the help of the next line of reasoning, reflected in the next thing he states, to get the correct answer $1 \text{ m}^3 = 0.000\ 000\ 001 \text{ km}^3$.

The second thing he says is that you can add 999 to 0.001 to get 1. In the last statement he shifts his focus from $1 \text{ m}^3 = 0.000\ 0001 \text{ km}^3$ and focuses instead on how to get to a length of 1 m by adding 999 units to 0.001 m.

C returns to look at where he has written $1 \text{ m}^3 = 0.000\ 000\ 1 \text{ km}^3$

C: But here it is not completely correct. It should be 8 zeroes. He writes 0.000 000 001 in his notebook.

ID: How do you know that there should be 8 zeroes?

C: I don’t remember how I calculated, but there should be 8 zeroes and there it should be 12 zeroes. (C points at where he has written $1 \text{ dm}^3 = 0.000\ 000\ 0001 \text{ km}^3$).

ID: If you use the same method to help you that you told about before, from kilometres to cubic kilometres. How do you do it then?

Silence

ID continues: You have mentioned two things. You have said that ‘one notch’ means that you ‘go down three steps’, and you have said that one can add 999 there.

Pause

If you look at km^3 , then how do you think about it? How do you think similarly here? (ID points at $1 \text{ m}^3 = 0.000\ 0001 \text{ km}^3$). Or do you only remember a certain number of zeroes?

C: No, no, you think like this. Here there are metres, then you add four zeroes. (referring to the transition between m and km).

ID: Why exactly four zeroes?

C: Because there are 1000 metres in 1 kilometre, then you add like that. You divide with 'two notches', it becomes 1000 000.

C understands the transition between 1 m and 0.001 m, which is equal to 1 km, as a view-turn. He reads the number 0.001 from right to left, including the first digit zero, and gets 1000 km. Then he knows that the transition from 0.001 m to km^3 means 'two notches'. He arrives correctly at $0.000\ 000\ 001 \text{ km}^3$. To check that this transition is correct, he could again read from right to left, including the first digit zero, and get $1000\ 000\ 000 \text{ m}^3$, which is equivalent to 1 km^3 .

As part of a generalised conclusion, C understands the transition between units of volume as 'one notch' or 'four zeroes'. He looks at the transitions as compositions where the parts can be view-turned and reads the numbers from right to left. 1 dm^3 becomes 0.0001 m^3 . He showed me during the interview that he writes 00001, a view-turned 1 with four zeroes. The 'point' is put afterwards. He only needs to know that there are 10 dm in 1 metre. C reads from right to left and then it becomes 1000, which for C means that 1000 dm^3 is 1 m^3 . The flexibility consists of a view-turn of reading from right to left and from left to right, when appropriate. It also means a separation of variant aspects from invariant. The notches are invariant and the transitions varies; from units of length to units of volume. There is also a simultaneous focus on how many numbers of digit 9 should be added to make a full unit. However, the flexibility is based on a reduction of parts from wholes, and the contextual relevance is not considered here, although the thinking about the notion that there is room for 999 units in a 0.001 unit perhaps is derived from reflections on previous experiences of working with concrete material. This way of seeing has become abstracted. C has an independent authorship, which means that he can correct his own 'mistakes'. He says: 'But here it is not completely correct. It should be 8 zeroes.' And then he writes 0. 000 000 001 in his notebook. Both agency and authorship are independent.

-∞-Exemplar from an interview with S -∞-

Here we enter the dialogue when S has talked about what she remembers from the lesson on factorisation and after we have completed that part of the interview. I introduced another theme of discussion:

ID: Sometimes M says 'Now you should discover something'. What does she mean by 'discover'?

S: Yes, well, 'discover' one should perhaps do to get it easier and to see the system, the pattern.

ID: Can you remember a pattern right now?

S: Eh, well, no.

ID: Is there something else one can 'discover', something you remember, where you learnt something? When you felt that you learnt something?

S: Yes in grade four, when we had multiplication table. Nine times six and nine times seven, the difference is nine between them.

ID: Yes, that is a pattern

Do you remember when you worked with factorisation?

S: Yes, there was an example of 3 times 3, times 3. 3 times 3 is 9. And 9 times 3 is 27.

ID: Do you remember when you have worked with this, when M didn't talk about factorisation?

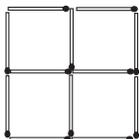
S: It was when we were working with cubes. Length times width times height. If the length was 5, width 2 and height is 2, then it becomes 20.

S gives an example of a pattern, a composition wherein the factors are multiplied. This is a valuable insight, since she can use this pattern if she understands the relation between the factors in the two compositions she mentions; $3 \times 3 \times 3$ and Length \times width \times height. We cannot, however, tell whether she understands that the relation between the factors in $3 \times 3 \times 3$ can be seen as a numerical expression of a cube- relation, which also can be seen as a volume. The flexibility lies in the shift in focus between the relation between the factors and the product, on the one hand, and a relation between length, width and height and the product, on the other. From the interview with S, I found that the products in both cases are seen as number-products. These compositions were suggested to be seen as patterns to be discovered. I observed that the pupils in the Swedish study worked with patterns and discoveries in mathematics. The ways of seeing, making a picture of mathematical patterns has influenced S to author her knowledge.

-∞-Exemplar from an interview with D -∞-

The next exemplar is taken from an extract from the same interview with D where an extract was used as an exemplar of associative flexible experiencing (see p. 87-89 above). Thereafter follows an exemplar from an interview with N, also from the Indian study. We can see that in both these exemplars, the focus is on formulas and what is given in the example:

I have placed four squares with match-sticks. These squares can be seen as inscribed in a bigger square. The small square's side is one match-stick long, i.e. 2 cm.



ID: How large is the area of the big square?

D writes:

The side= a^2

$2 \times 2 = 4$

$4 \times 2 = 8$

$8 \times 2 = 16$

D: We know that one side is 2 cm. And in the *sutra* (formula) it says that a side is squared. a^2, a^2, a^2, a^2 becomes $4a^2$.

My reflection: D wrote that the side is equal to a^2 . It should be the area that can be expressed as a^2 . But listening to D, she seems to have an idea of that area can be expressed as a squared side. She also points to each of the squares and counts a^2, a^2, a^2, a^2 . She gets $4 a^2$, four squares. A nice transition between the concrete squares and the algebraic expression, as well as an understanding of what the area of squares represents.

ID: If the side is 3 cm instead, how large is the area then?

D: Then I take 3×4 .

D writes: $3 \times 4 = 12$

My reflection: Now she might have confused the side and the area? 3 should have been squared to be consistent with previous reasoning.

ID: If the square with the side 2 cm was a side in a cube, how large would the volume be?

D writes:

$$a^3$$

$$2^3$$

$$2 \times 2 \times 2$$

$$8$$

D understands the algebraic formalisation of area and volume. The flexibility lies in that she can see the algebraic relation between the length of one side and the area or volume. She could not, however, deal with a change in the length of the side in the square and the transition to area. When D works with 3 cm as the given length of the side in the square, she all of a sudden makes a break in her consistent understanding. The length of the side is perhaps confused with the area. This is a good example, as it seems to me, of what happens when the compositions are focused and separated from context. D authors her knowledge with the help of algebraic expressions.

-∞-Exemplar from an interview with N -∞-

This extract comes from an interview with N. N has just mentioned that she is in the habit of making, what she and many pupils at the school call, ‘opposite-questions’ (they used the term in English). I wondered where they had derived the expression from. In the textbook *Geometry and Application* (p. 69) I found a few lines that perhaps had influenced this way of thinking:

For an example in a triangle ABC (figure 6.1) the angle B is the right angle,
 $AC^2 = AB^2 + BC^2$

Any *opposite* of this example is also true.

Now we enter the interview:

ID: How do you make these ‘opposite-questions’ when you practice for exams?

N: There is a question about *this* side (N points to one side of the right-angled triangle in her textbook) and *that* side is given, (N points to another side of the right-angled triangle), then I can take the formula $a^2 = b^2 + c^2$ and then it comes out!

But it can be that *this* side and *this* side that is given, then *this* and *this* side are *related to* this side. Then it comes out. These are opposite questions.

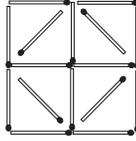
In this extract, N pointed out that two different questions can be posed, given Pythagoras' theorem. Later during the interview, N told me that there are three possible variants of questions that can be asked from this theorem. N uses the expression *saman*, which in Oriya means 'equal to', to describe the relationship between a, b and c in the theorem. N also uses the expression *bhariba* for what comes out as a result of a calculation. The language use and the understanding that three questions can emerge from this composition, can be seen as an indication that N understands Pythagoras' theorem as a relationship between a, b and c. She sees the theorem as three different relationships, depending on if side a, b or c is focused. The flexibility lies in her seeing the composition as a view-turn. N also shifts her focus between the parts of the composition, which are varied and which are invariant. This understanding of the theorem, and many other formulas, is something she uses when she practices for exams. She tells me that it is important to practice many different questions, which they might not have met in the textbook, and where she has to figure out herself how they can be formulated. As a response to the high grade of formalisation of mathematics in the classroom, N:s mode of knowing is mainly compositional. Her authoring is based on what meaning she finds in the mathematical formulas, and what the situation demands of her. Within this category I have also placed pupils who were used to dealing with the high level of formalisation, and succeeded to shift between the mathematical logic expressed in algebraic terms, and the meaning in relation to 'area'. I chose not to present extracts with these pupils here, since these exemplars would not contribute further to the illustration.

1.1.3 Three exemplars of contextual flexible experiencing

Three exemplars of *contextual flexible experiencing* are presented in the following extracts, and were selected as the most illustrative. In several other cases of contextual flexible experiencing observed in the material, there were combinations of compositional and contextual flexible experiencing, which represented the same kind of content- and context-related flexibility as in the exemplars described below. The first extract comes from an interview with R from the Indian study, while the two other extracts are taken from observation notes from the Swedish study and from an interview with Al.

-∞-Exemplar from an interview with R -∞-

I place this figure with the help of matchsticks in front of us. The length of one match- stick is 2 cm.



ID: How many triangles are there?

R: 8

ID: Show.

R: 1, 2, 3, 4, and then this base, this side, side, one triangle and this is a triangle. 5, 6...7, 8

ID: Or more?

R: Yes, there are more... 12!

ID: And if one matchstick is 2 cm long, how large is the area that the triangles occupy?

R: Total area...

R is writing and telling me what she is writing.

Finally, R says: 24 square centimetres.

ID: 24 square centimetres, how do you know that?

R: The formula for the triangle should be used. A half times base times height. That is a half times 2 times 2.

A triangle's area is 2 square centimetres and 12 triangles' area 2 times 12 square centimetres.

ID: To find out the area, how many triangles are there?

My reflection: R still counts with 12 triangles, even though there are only 8 to be considered in the area that they occupy.

R: There are 12 here. To find the area I need to find one triangle's area. One triangle is 2 square centimetres and 12 triangles' area is...

R is thinking. Pause.

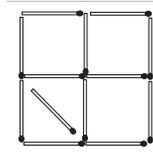
ID: How many triangles are there in this square?

I have placed a square in front of us made of four match sticks.



R: If there are two diagonals there are four triangles, if there is one diagonal, there are two triangles.

I place a square in front of us which looks like this:



R: There is one diagonal...

ID: Yes. How large is the area now?

R: The whole square's area or one square's?

ID: The area of one triangle.

R: The triangle's? There is no triangle. After a short pause. In the same way as before... then the quadrangle is a square.

ID: Yes, a square.

R: In a square there are two sides of equal length and the angles are 90 degrees. Then the formula (R says *formula* in English) is half times the side, times the side. Then it becomes 2 cm, 2 cm. 2 square centimetres comes from this for one triangle. For the whole quadrangle it becomes 2 times 2, that is two triangles, 2 times 2, four square centimetres!

ID: So one square has an area of 4 square centimetres and the total area of the big square?

R: Totally, one takes the sum of all squares, that means that one takes 2, 2, 2, 2...8 triangles.

One triangle's area is 2 square centimetres and eight triangles' areas are 8 times 2, then it becomes 16 square centimetres.

I put four squares in a row; each side is one matchstick long:



ID: What is the total area of the squares?

R: 32 square centimetres.

R calculated the area of one triangle. She stated that $\frac{1}{2}$ multiplied by base multiplied by height is the area of a triangle. The area of one triangle was 2 cm² and that 12 multiplied by 2 (the area of 12 triangles) is 24 cm². By the end of the

interview she corrected her mistake and wrote 8 multiplied by 2, and got 16 cm² instead, which is right. For the second question she started to calculate the area of the triangle and multiplied that with 2 to arrive at the area of one square; 4 cm². Thereafter she multiplied with 8, which was the number of triangles. When she corrected her solution by the end of the interview, she multiplied with 4, the number of squares, and got the right answer (see below how she was writing).

My reflection: in this calculation she has not distinguished the area of a square from the area of a triangle although she has an understanding of what area is. Her answer becomes double the area size in the end, since she counts with 8 triangles, instead of 4 squares and with double the area of the triangle again, as a consequence of her first mistake to calculate the area of triangles. R speaks with conviction. I finally gave her the task of deciding how big the area of the square in the square, where every side is two match-sticks long, i.e. 4 cm long.



R: 16 square centimetres!

R calculates this as a squared side (see below how she was writing).

ID: You have found that the area of this square is 24 square centimetres (the first question), and 32 square centimetres for the area of this square (in the second question), and 16 square centimetres in this (the last question).

R goes back to the tasks and explains in detail how she calculated. She corrects her own mistakes. About the four squares in the row she explains to me that it is not really wrong.

R: This is right, because it can be like that.

ID: Can you make me understand?

R: If it is tables in a row. If there are tables in a room, then they 'take up more space' than if the tables are put in a square.

Here follows what she wrote:

$$\frac{1}{2} \times 2 \times 2 = 2 \text{ ବର୍ଗ.ମ.}$$

$$\Rightarrow \frac{1}{2} \times 2 \times 2$$

$$\Rightarrow 2 \text{ ବର୍ଗ.ମ.}$$

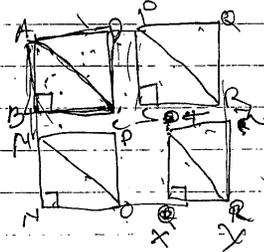
$$4 \times 2 = 8 \text{ ବର୍ଗ.ମ.}$$

ΔABC ର କ୍ଷେତ୍ରଫଳ

$$= \frac{1}{2} \times \text{ସମଲୋଢ଼ୀ ସଂକ୍ଷୁଦ୍ଧ ବାହୁ}$$

$$= \frac{1}{2} \times 2 \times 2$$

$$= 2 \text{ ବର୍ଗ.ମ.}$$



$$AB$$
 ର ଚତୁର୍ଥାଂଶ କ୍ଷେତ୍ରଫଳ = $2 \times 2 = 4 \text{ ବର୍ଗ.ମ.}$

$$\text{ସମସ୍ତ ୪ ଚତୁର୍ଥାଂଶ କ୍ଷେତ୍ରଫଳ} = 4 \times 4 = 16 \text{ ବର୍ଗ.ମ.}$$

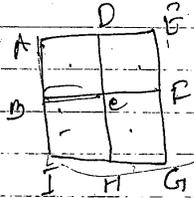
\therefore ଚତୁର୍ଥାଂଶ କ୍ଷେତ୍ରଫଳ ଦ୍ଵାରା ଗଠିତ Δ ର କ୍ଷେତ୍ରଫଳ କେତେ ?

$ABCD$ ଚତୁର୍ଥାଂଶ କ୍ଷେତ୍ରଫଳ

$$= (AB)^2$$

$$= (2)^2$$

$$= 4 \text{ ବର୍ଗ.ମ.}$$



$AECE$ ଚତୁର୍ଥାଂଶ କ୍ଷେତ୍ରଫଳ

$$= 4 \times 4$$

$$= 16 \text{ ବର୍ଗ.ମ.}$$

Here, I take an example from classroom work in the Swedish school class, with relevance to this particular mode of knowing. Teacher M had given a short introduction to negative numbers, asking the pupils where they had encountered negative numbers, and said that this was something they maybe did not know much about now, but that they should know more about it in class 7.

M instructed the pupils to make a story. She wrote:

5 - 8911

M: What are your stories? She circulated around the classroom and came to the group discussing how a plane could crash and end up below the water surface if it crashed into the sea.

S: Is there such a depth?

There seems to be some agreement on that there is, but nobody comes with a concrete suggestion for where it could be.

S writes and R, P and D explain how it can be formulated. S takes some phrases and adds her own. In the end it looks like this (also see below the original writing of S.

An airplane is going with speed 5 meters above the sea level. Then it crashes down 8911 meters below the water surface. How many metres below the water surface is it then?

Answer: The airplane is then 8906 metres below the water surface.

$$5 \text{ m} - 8911 \text{ m} = 8906 \text{ m}$$

S, R, P, D

My reflection: They forgot to write the negative sign, which would indicate that it is a negative number, or a negative height (- 8906 m).

Here follows the text S wrote:



ett ~~plan~~ flygar
plan



ett plan flyger 5 meter
över havet. sedan störtar
det 8911 meter ner. Hur
många meter under
vatterytan är det sedan?

Övar: Flygplanet är då
8906 meter under vatten
ytan.

$$5\text{m} - 8911\text{m} = 8906\text{m}$$

Negative number means to the pupils S, R, P and D 'descending' in height below the water surface and that there is a reference line, in this case represented by the water level, which indicates a border between a higher level and lower level. However, it is difficult to know whether their understanding of negative numbers follows this thinking about loss of height. The pupils have measured temperature, and talked about minus temperatures. The reference point was in that example zero degrees centigrade. In their own example, they figured that an airplane would fly at a height of 5 m above sea-level (the reference line), and crash into the sea to reach a level below the water surface of 8906 m. They were encouraged by the teacher to find such a depth, using an atlas. They found the Marianas Trench in the Pacific Ocean. The shift between understanding the value decrease and a plane that crashes and hits the bottom of the ocean illustrates a contextual flexibility in knowing.

AI had written a math story to the numbers 1 - 1/5.

ID: How did you solve this problem with a story?

AI: I got 40. Because I had 50 candies from the beginning and then they ate 1/5. 40 were left.

Here AI put the notion of ‘one whole’ into context. She could see the number ‘1’ corresponding to a visualised whole, represented by 50 candies. AI understands a whole in terms of 1 as a unit of 50 candies. She shifts between the number one, and unit 1 represented by the 50 candies.

In *contextual flexible experiencing*, aspects are contextually discerned and fused. AI derives from the number 1 the contextual understanding that 1 is one whole represented by 50 candies. With this understanding it would perhaps not be difficult to solve $5 - 1/5$.

-∞-

2. Conclusions

Many researchers in mathematics education have expressed a need for re-thinking our theoretical conceptualisations of ‘knowing school mathematics’, and how its experiential, contextual and discursive aspects can be related. In the present thesis, I have attempted to discuss some of the conditions for an educational theory on knowing school mathematics, which breaks the boundaries of dualism of knower and known. In this part of the result section, I have described the outcome space of three modes of knowing.

The general meaning of *flexibility in knowing* is to see, deal with, assess and to understand which aspects are critical in the process of delimiting parts and wholes, and understanding their relations. It also means to be on a look-out for new aspects within changing wholes. The theoretical and empirical exploration led to three categories of flexibility in knowing school mathematics.

The *associative* way of experiencing means that the learner *arbitrarily* delimits units from a mathematical problem, or else relevant parts of the problem are not discerned at all. Units are experienced as arbitrarily associated, but where shifts in how one understands part-to-part relations are arbitrary.

In the case of V, the question and the answer were experienced as an undifferentiated whole. This experiencing is associative in that V does not see

any meaning of discerning parts, and has to rely solely on memorisation and being procedural in her ways of cruising through the task, catching hold of whatever she can to be able to remember and reproduce the subject matter in an exact and detailed manner. The mathematical formalisation is by V experienced as a foreign language, which she cannot use or understand, and which she does not feel at home with. Still, she struggles to work with it.

D helps to illustrate an arbitrary way of delimiting parts, but D:s focus, in contrast to V, is on mathematical symbolism and delimitation of whatever aspects she associates with the current problem. In the first calculation of the area of the rectangle, D makes calculations on the basis of the fact that the length is 3 cm. In that task, D had an idea that the length and the height should be multiplied. Why does she then, later introduce a cube-relation to the calculation of the area of the triangle? We cannot know for sure whether she discerned the relation between 'side' and 'area' of a triangle as a whole, and why she meant that the relation could be expressed as 4^3 . That some of the sides in the quadrangle should be cubed, to arrive at the area of a triangle, makes no sense according to mathematical logic. It is D:s own vague idea that area is related to the cube of the side. 'The square of the side' would have made more sense, since the problem deals with area. Here we can only guess what D was thinking. Perhaps she transformed a part of a postulate, 'sum of all three sides' in the triangle, to a cube-relation.

Interestingly enough, the high degree of formalisation seems to be a condition for the associative mode of knowing. This was assumed by teacher A also, who could trace the pupils' mistakes back to a rigid memorisation of meaningless details, such as denotations in figures. The dominance of mathematical logic in formalised versions contributes to some pupils' dependent authorship and agency. For V and D, the mathematical logic is difficult to understand, so they choose procedural ways to come closer to the mathematical truth. These strategic ways are based on memorisation and repetition with the purpose of reproduction only, in the case of V, and with some purpose of understanding, in the case of D, although the mode of knowing makes her focus on aspects in such an associative way, that she does not gain a holistic picture. Her understanding remains undifferentiated and she would need help to understand how to change her mode of experiencing school mathematics to more adequate approaches.

Among most of the pupils in the Indian school class, there seemed to be a sense that mathematical relations can be defined and derived from what is given. D tried to do this, although in a very arbitrary manner. I wondered why

the associative mode of knowing was not found in the Swedish material. Perhaps it was due to the selection of class and pupils. If I had selected other pupils in the same class or in another class in Sweden, I might have found this mode of knowing. In the Swedish study, perhaps the classroom discourse supported by teacher M made all pupils experience themselves as authors of their own knowledge; hence the associative way of understanding problems did not appear to them as a mode of knowing. Teacher M's insistence on presenting mathematics in a context, had as a consequence that the pupils could experience patterns and make a picture of the mathematical logic underlying the use of mathematics. The pupils were encouraged to be inventors of mathematics, rather than to discover given truths. Contextualisation of mathematical problems probably also contributed to the fact that the pupils did not understand parts as loosely connected to one another in a mathematical problem.

In the mode of *compositional* flexible experiencing, critical aspects are experienced as *compositions within a whole*. The aspects that were discerned by C and S were numbers. The numbers had structural aspects, as well as referential. In the case of C, he sees 0.001 both as a part where there are '999 parts missing up to 1.000' in relation to dm^3 and m^3 ; and as 'four steps from 1.000' in relation to m^3 and dm^3 . In his understanding of number-relations, he shifts between these two compositional relations. The numbers are *view-turned*, in the sense that they can be mirrored in each other by a process of restructuring.

S, also from the Swedish school class, relates the examples from factorisation to similar calculations for the volume of a cube. S observed that the numbers were used in similar ways in '3x3x3' and in the calculation of 'length multiplied by width multiplied by height'. The numbers were focused, and through an independent authorship C and S arrived at 'original' conclusions on number relations. It is, however, not possible to draw further conclusions on how S experienced the number-relations in this case. She was perhaps momentarily focusing the pattern of relations between three units through multiplication, more than the meaning of a cube-relation. C, on the other hand, seems to have developed a more stable number-relation which concerns relations between the squared centimetres or kilometres, and how many parts can be included into 1 unit; here the unit is represented by any square unit.

In the Indian study, in the case of D, the pupil shifts between numeric and algebraic calculations and formalisation. N understands the compositions in Pythagoras' theorem as a mathematical relation. The view-turn in N's case, is a view-turn among the parts of a formula. This can be traced back to the

teaching practice that the teacher presents, from the textbook, examples which focus on alternative situations where the mathematical relation is used in different ways (see the example of Mensuration, discussed on page 164 in the present thesis).

As for *contextual flexible experiencing*, parts are delimited and given meaning in relation to a relevant *context*. In the case of R from the Indian study, she understands 'area' and R can flexibly move between examples, depending on which context she means 'area' is related to. Here, R transcended the limits of the context in which the problem was given and, although there was a high degree of formalisation in her written solutions, she imagined another context. If the squares were tables, she figured, the tables would occupy a larger area than the total calculated area of the squares in a row.

Se, R, P and D, a group of pupils working together in the Swedish context, discussed negative number and arrived at two different understandings, related to the task given and to an imagined situation. They created a story of '5 – 8911', and imagined a 'loss in height' and a 'position below the water surface' Teacher M provided the 'real' context.

The three modes of knowing depended on how the learner constituted her/himself as a learner and experienced the learning context. We see that the pupils bring into the interview situation and the classroom context, different intentions and experienced purposes of mathematical activity. We can see that contextual and discursive aspects of learning, as well as teaching approaches, influence how the pupils choose focus in mathematical activity in both the studies.

In the Swedish study, the mathematical situations were set by teacher M to engage the pupils in experiences of mathematics in different contexts. During the conversations with the selected pupils, I found that both the compositional and contextual mode of knowing were related to *a reduction* of parts from wholes. C focused on specific parts of compositions in order to make transitions between units of volume. When M gave the pupils the task to make up a story about the mathematical manipulation 5 – 8911, the way of giving the task did not encourage the pupils to understand negative numbers in a specific context. The pupils' learning approaches, as original thinkers, and as authors, made them explore 'minus' in 5 – 8911 as 'loosing height' and 'an airplane being below the water surface'.

In the Indian study, the *conversation* in the field of learning both frames how the learners work and learn, and how the teachers teach. Textbook content and

teacher A's presentation of it, gave the pupils opportunities to vary and understand the parts of problems and *sutras* (formula). There was a homogeneity in how the mathematical content was presented and negotiated, but a heterogeneity in how the pupils understood content and made sense of it. In one exemplar, a pupil focused the relation between question and answer, no matter what answer is given to the question. Another example from this study is when N focused on parts of formulas, *sutras*, and view-turned these to explore the relations in Pythagoras' theorem. R shifted between understanding 'area' as some 'inherent quality' of an object, to something that 'takes up space'. She calculated areas of a square with four inscribed squares. When she calculated the area of the equally sized four squares, separated by a distance and arranged in one line, she came up with another answer, explaining: *In a room objects take up more space. A table occupies a bigger space than the area of the table itself.* Here R calculated the first areas of the squares wrongly. She should have got the same areas from calculation. She shifted her focus from mathematical logic and content to a reality-context and gave 'area' a more contextual meaning. Interestingly, R is considered as a high-attaining pupil and would calculate these areas without any difficulty at an examination.

This we can compare with an understanding referred by Scribner (1984) among dairy workers. Men taking an inventory in the dairy warehouse used the physical environment as part of their arithmetic calculations. Because they knew exactly how many boxes filled a certain space, they subtracted from that number the number of the boxes they estimated were missing from the cube the boxes would form if the space were completely filled. Scribner's dairy workers got reliable arithmetic results, although the calculation process did not involve mathematical symbols alone, but was based on contextual understanding.

I saw in the Indian study that even though the pupils were presented externally authored knowledge, most of them did not consider the mathematics they dealt with as purely de-contextualised examples to reproduce, but as examples to be subjectively experienced with the help of mathematical logic. Mathematical Truth was a means to an end. They were autonomous learners, who negotiated meanings while making sense of the content, individually or with each other during breaks and out of school.

It is important to understand the variation within each category of knowing in the contexts of learning. The description of specific features of the educational/classroom discourses which were seen as related to agency and authorship is followed by a description of modes of being a learner in the

following chapters, called 2: *The Swedish school class context* and 3: *The Indian school class context*. The understanding of these features of the learning context contributes to a holistic understanding of the three modes of knowing. In part V *Conclusions and discussion*, the relation between the learning context and the three modes of knowing is discussed in greater detail.

2: The Swedish school class context

1. Discourses on being a learner

In the national curriculum for the compulsory school system in Sweden, LpO 94, discourses on learner-centred education emphasize aspects of learner identities, which are considered valuable for the work at school, as well as for developing skills needed for life-long learning and participation in society.

1.1 Personal growth and social responsibility

Within the discourses on being a learner, there are more specific themes concerning personal growth and social responsibility that are relevant in relation to the present study. It is stated in the curriculum for the compulsory school system (LpO 94, English version) that:

- The school should be open to different ideas, encourage their expression, emphasize the importance of forming personal standpoints and provide pupils with opportunities for doing so. Education should be objective and encompass a range of different approaches, so that all parents feel able to send their children to school, confident that they will not be prejudiced in favour of a particular view (p. 4).
- The school should stimulate each pupil towards self-development and personal growth. The pupils should meet respect for their person and work at school. The school should be a living community that provides security and generates a will and desire to learn. Since it works in an environment with many sources of knowledge, the school should endeavour to try to create the best conditions for the pupils' development, thinking and learning. Every pupil has the right to develop at school, to feel joy of growth and experience the satisfaction that comes from making progress and overcoming difficulties (p. 7).
- Education in the school should be non-denominational. The task of the school is to encourage all pupils to discover their own uniqueness as individuals, and thereby actively participate in social life by giving of their best in responsible freedom (p. 3).

Work with different individuals' values and perspectives should, according to Swedish education policy, permeate the whole learning environment. In

practice, however, teachers face problems connected to the aim of critically examining taken-for-granted values and contributions from different cultural perspectives, since the competence and knowledge needed for this is lacking in their teacher education (Lahdenperä 2000). The individual pupil's development is the centre from which the work at school is supposed to be organised. There is an emphasis on self-development, which includes the expression of ideas and forming personal standpoints, development of skills, and participation in social life. Social responsibility is thus seen as a competence which emerges from an increasing individualisation.

1.2 Learning responsibilities

Within discourses on learner-centred education, 'responsible freedom' is a balancing factor between the respect for individual differences in needs and interests, which must be promoted and protected, and wider social responsibilities and interests (Entwistle 1970). Within the discourse outlined in the Swedish national curriculum, LpO 94 (English version), responsible freedom is encouraged in relation to the education that the pupils receive:

- By participating in the planning and evaluation of their daily education, and exercising choices over courses, subjects, themes and activities, pupils will develop their ability to exercise influence and take responsibility (p. 5).
- Pupils should be able to keep their bearings in a complex reality, where there is a vast flow of information and where the rate of change is rapid. This is why methods of acquiring and using new knowledge and skills are important. It is also necessary for pupils to develop their ability to critically examine facts and relationships and appreciate the consequences of the various alternatives facing them. Language, learning and personal identity are all closely related. By providing a wealth of opportunities for discussion, reading and writing, all pupils should be able to develop their ability to communicate and thus enhance confidence in their own language abilities (p. 6).
- It is not in itself sufficient that education imparts knowledge of fundamental democratic values. It must also be carried out using democratic working methods, and prepare pupils for active participation in civic life. Education should be adapted to each pupil's circumstances and needs. Based on pupils' background, earlier experiences, language, and knowledge, it should promote the pupils' further learning and acquisition of knowledge (p. 4).

Above all, Swedish education is organised to enhance abilities which are valuable in a technological society, where information needs to be critically approached and understood.

1.3 Teaching responsibilities

Within the framework set by this discourse, education is to be organised by the teachers, so that all the learners are provided with the best possibilities to reach prescribed goals as stated in the curriculum (LpO 94), as well as in the syllabi for individual subjects.

- Included in the professional responsibility of teachers, there is a necessity of constant examination of learning goals, following up and evaluating results, as well as testing and developing new methods. Work of this kind has to be carried out in close contact with the home and with the local community (p. 7).

To sum up, the centre of Swedish education goals is the development of the individual child's personality and individuality, as well as a responsible freedom in relation to participation in society. This also requires an active and responsible attitude from teachers, and critical reflection on their own work.

2. Localized epistemologies in the context of the Swedish study

The rapid transformation Sweden has undergone since the beginning of last century, from a rural society to an industrial nation, and then to a highly technological knowledge society, has had considerable impacts on people's lives. Science in a way can be said to have moved into almost everybody's home, with electrification, with the phone and the radio, the computer and internet. The world is said to have become smaller through international networks of transportation and communication. There has always been reflexivity between society and science. Society has the possibility to 'speak back' to science, and to contribute to the development of science, for instance in motivating the production of certain tools (Nowotny et al. 2001). The use of tools, e.g. computers and calculators, however, demands lesser and lesser mathematical knowledge and skill among the population at large, and instead places the demands on an increasing level of expertise among the professional groups who develop the tools (Knorr-Cetina 1997). Earlier 'production' and 'consumption' discourses about products have been partly replaced during the 1990s by an economic discourse focused on 'service'. There are still discourses celebrating the individual as a consumer with a freedom to choose among products, thereby preparing the ground for new meanings of individualisation -

what Knorr-Cetina (1997) calls an object-centred sociality. These meanings of individualisation are, according to Knorr-Cetina, related to what Rheinberger terms 'epistemic things', defined as 'any scientific objects of investigation that are at the center of a research process and in the process of being materially defined'. Knorr-Cetina mentions the example of computer and computer programmes:

...they appear on the market in continually changing 'updates' ...and 'versions'...These objects are both present (ready-to-be-used) and absent (subject to further research), the 'same' and yet not the same (p.10).

As a consequence, objects of knowledge - 'epistemic things' - are always in a process of being developed. The process entails a chain of wanting, not based simply on motives and intentions, but on a subjectivity related to the object of knowledge. The bonding to and identification with epistemic things. One of her conclusions is that science has provided a new context, where the object is reflected upon and also contributes to situated selves:

...I considered the spread of expert contexts and knowledge cultures throughout society...as a possible driving force behind the rise of an object-centered sociality. The pervasive presence of such cultures (of expert selves, of objects having the qualities of objects of knowledge and so on) implies a reordering of social relationships around objects of knowledge (p. 23).

In formal education, quantitative and qualitative traditions have been described by Cole (quoted in Biggs et al., pp.161-162). The quantitative tradition maintains the view that learning is the aggregation of content. The contents of learning can be evaluated by measuring the amount of information one is able to recall. The qualitative tradition instead sees learning as a cumulative process of interpretation and incorporation of new material.

In education, the individual is positioned in the creative process as a subject of 'personal expressive value' (Sinha 1999). In the contemporary multicultural society of Sweden, a diversity of images and ways of being is offered, and learning takes place at different sites and in different ways compared to before, which also influences how fields of learning at school are constituted (Brömssen 2003). Globalisation and multiculturalism have had political,

economic, cultural and educational impacts. As the world has ‘moved in’ to Sweden, it is becoming increasingly difficult for the urban individual to praise individual independence or to live with the illusion of being disconnected from the world. The Western concept of person, ‘the independent construal of the self’, gives rise to processes like ‘self-actualisation’, ‘realising oneself’, ‘expressing one’s unique configuration of needs, rights and capacities’ or ‘developing one’s distinct potential’ (Markus & Kitayama 1991, p. 226).

In a highly technological society like Sweden, mathematics education is considered to be crucial for participation in democratic and economic decisions and for development of new technology. Mathematics is also considered to be essential for everybody’s life-long learning (Mouwitz 2004). In addition to such aims, mathematical skills are often evaluated at schools on the basis of how much of the textbook content has been covered in a specified time. The majority of pupils regard mathematics as an important and useful subject, which they think they will make use of in the future. At the same time, the subject is experienced as being difficult and quite uninteresting, as lessons proceed slowly. Communication and reflection does not seem to be a priority, rather individuals work with mathematical tasks in isolation from the teacher and class friends (NU03, p. 36).

On the basis of research conducted over a period of 25 years, Engström & Magne (2003) discusses different levels of attainment among pupils in a Swedish average town. Almost every sixth pupil had problems following the mathematics taught. And it is the pupils who diverge from the norm of mathematics learning, those who either achieve low or high, who most frequently give up on participating in the learning practice. Another conclusion is that pupils’ level of attainment in mathematics has been stable, despite the fact that there have been three different curriculum plans for the compulsory school for the period from 1969 to 1994.

The goals in terms of learning outcomes which the pupils should have reached by the end of the fifth and ninth school year are described in the syllabus for mathematics. For the fifth year, the syllabus (English version) says:

Pupils should have acquired the basic knowledge in mathematics to be able to describe and manage situations, and also solve concrete problems in their immediate environment (p. 25).

For the ninth year, the learning goals are:

Pupils should have acquired the basic knowledge in mathematics to be able to describe and manage situations, as well as solve problems that occur regularly in home and society, which is needed as a foundation for further education. (pp. 25-26)

Here, we can see that mathematical knowledge is to be dealt with in relation to real life situations. Mathematics is described as an activity related to problem-solving:

Problem solving has always occupied a central place in the subject of mathematics. Many problems that are directly connected to concrete situations can be solved without using mathematical expressions and methods. Other problems need to be removed from their context, and be provided with a mathematical interpretation and solved with the help of mathematical concepts and methods. The results can thereafter be interpreted and evaluated in relation to the original context. Mathematics may also be used to solve problems, which are directly linked to concrete reality. In order to successfully apply mathematics, a balance is required between on the one hand creative, problem solving activities, and on the other knowledge about mathematical concepts, methods and forms of expression. This applies to all pupils, not only those who need special support, but also those who need special challenges. Mathematics is closely connected with other school subjects. Pupils obtain experiences from the surrounding world and can thus use this as a basis for expanding their mathematical skills (pp. 24-25).

In the Swedish national curriculum (LpO 94), knowledge in mathematics is described in a more multifaceted way, compared to earlier curricula. The national curricula since the 1960s have had a basically utilitarian perspective on mathematics (Petersson 1990). Mathematics was pictured as a tool for describing reality, needed to calculate consequences of one's actions. The use of mathematics in various reality-based situations was considered of high value for further studies and in life. The pupils' skills and their skills' utility aspects were to be stressed in education. Focus was on problem-solving, and on

calculating or applying certain procedures and skills in different situations. However, in the latest two national curricula, Lgr 80 and LpO 94, other processes and competences are described. Goal-orientation without detailed content; focus on processes to be developed - for instance problem-solving, interpretations, analysis, values, communication - and to be able to see the role of mathematics in society and everyday contexts, are all aims that reflect another view on mathematical competency, compared to earlier national curricula in Sweden (Samuelsson 2003).

In practice, despite official policy goals, learning in mathematics at Swedish schools still often involves pupils in reproduction of textbook examples and memorising procedures, because the majority of teachers insist on following the given textbook examples and base their teaching on the way mathematics is presented in the texts. The Swedish National Agency of Education (Skolverket) has in the report *Lusten att lära: Fokus på matematik* (Wanting to learn: Focus on Mathematics, Swedish National Agency of Education 2003) described how teachers in general do not take the possibilities to interpret the various goals of education in order to stage conditions for learning so that pupils can be inspired to learn mathematics.

In spring 2003 started a research project (Wistedt 2005; Sollervall & Wistedt 2004) called *Pedagogik för elever med förmåga och fallenhet för matematik* (Gifted Education in Mathematics), the research focus is on the heterogeneity of gifted students. The understanding of the term 'gifted' is in relation to capabilities that are valuable for mathematics learning. A definition of *mathematical aptitude* is a dynamic capability taken from Krutetskii, who conducted a study on specially gifted children and adults 1955-1966. It is a capability connected to the specific activity; created during the activity and evolving with work. Success in an activity is related to a variety of capabilities, and lack of capability in one area is compensated by strength in ability in other areas (quoted by Sollervall & Wistedt 2004, p. 3). The authors note that the ability to think quickly, to calculate, and memory for symbols and numbers are less important abilities for mathematics learning, but these are the abilities that are usually valued in education.

The aim of the project is to study mathematical aptitude empirically. Researchers, teachers and teacher students develop the treatment of mathematical problems together, so that research-based knowledge and proven experience (*beprövad erfarenhet*) is combined.

In her classroom discourse, teacher M encouraged the pupils to experience authorship and agency. She had an awareness of how the classroom discourse constitutes the learners, but at the same time has to resist dominating discourses on mathematics. Teacher M had informed the parents about her way of teaching at the beginning of class four, when she took over the class from another teacher. The parents posed many questions, especially concerning M:s usage of textbook material. Many parents “had gone in the old school”, M says, and wanted to see the children’s results and progress. I asked M what she meant by saying that the parents had gone in ‘the old school’. She gave me a text, which she had prepared for a speech during a re-union meeting with her classmates from school. The text was entitled *Realskola-grundskola (Realskola was an optional form of schooling during the period 1905-1972, which pupils could apply to after their 4th, 5th or 6th year of the type of school where most children studied, the so-called Folkskolan. Grundskola is the term for the nine year compulsory school system, which has existed in Sweden since 1972)*. I have selected the following passage to illustrate how M experienced her time in the ‘old school’:

School was considered to be important; it was the transmitter of knowledge. Teaching in those days was all about a transmissive mode of education. The teacher was seen to be a large container full of knowledge. This knowledge was supposed to be dispensed to the pupils in appropriate doses. The pupil was the empty container whom the teacher was supposed to dispense knowledge to. The focus of instruction was on what the teacher was supposed to teach. Some also had the intention of finding good methods. Instruction was placed at a high level of abstraction. The teacher had already found the quickest and ‘best’ method, and ‘went through’ this at the blackboard. If a pupil put up his hand and didn’t understand, the teacher might well answer: Weren’t you paying attention? The ability to make abstractions was important. Pupils who had trouble following were not considered to have sufficient aptitude for secondary levels. Instruction in those days focused teaching to a greater extent than learning aiming at understanding.

The parents who spoke with M when she started teaching the class felt that it was difficult to know whether the child learned enough to manage college studies if he/she did not work with a textbook, and the parents had a wish to know more about what the children did and about their study progress, preferably through written examinations. So, M met the parents’ wishes for

information by providing, together with the children, weekly reports in a notebook. There was also weekly communication through this notebook. Teacher M rarely used textbooks. She planned, evaluated and helped the pupils in their learning process as she designed various tasks and themes. She said about homework:

Homework is a very burdened concept, if one only thinks that homework is something one gives only to cover a certain amount of tasks within a specified time. That, I think, is serving a bad cause (Swedish: *av ondo*). On the other hand, I always write on the whiteboard: 'For Monday', 'For Tuesday'.

She explained that the tasks she gives to be solved at home are given so that the pupils come back to school with an experience, instead of having the teacher just talk about it. She tells about a few instances of such tasks:

For instance, if we now say that the parents are asking for a few weeks' leave from school. Then I can give them some tasks that can add to the teaching. They get an assignment and take their small notebook with them and then we discuss afterwards what they can do with it. They sometimes collect information about the places where they have been, about Kaknästornet, Notre Dame, what one has seen, a map of the underground in England, there we can work with the concept of scales and so. They should always send a postcard. Is there a time difference? When is sunrise, sunset? How much does milk cost? What does it cost to send a letter?

All pupils experienced a difference in how they are working with and understand mathematics compared to before class four. Before teacher M took over as mathematics teacher, mathematics lessons were, according to many pupils: *A competition and a matter of being the fastest*. A difference compared to earlier experiences was also experienced in similar ways to what S expresses:

S: She asks how do you think and she encourages us to tell her about it.

ID: What does she want you to tell about?

S: The steps in between.

Teacher M encouraged the pupils in various ways to be *authors of their knowledge* and to have an *independent agency* as a learner. The pupils in the study observe that teacher M is interested in their ways of thinking, they talk about how they appreciate the final product after a completed project, and that their documentation has a personal value. The pupils' experience is illustrated by the following statements:

C: Now I know everything about Time.

K: I can go back to it.

B: We learnt 'maths in town', time tables and such things. In class three we didn't learn anything. M explains more.

('Maths in town' is part of the Time-documentation, see Appendix 6).

Al: I have done such things that I don't know what it was. M explains in more detail. It is more fun with maths, it is still boring, but more fun than before.

Teacher M told me about how mathematics can be made enjoyable:

Teacher M: Another thing which is important also, I think, is that the teacher has to radiate that feeling that she thinks it is enjoyable herself, How otherwise will they think it is fun when they sit there, awfully boring!

ID: Yes...

Teacher M: This one has to be able as a teacher to communicate to children.

ID: Exciting.

Teacher M: Yes, it is maybe not always so fun and exciting, but then they at least have got an assignment where they can feel that they succeed and it is maybe more important because - 'yes!' they raise their fist 'yes' and it is maybe more important than that it is fun. That is why I call my first report meaningful learning, because I think it is very important that one doesn't have fun on maths lessons but that one makes maths fun, which is not the same thing!

Teacher M believes that the work with mathematics demands attention and concentration from the pupils, but pupils can still have fun while doing mathematics. M found in earlier evaluations (one of which was published in a
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report as part of an in-service teacher course) that pupils think it is important that mathematical tasks are:

- Reality-connected (verklighetsanknutna)
- Are designed so that one can make a picture of them in one's mind (är sådana att man kan skapa sig en bild)
- Are designed so that one can draw or build something (är av ett sådant slag, att man får rita eller bygga)
- Preferably can be discussed in groups (gärna kan diskuteras i grupp)
- Have the right degree of difficulty (har rätt svårighetsgrad)
- Are of various kinds (är av varierande slag)

These results were derived from interviews and a questionnaire with class seven pupils and college pupils, whom teacher M had previously been mathematics teacher for during classes four through six. All the listed points show that the pupils prefer to be active and reflective about the mathematical content. The conclusions drawn in her report have clearly inspired her to develop her *pedagogical power* (cf Jaworski et al. 1999).

Teacher M worked with mathematics as a subject with an epistemological core which can be reflected upon and negotiated. She worked extensively with issues concerning language, various language meanings, and meanings of mathematical concepts. Pupils formulated their thoughts together. This often led to pupils making guesses, trying to find a plausible answer, or writing down new mathematical words in their 'thought word book'. An interesting example of how teacher M worked is when Teacher M posed a question to the class, which they discovered could have quite different answers depending on what they thought the question really meant. The original question was: *How many dices did we use yesterday?* After several rounds of discussion, the pupils could agree on what questions they had answered. The interesting point here is that teacher M and the pupils were engaged in finding out the meaning of the words. The pupils perhaps got a feeling of how important it is that the question is precisely stated if it demands a mathematical solution. This example is further discussed in the next section (3. *Classroom discourse*).

Another time, the whole class had been engaged in making their own temperature measurements and diagrams. They also checked in the daily newspaper to see how weather forecast charts corresponded with their own diagrams. Once, the pupils discovered a mistake in the scale in a published weather forecast chart. Together with teacher M, they sent a letter to the newspaper. This example is further discussed in the next section (3. *Classroom discourse*) as well.

Several mathematical games were produced by teacher M and her pupils. The ‘fraction-games’, for instance, were dice-boards, where fractions were added to fractions. The pupils were also encouraged to make developments of these fraction game boards (Appendix 7). This example will be discussed in the next section (3. *Classroom discourse*).

3. *Classroom discourse*

Perhaps one of the most important aspects of M:s teaching was how she made the pupils feel a responsibility to learn. Over a period of thirty years, she had developed her personal style of mathematics teaching. The main way she presented different mathematical problems, was that she used to introduce the problems in a context. The discussions in the classroom were planned by M in relation to, as she said: ‘structure and educational goals’. Her main aim was to arrange the pupils’ project work and tasks so that a specific content would be included, and the meaning of the content and different ways of solving problems and tasks could be discussed. The classroom discourse involved collaborative meaning construction. The pupils were encouraged to discuss mathematics in various ways, and such discussions were initiated and guided by the teacher. The discussions sometimes had an aim to explore mathematical patterns, to negotiate solutions, to reflect over a question, to discuss hypotheses and to argue for one’s thoughts and reflect over the mathematical language.

Teacher M used positive powerful words to make the pupils understand her aims. For instance, the expression *plausible reasoning* was used to underline that mathematics is not about finding out ‘right answers’, but about understanding. She often said that the pupils would do an *exploration*. The expression was used to direct the pupils’ attention to the idea that there is something to discover, which needed a special kind of attention. M also gave the pupils longer periods of *project work*, which were designed in a manner to give rise to discussions and collaborative work. There are three project assignments I would like to describe briefly, in order to clarify how teacher M encouraged the pupils to develop their authorship. The first project was initiated during a lesson as a result of an unforeseen incident, the fact that a question posed by teacher M, was obviously understood in many different ways by the pupils. In this example we can see that teacher M is open to listen to the pupils’ understanding, and that the pupils are used to teacher M giving them this kind of opportunities to explore language use in relation to mathematics. This description is based on teacher M:s own notes from her lesson, which began with the question she posed: ‘How many dices did we have yesterday?’:

We had practised multiplication with the help of dices. We had used three different kinds (with maximum 6 dots, 10 dots and 20 dots). Each dice had three colours (white, green and red). The pupils sat in groups of three with two dices at the same time. I walked around in the classroom and changed the dices according to what progression in level of difficulty I thought was necessary to introduce. The day after I asked: 'How many dices did we have yesterday?' The pupils got each one paper so that they could individually write their answer. Then they could compare and discuss in groups. And as usual we listened to the different solutions and the thinking behind. What had they written? Yes, it was far from what I had thought they would have reflected upon. There were 8 different answers. We wrote these answers and the explanations for these on the board. In each case we discussed whether that was an answer to this question (How many dices did we have yesterday?) or to any other question.

The pupils who had answered 2 dices, said that each group had had two dices at the same time. A discussion led to the conclusion that this was an answer to the question : How many dices did each group use at the same time?

Some pupils thought that it should be 3 dices. They explained that they had used three different kinds of dices; one with highest 6 dots as the highest (6 sides), one with 10 dots as the highest (10 sides) and one with 20 dots as the highest (20 sides). After reasoning about this they thought that they actually had answered the question How many different kinds of dices did we use?

A few pupils thought that 6 dices was the right answer. They had answered the question: How many dices did each group use?

A couple of pupils gave the answer 9 dices. There were three different dices and three colours of each. These pupils formulated the question they had answered: How many different kinds of dices had we used?

Another alternative was expressed in the answer 14 dices. All the seven groups used two dices at the same time was the explanation to this. The question they had answered was, after discussion agreed to be: How many dices did the groups use at the same time?

There were also pupils who answered 21 dices. We were seven groups and there were three different dices. What question they had answered created some problem. There was at last an agreement on that the question was: How many different dices did we use in all the groups?

Another answer was 63 dices. They had answered the question: How many different dices, sorts and colours did the groups use in total?

The one who guessed 75 dices was closest to the right answer. Teacher M let the pupils count the dices, which she had put in three plastic bags, to arrive at the correct answer, which was 80.

The next example is based on a project on Meteorology. The pupils had measured the outdoor temperature for a couple of weeks and drawn their temperature diagrams. When the pupils discovered a mistake in one of the diagrams given in the daily newspaper, the pupils, with the help of M, wrote a letter to the newspaper. They received a letter from the newspaper with an explanation to why the diagram for the month was wrong, including a complete description of how the graphics is made by inserting values given by the SMHI (Swedish Meteorological and Hydrological Institute). The values for Normal rainfall and Normal temperature during the month were wrong, which the pupils in M:s class had discovered. They had also noticed that the graph for normal rainfall was incorrect. The author of the letter, representing the newspaper on this issue, thanked the pupils that they had discovered the mistakes and gave them praise for this, and for the fact that they had, through their teacher, written the letter, so that they would be more observant of these matters in the future.

The third example I would like to describe is related to the use of fraction boards. The pupils generally get approximately 2 minutes for these math games. The principle of the games is that there are fractions written on a game board (see Appendix 7). The pupils are instructed to throw a dice and note down the sum of fractions (number of points). The one out of two players who

gets the highest sum of fractions wins. There are three levels of difficulty. The lowest level is the one with these fractions: $1/2$; $1/4$; $1/8$; $1/16$; $2/8$; $2/16$. The second level has fractions which can be converted to decimals, if required. These include $1/2$; $2/10$; $3/8$; and $3/5$. The third level includes halves, thirds and quarters. Especially the thirds are harder to count with. One of teacher M's project tasks is to make a fraction board of one's own, and to use it as a game board at home. The fraction game works like this. The pupils play in pairs. First a player throws the dice and the number of dots corresponds to the number of moves on the board. The fractions are noted down after each move and by the end of the game, after five minutes or so, the sums of the fractions are calculated. The player who has the highest sum gets the highest score and wins the game.

All these examples are also examples of opportunities for authorship, which M offers. In the first example, the authorship constitutes a space of reflection over verbal language and mathematical exactness. In the second example, the pupils not only produced their own knowledge of temperature and constructions of diagrams, but also understood the importance of being exact. The pupils could see how important it is to be as exact as possible when making diagrams of rainfall and temperature. At the same time, the pupils could see that estimated figures for temperature and rainfall can be used for a weather prognosis. Their authoring of their knowledge empowered them to address key concerns as questions to the newspaper. In the example of fraction games, specifically the opportunity to develop a new fraction game board provided a ground for mathematical authoring and understanding of fractions.

3.1 Teacher M's pedagogic power

Teacher M has knowledge in pedagogic theories and research, and continuously reflects upon her way of teaching. She clearly also has developed powerful ways to implement her knowledge of pedagogic and didactical theories in her teaching. During the interviews, for instance, she talks about constructivism, referring to Engström's (Engström 1998) book *Matematik och reflektion* (Mathematics and Reflection, 1998). She mentions Vygotskij's theory on the proximal development zone, which she has written about in her report, and refers to it both during interviews and during observations. The didactical model developed by Malmer (2002) represented a practical tool for her to approach the mathematical content, starting from the pupils questions, or if they 'got stuck' in the middle of a task.

I asked M to tell me more about the model for understanding mathematics, which Malmer (2002) has developed. M had page 31 (Appendix 8) of Malmer's

(Malmer 2002)book *Bra matematik för alla. Nödvändig för elever med inlärningsvärigheter* (Good mathematics for all – but necessary for students with learning difficulties)in front of her and was looking at the model. Here follows an extract of what M thinks is important to consider in relation to the model, which she many times referred to during the interviews:

You have to have a context. The context can be something they have been working with or seen the day before, something they have experienced before, something you have created together with them. In this you know that there is certain mathematics which you want them to know, and then you have to check their previous understanding and that you do by using language. It is like a game of chess - it is really! If I make a move they have to think, you challenge them with kinds of questions which are not those yes-no questions. These somehow challenge them to think one step more. And then I discuss the questions with them - that is the first step. (She points at Göra Pröva (Do/Try) in the model). And then you have to put (försätta dem) them in a situation where they have to Do/Try. Then the most important thing is what they themselves ask. Yes, well, 'do/try' Malmer calls it, it is this explorative learning method, how you pose these didactical questions 'What', 'How' and 'When'. So that they need to try and then that they come to something in this trying. So, hopefully they make something visible to themselves. Otherwise you have to do that with questions so that they take one more step. And then the most important thing is that you have to, together, perhaps the whole group, gather up what they have concluded and understood. So that, some of them think about the concrete, and you see if they can see something. And it is all about, so to speak, if you arrive at the triangle's area, which you can generalise from what they have done. Then you can apply this with the children and then you have done it and seen it. That is what I mean by explorative learning methods. It is often during practical exercises they see things.

Teacher M often returns to the idea that it is important to provide a context or build a context from where learning can start. Learning is seen as a process, where the mathematical ideas and patterns are *discovered in some kind of context*. M

handed me a copy of a text from Engström's *Matematik och Reflektion* (Mathematics and Reflection during my observations in her classroom. I looked at these pages when I prepared for the interview. Engström (pp. 11-12) states that constructivist instruction:

- Takes its point of departure in the standpoint that the pupil uses what he/she already knows to develop solutions that carry personal meaning;
- Stimulates pupils to reflect over their mathematical activities;
- Is characterised by a substantial proportion of practical activities, that allow the pupils to construct their own mathematics;
- Provides ample opportunities for group discussions, that let the pupils confront their views with those of other pupils, which develops the pupils' capacity to motivate and argue for their ideas;
- Sees learning as a problem-solving activity, where the pupils' own issues and ways of formulating questions is provided considerable space;
- Is grounded in the reality of the pupils, not in fictitious situations.

During the third interview, I asked teacher M to tell me more about her thoughts about this text. She said:

When I read this I thought: This is what I am doing! This is what I have been thinking. She looked at the points concluded by Engström on Constructivist teaching and commented: It is important to know the children's previous understanding, to give them inspiration all the time. One 'sees learning as a problem solving activity where the pupils' own questions and ways to formulate problems are given a lot of space' (M reads the text). This is a little difficult, that they should formulate their own questions, it is like with all socialisation, you have to conquer new domains successively and maybe they become larger. They start, those who can, but it takes time and it is hard. And it also depends on what you mean by their own questions, it is not like sitting and doing calculations. Sometimes I can involve them in my world. I have used that often, my travels and so, I have sent postcards and if they have been abroad we have exchanged experiences.

Teacher M says that she works with 'constructivism' in the sense that the context and questions are developed so that the pupils can create their own pictures and understanding. Meaningful learning, M explains, is about

providing learning opportunities that relate to the children's *own* reality. This reality can be the reality that is constructed in the learning context, or the reality that is somehow 'out there', but again directly related to them.

3.2 Teacher M:s approach to teaching

The class had been working with the theme *Time*, for three years. I observed a few lessons when the pupils were working with a calendar, and when they were making their final documentation. During the theme work *Fractions*, I saw many instances of classroom conversations, between pupils working in pairs, fours, or when the whole class discussed solutions on the whiteboard. Teacher M:s approaches to teaching can be described in relation to the following three features:

- A common reference frame
- Negotiations on meaning
- To make structure of learning and work goals visible

Teacher M has worked as a teacher for thirty years, and during this time developed the mathematical content, related to the learning aims and structure of her teaching, and to the national curriculum (LpO 94). She has thought deeply about how to best arrange learning situations and how to assess learning. She has found different ways of evaluating and refining her teaching. She feels that creating a common reference frame makes pupils understand mathematics as part of their *shared experiences*. The mathematical content should, according to M, be arranged so that the pupils can relate to it, personally and as a group, and the process of reaching learning goals should be an important part of teaching mathematics, rather than focusing on results only:

If one uses the textbook and follows it, and if one hasn't read through all the tasks, and not thoroughly looked at them, what is it then one is helping them (the pupils) with? But if I have given them an assignment, then I know the goals. I can see their process and then I can intervene and support and make them take one step further.

Teacher M argues that readymade tasks in the textbooks often include aims that do not make the pupils understand the content. She does not feel that doing such tasks supports learning goals she sees as important, and which are emphasized in the curriculum and syllabus. She explains that the pupils need challenges in their processes of coming to know:

And then you can give them a... possibly a new question, which is like a step onwards, that is why it is important to have a clear idea of the goals yourself....

M contextualises and expands the mathematical content when she plans project work and guides her pupils. Referring to Lindqvist's interpretation of Vygotskij (Lindqvist 1999), M writes in her report about what possibilities there are to work with the pupils' interest for mathematics:

Vygotskij claims that there are three important pedagogical conclusions that can be drawn in connection with children's interest and content. 1) The subject matter should be related to children's interests and the content taught at school. 2) Instruction should not involve repetition, but instead, subject matter should be expanded and deepened. 3) There should be a connection between the content and real life.

M is interested in evaluating the project work she gives the pupils. And such evaluations have led her to deeper insights about her teaching:

...if you construct a shared frame of reference that you think through. Obviously, the first time you do it you haven't thought about – no, I say – but then you realise more and more that Ah! That is something I can use, and next time you do it, that is something you know, so it is incredibly important I feel, that you evaluate yourself so that you think: Was this any good? How did they perceive it? Because we were working with 'Time', well I was still disappointed, because only 60% had been positive in the larger evaluation we made a couple of years ago, because I felt we had been using what the pupils had been thinking about. That was reality, but it wasn't exactly *their* reality, and I suppose that is what she writes about, Inger Wistedt, have you read it? (yes, I say, *Rum för Lärande* - Room for Learning).

The context for the mathematical problem is important for M when she plans project work and mathematical tasks. Teacher M: s intention is to stage a reference frame so that the context is experienced by the pupils as part of their own context. A context can be constructed, M explains, around an open-ended

question, where the pupils are encouraged to discover the context that is relevant to them:

A set of ordinary calendars, the ones that are decorated with a wreath of lingonberries, became part of our learning materials. At the end of some lessons, the pupils were given the opportunity to get acquainted with the contents. In exploratory group conversations, the pupils were able to complete each others' observations. At the end of the term, we had group conversations on the following topics: Today is June 6. What do we learn about today by studying the calendar? A compilation showed what the pupils had concluded: It was Wednesday, week 23 of the sixth month, a flag day, Sweden's national holiday, the day of the Swedish flag, the name day of the King. The sun rose in Lund at 04.29, and set at 21.43. The full moon appeared at 22.12 and set at 04.53. The sun and the moon were not visible at the same time, the sun did not set at Kiruna (the northernmost city of Sweden). Some groups had also calculated how many days were left till midsummer, till the next full moon, till various pupils' birthdays and holiday trips, till the solar eclipse and the beginning of the autumn term.

Now we had created a shared frame of reference, which we were going to refer to during the next school year.

Here, the pupils can know about the chronological time and relate it to their own interests and experiences of time, as well as the relation of time to the movement of the sun. Mathematical content is integrated with the science subjects, which M also teaches:

Yes, just that, that is what you get and then the children don't know about it, because then you have fun when you go swimming in the daytime and that the sky is blue, and what does it look like when the sun sets, that it is all red, and then you can talk a bit about optics, and at the same time what time it was then, and how does time progress, what time is it when you go swimming some other time, so you can do so, if you construct a shared frame of reference.

And she believes that pupils should get many opportunities to express their thoughts, so that they can reflect on their solutions and discuss them:

During the autumn term in fourth grade, we worked mathematically using the pupils' reality as a point of departure whenever it was possible. An evaluation in February showed that the pupils felt motivated to work in an exploratory manner with mathematical tasks related to their own daily life. A large part of the lessons contained tasks where thoughts were verbalised, and reflections about new problems, and the pupils discussed their solutions. The pupils have made a lot of progress in terms of daring to share their reflections and presenting arguments for their solutions. In one of the classes, girls are the majority, and in the other there are the same number of boys and girls. The classes are different. They do not need the same time to learn. The content has mostly been the same in both classes. The contexts have sometimes been different, concerning the pupils' frames of reference.

Teacher M usually contextualised mathematics so that the learners could understand *mathematical ideas as universal phenomena*. In the following extract, we can see how M constructs the context and encourages the pupils to discuss their thoughts. In this extract M stands for the teacher and P for when a pupil was talking. T and B are pupils who made more than one statement. M started a lesson at 8.30 one late autumn day like this:

Teacher M: Has the sun gone up when the light looks like this (så här ljusst)?

P: Yes, sun has just risen.

Teacher M: When does the sun rise in our city?

P: 8.10.

Teacher M: Can you point out in which direction the sun rises.

Almost all pupils point towards the rear end of the classroom.

Teacher M: Is that the east?

The pupils nod and continue to listen.

Teacher M: What time is it for S in Thailand now?

P: It is afternoon.

Teacher M: What is he doing?

P: He is bathing at the beach.
What time is it for M in Uruguay?

P: It is night.

Teacher M: What is he doing?

P: He is sleeping.

Teacher M: What time is it for Y in Sri Lanka?

P: It is lunch time.

Teacher M: What is she doing?

P: She is eating.

Teacher M: When I was in Tanzania the sunlight was always red. Why is it like that? Have you read about Optics with P yet? It is because the sunrays hit the molecules in the air from a sharp angle in Africa.
Why isn't the sun green or blue? What does light consist of?

T: Seven colours.

Teacher M: Have you separated colours in something? With a prism?
Do you remember when we went bathing during our school camp? What colour did the sea have in the evening?

B: Black.

Teacher M: Shallow water is green during daytime.

Now we work with our tasks and you get follow-up questions depending on what you conclude. This is a way to learn about chronological time.

Teacher M proceeds to walk around in the classroom. She asks two pupils working together:

Teacher M: When is winter solstice?

Ma: I checked all the days and then I took the middle one and then I looked at each day.

Teacher M: Did you come to an answer?

T: No.

Teacher M: Formulate it together and come with a suggestion.

She asks another pair working together.

Teacher M: Why is winter solstice the 19th of December, B?

B: What?

Teacher M just looks at him.

B: I said 'what?'

Teacher M: Why is winter solstice the 19th of December, B?

B: I don't know

Teacher M: Then I am waiting.

Then you have to formulate.

B: What should I write?

Teacher M is silent.

B: All days between 19-24 December are short days. 19th December is the shortest day.

Teacher M: Is that right? Have you looked in the calendar?

B: Yes, I have looked together with R and calculated and checked it. R is silent.

Teacher M: If you agree on it, then it is good.

Teacher M: s mathematical epistemology is based on the notion that 'objectivity' can be achieved through the pupils' own construction of knowledge. She believes that it is very important that the learning situations are arranged around the pupils' own understanding and context:

The new education plan suits me well. I am not, like, the one who just follows one track, if one doesn't really reach the goals in the children's world, yes I say so, then one can keep the goals and have the thoughts of the children as a context even if one had something else as a context or thoughts. The really important thing is that one has the goals and that one knows what one wants to proceed with, and take the path that is pupil-oriented.

M stresses that: *To make the children understand, that's what the teaching assignment is all about!* She explains her ideas, using the metaphor of different roads to the same goal:

Well, there's this thing about goals. Of course you need the goals. Then, if you want to get from Lund to Malmo, there are many routes. If you take the quickest, it will naturally be the motorway, but you can also take a by-road through the beech woods, and have a picnic, and look at the flowers. And you will get a completely different experience. The goal is the same, but the question is, what do you want to have along the way? You have to know the goal, and that is why the new national curriculum is so good.

She also expresses about 'heartfelt' experiences:

Experiences are the important thing (with emphasis), because that is what stays in your heart and your mind, and many children, especially those with difficulties to concentrate, can be helped by having been in a situation they can remember that they experienced something, and they have seen all sorts of things on the walls, you can refer to some kind of picture on an overhead or a slide they can remember, and then we have the special needs teacher there, what do you remember? and then you connect to those feelings and then, oh, now what was this about light? do you remember that? how long did it take before the sun set when you went swimming? so that you are creating a positive atmosphere.

M often gave the pupils questions in order to promote understanding. She calls these questions ‘steps in-between’, and argues that such steps make the pupils take a bigger leap towards understanding:

Two pupils were working with the task $1/5 + 1/15$.

M asked: Which one is biggest?

S: $1/5$.

M placed the two fraction-sticks, with the proportional lengths

$1/5$ and $1/15$, on the table in front of the pupils.

The pupils placed the fraction-stick $1/5$ over the fraction-stick $1/15$.

They counted 1, 2, 3, and proceeded to write and talk about the solution.

M encouraged discussions in connection with the production of and work with concrete material; helped pupils visualise mathematical concepts, and worked with documentation of thematic project work. She says that it is important that pupils are free to express themselves, and see that mathematics can be negotiated and the meaning agreed upon:

Precisely this thing about daring to make hypotheses, and daring to test, and if you have done it in a very

simple way in the natural science subjects you can do it in mathematics, so in that case they know how they are thinking and they dare and that is the way we have always done, so they formulate hypotheses and then they have created their own problems, then they have carried them out and reported their results, and then at the end we have made a joint summary, always so that the children get everything we have written, so we should agree then, so that I feel it is a good point of departure, and that it is what we also do in maths, with joint discussions, and that does develop language and feeling free and open-minded.

M used concrete material when the pupils were beginning to work with a mathematical idea, and then proceeded to encourage the pupils to try to make a picture of what they knew:

Teacher M: Those of you who don't have 12th-parts, can maybe figure out how to do?

T: I know what to do!

Teacher M: What do you think he has done? He has folded 6th-parts. Can one think without doing it?

Teacher M stresses that it is important that the pupils try to express their thoughts verbally. Sometimes she helps them with words, and asks them if she has understood what they meant. It appears from her teaching methods that M thinks she can get information about the pupils' thoughts and a good assessment of their knowledge in classroom talk. The theme documentation is another assessment method. A third method for assessment is diagnostic tests on arithmetic.

Here follows an extract of a lesson on fractions, which nicely illustrates how M gives the pupils a picture, so that they can understand the meaning of fractions. The extract also shows how M works with narration and negotiation (M is teacher M, while P stands for different pupils, if not otherwise mentioned).

Teacher M writes on the whiteboard: $2/6$; $4/9$; $3/5$; $2/4$

Teacher M: What can you tell about these fractions?

How can one re-write them so that it is easy to compare them?

P: One can divide into twos.

Teacher M: How?

P: Shadow 2 squares out of four.

M draws a quadrangle with two shadowed parts out of four parts and next to it a similar quadrangle which is divided into two parts, whereof one part is shadowed:

Teacher M: Are these two different?

P: No, I think they are the same.

P: I think it is equally right.

P: It is the same as half.

Teacher M: Then we can put an equal sign between them, can't we?

If I compare these two figures, I think they are the same. The shadowed space is same. Can you compare them?

Teacher M makes a pause here and all pupils sit in silence and look at the board, as if they were trying to make a picture in their minds.

M: Now, we'll do like this. I want to explain. If we do like this, then what do you think, if you only look at this picture?

Teacher M draws a figure of a pizza with 3 shadowed pizza slices out of a total of 8 slices.

Teacher M: Now I pose my question, because I want to 'trick' you a little. What is 8 minus $3/8$?

P: $5/8$ of 8 are left.

Teacher M: Oh, you haven't eaten them all, have you?

P (same pupil again): It is $5/8$ that is not shadowed.

Teacher M: As it says in the book, ' $1-3/8$ '. What does that mean?

B: What?

Teacher M: Yes, it says like that in the book. What does $1-3/8$ mean?

P: One should take $1/8$ away so that $2/8$ are left.

My reflection: This pupil is perhaps thinking that one can take a first step towards the solution, seeing that $2/8$ are equal to $1/4$. $2/4$ was illustrated and dealt with at the beginning of the extract.

Teacher M: One whole and $3/8$, yes, I can see that...

Teacher M: Now, we have got to have a picture in our mind otherwise we only have numbers.

What stories do you have?

P: One has a cake and then one eats $3/8$

P: If I go to the bakery and want to buy cake slices and there are only $\frac{5}{8}$ left of a whole cake.

Teacher M: Write and draw on the back (of the paper). Do it quickly. It has to be about reality, and this is important for you, so that we can play the (fraction) game afterwards.

Teacher M walks around and looks at the pupil's work. She asks them:

Teacher M: 'What stories do you have?'

M frequently works with visualisation practice, giving the pupils sometimes 'half a minute' to try to make a picture about what they are working with. Here follows an extract, where the pupils are presented with a problem illustrated by pictures, and where M:s aim is to make the pupils understand the relation between numerator, denominator and shares:

Teacher M: Now I shall pose a question very clearly to you. What is the amount of girls, expressed in shares?

Teacher M has written three F:s (for girl; 'flicka' in Swedish) and five P:s (for boy; 'pojke' in Swedish)

P: $\frac{3}{8}$

Teacher M: Yes, and that really means..?

P: 3 of 8

Teacher M: How many are girls?

P: 3 girls

Teacher M: There are three girls. So 3 are girls

Teacher M writes on whiteboard:

Ans: (3 of 8) $\frac{3}{8}$ are girls. Three are girls.

M makes 18 dots on whiteboard

Teacher M: If $\frac{1}{3}$ of the 18 have red jackets. How big a part has red jackets?

P: $\frac{6}{18}$.

Teacher M: How did you get that?

P: I counted them.

Teacher M: Now, you have to imagine the picture. Visualise it.

P: 6 multiplied by 3

Teacher M: Yes, but one can also take $\frac{1}{3}$ of 18 which makes:

Writes on whiteboard: $18/3=6$

Teacher M: I have mentioned it earlier. It will come later.

If I write (she writes $3/8$ on the whiteboard....) $3/8$, then which is the denominator?

And what does it mean?

P: What I should take three times.

Teacher M: Yes that was well said, but I don't really know if I understand.

Same P again: How many parts there are of something.

Teacher M wanted to take the pupils one step further and see the meaning of the denominator as a mathematical term, what 'denominator' means in the different examples, and how to use it in calculations. M often informs the pupils of the curricular goals and she also says: you maybe do not know much about it now, but you should know it in class 7 or 8, according to the syllabus.

3.3 Learners' modes of being a learner

The pupils said that they 'have to think more' since M became their teacher. Most of the pupils mentioned that it is important to think in 'intermediary steps' (*mellanted*) until 'one knows it'. They also felt that discussions in class, talking about the steps between stating the problem and reaching the solution, are a way to practice so that one can gain 'real knowledge'.

When the pupils talked about their documentation, they all referred to how they '*did it*' or what '*the steps*' were. But at the same time, most pupils considered that the work itself is not just something to be 'done'. It was valuable for them to be able to concretely 'see' their *learning*.

S: One understands more now, one learns more. Before it was about who did something and then it was finished.

ID: Do you feel the same about your documentation on 'Time'?

S: It is not most important that it is done. When we did 'Maths in town' we went to the city centre and we had our tasks. We had done them before, but then we had not practiced. Now we practiced and then we should see if we had learnt something.

ID: What did you do?

S: We went to bus stops and looked at bus time tables and looked at clocks with digital time...

ID: Can you tell me about some other things from your documentation.

S: The circle diagram was difficult. We did it in class. I made a circle and then I made a table. There were a lot to think about when I was writing about 'My 24 hours'.

In Vasaloppet there were time-differences. That I understood, so that was fun. One 'added up' to one whole minute and then one 'took' the hours...

A explained in more detail how he filled in his circle diagram for 'My 24 hours'. ('My 24 hours' is included in the pupils' Time-documentation, see Appendix 4):

A: At the beginning there was, like 1380 (minutes) so I had to add up (to 1440 minutes in 24 hours). It surely looks strange. The others had a lot of small things which took like 25 minutes, I didn't care about that and added it to 'Sleep'. It turned out to become a diagram after all.

Other pupils noted down, in a very detailed manner, their activities during a 24 hour-period. The next step was to categorise all the described activities in large descriptive titles, so that these could be transformed to sectors in a circle diagram.

During our conversations, the pupils focused on the particulars within the studied mathematical idea. For some pupils, the mathematical exactness was not so important. A (above) understands the context of the task, but feels that it is acceptable that he further de-contextualises the problem.

According to A the diagram 'turns out right anyway', even if it means that he sleeps 14 hours, according to his own diagram. He could have made the diagram more contextually correct, but chooses not to. He feels that it is not necessary, taken into consideration what the problem demands of him to do. A interprets the aim of the task to be 'productive', rather than 'creative' (Sinha 1999). In this example he has focused more on the mathematical content, than on the context. Other pupils focused on both the context and content. As Resnick argues, there is an implicit value of 'pure-thought' activities in education, where thought should proceed independently. By contrast, most activities outside school require the aid of physical and cognitive tools. The Swedish pupils in the study seem to have understood the context of the tasks they dealt with as equally important as the content. However, A is an example of a pupil who chose to focus more on the mathematical content, rather than on the context.

Teacher M encouraged the pupils to explore and to talk about the mathematical content. For the pupils, this meant that they tried to understand to what extent they needed to be creative and original in their answers, and to what extent they could make a problem-solving plan, which were important aspects of their individual agency. As the pupils were constantly encouraged to discuss, even when they worked individually, the solutions to problems or tasks were collaboratively agreed upon:

ID: What does M mean when she says that you should reason about something?

S: Reason can be to believe, to guess.

ID: Can one guess in maths?

S: If one hasn't practiced, one can, I think.

ID: Just before we came here (to this room) M said something about that 'All of you have the courage to guess', 'you have the courage to talk' - what does she mean by that?

S: I think she means that we perhaps don't need to know it instantly, and that before we have worked with it we can guess, and later we should know it, when we have worked with it.

The pupils seemed to constitute themselves as participants in a classroom discourse, where they were able to voice their own thoughts and ideas. The classroom discourse was experienced as giving opportunities for collaboratively authored knowledge:

Teacher M: Does anybody know what T does when he doesn't have 12th parts?

B: He cuts 6th parts into halves.

B: How does one make 85th parts, then?

Teacher M: How do you reason about that, B?

B sits for a while in silence. Then he brings a calculator, a ruler and a paper and tries to solve it.

By the end of the lesson teacher M says:

Teacher M: T was thinking about how to make an 85th part. How do you think he did it?

P: He measured with a ruler and divided into 85 parts.

P: they must have become small.

P: He perhaps took this side of the paper (points to the longer side of a blank paper).

Teacher M: How did you do it, B?

B: I did like that, but I took a paper with squares (cm²-squares). I took 22 squares and divided each square into 4. Then I didn't get exactly 85.

Teacher M: How can we get exactly 85?

Silence.

Teacher M: would you like to hear my suggestion?

All pupils: Yes.

Teacher M: I took 5 rows of eleven squares and 3 rows of ten squares.

Here, teacher M is part of the collaborative authoring. The next extract illustrates how the pupil's authorship is both collaborative and independent.

M has given a question on a paper, which says: 'How much is $1/5$ minus $2/15$?'

Teacher M asks A and T: 'Can you make 5th-parts into 15th-parts?'

Pupil A counts aloud 1, 2, 3 and 4, 5, 6 and 7, 8, 9 and 10, 11, 12 and 13, 14, 15.

T: Then we have got 3, 3, 3, 3 and 3.

A writes $3/12$, crases and writes: $3/15 - 2/15 = 1/15$

A: Yes, now we are finished. T agrees.

My reflection: $1/5$ corresponds to $3/15$. They seem to work with the idea that $1/15$ should be divided into groups of threes to arrive at $1/5$. They are actually working with the lesser known denominator, without formally writing it down.

In the following extract, teacher M has asked the pupils to discuss a task in groups. Then she walks around and gathers them for a short summing up.

Teacher M: I want to speak with A, D, J, S, Al and K.

Teacher M: How did you solve $1/2 + 1/3 + 1/6$?

The pupils have written: $1/2 + 1/3 + 1/6 = 1$

D: We saw that $1/3 + 1/6$ is $1/2$

Teacher M: Now I want to speak with S, Sa, R and B.

Teacher M: We cannot count with the help of paper all our lives (M is referring to fraction-sticks the pupils have cut out from paper).

How did you solve $1/3 + 1/6$? How many 6th parts are $1/3$?

S: 2

Teacher M reads: $2/6 + 1/6 = 3/6$. Is that the same?

S: Yes.

Teacher M reads: $1/8 + 1/4 = 3/8$ and asks: How has Sa been thinking? (directing her attention towards Sa)

Sa: That $1/4$ is $2/8$

Teacher M: And $5/4 + 10/12$? $5/4$ and $10/12$, is that the same? Silence.

Place it! (using fraction sticks).

R: No, $5/4$ is more.

B: More than 1.

The pupils gain knowledge on relations between fractions, at the same time as M increases the level of formalisation. The pupils talk about their agency in relation to ‘courage to work’ with mathematical tasks and with respect to ‘learning’:

A: If I ask if it is right, she says: ‘Compare, see if it is right.’ It is not she who should check if it is right. She should not do it for you, one has to do it oneself so that one learns. So she doesn’t stand there and check.

ID: Sometimes M says that you should discover something, what does she mean?

A: There is a big difference between discovering and writing. When one writes in maths books without thinking, but when she says it in that way one learns more, I think. In class 1, 2, 3 we had a teacher like that. We learned nothing. She said: ‘Now you get a paper, write something, when you are finished, come forward to me’. Then in class four we got no papers. ‘How does one do now?’ we were thinking.

ID: Have you learnt more mathematics now?

A: Yes, I have learnt a lot. Our teacher doesn’t only give us papers, if one doesn’t find the mistakes, one has to work harder.

C explains how pupils get the courage to work, when teacher M focuses more on what is right than on what is wrong:

ID: Does M say that ‘you are wrong’ sometimes?

C: Yes...

ID: How does she say it? Does she say: you are wrong?

C: No, no. Here I maybe am right on km but not on mil (one Swedish mil is 10 km). Then she says that I am right up to here, but that I have thought wrongly about

the other, so that you sort of don't have to begin from the beginning; that can be good so that you get the courage to go on.

It seems that the pupils experience that correction and keeping track of solving procedures is their own responsibility. The pupils' authorship in the classroom is mostly collaborative but nevertheless, the individual pupil experiences it as independent in relation to other pupils and the teacher.

4. Conclusions

The teacher's intentions expressed in classroom talk and activity contribute to the pupils' learning approaches and learner identities, but not in a directly corresponding way.

The results show a certain contradiction between the teacher's intention of working with a negotiable epistemology, and the pupils' experiences of the tension between pure and experience-based mathematics. While this tension is not directly expressed during interviews, my interpretation is based on the observation that the pupils during the interviews generally talk about *what they did*, about calculations, and about steps.

Despite this focus on procedure in their explanations, the resistance to the absolutist view on mathematics that they experienced with their previous teacher is strong. All the pupils talked about how meaningful their documentations on their thematic project works were and that M:s way of teaching is based on that they have to think and take responsibility for their solutions, build up argumentations for their problem-solving and validate their conclusions in collaboration with other pupils. Teacher M encourages authorship, as we have seen for instance in exercises such as 'How many dices did we have yesterday?', 'Meteorology' and 'Fraction games'. These possibilities for authorship specifically open up for reflection on the inexactness of verbal and written language and the exactness in mathematical concepts and thinking. The pupils respond to these possibilities in maintaining the fallibilist classroom discourse.

The pupils developed a relational agency in response to offered possibilities for authorship, and to their modes of knowing, based on agentive (empowered) authorship. The tension between a fallibilist and an absolutist view is thus overridden by teacher M and the pupils, and a space for reflection is opened, where the ontological status of mathematics is not omnipotent any more

(Harvey 1989, quoted by Dahlin 2001) but closely connected to our experiences of it; through language use and reasoning.

A, who I selected to illustrate authorship, speaks about the context as less important for his diagram. He ignores the fact that, according to the diagram, he slept 14 hours a day, although in 'reality', he goes to bed late, and he practices sports, meets friend, does homework, eats dinner etc. in his after school time. Perhaps he understood the task as a 'school problem', and that is why he only included 'eat breakfast', 'school' and 'dinner' and 'sleep' in his diagram. When we look at how he approached the content, his intention was mainly directed towards the calculations for the process of constructing the diagram as the most important. That was what he talked about, and found important to tell me about during the interview. The task demanded of the pupils to observe and note down activities during 24 hours. The description was thereafter to be categorised and labelled. 'Sleep', 'School', 'Leisure time activities', 'Family' and 'Other activities', were commonly used categories.

Most of the exercises and tasks the teacher prepared demanded of the pupils to put mathematics in context, either in a wider reality context or in a context where mathematical-logical relations were to be explored. Great importance seems to be attached to the aim that the pupils understand both context and content in the Swedish school class context, and that is perhaps also the greatest challenge to both the teacher and pupils: to maintain a classroom discourse based on plausible reasoning, explorations and developing given mathematical examples, and at the same time bridge the gap between what the pupils understand as context, and what they experience as mathematical content.

The negotiable epistemology of mathematics encourages reasoning and negotiations (Burton 1999). The context sometimes demands of the pupils to be original in their ideas about the context, which may lead to instances where the mathematical content is thought of as background instead of the context. Here the curriculum (LpO 94) contains contradictory perspectives: socio-cultural in the description concerning mathematics, combined with a cognitive perspective in the description of goals to achieve (Kroksmark 2003). Taken together, discourses on mathematics in society and the debate on mathematics among mathematicians, philosophers and researchers in mathematics education, all contribute to the experienced tension between pure and experience-based mathematics. Despite such tensions, teacher M:s emphasis on understanding mathematics in context, and on a holistic approach to learning, perhaps has an impact on the pupils' experiences of mathematics in a longer

perspective. Most of the pupils who have encountered teacher M:s teaching probably are equipped with knowledge, skills and understanding of mathematics which will be useful to their lives. It seems that the pupils' intentions in the field of learning converge with the teacher's intentions at this point of intersection. From both sides, there is a willingness, joy and responsibility, to understand and learn mathematics.

3: The Indian school class context

1. Educational discourses

First, the discourses *in* and *out of* school are described, which were found to influence the pupils' modes of being learners in the studied mathematics classroom. The discourses in relation to learner-centred education, present in policy documents, in educational debates, in teachers' discussions and in the context of the provincial town, have been considered here. Thereafter, the classroom discourse is considered, together with localised epistemologies.

1.1 Discourses on being a learner

In the Indian context, being an *autonomous learner* was offered as a discursive possibility in society generally. It is considered valuable to cultivate patience and concentration, which are important aspects of being an autonomous learner. In the local provincial town culture, there are many examples of meditative, repetitive and detailed practices, constituting people's modes of being through a prolonged narrowing of attention. These practices promote appreciation for details, repetition, recitation, conversation and silence. But children do not only learn to be autonomous; they also form identities as interdependent relational individuals, in a closely knit social web demanding well developed social and communicative skills. In the extended family, who most often live closely together, plenty of opportunities for interaction which refine children's abilities to be listeners as well as discoverers are given; for example grandmother's story-telling/a story for breakfast, lunch, snack and dinner, religious rituals at home, listening and participating in everyday talk, reflecting upon questions on issues that schoolchildren are supposed to know or are asked to answer, participating in debates, and so on. Children's exposure to a variety in people's beliefs and opinions probably stimulates a type of thinking that is open to argumentation, as well as a curiosity for what lies beyond what we are able to perceive ourselves. In a *communicative* society, there is a continuity with the past, which can be made sense of through repetition. There are still people in the rural areas of Orissa who embody vast knowledge, which has been transmitted as narratives over generations. My husband told me about men in his village, who embodied the mathematical work of Lila Bhati Sutra and other verbal knowledge traditions in Oriya and English. These people were respected and 'feared' for the same reason:

They could stop you anywhere in the road to ask you any question on mathematics or English or Oriya, and

you would be feeling ashamed if you, as a school-going child, with responsibility to the community, could not answer the question. Sometimes, if you could not answer immediately, you would be stopped many times until you could answer.

Many parts of everyday practices demand that people learn from repetition. At festival days, to take an example with mathematical relevance, girls decorate the yards in front of the home with rice-colour and make complicated symmetrical patterns. These patterns the girls have learnt from each other, and as the patterns are based on mathematical series, they can be developed infinitely. At school, I saw that the pupils learn how to solve mathematical problems through repetition, and through keeping attention alive by changing focus for each time the material is reproduced. Speaking out loud, 'reciting', is a way of *calling into focus*. This is a very common practice at the primary school level. At secondary school level, tasks and examples are 'copied', written down - in fact a form of *silent recitation* - and repeated, from the textbook or from the blackboard or even from the pupils' own notebooks, as I saw in my study.

Among parents, there is an emphasis on the ambition that their children should study and study well. However, due to various reasons discussed below, parents have become more and more convinced that education means to be able to *quote textbook knowledge*. Schooling has become more valuable than informal learning.

Right from early childhood, many children especially those belonging to middle classes, are made to slog through home work, tuitions and coaching classes of different kinds. Leisure has become a highly scarce commodity in the child's, especially the urban child's life. The child's innate nature and capacities have no opportunity to find expression in a daily routine which permits no time to play, to enjoy simple pleasures, and to explore the world. (NCERT, p. 1)

In the report *Learning Without Burden* prepared by the National Advisory Committee (1993), appointed by the Ministry of Human Resource Development, is pointed out that since covering the syllabus has become an end in itself, classroom teaching has become a ritual, and in several states, among these Orissa, children are encouraged to attend after-school tuition, since there is no time to cover the whole textbook content. Everyday mathematics at secondary level is de-contextualised in a way that makes it

difficult for the pupils to understand them as reality-based, since the book-version of reality is alien to their own. In the mentioned report, there are several examples of how textbook problems can be misinterpreted or not fully understood, because the textbook context is too foreign to children with a rural background. The problem of readability in textbooks is also overwhelming, since the pupils often only are supposed to try to understand the given text. From a class five textbook in mathematics, the report of the National Advisory Committee (p. 3) cites the following example:

We find that while dividing a decimal by a multiple of 10,000 or 1,000, we first move the decimal point to the left as many places as there are zeroes in the number and then divide the resulting decimal by the second factor of the divisor.

The report (p. 3) also points out that textbooks, are written with the purpose of conveying information, rather than aiming to make children think. For instance in a textbook for class ten, the term pH is defined as *the negative logarithm to the base 10 of the hydrogen ion concentration in ions per litre or moles per litre*. Another problem (NCERT, p. 9) is that far too many abstractions are introduced all at once. In class one, pupils are supposed to go up to 100; in class two, up to 1000; in class three, up to 10 000; in class four, up to a million; and in class five, up to one crore (100 million). Volume, weight and length are introduced at the age of seven or eight years. NCERT concludes about mathematics teaching that:

The tendency to follow the logic of the discipline of mathematic rather than psychology of learning as the basis of curriculum becomes even more dominant. Mathematics, thus acquires the image of an esoteric discipline which has little application in the real life of the child (p. 9).

The discourse on *autonomy* is present in several educational and policy documents. A.N Maheshwari (2004), former Chairperson of the National Council for Teacher Education (NCTE), has in consultation with some teachers, heads of schools and educational experts in New Delhi identified what they call *life skills*, which can be interpreted from statements of the curriculum, and has described them in stages. These are skills that every student should learn at school, which have been classified in broad categories by Maheshwari. Here I choose to look at the category *learning to learn*, since it

includes skills which need to be cultivated to become an *autonomous learner*. Important skills mentioned by Maheshwari? are:

At primary level:

- Be able to take interest in learning, become inquisitive and acquire self motivation. - Become an autonomous learner.
- Be able to complete time-bound tasks/ assignments in time.
- Be able to sit still, if required.
- Be able to listen attentively to a story / narrate it by recalling it from memory.
- Be able to concentrate on a given task and remain focused on the theme of a discussion etc.

At secondary level:

- Be able to understand problems.
- Be able to analyse them.
- Find their solution within the available resources.
- Be able to participate in group activities, such as peer group learning.
- Be able to carry out assignments through cooperative effort.
- Be supportive.
- Be able to appreciate better achievement and have a healthy attitude towards taking part in competitive activities.

Notably, abilities to gain procedural knowledge and skills are focused in Maheshwari's interpretation of the life skills mentioned in the curriculum, such as being able to complete time-bound tasks/ assignments in time, sitting still, listening attentively, and so on. At primary level, there is more information about what behaviour is appreciated, and what kind of learner identity is accepted and valued, rather than indications about how these abilities can be developed into competencies. Even the statement *Be able to understand problems* at secondary level can in this context be interpreted in a procedural way. The procedural view on how to gain knowledge and the significance of character-building is also found in the text of the National report on the development of education in India (1988, p. 25):

The content and process of education at the upper primary stage are in post-independent India directed to the consolidation of knowledge, skills, attitudes and values acquired at the lower primary stage and acquisition of wider knowledge base in the context of a broad based general education. (...)The curriculum at the secondary stage is

designed to equip the children either to take up higher course of study to prepare for entrance into the first degree in a college or to equip them for vocational instruction at the higher secondary stage or to enter the work force. Since secondary education continues to be terminal for a large majority of children who are likely to enter the world of work, the emphasis is on bringing their education to a standard which would enable them to enter life with self-confidence along with useful knowledge and skills, proper work habits, attitudes and character which would contribute to their productivity and well-being.

The latest National report on the development of education in India (2004), contains similar formulations, but there is also a new emphasis on the aim that education should give competences for life and that it should not only 'provide bookish knowledge to the students but also develop skills and competences so as to enable them to earn their livelihood' (p. 75).

In the empirical material from the school class context in Balasore, Orissa, I saw that, although the pupils are not encouraged to actively engage in peer group learning during lessons, they are highly engaged in peer group discussions during recess and when the teacher is temporarily absent. This is known and appreciated by the teacher:

Friends can very well understand each other. They have respect for us. They are thinking 'I haven't understood', then they can become shy. That's why it's easier that they talk with each other.

From conversations with the teachers, I learnt that they did not receive their salaries on a regular basis. Sometimes there is an interval of six months or more. Overcrowded classrooms, irregular attendance, and dropout for a number of reasons, are some of the problems mentioned. These might be additional reasons for the teacher to put the primary responsibility to learn on the pupils.

When teacher A tells about how she makes 'weak' and 'bright' pupils understand, she mentions aspects which are important for being an autonomous learner (the concepts 'weak' and 'bright' pupils were used by the teacher and therefore I use the same terms):

ID: How do you get weak pupils to understand?

Teacher A: I tell them that I have already explained (translated literally: given understanding) and that understanding will emerge (translated literally: it will come). But for many, understanding doesn't come, so I show them on the blackboard and try to make them understand. I say that we have a problem and they say it is dawning upon them (translated literally: it is coming). Then they can do it themselves and for the weak pupils I show on the blackboard and ask so that they themselves can understand and then I say that those who don't understand: can you make the others understand during recess, talking with each other? 'Those who have already read through the problem, can you get those who haven't understood to understand?'

In this class, everybody can talk with each other. Yes, and they can discuss with each other or ask me. I cannot give so much to the weak pupils, because we always have to hurry to finish this course. This course has to be finished so I cannot give much time to weak pupils.

Teacher A feels that her explanation and the pupils' work with the task will, combined, give rise to understanding. Later during the interview, teacher A talked about the better-achieving pupils:

Teacher A: There are a few clever pupils, they are intelligent and they ask questions.

ID: Do they ask many questions?

Teacher A: Sometimes they pose good questions and tell right, but they don't get good results, they have wisdom (buddhi). They are intelligent, but they cannot 'apply their intelligence' (she used the expression in English) in a correct way.

Teacher A is mainly concerned about providing the pupils with tools for their learning to take place, while the responsibility to find out 'the right ways to do the tasks' and to 'understand the content' is the pupils' own. Teacher A can only give the same presentation over and over again, until the pupils say that they understand. Then it is up to the pupils to practice. I asked A how the pupils in the class solve problems:

Teacher A: Weak pupils try to do it, but forget and bright pupils I ask, and then they suggest that 'this becomes this' and then I say 'yes, this is right', then I

explain what became wrong, if a weak pupil replies, and that this became wrong, and that we should solve it in this way. In these different ways I do.

Autonomy is offered as a discursive possibility, also in the classroom context. However, the autonomous learner is dependent upon Teacher A's presentation and the textbook examples. Struggle with textbook content often results in 'mugging up' (memorisation without understanding). These aspects are important to children's ways of constituting themselves as learners.

Connected to autonomy and interdependence is the aspect of *struggle*. I saw that the way the girls struggled at home with various tasks, corresponded with a dedicated struggle with school work. Children participate to a high degree in everyday activities, and are constituted as responsible young persons. Girls take care of younger children and some parts of household work. They have duties during religious festivals. Teachers told me that some girls sometimes fail to attend class during *puja* days. In rural areas, constant electric power cuts make study conditions problematic, and a space for doing study can be difficult to arrange. An important aspect I observed was that children study hard in the mornings before school, as well as in the evenings after school, at private tuition, alone or with the help of someone at home. I found that most of the pupils had a commitment to their study, with the aim of doing something with their lives, and to do good for others and the country. They had aims to become *inventors of medicine to cure diseases*, or *to become famous so that they could do something for the society*. 'To be good human beings' was a dominating wish.

Education in the highly competitive educational system of India is a key to social advancement. There is even a general belief, across socioeconomic backgrounds, that the score in the final examination is what really matters. In the provincial town of Orissa, children cultivate feelings of responsibility towards the family and are convinced that to study is to do one's duty as a child. There is proverb in Orissa, which is used in everyday life to underline the important stages of child's development and the view on the duty of a child. The original was written in sanskrit by Chanakya, who lived between 350- 283 BC. He was an advisor, economist and Prime minister during emperor Chandragupta's ruling. The proverb goes like this:

Treat your kid like a darling for the first five years. For the next five years give him discipline. By the time he turns sixteen, treat him like a friend. Your grown up children are your best friends.

It is possible that the *struggle* pupils experience with study, recognisable from other aspects of life and combined with a sense of doing one's duty, is an important feature of their modes of being a learner. And this experienced struggle, which permeates life, maybe inspires the pupils to actively engage in their processes of coming to know?

Here one teacher voices the teacher's responsibility, which illustratively summarises similar opinions expressed by all the teachers at the studied school:

In the context of India, particularly of Orissa, we as teachers face manifold problems. But as we play a major role in educating the children of our nation, we try our best to get best results.

For class 9, there are six units included in the syllabus for mathematics (NCERT, p.58):

1. Number systems
2. Algebra
3. Coordinate Geometry
4. Geometry
5. Mensuration
6. Statistics and Probability

I observed lessons covering unit 4. *Geometry*, which includes Introduction to Euclid's Geometry, Lines and Algebra, Triangles, Quadrilaterals and Area, and unit 5. *Mensuration*, which includes Areas and Surface Areas and Volumes. General guidelines for instruction given in the syllabus are (ibid., p. 58):

1. All concepts/identities must be illustrated by situational examples.
2. The language of 'word problems' must be clear , simple, and unambiguous.
3. All proofs to be produced in a non-didactical manner, allowing the learner to see flow of reason. Wherever possible give more than one proof.
4. Motivate most results. Prove explicitly those where a short and clear argument reinforces mathematical thinking and reasoning. There must be emphasis on correct way of expressing their arguments.
5. The reason for doing ruler and compass instruction is to motivate and illustrate logical argument and reasoning. All constructions must

include an analysis of the construction, and proof for the steps taken to do the required construction must be given.

A general description of proofs in mathematics and an introduction to mathematical modelling give further information to teachers for correct instruction of the subject matter. Here follow a few lines on *Proofs in Mathematics* (ibid., p. 62):

What a statement is; when is a statement mathematically valid. Explanation of axiom/postulate through familiar examples. Difference between axiom, conjecture and theorem. The concept and nature of a 'proof' (emphasize deductive nature of the proof, the assumptions, the hypothesis, the logical argument) and writing a proof.

Introduction to Mathematical modelling (ibid., p. 62) is presented in the following terms:

The concept of mathematical modelling, review of work done in earlier classes while looking at situational problems, aims of mathematical modelling, discussing the broad stages of modelling-real-life situations, setting up of hypothesis, determining an appropriate model, solving the mathematical problem equivalent, analysing the conclusions and their real-life interpretation, validating the model.

The guidelines are teacher-centred, and the focus is on the objective that the presentation of examples should be straightforward, involving a flow of mathematical logic and reasoning. However, research in education indicates that just making learners work is not a particularly effective way of getting them to think about mathematics (Boaler 1997). Nevertheless, I found that many pupils worked hard with the given material in ways which promoted understanding. There were other pupils who did not understand the method of deduction or the mathematical logic, but still worked hard to find strategic ways to do the tasks. I shall come back to this later. What is interesting in relation to discourses is that perhaps the pupils' ways of constituting themselves as learners *at school* are not very different from the modes of being a learner *out of school*. The pupils in the study are part of a discursive web with ways of listening, talking and acting, which perhaps enable them to find meaning from their teacher's presentation, based on deduction. These modes

of being a learner also enhance their processes of coming to know, in a way the pupils experience as valuable. In the following, I shall explore how localised epistemologies possibly influence learning approaches.

2. Localized epistemologies in the Indian context

Sen (2005) holds that encounters of masses of arguments and counterarguments, an acceptance to a variety of perspectives, combined with a general liking to speak, are important general aspects of how people interact with each other in contemporary India.

He argues that the heterogeneity of Indian life should be considered in contemporary India, and within this heterogeneity, he investigates what unites the sub-continental epistemology. Sen observes that 'objective knowledge' is not put in contradiction to an active engagement and subjectively experienced knowledge. Important examples are given by Sen of how images of India originating from the West have influenced how 'India' is understood both in India and abroad.

In the discourses on Indian epistemology and ontology, more importance is given to the country's spiritual and religious traditions, than other, atheistic or rationalistic philosophies of the Indian subcontinent. Crook (1996) believes that we cannot estimate the degree of critical engagement in Indian argumentative tradition only based on sources from ancient times. We cannot either understand to what extent non-Brahmins (lower castes) were excluded from intellectual discourse. Crook argues that if we take into account people's knowledge production and knowledge base outside educational institutions, India is as literate as Japan and Europe.

In the provincial town of Orissa, where my study was carried out, the literacy rate was 71 percent year 2001 (Orissa human development report 2001). The disparity across gender and castes is high. The literacy ratio between men and women is 82%/60%. And among the scheduled tribes, which make up 11 percent of the population, the literacy rate ratio between men and women is 30%/7%.

The indigenous education, prior to the colonial period, was village based and village controlled (Alexander 2001, p. 89). We can see that the information order changed with the British rule, in that texts became available and written information became more valuable. Bayly (1996) points out that sadly enough, the British way of storing and ordering information engaged many Indians in lower clerical jobs, which demanded little or no comprehension of the content. The British failed in understanding, had lack of interest in, or even prejudiced

preconception of, Indian epistemology, Bayly explains, drawing a vivid picture of the impact the British colonial interests had on the educated and higher levels of society:

The knowledgeable man of the Indo-Islamic order was remade in the course of a generation to become the 'native servant of government' educated in Milton and Shakespeare, friend to Copernicus, and reader of *The Times* (p. 38).

The Macaulayan education system, which constituted the colonial education system, was described by Macaulay in *His Minute for Education* (Alexander 2001):

It is impossible for us, with our limited means, to attempt to educate the body of the people. We must at present do our best to form a class who may be interpreters between us and the millions we govern; a class of persons, Indian in blood and colour, but English in tastes, opinions, morals and intellect (p. 90).

The purpose of education would be to produce clerks and administrators. In 1928, almost twenty years before India's independence, criticising this way of teaching, Tagore, the founder of an educational institution in India that still serves as a model, wrote:

The purpose of education from the very beginning is to think: with how little we can manage our lives. The balanced growth of body and mind is possible only in a situation which has minimum influence from outside. Such a situation gives impetus to human creativity (p. 48):

Tagore further stresses that children grow up in the country:

(...) under dictates of others and are too keen to be moulded in their images, resulting in our becoming ideal office workers of lowest level (p. 48).

Printing and book trade made up the context of the information society, which created a new information order. Most school masters continued to work with purely oral means of instruction, but the existence of a text, especially a religious text, made the emergence of more centralised and standardised

systems of knowledge in both the religious and secular spheres possible (Crook 1996, p. 305). During the early years of post-independence, there were still teachers by family tradition, who lived and were teaching on the verandas of people's homes and were treated as respected *gurus*. They embodied vast knowledge and mostly relied on oral transmission of knowledge.

Still today, many instances of oral knowledge reproduction are given at schools. At primary level for instance, children learn addition, subtraction and multiplication tables by heart. Mathematics in classrooms are further usually practiced as response to teachers talk, performed in unison. At later stage, as I saw, pupils practice to memorise many terms and formulas, *sutras*. These tables have even been given a special name, *Paanika*, and are seen by people as basic mathematical knowledge, which should be learnt at an early stage. Rhymes with mathematical content are practiced at primary school from *Mana Sankha*, a collection of narratives, originally orally transmitted and dealing with everyday mathematical problems. 'Sutra' originally comes from Sanskrit and literally means 'thread'. Important mathematical threads are the *Baudhayana Sulba sutra*, from 800 BC, which includes Pythagorean triples expressed in algebraic relation, and *Apastamba Sulba sutra*, from 600 BC, which contains one solution to the general linear equation (Wikipedia 2008).

I found that when Teacher A asked for a *sutra* (which we can translate into formulas and theoretical models and the thinking behind these theoretical constructs), the pupils spoke in unison to state in detail the required *sutra*. Here follows a vignette to illustrate the context of when *sutras* are used:

Teacher A: Let us do some problem solving in geometry (*ankha kasibha*). Ok. Let us first look at question nb 1. Well, well, let us see it. In a cuboid oil-tank there is a length, width and height, they are 25 cm, 25 cm and 32 cm. How many litres of oil can it contain?

Teacher A writes:
Length=a= 25 cm
Width= b= 25 cm
Height= c= 32 cm

Teacher A: Let us look at the solution. What is the *sutra* of cuboid? J, You cannot do anything if you behave like this, how can you pass mathematics? (J is a pupil who was talking at the same time as teacher). J, say, what is the formula?

J is silent.

Teacher A: What is the total volume?

L to P: I want to do it!

Teacher A: Ok, I can do it for you.

L: No, no, I would like to do it!

Teacher A: 25 multiplied by 25, What happens?

Unison: 25 x 25 is 625

Teacher A: Will it work in this way?

Unison: Yes, yes

Teacher A: Then you work!

Everybody starts counting and small-talking

Pause

Teacher A writes on the blackboard

The volume of the oil-tank is $a \cdot b \cdot c = 25 \times 25 \times 32 = 20\,000 \text{ cm}^3$

$1000 \text{ cm}^3 = 1 \text{ liter}$

$20\,000 \text{ cm}^3 = 20 \text{ litres}$

In this oil-tank 20 litres can be stored

Sutras are not prescribed as specific statements, but with an intention that the pupils will understand how these items are applicable in multiple and novel situations. This way of dialoguing was by the pupils experienced as an opportunity for reflection. Teacher A is using the flow, or interruption of the flow, in the classroom conversation to make the pupils focus on the direction, from the beginning to the completion of the task. The space for reflection and expression of alternative solutions is limited, though.

Mohanty (1994, p. 39) discusses word-generated knowledge as a distinctive feature of Indian epistemology, - at least recognised by most schools of philosophy - and which has influence on school education. He investigates the relation between *sabda* and *sabdabodha*. 'Sabda', in the sense of 'word' or 'utterance' and 'sabdabodha' with the meaning of 'knowing on the basis of understanding the meaning of an utterance'. Mohanty believes that the non-distinction between propositional thought and objects we direct our thought towards, can be understood in a text as a relational entity between the meaning of the word and the meaning of the sentence:

Just as a word refers to an entity, so does a sentence refer to a complex relational entity. To know the meaning of a word is to know what relational entity it designates (p. 47).

The relational structure might correspond to an ontological structure, Mohanty argues. He takes the example “The cow is white!”. The word ‘cow’ means, denotes, refers to, either the universal cowness, or a particular cow possessing ‘a universal cowness’. The sentence “The cow is white” expresses, means or refers to a relational structure, of which the components are the referents of “cow and “white” (and the implied relation between them). If the cow over there is in fact white, there is an ontological structure: the individual over there, which possesses ‘cowness’, is characterised by an instance of the colour white, i.e. by a colour-particular, which again possesses the universal whiteness (Mohanty 1994, p. 47).

There is a similar line of reasoning in Wittgenstein’s theory on language-game (1968). Wittgenstein states that there is a relation between meaning and use. The use of the language is seen as producing meaning. But, contrary to Mohanty’s reasoning, the ontological structure in the sentence is in Wittgenstein’s theory understood in the context in which the language is used.

The lexical meaning of the mathematical language, represented in *sutras*, is by most of the pupils understood in the way Mohanty describes as an ‘ontological structure’. The two pupils who could be described in the analysis by the category *associative mode of knowing* only worked with the text, the *level of words*. Those pupils who could be described by the category *compositional mode of knowing* understood the *ontological structure* as well. Through their understanding of the ontological structure, at the level of meaning, they found out critical aspects for solving the mathematical task. Finally, the pupils who could be described by the category *contextual flexible mode of knowing* moved beyond the text and the ontological structure. They found out the *relevance structure*, based on experiences of what the learning situation demanded (Marton & Booth 1997).

The non-distinction between propositional thought and objects we direct our thought towards, as described by Mohanty, is possibly another aspect of what makes pupils approach mathematical problems holistically. An example cited by teacher A is the postulate which states that ‘The sum of three angles in any triangle is 180 degrees’. It is presented by teacher A as an ontological structure in which the three angles’ sizes are related to the sum 180. Teacher A believes that when the ontological structure is understood, the significance of the theorem is also understood.

When the pupils all say that mathematics is about *understanding*, it is quite likely understanding of the relation between words and ontology that they are referring to. But, as we shall see, some pupils reach another level, beyond the words, and explore critical aspects in ways that make sense in relation to mathematical logic.

Word-generated knowledge is appreciated, perhaps due to the argumentative tradition in India. Conversation among people in the provincial town of Orissa, where the present study took place, is an important part of how people constitute themselves in relation to each other. The relational individual is constituted as part of a network of family members, friends, colleagues, and so on. Interrelatedness in communicative practice means that people mostly do not state or represent a specific opinion as a stable element of their identity. The same person can during a conversation 'try out' different opinions for the sake of argument, and people contradict each other as well as themselves. And for the pure enjoyment of conversing, these discussions sometimes last for long periods of time.

Perhaps the access to multiple storylines and the importance of conversation *out of school* make a ground for the pupils' relational agency towards each other, the teacher and the textbook *at school*? Their relational agency influences their experiences of learning. The pupils develop a conversational epistemology *out of school*, which it is reasonable to suppose contributes to an approach to learning as a process of recitation and repetition *at school*. The recitation, repetition, practice and reproduction are ways that the pupils use to gain understanding.

The classroom discourse, however, clearly limits the variation of ways that pupils possibly can constitute themselves as learners and how they come to know, due to the asymmetry in conversations, and the fact that the course content is generally treated as 'objective' (unquestionable) in the classroom discourse. Teacher A talked about how important it is for the children to learn from each other outside the classroom. She said: 'They don't need to be afraid'.

This statement understood in the context, carries a meaning related to the *power distance* (Hofstede 1980) between the teacher and the pupil. Out of school, a child is used to listening attentively to everybody who is older than her/himself, and in showing respect to elders. This involves the child in a practice of listening and reflecting, rather than talking and questioning.

Another kind of limitation relates to the scope for collaborative learning. While to some degree, learning at school is organised in order to be collaborative, at

the same time school learning is evaluated on the basis of individual achievement. The relational agency, which the pupils develop in and out of school, contains a varying degree of independence. This is why I looked closer at the relational agency and pondered on certain fundamental questions:

- Why do all the pupils struggle so hard to understand the mathematical content?
- Why do some pupils continue to struggle for understanding although they do not reach the meaning of mathematical logic?
- Why do some of the pupils repeat and practice, and are able to write the whole problem solved from beginning to end, even though they claim they do not understand it fully?

I took inspiration for my analysis from Indian epistemology, as interpreted by Potter, in his discussion of Mohanty's *Gangesa's Theory of Truth*, where 'Knowledge' does not correlate with 'Truth'. Knowledge is neither 'justified true belief', where belief means the disposition to respond in appropriate ways when stimulated, nor a belief which corresponds to reality (Potter 1984, p. 324). He concludes that the debate on knowledge (in the *svatab/paratab* debate) is to be concerned with a *workable awareness*, which leads to the satisfaction of a motivating purpose (p. 319):

The opposite of "workable" is not "false" but "not workable" (*aprama*), which term is intended to characterize all kinds of awarenesses which cannot lead to the satisfaction of a motivating purpose. The term *aprama* ranges, as we saw, over doubtings, errors and reduction ad absurdum arguments. When one is in doubt, he is not satisfying a purpose (doubting is not a purpose). Errors (i.e. perceptual errors, like the mirage) frustrate our purposes by misleading us. Finally, in a *reductio* argument the purpose is to prove one's own position (so we're told) but what the reduction (*tarka*) does is merely to convict the opponent of a fault, which does not (at least by itself) effect any proof (unlike a proper inference, where the conclusion does indeed prove just what was intended to be proved).

Potter suggests that Indian thought adopts a 'utility' reading of 'truth', and that 'knowledge' is related to purpose. Hence, since 'truth' *means* serving a purpose, there is no contradiction between truth and purpose. Potter further suggests that there is an agreement in all systems of Indian philosophy that the supreme

human purpose is liberation, and that there is a fixed, though context-sensitive, value system which coheres with that highest purpose (p. 323). The Western distinction between cognitivism and non-cognitivism in dealing with the fact-value gap is absent in Indian philosophy, writes Potter (p. 324) when he discusses the significance of the notion of JTB ('justified true belief):

The JTB account of knowledge is perhaps doomed as a futile attempt to provide a foundation in the absence of normative convictions which would constitute the proper, but now abandoned, core meaning of "knowing". Modern epistemology, getting the wrong message from Plato, perhaps, hoped that that core normativeness could be found in the necessities of formal (mathematical, logical) "truth", that is in consistency or coherence. We are now discovering that that is a forlorn hope, that inquiry is adrift without a recognition at least of the worth of what the inquiry is *for*.

In letting these aspects of Indian thought inspire my analysis, I made an attempt to come closer to understanding why some pupils go on struggling with mathematics, although some of them were clearly disenchanted with mathematical logic. Pupils with mode of being a *disenchanted learner* still had a motivating purpose of *doing work*. The learner could, with this purpose of doing work as part of h/er/is own truth, work with mathematics in a way that amounts to striving to find out 'what works' in relation to the given task. The procedural modes of knowing did not seem to be connected with the purpose of finding truth or following mathematical logic. Work was perhaps their truth. It can also be argued that based on the way teacher A gives information about a phenomenon x, the process of reproduction of the information concerning x is understood by teacher as well as pupils, as *both* work and truth. 'Understanding' the information, in this case, amounts to understanding an ontological relation in the mathematical text. A few pupils also understand what critical aspects they can vary the meaning of. The pupils' struggle with 'doing work' also has another, perhaps more important, higher motivating purpose. They want to become successful pupils, and do their duty as children as a responsibility towards family and community.

As we shall see in next section, the pupils had a notion of their processes of coming to know as part of *work processes*: of working through the problem, varying it, repeating it and striving to achieve exactness and correctness. 'Correctness' was experienced as ways of having/grasping a picture of the

whole problem, and different important parts of it. In teacher A's classroom, the pupils were concerned with looking for new and different aspects of the given structured content. What-questions dominate the classroom discourse: *What is given? What is the theory behind? What is the meaning of the solution?*

As part of the conclusions, we can see that a contextualised, conversational way of thinking and learning, which to some extent still exists in Orissa society at large, is today neglected in school practices. There is also a delicate balance between being competitive, as required in discourses that permeate Indian national education policy documents, and being interdependent selves. The interdependent construal of self, as observed by Markus & Kitayama (1991), actually implies a high degree of self-control and agency to effectively adjust oneself to various interpersonal contingencies (p. 228).

Teacher A focuses on a single 'method' for teaching, on 'right' answers and on 'good' results. Teachers at the school in the Indian study in general think that *the textbook is the course*, although in the introduction of many textbooks nowadays it is stated that the examples in the book can be expanded by the teacher, and can function as help for teachers in planning their teaching. Most of the textbooks are written in an abstract language which the learner is expected to relate to. The problem is that pupils can not always create their own meaning in relation to the content. Maheshwari at NCTE Delhi, India (2004) observes that children often don't recognise the context related to in textbooks, especially not those who come from rural settings. He also states that it has become necessary to investigate the learning outcomes, so that parents, teachers and children can understand them. He continues:

But the learning outcomes shall not be limited to mere acquisition of knowledge of concepts and facts of these subjects (languages, mathematics, science, social sciences, art and aesthetics, work experience, health and physical education). Though the textbooks may continue to remain the principal source of teaching-learning, the task of the teacher should go beyond just covering of the prescribed syllabi and emphasize a better appreciation of the life's environment. Therefore, a new type of teacher will be required (p. 3).

That a new kind of teacher is required is a thought provoking statement! In the same text, he explains:

The teacher educators will have to provide a broad exposure to a variety of strategies that can be employed for ensuring that each child is able to acquire skills, attitudes and values relevant as envisioned for the concerned stage of school education. (p. 3)

Maheshwari recognises that there is a need to provide skills in teacher education and in-service courses which would help teachers to teach more holistically. He also proposes that evaluation could be the responsibility of the school, rather than falling under the Boards of School Education that conduct public examinations for purposes of certification.

When Teacher A talked about the pupils' studies, she was mostly talking about how they must train their brains, listen and practice. She stressed that the children did not try hard enough in general, and that they mostly did not practice until it is time for examination.

They do not understand that they need more practice than that.

To illustrate other common beliefs held by teachers at this school, the following are extracts from texts written by 8 teachers. They were asked to write about *How do children learn?*:

A class either sails or sinks with the teacher. It should be in the mind of the teacher. Accordingly a teacher to get fruitful results of his teaching should set questions in a judicious manner to enable all types of students to answer.

A teacher should know if possible the private life of a student. Accordingly he should try to help them and it will create more love, affection and regard for the teacher. Then teaching-learning process will be more effective.

Education is a two-way traffic. So a student should be given maximum freedom to express himself before a teacher. Then the teacher should begin from the best point spoken by a student.

Finally, the teachers wrote about how communication in the classroom could be enhanced through dialogue, by posing questions so that the pupils would feel free to express themselves and by creating a good atmosphere.

3. Classroom discourse

The classroom discourse was centred around a conversation which supported deduction. The process of reasoning was to move from the general formula to the specific problem. The conclusions were to be drawn as a result of the premises given. I saw that the focus in classroom discourse was on the aim that the pupils should do their tasks correctly according to textbook examples. Observe that ‘doing the tasks correctly’ does not mean in this context that the pupils would simply reproduce. ‘Doing tasks’ had other experienced qualities, which were related to ways of gaining understanding through repetition of critical aspects. Here are additional extracts from texts written for this study by 8 teachers at the Balasore school. They were asked to write about ‘*What does problem-solving in mathematics mean?*’:

We try our best to teach with the best results.

The aim of teaching is to transmit knowledge to the student for best results. The method that is best is to follow a question-answer type of teaching method.

A teacher should follow a question-answer type of teaching method, so that the participation of students will be more and then it will be easier for the teacher to lead the students in the teaching process.

There should be multiple-choice type of questions which is easy to begin with for the highly intelligent students as well as for the below average students.

The method of teaching is to *transmit knowledge*, and by that the teachers mean to *pose questions* and *receive answers*.

3.1 Teacher A:s pedagogical power

In any learning context, discursive and contextual possibilities emerge at every learning situation. A teacher’s pedagogic commitment towards teaching and to pupils’ learning is reflected in her/his ways of approaching teaching and learning in a general and in a situation-specific way. I have here used the term *pedagogical power* which, Jaworski, Wood and Dawson (1999) define as the pedagogical knowledge the teacher uses to solve mathematical problems.

Teacher A has perfected her own way of teaching. She was well prepared on the subject matter, she had focus, and she had a method. She used a method of teaching for mathematics, which went along the lines of presenting the

examples in sequence, and asking the pupils for specific facts, in terms of required tools, formulas or denotations. Then she proceeded to solve the problem and to state the answer, and finally asked the pupils for the significance of the answer. She asked the pupils in a manner so that they could reflect upon what they knew, and what would be next in the process of coming closer to a solution. Teacher A:s main aim was to prepare the pupils for examination and to hand over samples of questions and examples.

Teacher A believed that deduction is the base for problem-solving, and that the pupils should understand this base. So when she presented an example, she did it as a sequence of figuring out:

- What are the conditions?
- What are the data?
- What is the proof?
- What is the sutra?
- What is the solution? (reconnecting with the mathematical problem, in order to answer the implicit question ‘What was the solution really about?’)

3.2 Teacher A:s approach to teaching

The teachers at the studied school all wrote about problem-solving in similar ways, as a response to the two questions I gave them (*What does it mean to work with problem solving at school?* and *What does problem solving in mathematics mean in your classroom?*), which made me realise that for them, *deduction* is a major guideline as how to solve problems and to present mathematical tasks. I have chosen to refer to what teacher A writes, as I think it illustrated best the general trend in the teachers’ answers:

Mathematics is a subject of problems. Its teaching and learning demands solving of innumerable problems. Efficiency and an ability in solving problems is a guarantee for success in learning this subject.

PROCEDURE

- 1. Recognising the problem or sensing the problem**
- 2. Interpreting, defining and delimiting the problem**
- 3. Gathering the data in a systematic manner**
- 4. Organising and evaluation of the data**
- 5. Formulating tentative solutions**
- 6. Arriving at the true or correct solution**
- 7. Verifying the result**

It involves scientific thinking as a process of learning.

The mathematical content, described in the textbook and presented and dealt with in class, follows a particular mathematical logical structure, which offers a varied view on the postulated truth. Let me explain this. I have chosen to examine an extract from the textbook used in the class. First, one can read a text which gives a general introduction to the subject, in this case Pythagoras' theorem. Thereafter follows a specific definition on the theorem, and an example. Lastly, three sequels are described and the proof ends the description of the theorem. In the following extract from the textbook used in the class (pp. 68-69) it is stated that (*my translation*):

For example, in a triangle ABC (figure 6.1) the angle B is the right angle, $AC^2 = AB^2 + BC^2$

Any opposite of this example is also true. In any triangle, if the square of one side becomes equal to the square of other two sides, the angle opposite to the longest side is a right angle.

In other words, the textbook writers have the intention of illustrating how Pythagoras' theorem can be thought of as a relationship, which corresponds to three different problems, depending on which of the three sides are unknown. Here follow the three alternative problems, which can be seen as illustrating the relation between the three sides (*my translation*):

Anusidhanta (Corollary):

1. **If in any rectangle the near-standing sides are a and b (figure 6.2), then its hypotenuse is $\sqrt{a^2 + b^2}$ (both the sides are of equal length).**
- 2.
3. **If the sides of a square is a, then the hypotenuse root of $a^2 + a^2 = a\sqrt{2}$**
- 4.
5. **In the Samakuni samadibaho (equilateral triangle) if the near-standing sides are a, then the hypotenuse is $\sqrt{3}a$**
- 6.
7. **If in an equilateral triangle every side is a, then its height is $\frac{\sqrt{3}}{2} \cdot a$**

The intentions, implicit in the text of the textbook, are that Pythagoras' theorem can be understood in three ways, which are called 'opposite' ways in

the text. These three ways are useful depending on what is demanded of the mathematical situation. This intention of making pupils understand the relationships between variables in sutras is implicit in teacher A:s teaching. She does not explicitly comment on the variation. Teacher A:s main purpose with her teaching was that pupils would themselves reach an understanding of sutras and their roles in the light of mathematical examples. Her intentions are:

- To give presentations of examples which provide mathematical logic.
- To give presentations which are structured, according to textbook examples, so that the varied view on the content might be experienced by the pupils.
- To pose rhetoric questions in order that the pupils learn mathematical deduction.

The aim was to state the question, give the data and conditions and to pose questions to encourage reflections on what is necessary for devising a plan for solving the problem. In the process of solving the problem, the aim is to make the pupils understand deduction:

Teacher A: What is given in dhata (premise)?

Unison: ABCD is a quadrilateral.

Teacher A: What is more given?

Unison: AB is equal to DC.

Teacher A: AB is equal to DC.

Unison: AB is equal to DC.

P: AD is equal to BC.

Teacher A: AD is equal to BC.

My reflection: Do the pupils respond to A:s statement for confirmation, an implicit question?

Unison: Yes, yes.

Teacher A: What did you understand about what 'equal' means? Let us write.

Teacher A: ABCD is a quadrilateral, where AB is equal to DC, where AB is parallel to DC.

Teacher A repeats this statement (ABCD is a quadrilateral, where AB is equal to DC, where AB is parallel to DC.)

Unison: AB is equal to DC, AB is parallel to DC.

Teacher A: What is going to be proved (pramarnha)?

Teacher A: The pramarnha is ABCD is a parallelogram. ABCD is a parallelogram.

Unison: ABCD is a parallelogram.

Teacher A: Then I am writing 'ABCD is a parallelogram'. Now we are going to draw the figure. What are the possibilities? (Translated literally: How does it become easy to draw?).

P: Yes, yes.

Unison: Yes, yes.

Teacher A: What are you thinking? In which way is it possible?

P: Let us draw two angles?

Unison: We join the angles.

Teacher A: Ok, let us draw the angles and we join with BD, all right?

And draw and join BC and DC. What do we see here?

Unison: ABCD is a parallelogram.

Teacher A: Then now, what do we mean by a parallelogram?

P: In any quadrilateral, where opposite sides are parallel and equal.

Unison: Opposite sides are equal and parallel

Teacher A: Here, you can see in this figure opposite sides are equal and parallel.

Unison: Yes, yes.

Teacher A: What more is required here?

Two opposites, AB is equal to DC and parallel to each other and BD is a shared side, then it is a parallelogram.

Pause

Teacher A: In a quadrilateral if the opposite sides are parallel to each other, then we have taken one side. We can now prove that other sides are parallel to each other...

P: We can prove this is a parallelogram.

Teacher A: To prove that a quadrilateral is a parallelogram, AB is equal to DC, it becomes parallel, what are the conditions left behind it?

Teacher A: The angle created by side AB and BC is the angle ABC.

P: Yes.

Teacher A: Yes, what do you mean by it? Sit down (the pupil stood up to answer the question).

Teacher A: When two straight lines are parallel to each other, then the angles opposite to each other are equal

P: Yes.

Teacher A: Now it is our work to make these two angles equal. And if we make these two angles equal, and we know that AB is parallel to DC...

Pause.

Teacher A: How do you make the angles equal? Can we measure them?

Can we say that: A: The angle ABD is equal to BDC?

Teacher A is writing three conclusive points on the blackboard

P: The triangle ABD is equal to triangle BCD.

Teacher A: The triangle ABD and the triangle BCD are congruent.

Unison: The triangle ABD and the triangle BCD are congruent.

Teacher A: What are the conditions for congruency?

From this extract we can see that teacher A's intention was to make a clear presentation and to make the children understand both the example from beginning to end, as well as the method for solving the problem. Teacher A presented examples from the textbook as objective truths. Her questions, although mostly demanding absolutely exact and correct answers, were experienced by most pupils as questions which open for reflections, as we shall see in next section. Even though I pointed out to teacher A, after observing a couple of lessons, that I wanted to come to one more lesson to especially observe the interaction between the teacher and the pupils while solving problems, the same pattern of teaching was followed at that instance. It made me wonder, why does she see the interaction and I don't? It may be that A was more focused on her own presentation and on the pupils' interaction with her presentation in the process of unfolding the solution, rather than on dialoguing. A says:

When I give them understanding (*explain*), they learn a problem-solving attitude (*she uses the English term*). They get to know how they should solve the problem so that they can solve it. That I try to get them to understand.

And again in English she says:

How to solve it, what is given? What to find out, how to find out, this I try to teach them.

In relation to the narrowing down of the space of possible plans for solving mathematical problems, by external authors, teachers and textbook writers, the pupils try hard to cultivate patience to reach understanding in their own ways, as we shall see in the next section. There is an experienced direction: starting from the question, through the completed solution, to the answer and back again. When talking about how they solved a problem, the pupils narrated the whole procedure from beginning to the end. Some of these pupils said that they did not yet understand the problem fully, yet they could state the whole direction from Q (question) to A (answer). They varied the way they repeated the problem every time they practiced, so that understanding slowly emerged. I shall return to this later.

Teachers at the school told me that boys in general are more trouble-makers, as they usually pose why-questions, while girls are studious and try to learn. Teacher A told me about one difficult pupil who could have done well in her exams, if she only would stop asking why-questions and practice instead. Teacher A had even talked to her parents about the problem.

ID: There is one girl, whom I haven't talked with, she often asks questions like: this can be solved in this way also...

Teacher A: Yes., now I know.

ID: Is she a good student?

Teacher A: She doesn't do well in the exams. We have discussed about her in the staff room. She wants to speak a lot and asks 'Why can't we do like this?' But during the exams she cannot express. I do not know exactly why she does like this. She thinks that she knows. What she has in her head is right. And what is new she doesn't accept. Or she thinks that yes, this was right because it is similar to what she knows.

She makes trouble. Everybody thinks so. She talks a lot and doesn't score well. At the half-yearly exams her father became very angry. I told her that although she knows so much she cannot write on the exams. If she could write it, she would get more points. She has problems to express or problems to write. What she is thinking she cannot write and she has not practiced to write. It can be that one has to practice to write before the exam. Our children don't do that. To write oneself before the exam. To write question and answer without looking. They write directly at the exam.

It seems that teacher A believes that there can be a gap between what the pupils know, and how they are able to express their knowledge. A frequently returns to this in the interviews. She stresses that it is imperative that the pupils learn to understand the mathematics in the tasks, so that they can use it to express themselves. She feels that understanding is reached only after a great deal of practice. The presentation is usually written down by A on the blackboard, and her talk was interspersed by the pupils speaking out in unison after an individual had spoken. Sometimes the whole class spoke in unison following A:s presentation. It seems that only when pupils were not sure about

the next step, a few hands came up, and someone was asked to stand and make a try. After the right answer had been uttered, the whole class resumed in unison. At some instances, children were asked to present their solution on the blackboard, and friends could take over when someone failed to remember the correct way. These ways of responding are of course understood as ways 'one usually responds to Teacher A:s requests during lessons'. Teacher A usually concluded the presentations with instructions to the pupils to 'do' the tasks. Here, 'doing' (*kehariba*) includes all kinds of meanings, such as: write, think, understand and do it correctly.

It is assumed that the children should work hard to understand the way to solve a problem in a single correct manner. This means extensive practice and serious attempts to 'see' the proper ways. 'Seeing' is a mental process of listening to the teacher's words, reading and repeatedly copying the text, so that the re-constitution of the already given is correct in detail and sequence. By striving for this kind of exactness, the children are through hard work able to reach better and better understanding, so that finally the 'best' understanding is reached. The focus is on the *direction* from beginning to end, which should be repeated until one 'sees the whole'. The pupils can themselves try to understand part-relations to a whole. The exactness is a direction, rather than a goal in itself, and the work for understanding is the responsibility of the children, which they attempt to achieve through listening, practicing patience, concentration of mind, collaborative learning and repeated practice. The general taken-for-granted assumption on teaching which Teacher A shares with many teachers in Orissa, is that her presentations should be perfect, in order for the children to understand the content clearly. Learning is a process of repetitive practice. And at the same time, the children practice concentration and the ability to listen, follow an example, memorise, practice and repeat the content, until understanding is reached. The teacher expects the children to learn the examples and tasks exactly in detail and sequence, and that the direction in the presentation is focused, more than different parts or steps in the example/task.

The teachers at the selected school were asked by me to write about the following topics: What does it mean to work with problem solving at school? and '*What does problem solving in mathematics mean in your classroom?*'.

I have chosen here to refer to teacher A:s statements, as I think that it shows the general trend among all statements. (Note that she uses the term 'Ans.', which stands for 'Answer'). Teacher A writes:

Ans. Take for example that finding the volume of a cylinder is a problem before the class. Its formula has to be developed on the basis of the earlier formula for the volume of a cuboid. While analysing the problem, it gets connected with the previous knowledge that volume of any regular solid can be found by multiplying the area of its base with the height of the object. The given information is so organised that it becomes the required information. The area of the base of the cylinder is found by an already known formula and method. Then the required formula is obtained by multiplying this area with the given height. For the purpose of verification it is applied to a number of similar problems or situations and the results are checked. The solutions to the problems always come from the students. The teacher remains in the background and directs or guides the student activity from that position.

Here we can see that A has reflected upon the content in the textbook. She believes that the information in the mathematics book is arranged in such a way that formulas relevant for any problem can be derived from general formulas given there. This, she feels, is also the basis for deduction. As we have already seen, the examples in the textbook often are solved with formulas which can be derived from other formulas and these, in turn, have been used in previous examples.

Wholes are distinctly and precisely presented, and every example is expected to be experienced by the students as an organised whole in one specific way. Teacher A expects the student to work with deduction for exactness and understanding.

We can relate to the earlier discussed absence of distinction between mathematical ‘work’ and mathematical ‘truth’. The sutras are to be understood in the context of the specific problem. But again, the sutras should be understood so that these can be used in different problems in different ways. So, sutras are tools *for* analysing problems, with an aim to understand mathematics.

I asked teacher A: I have seen that better achieving pupils look at their earlier notes and don’t want to write the example down once again. Have you seen that?

Teacher A: Yes

ID: You usually say that they should write it once more, write it again, they should practice, is that why you tell them...

Teacher A: Yes, that is a problem. They have gone to tuition and practiced at home. They look at the sample and tell from their notes, but they might not have understood it. If their notebooks are closed or I take their notebooks, there are just a few who can tell. Bright children can tell if we argue with them a lot. It is a bad habit (A says *bad habit* in English). It can come to the bright pupils at an early stage (they can understand). Then they look a little, and close the book. That's the way they do it. And they don't want to write. They might be right, they might do it, but I say 'No, write, practice, see if you can do the test questions without looking'.

She continues to discuss whether memorising is important:

Teacher A: I think, but I am not sure, that one has to remember many things in mathematics. No, no, one doesn't need to remember so much. One has to use the brain, isn't it? A little one has to keep in memory. Suppose, different formulas or, to take an example, in a triangle, the sum of three angles is 180. That one has to remember, isn't it? But how it becomes 180, one doesn't need to remember. That one has to solve (she says 'make a solution', solve kariba, in Oriya). Or if one solves a mensuration problem, then we must know that in a cube the total surface area (she uses the English term surface area). The formula and the volume formula, but when one is solving a problem one doesn't need to remember anything. What is given, one has to look at, what can one get out of the problem one has to look at and solve. It is of no use to 'massage it in' (learn by rote).

In mathematics one doesn't need to keep so much in one's memory. The children don't understand that, they think they have to remember everything and they try to remember and that's why they get wrong on the exams. Take for an example a geometrical problem. There are the denotations ABCD suppose, and in the exam they remember that the angle ABC is equal to the angle BCD, which is wrong. I know what their problem is. I tell then that one can not massage it in. One cannot learn mathematics by rote, one needs understanding. How can one understand? All don't have the understanding. There are weak pupils especially in mathematics and especially among girls. Of 47 children, 15 understand mathematics.

Teacher A thinks it is problematic that the pupils do not understand that they cannot memorise by rote, but that they have to look at the conditions and data provided to figure out what formula to use. Interestingly enough, the dominating aim is *to understand the principle*, for instance that the sum of three angles is 180 degrees in a triangle. The secondary aim is to understand *what the*

conditions for the principle are. A considers this to be a problem related to the principle. The purpose is thus not to prove the principal statement, but to explore the examples related to the principles.

3.3 Learners' modes of being a learner

In this section, attention will be on how the pupils constitute themselves as learners in relation to the culturally and socially constructed and given meanings in discursive formations and practices that are available to them. My assumptions are in line with poststructuralist assumptions stating that the subject is not only created/constituted by discursive meanings, but also creates her/himself through them: by taking them up, challenging them and deconstructing the meanings, as a subject or as part of a collective (Taguchi 2004). In this way, the learner's identities are possibilities for mediating agency and authorship. The subject exists as a process, and is both *constituted* and *constitutive of* discourses (Davies 2000, p. 139). My use of the notions of *agency* and *authorship* is inspired by Burton's research and epistemological explorations (cf Burton 1995). This way of viewing 'knowing' involves *who* comes to know, as well as *what* one comes to know, and the subjectively authored known.

3.3.1 Mathematical logic

The pupils believed that it is most important to have a focus on the *logic* of mathematics. Formulas (*sutras*), had meaning in relation to mathematical problems. All pupils expressed that they practice mathematics for *understanding*. When I asked them further about what they meant by this, a varied picture emerged and the conversation led to asking the pupils to choose a problem which had been especially difficult to solve.

The mathematical word problems were not considered as purely objectively given, but as distinctly given exercises on mathematical logic. This became evident when many pupils explained about their repetitive practice, that they varied the way they 'saw' the content each time they solved a problem. Many pupils also talked about how they choose formulas for different problems and how they derive formulas from others, and how they view *sutras* as mathematical relations between variables.

In this context, it is important to note that in Oriya, the language used in the classroom instruction, there are two expressions for the mathematical equals sign, ' = '. These expressions are consequently used in their two different meanings by teacher and by pupils. When ' = ' is used in a mathematical relation, it is referred to as *saman*, which has a *relational meaning* of the use 'equal to'. In Pythagoras' theorem, the relation of a, b and c is described in terms of a²

saman square root of the sum of b^2 and c^2 . When pupils talk about the square root of 289, then the use of '=' means that the result follows from calculation and they say that from square root of 289 emerges 17, they use *bhariba* and say/write 17.

3.3.2 Problem solving procedures

A few pupils also mentioned how they re-phrase questions, so that they can see possible questions to which a *sutra* can be applied. I asked the pupils to choose a problem that is difficult, and which they do not yet understand. All the pupils talked about their solutions to the selected problem in a similar way to Sa in the following extract:

ID: Can you show me an example of how you solve a difficult problem and how you come to understand it?

Sa reads the task out loud first.

Sa: First I have to understand the general meaning of the problem (*sadhana gatana*). After that I draw the figure and then I do it.

Sa continues after she has read through in a half whispering mode:

Sa: There two sides are of equal length. Then *pramanya* (proof)...*pramanya* means...what *pramanya* one should give.

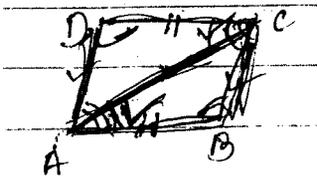
ABCD is a quadrangle (Sa speaks and writes simultaneously). There comes a proof after that. No, I forgot the *dhata* (premise).

AB is equal to CD (she writes $AB=CD$).

My reflection: Sa thinks aloud. Here, saman means 'equal length', and samantara means 'common side'. She realizes that she should make a figure to understand the relations between sides in the quadrangle and the congruent triangles.

Sa: There is both *saman* and *samantara*. Then I have to draw a figure.

She draws:



Sa continues:

Angle AC... and then...

In the triangle ABC, (angle) AB is equal to AD (she says *is equal to* in English).

AC is a common side in both the triangles and the triangle ADC, I haven't done this one and I have forgotten...

AB is equal to DC and AC is a common side in both the triangles.

Atha (that means) ABC is equal to ADC and ABC is *sarbasaman* (congruent to) ADC.

Then we have got the triangle ABC is equal to triangle ADC, they are congruent triangles, that means, AB is equal to DC.

Here we can see that the solution of a problem follows a certain pattern. The questions that the pupils ask themselves are similar to those posed by Polya (1985) in *How to solve it*, where he generalises problem-solving procedures in mathematics. The general procedures he suggests involve the following questions: *What is the unknown? What are the data? What is the condition?*

3.3.2 The importance of repetition

The pupils I interviewed made reflected guesses on which formula could be used. All of the pupils talked about the importance of repetition, and explained what they did when they did repetition. Sa explains:

ID: How do you learn mathematics?

Sa: It is given in the books. One has to practice.

ID: How do you memorise all?

Sa: In the theorem you mean?

ID: Yes

Sa: First we learn words, then sentences, in class one we learn ah, kha, gha (letters of the Oriya alphabet) After many times' repetition, one can learn the theorem. Mathematics is to understand. One doesn't need to

memorise much. One should not just ‘massage it in’ (learn by rote). If one tries to remember, for instance, that ABC is equal to ADC , or that AB is equal to DC , if one writes that on an exam, and that BD is the common side of the two triangles, and understands exactly, then one doesn’t need to remember anything.

The premise, the mathematics and the proof is needed.

ID: How many times do you have to write?

Sa: This one doesn’t need to learn by heart. (she reads the task aloud again)

One has to know what is demanded in the task, what one should do. One has to understand the proof and then one understands.

All pupils stated that understanding comes first, and then memorisation. A few pupils also felt that in order to be able to really ‘make a picture’ about the problem so that they could ‘see’ whether it all came out right, they have to practice and do the problem several times, at least two, three times. These pupils also talked about how important it is for them to make other friends understand. Teaching the friends is good for their own and their friends’ understanding.

‘Doing tasks’ and repetition was crucial to understanding the mathematical logic. The understanding these pupils aimed at was concerned with *how to solve* the present mathematical problem. Most of the pupils compared how the formula had been used in one problem, and tried to figure out how to use the formula in another way in a new problem. In a similar fashion, some pupils visualised and ‘saw’ if the task had been done correctly or not. These processes of coming to know were subjective, in the sense that the possibilities were subjectively experienced, and the ways of doing and understanding were subjectively constituted.

3.3.2.1 Applying formulas

As stated earlier, the pupils experienced the mathematical word problems not as purely objectively given, but as distinctly given exercises on mathematical logic. The logic was like ‘ground rules’ in their game of finding out possibilities to arrive at a solution.

Rather than considering logic, some pupils focused on what *sutras* or formulae were important in the present problem. N says when she speaks about a problem which is difficult to her:

N: Formula does not come to me. I didn't take the book with me. It is there.

ID: You can see if you can find it. Here, you can look in the book.

She turns the pages for a while, and then she says:

N: This formula, this formula and this formula!

ID: Do you need these three?

N: No, not all! I have to see what I can use.

N attempts on a written solution. After she has finished writing, she says:

N: When one knows it, one knows it.

N tries to figure out how to derive from the formulas she has found, one particular relation that works in the problem. This is a common statement among the pupils who were participating in the Indian study. With the statement 'When one knows it, one knows it', N probably refers to her experience that she can sense when she has understood something, and that is when everything falls in its right place.

3.3.2.2 Memorizing questions and answers

Although most pupils expressed the idea that memorising was secondary to understanding, there was one pupil, V, who looked at the question and answer as a direct relation which should be memorised, and not as a means for understanding the logic of mathematics. V memorised the wordings in the questions, and tried to keep in memory which questions belonged to which answers. This tendency of approaching the questions and answers, Q and A, was probably a result of the writing procedure as a method of practicing, used in the classroom. So-called *meaning books*, marketed exam-kits, which gives examples of Q and A which frequently appear in examinations, also probably encourage pupils to approach learning in this way. The lack of time for the pupils to really be able to direct the required attention to a wide range of theoretical constructs, combined with an approach to teaching that does not encourage pupils to discuss their understanding, all tend to encourage this kind of memorising without understanding. Even among the pupils who said they aimed at understanding, several pupils looked at the relation between Q and A as a relation between premises (*dhata*) given in the question and the meaning of the result. S says:

It is important to come to the proof from the *dhata*. One has to remember some things, like *sutras*, *sutras* one has to remember. For instance Pythagoras' theorem and that (she speaks about two tasks; 17 b and 19) in a rectangle there are all sides of similar length, one has to remember in order to understand.

The method of writing and memorising gives understanding, according to all pupils. Surprisingly, Sv writes down a whole problem, including question and answer, but cannot tell me about what she has written. She claims that she has to go through it two, three times more to understand it fully. This awareness is interesting, and tells about her wish to understand the problem through getting more and more 'acquainted' with it.

L also says when speaking about a difficult problem, that she has done the task twice, but not got really 'the habit of doing it'. She says she does not know if it is right, if she has written it rightly.

The following statement by D summarises well the pupils' intentions about practice and understanding:

In maths one can not remember anything without practising. If one practices many times one can. It is not enough that the teacher gives us understanding.

The pupils were all aware of that learning with understanding is essential. Having an understanding of something implies memory, just as memory implies understanding. Understanding is by the pupils experienced as a simultaneous act of reaching understanding and to have an understanding.

3.3.2.3 Understanding the problems

S is considered by the teacher to be a high-attaining pupil. But all pupils, just like S, say that mathematics is about *understanding*:

First one has to understand the problem and then remember the *sutra* and practice many times.

She works for understanding in the following way:

First one has to look at the problem and then close the book in this way and then it is only to copy the

question and then see what became right or wrong and thereafter to do it over and over again. Then it is enough, then one knows it.

She told me that Tuition Sir (the private teacher) often looks at the solution:

If I have not understood fully he shows it again and tries to make us understand. And then he shows us another problem which we work with. And if we do not understand we ask him.

S also speaks about understanding the process of deriving a proof or result from what is given:

First one has to get the proof (pramanha) from the conditions (dhata). One has to understand how to come to the proof from the data given. One has to look in the book so that one can remember that this is a rectangle with two sides of equal lengths. One has to remember the sutra to understand.

3.3.2 Agency and authorship

I wondered how the pupils could experience their processes of coming to know as something they should take active part in, considering that they were apparently encouraged just to reproduce textbook examples. The conversational epistemology, which made the pupils vary the content and engage in practices of listening and concentrating, perhaps contributed to the attention given to varying the problem, bringing in new points and creating new links, so that new possibilities of connecting elements are found, relevant to the problem.

There is also another point I would like to make. The agency and authorship, experienced by the pupils, had consequences for how the pupils constituted themselves as learners. I found two principal modes of being a learner: the *committed mode* and the *disenchanted mode of being a learner*. Most of the pupils were 'enchanted' by the work with mathematical logic and were trying to come to the solution of problems. When I asked S how she knows when an answer is right, she said: *Tuition Sir is there*. Behind this statement lies her experienced relational authorship. She means, that for her own correction, and when she needed to consult him for repeating what she should already know, 'Tuition Sir is there' to state the known. In fact, teacher A very seldom goes through her presentation a second time, and never gives alternative solutions. In this

relationship therefore also lies an implicit possibility for the pupil to write and understand problems with authorship.

There were two pupils in the study who represented a mode of being a *disenchanted learner*. Although they developed ways of coming to know, these were not guided by mathematical logic, and mathematics seemed to be something rather alienated to them. They did not know in which ways to learn for understanding the mathematical logic, although this was their expressed aim. They had a more dependent agency and had given up their authorship. They were instead struggling with simply 'doing their work' and memorizing as much as they could. The disenchanted mode of being a learner can be compared with the quiet disaffectionation expressed by pupils in UK observed and interviewed by Nardi & Steward (2003). The pupils in the UK-study saw rote-learning as meaningless and that teacher's presentation could only be tolerated. Pupils' wish to enjoy mathematics was the most important factor for appreciating and taking part in the mathematics classroom. All pupils who participated in the Indian study underlying this thesis, struggled to make sense of mathematics or with finding cues. Even the disenchanted pupil did her work, although she could not find mathematical meaning in the work.

The fact that most of the pupils engage themselves in textbook study most of their time, gives them little space for reflection and for authorship of their own knowledge. The limits of what they possibly can learn lies at the border of what the textbooks describe. Their meaning constitution deals with theoretical constructs and mathematical logic. The procedure of writing engages the pupils in a varied way of seeing the task, until a perception of it has been achieved. I found that most of the pupils could remember exactly, word by word, the *sutras* needed. They followed the method of deduction, in line with how the teacher had presented the tasks.

In order to be able to make the right choices, to claim what is important in relation to the rest, one has to understand what one is reading. During classroom talk, the sequencing and details of a mathematical example were focused, so that part-whole-relationships could be discerned, and exactness could be experienced as direction, rather than being an end in itself. So, even though there exists a power distance to the teacher, and the main aim of attending class is to be able to answer teacher's questions (and later examination questions), there is a need for the pupil to *memorise for understanding* as contrasted to rote memorisation.

Among the pupils described in the category *compositional mode of knowing*, there is a deep approach to learning (Marton & Säljö 1976; Marton & Booth 1997). The meaning of the mathematical *sutras* is looked at from different angles. When pupils adopt a *contextual flexible mode of knowing*, there is a deep approach to learning, where the learner is trying to look *through* the mathematical logic. They had found out that practising through varying the word problems made them see the problems in a new light, which enabled them to understand, so that they could memorise useful aspects while doing the problem.

4. Conclusions

The relation between the teachers intentions which she brings into her teaching practice and the learners' intentions, which they bring into their learning can be understood in terms of what aspects of educational and classroom discourses that can be found to have some influence on the learners' modes of being a learner and approaches to learning.

There are expectations on the children to be autonomous in and out of school. The practice of memorising is recognisable from other contexts. The pupils experience their ways of knowing as agentive in their subjective meaning-making, although the boundaries for thinking and reasoning about mathematics are very narrow. Teacher A focuses on the mathematical content from its logic, and considers it in the light of presentation of a sequence which is deductive. The pupils' and the teacher's intentions meet at the point of intersection of telling and re-telling. There is a struggle to understand and a responsibility to learn, that are both highly characteristic of this classroom. The relational agency is developed as a response to demands of making sense of theoretical constructs given in the textbook. The pupils authored their knowledge in a subjective process of understanding the ontological structure (Mohanty 1994). The authorship is reflected in the pupils' modes of knowing.

V Conclusions and discussion

In this part, first the method is discussed. Then the conclusions from the results will be discussed, regarding the three flexible modes of knowing that were identified and the relation of these modes of knowing to context, as well as possible implications in relation to mathematics education.

5. Method discussion

The aim of the thesis was to explore flexibility in knowing school mathematics. My aim was not to understand learning as an external relation between conditions for learning and the learning outcome. The collection of the data material (see page 45, for an overview of the material) was instead performed with the intention to focus internal relations in the experienced learning context. These were understood in terms of modes of being a learner and modes of knowing on the one hand, and the available discursive possibilities for understanding mathematics and for learner identity, on the other. It was possible to reach these results, using contextual analysis. Svensson (1978) developed the foundations for contextual analysis, an empirical and holistic research approach, where descriptions emerge from analysis of the empirical material, based on delimitation of objects of knowledge in context, focusing on main characteristics and relations of the objects.

Another methodological assumption made in the present thesis is, as is the case in all qualitative research, that the researcher her/himself is not merely a witness to the research process, but a participating subject (Kvale 1989). There is always a level of subjectivity involved in the interview process, as well as in the selection of important data from the whole material. In the creative research approach proposed by Giri (2005), subjectivity is seen as an important part of the process of understanding people's ways of thinking and being - Giri here includes both the people observed, and the researcher's own subjectivity. Compared with dominant traditions in mathematics education research, the focus is here on categories of description as a main result, which is a shift from the focus on categories as pre-defined assumptions (Svensson 1997).

Kvale (1989) has pointed out that in qualitative research, the reading of the text in the data material and the interpretation can be validated depending on question and perspective. My research question did not concern why the pupils made mistakes or the observation that their solutions were not entirely correct.

The empirical material was to be understood from a learner's/teacher's perspective. My reading was in other words experiential (Theman 1985). The pupils' statements, both expressed verbally and mathematically, were read in relation to how they thought about the consequences of their statements. In a phenomenographic research approach, empirical findings do not claim a definitive relation between the learning context as described by the researcher, and the experienced learning context. The thesis represents an attempt to understand and describe internal relations between the knower and the known and the experienced learning context. Each category represents a theoretical representation of flexibility in knowing (Marton & Booth 1997; Theman 1985). *Within* the three modes of knowing the variation is related to the context from which they were derived.

The intercultural perspective adopted in this thesis helped me to see and 'expect the unexpected'. One of the basic assumptions made within intercultural perspectives is that cultural perspectives contribute to how we interpret reality and that we can use multiple cultural perspectives to understand an empirical material. The comparisons made between the studies, conducted at two different locations of our world, have been made through an understanding of dualisms and non-dualisms. This implies that comparisons were carefully made, in order to achieve a shift in the value-binary of, among other aspects, *learner-centred teaching* versus *blackboard teaching*. In international discussions on mathematics education, such characteristics are routinely polarised in terms of 'good' or 'bad' standards of teaching. If I had looked at the two studies in a non-critical way, I could have drawn less relevant conclusions, based, for instance, on dichotomies of *traditional teaching* and *progressive teaching*. I might have observed that what was going on in the Indian classroom was purely a representation of the 'catechistic discourse': the teacher used the blackboard in front of the class to transmit correct knowledge content, and made pupils speak out in unison what they were asked to say. In the Swedish study, I might have drawn conclusions based on the assumption that the very varied way the teacher presented the content and the discussions fulfilled all conditions of 'progressive teaching' and that, as a result of that way of teaching, the learners would as a consequence of the teaching method be seen as constructive in their ways of understanding the content.

5.1 The learner-centered educational discourse

In the following, certain characteristic features of school mathematics and learner identities that can be found in educational discourses are outlined, with a special focus on learner-centred educational discourses, that teachers and policy documents maintain and contribute to in both Sweden and India. These

educational discourses are also part of an ongoing international rhetoric (Sugrue 1997). Taguchi (2004, p.66) pursues the observation that despite policy objectives aiming for learner-centred education, educational discourses reflect a number of leading and dominating ideas on what schools are for, as well as concerning what the organisational structure and 'school mathematics' should be. These assumptions are so strong that, even if educational plans speak of 'the competent child', according to Taguchi, notions with the potential to empower the pupils are not half as strong as still dominating discourses that aim at controlling or 'normalising' the pupil, involving continuous assessments of children's cognitive, social, emotional and motor development. In accordance with the dominant developmental psychological discourse, pre-school-teachers and teachers first of all identify the children's weaknesses, rather than their competences. Taguchi argues that this happens in spite of what teachers say at parent meetings, read in books and hear at seminars, since these notions are reproduced in embodied approaches and patterns of actions in pedagogic practice. She concludes that it may take a long time before the pedagogue's eyes focus the child and before pedagogic practice changes in other ways than on the surface, in relation to the visionary discourse of the educational plans. And even if that happens, in fact it will only amount to a new normative view and practice that is reproduced.

Sugrue (1997) questions the whole notion of learner-centred education, and considers that it is a social construction. He believes that in a global perspective, the notion of learner-centred teaching has to be understood in its implications for classroom actions, and that it has to be empirically investigated in its context to become more precise.

In both India and Sweden, discourses on learner-centred education can be traced back to educational thinkers at the beginning of last the century. Tagore and Key provide illustrations of such discourses that have had a far-reaching impact on pedagogical reflection.

Rabindranath Tagore in India, who won the Nobel Prize in literature in 1913, outlined a pedagogy for 'True Education' in his book *Philosophy of Education and Painting* (2001):

What is required for true education is the ashram in which prevails the living atmosphere and the background necessary for a holistic education. (p. 43)

Our pledge from the very beginning was that the children of our ashram will be always keen on keeping a constant

touch with the world around them. They will do research, examine things, collect interesting items. In other words, there will be a team of teachers whose vision has transcended the world of books, who can see, do research, and who have inquisitiveness about the universe, whose joy is in searching into expanding knowledge; and whose power to inspire others can create a co-operative community (p. 49).

In Sweden, Ellen Key formulated a proposal for 'real' education in her book *The Century of the Child* (reprinted 1996):

In the school I dream of, mathematics would, for instance, be studied in wintertime; they go well with the cold and clear winter air! But in spring and autumn, pupils would study outdoors in nature all day long; not each area of nature as a separate subject, but instead insights into geology, botany and fauna would be obtained in an inseparable connectedness. While the pupils by living observation learn details, they afterwards in a living textbook receive a summary with the broad outlines of what their senses have taught them, or else – on rainy days! – they themselves can compose a written and drawn account of it. General education is not to know the number of stamens or the number of bones in hundreds of flowers and skeletons, but to be able to follow the laws of life and evolution in the nature that surrounds us; to be able to observe and combine yourself nature's phenomena - that is general education; influencing feeling and imagination, as well as thought and character (p. 163, *my translation*).

These thinkers had visionary opinions about how to educate children in a holistic fashion, and shared a view that learning takes place parallel to personal growth, contributing to discourse on child-centred education. They strongly emphasise that education should care for the development of the whole child. Teachers should empower the child to think about aspects as part of a greater whole, and actively take part in their own quest for knowledge. The naturalistic discourse, which presupposes that the learner is a 'curious, 'spontaneous' and 'creative learner', argues Sinha (1999), is based on 'that child-centered discourses position learners as having a subjectivity which is absent from

cognitive discourses' (p. 42), where, he continues, naturalistic discourse presupposes learners as entirely viewed in terms of mental processes. Sinha contends that the naturalising discourse does not take into consideration 'the specific way in which the learner has to be positioned and constructed, in such a way that a particular learning situation is framed as an occasion for the display or manifestation of creativity' (p. 44). In the present thesis, an attempt has been made to understand the *educational discourses* which I found traces of in the empirical material, concerning how mathematics is described, how the learning goals are related to learner-centred educational goals, and how learner and teacher responsibilities are described.

Instead of interpreting what I could see *per se*, I adopted a second-order perspective and an intercultural research approach. I tried to look beyond the obvious. I wanted to see how the learners performed their learning, from their own perspective. What were their intentions, and what modes of being a learner and modes of knowing did they bring into the learning situation? These were the overall questions I asked myself. I saw that the object of knowledge was actively modelled by most of the pupils in the whole empirical material (except by the two individuals who displayed an associative mode of knowing). The object of knowledge was experienced through use of variation, in different ways. In the Swedish study, the character of the tasks that teacher M prepared demanded of the pupils to be both imaginative and to focus on the content at the same time. This encouraged the learners to be both creative and productive. Some pupils, like A, focused more on the mathematical part of the task.

In the Indian study, the classroom interaction demanded of the learners to be prepared to follow a flow of mathematical reasoning. This encouraged the learners to concentrate and to patiently strive for understanding. Some pupils, like V, had not reached understanding of content and did not follow the mathematical logic. Nevertheless, they still persisted in their work.

5.1.2 Modes of knowing in context

The general research focus in the phenomenographic research tradition is on the variation in how people experience parts of reality. From the point of view of variation theory, focus has been placed on the variation which is possible to experience based on the presentation of an object of knowledge. In the present thesis, the focus has instead been on *the process of using variation*. Specifically, the focus was on how the pupils in the two studies *flexibly shift in their ways of experiencing mathematics*: which aspects they discern from a whole, how they

delimit wholes, and how they understand the relationships between the parts and wholes.

The purpose of collecting material from classroom observations, interviews with pupils and teachers, combined with a varied range of written material from pupils and teachers, was done in order to focus on modes of knowing from different angles. The content of the interviews and observations varied, due to that the main focus was on modes of knowing and its relation to the experienced learning context.

The results show how the learners approached and gave meanings to the varied content in different ways. Differences occurred both within the studies and between the studies. The character of each category reflects a certain level of understanding of content and a specific way of focusing and using variation in content.

The *associative flexible mode of knowing* represents the least differentiated understanding among the modes of knowing. The *compositional flexible mode of knowing* means that the learner tries to derive critical aspects from compositions, e.g., number-patterns and formulas, and view-*turn* these. This was by far the most dominant mode of knowing. The *contextual flexible mode of knowing* represents an understanding of mathematical entities in relation to context in making sense of a problem. The context, mathematical and reality-based, is understood as a condition for understanding the mathematical content. R, for instance, calculated the areas and related it to a reality-context, which justified her understanding of mathematics reflected in three different answers about the same area. In the empirical material, there are important differences in the social and the cultural contexts, from which the learner derives meanings and intentions. It is thus important to discuss the character of the categories of description, so that they are not misinterpreted for their generalised homogeneous appearance. The three categories were found to best describe the whole outcome space. There are distinct differences between the three modes and variation within that need to be discussed.

Teacher M in the Swedish study encouraged her pupils to study the nature of mathematical ideas in different contexts. The classroom discourse supported the pupils in the effort to evaluate the purpose and logic of mathematics, and to make the theoretical constructs their own. The contradiction between the dominating absolutist view on mathematics, on the one hand, and the fallibilist approach which the pupils encountered in the classroom, on the other, is noticeable during the interviews. When the pupils are asked to talk *about* their work, they talk about *how they did it* and what the steps were. The content they

talked about was reduced to small parts and to simple arithmetic. Although the pupils performed their learning in a space where contradictory discourses were present, there were no pupils who had a quantitative view on learning, in the sense of that 'to be a good learner means to know more' (Cole, quoted in Tang & Biggs 1996, p.161). The collaborative work they engaged in was experienced by the learners to support processes which are valuable for them in a longer perspective. They learn to reason, to come up with hypotheses to verify their results, and to work together to create something new.

Although the teacher in the Swedish study consistently encouraged the pupils to understand mathematics in its context, the learners' intentions did not always correspond to the teacher's. Some tasks, e.g. the task where the pupils were to draw a circle diagram of activities during 24 hours, with the help of a time-chart they made, demanded from the pupils to reflect over both content and context. When the pupils were telling me about the circle diagram, they mostly told about what steps they had taken and not about their thinking behind the process of constructing it. There was a tendency to separate the focus on mathematical theoretical constructs from context during interviews. This tendency possibly was a result of that there was an experienced contradiction between how mathematics usually is understood, in terms of absolutist views, and the view of mathematics they worked with in their classroom, which is aligned with a fallibilist view on mathematics.

The social relationships in the classroom were mostly focused on the creation of objects of knowledge. The learners took the opportunity to constitute themselves as creative in their authorship and also to be productive. There was no experienced contradiction between being creative and productive. The pupils were empowered participants in creating mathematical ideas. Teacher M had successfully arranged learning situations (tasks, contexts of tasks and a reference frame), so that the learners could experience that they were owners of the learning goals.

In the localised Indian epistemologies, discourses on being an autonomous learner were part of a discursive web with conversation at its core. In the school class context, classroom discourse encouraged mathematical deduction through a *conversational approach*. The examples presented by teacher A were considered in interaction with the pupils. The communication demanded cooperation from the pupils, in the sense that they would understand when to answer and how to answer in unison. The pupils were asked to go through the task several times, and were expected to do so, as a form of repeated recitation.

Coming to know, understanding, and working with the tasks, were all parts of how the pupils experienced their knowing. Being autonomous learners meant for most of the pupils to repeatedly work with different aspects of a problem, or of a *sutra*. The conversion process from theoretical constructs to subjectively understood constructs, demands of the individual pupil to *independently work with variation*.

The teacher in the Indian study presented *sutras* in different examples, using a quite streamlined method of deduction. Her intention and the learners' intentions converged relatively well, in the sense that all pupils repeatedly went through the material they were supposed to make sense of. One pupil was seemingly disenchanted with the mathematical logic. But, although all the pupils in the study repeated the same content, the process of repetition varied. The pupils also understood the same content in different ways. The repetitive practice of working with mathematics, it seems, is consistent with how the learners understand the nature of learning and themselves as learners outside school. It was taken for granted that the pupils, through their subjective process of coming to know, should arrive at the exactly same solution as the one given in the text book. The theoretical constructs should be reproduced through a process of understanding the ontological structure.

In both classroom contexts, the pupils were able to experience agency. In the Swedish study, the figured learner identity (Holland et al, 1998) was both process- and product-oriented (Sinha 1999), and based on the culturally shared assumption about separation between self and others (Markus & Kitayama 1991, p. 226), being both autonomous individual learners and part of a collective argumentation in an otherwise individually-oriented educational discourse. The learners were *individually* supposed to contribute to mathematics through their hypotheses, work with concrete materials and discussions. A relational agency was developed in collaboration with the teacher and other pupils.

In the Indian material, modes of being a learner outside of school could also be related to the classroom work. The figured learner identity was based on the awareness that the child has a duty towards the family to achieve well at school. The dependency on external authors and the expected autonomy in performing learning, made most of the pupils cultivate a relational agency. The relational agency meant that the learner could cope with different discursive possibilities, something they were used to in everyday encounters. The learners were supposed to be competing, to be autonomous learners and also had an interdependent construal of self. The interdependent construal of self takes a

'high degree of self-control and agency to effectively adjust oneself to various interpersonal contingencies' (Markus & Kitayama 1991, p. 228). The power distance between teacher and pupils, and perhaps a feeling of *connectedness* between self and others, made pupils interact for mathematics learning out of class; during recess time or in the absence of the teacher.

A consistent relationship between 'mathematical work' and 'mathematical Truth' could be observed. Additionally, the pupils seem to work with mathematics and to gain understanding in the *process of repetition*. All the pupils meant that they have to make an effort to understand through 'mathematical work'. The results tie in with the findings of a study conducted by Dahlin & Watkins (2000), where German and Chinese students were interviewed about how they experience learning. They found that the Chinese students emphasized repetition, combined with attempts to discover new meanings in the material studied, in order to deepen their understanding. The German students, on the other hand, did not understand repetition as a process of gaining deeper understanding. Although Dahlin & Watkins do not claim their conclusions to be definitive, it is suggested that there is a consistency between the emphasis on *attentive effort* expressed by the Chinese students, and a traditional Confucian perspective on learning.

Marton, Watkins and Tang (1997) found four distinctly different ways of experiencing learning:

- Learning as committing to memory (words)
 - Learning as committing to memory (meaning)
 - Learning as understanding (meaning)
 - Learning as understanding (phenomenon)
- (p. 29).

A conclusion drawn in relation to *Learning as understanding meaning*, is that Chinese learners seem to be much better at seeing both sides of the notion that *having an understanding of something implies meaningful memory*, just as *meaningful memory implies understanding*. I observed a non-distinction between the process of work and truth (Mohanty 1994) in my Indian study. The mode of being a committed learner was found to be the dominating learner identity.

One of the interesting results of the present thesis is that although all the Indian pupils *express* themselves in similar terms about repetition, practice and making an effort, the way they make sense of the material through repetition is varied, depending on mode of knowing.

I did not see in the empirical material any gender difference in the interaction between teachers and girls and boys. Teacher M recognised that the pupils' different leisure time activities - in one class many girls were playing basketball - could be a context to dealing with mathematical problems. Teacher A's opinion that girls usually do not ask 'why-questions' perhaps reflected a prejudice against girls' abilities to think critically and analytically about the mathematical tasks, which she was unaware of.

An intercultural perspective helped me to see beyond often taken for granted theoretical assumptions on learning and learning environment. Biggs (1996) discusses the problems with the 3P-model (Presage, Process and Product), which he means underlies most of the educational research. He builds his argumentation on the fact that the 3 P:s are predefined and underpinned by Western assumptions. Within Presage, there is the student presage, defined as 'prior knowledge and experience relevant to the task, abilities, values and expectations concerning achievement, and approaches to learning as predispositions to engage in academic activities' (p. 52). There is also the teaching presage which involves personal factors, such as 'teaching competency and teaching style, conceptions of teaching and learning, and classroom climate established'. It also includes institutional factors such as 'course structure, curriculum content, methods of teaching and assessment' (p. 52). Process 'refers to the way students actually handle the task' and depends on 'perceptions of the teaching context, their motives and predispositions, and their decisions for immediate action, the approach to learning' (Biggs 1996, p. 52). The Product of learning can be 'quantitative recall' or 'correct' and 'abstract conceptualising' (p. 52). Biggs makes the point that the model has to be modified to be relevant in East Asian learning environments. He proposes that for these studies the student presage domain has to include attributions like effort, interest, strategy and cue-utilisation. The teaching presage should include language medium of instruction, and time on task. Influencing both these domains, argues Biggs, is the cultural context, which includes collectivism and value of education (with reference to figure 3.2 p. 62). The culturally modified 3P-model helped Biggs see that in Confucian heritage cultures (CHC), e.g. China, Taiwan, Hong Kong, Singapore, Korea and Japan:

- a) CHC teaching/learning environments, although characterized by 'large classes, external examinations, seemingly (to Westerners) cold-learning climates, and expository teaching', the students make an effort to seek meaning and to persist in 'engaging higher cognitive processes'.
- b) CHC students may be repetitive but there is no evidence that they rote learn more than 'their Western counterparts'.

- c) 'CHC students perform at a higher cognitive level in academic tasks'.
- d) 'CHC students see themselves as deep learners'. (p. 63).

The results of the thesis were thus derived from a contextual and experiential reading of the whole material. It occurs to me that Biggs' discussion on a culturally modified model for East Asian learning and learning environment can be taken one step further. I realise that the taken for granted assumptions not only affect the way I look at the Indian school class context, but also what I see in the Swedish school class context, and the way I compare the results from the contexts. I chose my first focus to be *flexibility in knowing* and the contextual analysis method helped me to understand the variation within categories of knowing and its contextual and discursive aspects. During the work underlying the present thesis, I used contextual analysis to understand the whole empirical material, with data collected from a Swedish and an Indian school class context. The flexibility in knowing was my focus and my view is that knowing can be seen as an internal relation between the knower and the known. Specifically flexibility in knowing was explored with a focus on how learners discern parts and delimit wholes and understand part-whole relationships. The learner's performance of learning, in terms of agency and authorship (Davies 2000; Burton 1995; Holland et al. 1998) was also focused. The research approach is based on an ontological epistemology of participation, which is situated in an ongoing scientific debate on conditions for a creative research approach. The arguments for this approach are based on the observation that distinctions made between objectivity and subjectivity, and between object and subject are culturally and historically founded, rather than inherent features of reality. A relational approach therefore has the potential to open up more productive fields of research in education. The view I hold in relation to knowing mathematics is that the ontology of mathematics is not seen as separate from its epistemology. Hence knowing mathematics can be understood as a relation between the knower and the known. In relation to the present thesis, *participation* can be understood as a meaning-making process, where the subject *makes sense* of the object of knowledge, and at the same time the meaning of content is approached, through the *intentions* the learner brings into the learning situation. The intentions are in this thesis seen as expressed in learner's language use, verbal or mathematical, based on agency and authorship. I used a particular dialogue model to encourage the pupils to make sense of the mathematical content, reflect on the nature of mathematics and the mathematical meanings, as well as to explore themselves as learners. This dialogue model has already been suggested to have implications to education,

in that it encourages the pupils to reflect over their language use and meaning (Anderberg 1999; Anderberg et al. 2005, 2006).

The analysis was done through the process of gathering information on flexibility, from the field and from literature. The following domains emerged from the process of contextual analysis:

Approaches to learning/teaching
Modes of being a learner
Modes of knowing

For the Swedish and the Indian study, I here wish to summarise the discussion of the conclusions on flexible modes of knowing in context. The summary is structured in the same way as in Biggs' discussion (see above). This is done since a direct comparison can be made between Biggs' discussion of generalised conclusions on East Asian learning and learning environment, and the discussion of conclusions made in the present thesis. It is my hope that the comparison can shed light over the difference between using predefined generalised conditions and doing contextual analysis.

The discussion of the conclusions on flexible modes of knowing in the Swedish school class context can be described in points a-d, (see Biggs 1996, and above, page 191-192 in the present thesis):

- a) The pupils in the Swedish school class context talked about their thematic project work in positive terms and as a contrast to 'before when mathematics was a competition and a matter of being the fastest' and when they did things without knowing what it was.
- b) The documentation of the thematic project work gave them a possibility to 'return' and to 'see' and remember the material. The fallibilist classroom discourse and its clear goals and structure, encourages the pupils to reflect over mathematical and verbal language use and about their reasoning. During interviews, though, there was an emphasis on re-telling what they had done and what the steps were.
- c) A main difficulty was to deal with tasks that demanded from the pupils to make sense of both content and context, and the pupils maintained, in collaboration with each other and the teacher, a fallibilist classroom discourse.
- d) The pupils did not explicitly express that they were working for understanding, but as a consequence of stating that plain recall and

meaningless tasks were not interesting, their intentions were clearly directed towards deep-level learning.

Conclusions on flexible modes of knowing in the Indian school class context can in the same way be summarised as follows:

- a) The pupils struggle to work with the mathematical tasks. The discursive possibility to be an autonomous learner is negotiated and the interdependent construal of self makes the pupils constitute a relational agency. The power distance between the teacher and pupil is compensated in that some pupils tutor the others.
- b) Repetition for meaning and memorisation are by the pupils considered to be two sides of the same coin. There is a perceived circularity between creating meaning and memorisation, as well as between mathematical work and mathematical truth. The work process is equal to the process of understanding content.
- c) Two pupils in the study are cue-seekers and have a surface-level approach to learning. One of them was even disenchanted with mathematics and could not make meaning of the content. She made an effort to indiscriminately memorise questions and answers. This pupil attempted to connect an answer to a question, through seeking cues from the task. The other pupil arbitrarily focused on cues from the task in formalisation of area into algebra. There were pupils who succeeded in using variation within formulas, and there were pupils who did not.
- d) All the pupils in the study expressed that they should understand mathematics more than memorise it. At the same time, work meant attentive effort with patience, concentration and focus, as well as memorisation and a 'perception' of when the solution is complete in detail.

Instead of focusing preconceptions on cultural differences and their effect on learning and learning environment, my attempt here has been to understand the information that came out from the interviews. I have tried to learn from the teachers and pupils involved in the work.

5.3 Implications for educational practice

The *associative mode of knowing* cannot be considered as an educationally valuable mode of knowing, since it rests on a surface-level approach (Marton & Säljö 1976), an atomistic approach (Svensson 1978), as well as a failure to understand mathematics in other ways than cue-seeking (Biggs 1996). The *compositional mode of knowing* and the *contextual mode of knowing* should preferably be encouraged in

mathematics education. Both these modes demand from the pupil a deep-level approach, as well as a holistic cognitive approach.

Nevertheless, these two modes have to be evaluated in the specific learning context, in relation to the mathematical content that is dealt with, and in relation to what the specific learning goals are. This conclusion is based on the fact that, although the pupils who represented the compositional and the contextual mode of knowing had valuable approaches to learning and content, the mathematical *thinking* behind the shifts in their understanding of parts and wholes was not always in line with what is considered to be mathematically correct.

The pupils in both class contexts engaged themselves in maintaining the view of mathematics that the classroom discourse offered. They participated in the mathematical work, developing a relational agency to the teacher, class-friends and to the mathematical content. The performance of learning was based on their own authorship (except for the case of one pupil). The use of formative assessments could engage the individual pupil further in their authorship. These assessments should include the pupil's dominant modes of knowing, always in relation to what learning goal is involved in a task.

Both the compositional and the contextual modes of knowing are educationally valuable, and the mathematical thinking involved in these modes can be improved, based on the teacher's regular observations of the individual pupil's learning. This means involving the pupil in reflection over language use and mathematical knowledge production.

A *pedagogy of learning* needs to take its point of departure in the conditions for learning rather than teaching, to develop conditions for learning based on content and on an understanding of how the learning context is experienced by the individual learner, as discussed by Marton, Runesson & Tsui (2004). In practical terms, some lesson time could, for instance, be devoted to a focus on students acquiring the skills of enquiry-based learning, rather than an exclusive focus on the content of their learning (Burton 2004). Work with this kind of tasks encourages pupils to re-tell the mathematical content in their own words, and to ask themselves questions on mathematical epistemology and ontology. To help teachers evaluate knowing in mathematics, a dialogue model (Anderberg et al. 2006) similar to the one used in this thesis could be developed, placing emphasis on learning about what modes of knowing the individual pupil's ways of understanding mathematics are dominated by.

A pedagogy of learning also implies a movement away from a focus restricted to teaching methods and ‘best practices’, to a focus on *learning* of a certain content. Above all, it involves the challenge of how to stage learning of that content in order to empower the pupils, combined with the evaluation of modes of knowing, so that pupils who do not fit into the norm of the average-achievers, can also understand and appreciate school mathematics.

Marton and Neuman (1990) and Marton et al (1994) observed that ‘sudden restructuring’ is crucial for learning arithmetic skills among school starters. The present thesis contributes to the picture of how pupils flexibly shift their focus when dealing with mathematical content. The empirical basis for the conclusions drawn in the present thesis was limited to two school class contexts, and the content which the pupils dealt with varied. Future research could be concerned with empirical investigations in several mathematics classrooms, to further explore meanings of flexible modes of knowing. The studies could be focused on a specific content.

In a longer perspective, a researcher-teacher collaboration project could define qualities of knowing to be used in the evaluation of pupils’ attainment in mathematics.

To conclude, there is an inspired hope that by exploring modes of knowing, the thesis will contribute to developing teaching practices, where learners at school are given contextually and conversationally relevant pedagogical challenges that make them understand and appreciate mathematics, and participate in mathematics learning in ways to enhance their future development as empowered and creative learners.

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Appendix 1

Interview guide for interview with teacher M.

I once heard about your pet ideas. They were:

Learning environment

Manner of working

Time

Communication and conversation

Could you develop some of your thoughts on these topics?

You refer to Engström and Malmer when you talk about your teaching. Could you tell me more?

When you talk with pupils about

things they don't express quite right

when they claim they have made a mistake

when they suggest a solution

I have observed that you have a special way of approaching the pupils. Could you develop your thoughts about this?

What do you mean by a 'shared frame of reference'?

What do you feel the pupils should know in mathematics (Time, Fractions)?

Tell me more about Maths in town, Vasaloppet, My 24 hours, A day at work.

Appendix 2

Guides for interviews with the pupils in the Swedish study.

Interview occasion 1.

What do you think about mathematics?

How do you learn mathematics?

Can you tell me about something you have been working with in mathematics recently?

What do you think about your theme documentation?

Tell me more about how you worked with one example that you choose.

What does teacher M mean when she wants you to explore, discover, give you a project work? What does teacher M mean when she asks 'Is that reasoning plausible?'

How does teacher M help you to understand mathematics/this example?

Appendix 3

Handout to the pupils in the Sweidsh study. Fractions (Bråk)

What is meant by a whole?

What is meant by one fifth?

What is meant by $1 - 1/5$?

Draw and explain!

Write a story for

$$1/5 + 1/15$$

And show how much it makes

Emma has a rope that is 20 metres long. She cuts off one fourth.

How long is the rope?

How many metres does she cut off?

How many metres are then left of the rope?

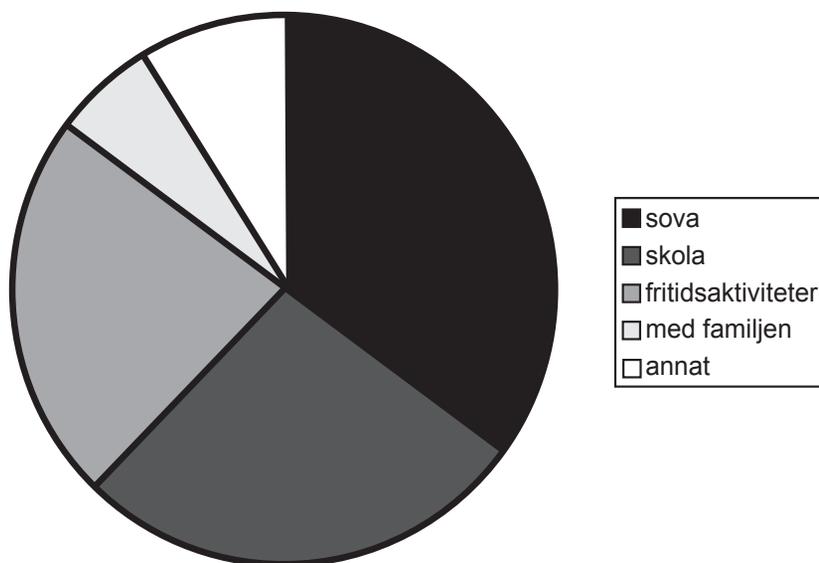
What did you learn in mathematics today?

How did you learn?

Appendix 4
My 24 hours

MITT DYGN	minuter
sova	480
skola	420
fritidsaktiviteter	328
med familjen	89
annat	123
summa	1440 minuter

Mitt dygn



Appendix 5

Vasaloppet

Do you remember that the first Sunday of March the Vasaloppet ski race took place? The winner of the year had the time 04.01.22. What do the numbers mean? Our computer teacher has participated many times. This year he ended in the 3964th position with the time 06.52.52. How far behind the winner was he? I think he is a great skier.

Appendix 6

Example of a task within 'Maths in town'.

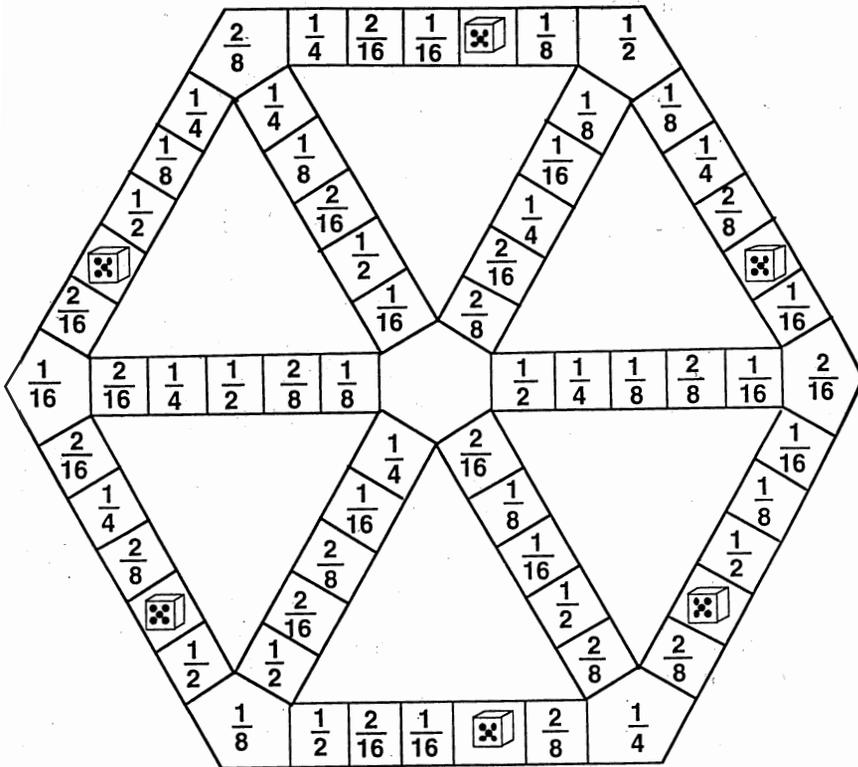
You are standing at the railway station. Look at the sign on the entrance door to the customer service. Study the opening hours.

How many hours per week does a person work, who works Thursday, Friday and Saturday every week? (Don't forget to subtract the breaks!)

Estimate how many percent that is of a 40 hour working week? (If you like, you can calculate it when you get back to school)

Appendix 7

Example of fraction game board.



Appendix 8

The didactical model by Gudrun Malmer.

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Page 31 in Malmer, G. (2002). Bra matematik för alla. Nödvändig för elever med inlärningssvårigheter. Lund: Studentlitteratur. (Good mathematics for all – but necessary for students with learning difficulties)

Level 1. EXPERIENCE, VOCABULARY, ASSOCIATIONS recognize, to have experienced. THINK / TALK

Level 2. CONCRETE ACTION practical work with completely concrete materials and with ready-made materials (eg building blocks, rods).
DO/TRY

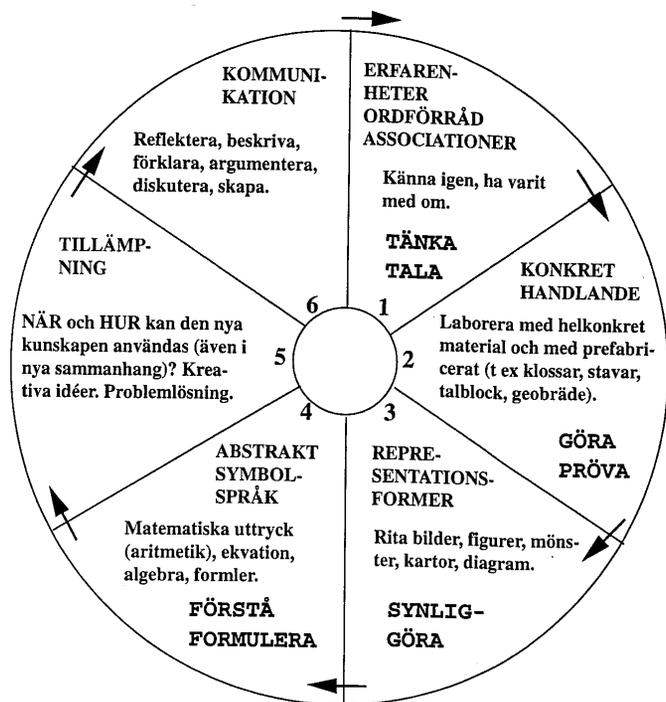
Level 3. FORMS OF REPRESENTATION draw picture, figures, patterns, maps, diagrams.
MAKE VISIBLE

Level 4. ABSTRACT SYMBOLIC LANGUAGE mathematical expressions (arithmetic), equations, algebra, formulas.
UNDERSTAND / FORMULATE

Level 5. APPLICATION, WHEN and HOW can the new knowledge be used (also in new contexts) Creative ideas. Problem-solving.

Level 6.

COMMUNICATION reflect, describe, explain, argue for, discuss, create .



Appendix 8

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 Page 31 in Malmer, G. (2002). Bra matematik för alla. Nödvändig för elever med inlärningssvårigheter. Lund: Studentlitteratur. (Good mathematics for all – but necessary for students with learning difficulties)

Appendix 9

Interview guide for the interview with teacher A.

What do you think about students' reflections in mathematics?

How do children learn?

Why can't some of the children learn mathematics?

You say that the students should learn deduction. What do you mean with deduction?

How do you teach geometry/mensuration? Can you give some examples?

How do the students solve the problems?

How do you think the students think about this problem? How do they learn it?

What is important in the students' learning of mathematics?

Why is it important to practice and memorize? Why is concentration important?

How do you think about the interaction in the classroom?

How do you think about your teaching?

How do you think the students think about mathematics?

Appendix 10

Guides for interviews with the pupils in the Indian study.

Interview occasion 1.

What do you think about mathematics?

How do you learn mathematics?

Can you tell me about something you have been working with in mathematics recently?

Interview occasion 2.

How many triangles are there in the square?

If one side in the square is 1 matchstick long and we say that its length is 2 cm, then what is the area of the whole square?

What is the circumference?

Can you build a square with double the area?

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