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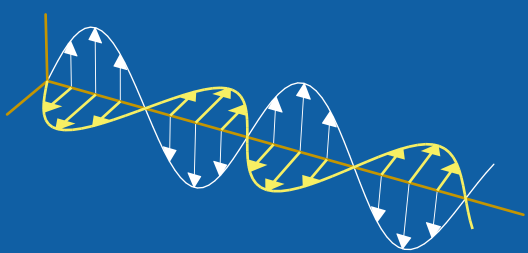

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Abstract

A time domain description of linear macroscopic electromagnetic phenomena is considered. We show that it is always possible to model the constitutive relations with a symmetric, positive definite optical response together with a convolution integral. An initial-boundary value problem is formulated for the macroscopic Maxwell equations together with a reflection operator modeling the exterior region. It is shown that the initial-boundary value problem is well-posed. Dissipation and finite speed of propagation is also considered.

1 Introduction

To describe electromagnetic phenomena in a general material we need constitutive relations that model the interaction between the electromagnetic field and the electromagnetic properties of the material [8, 9, 20]. In a time harmonic application, the constitutive relations are only a complex-valued number/matrix, typically found through measurements. Today, it is, however, popular to use time domain methods, *e.g.*, FDTD [30], to solve electromagnetic problems. This requires a much more involved model to fully describe all observed phenomena. The general linear model is the bi-anisotropic constitutive relation [22, 23].
We show that it is always possible to restrict the analysis to the case of constitutive relations with a symmetric, positive definite optical response together with a smooth convolution integral. This class of constitutive relations gives well-posed equations, *i.e.*, there exists a solution, the solution is unique and the solution depends continuously on the data [2, 24].

We consider a general initial-boundary value problem for the Maxwell equations. The boundary conditions are expressed with the split-fields [5, 6] together with a non-local reflection operator modeling the exterior region. The split-fields are especially useful in the inverse parameter reconstruction problem, see Refs.18, 25.

Well-posedness of the initial-boundary value problem follows roughly from the fact that the macroscopic Maxwell equations constitute a symmetric, hyperbolic system in the norm induced by the optical response. This is easily proved with energy methods, see Refs. 21, 24 and the references there given. For the Maxwell equations this corresponds to the use of the electromagnetic energy and the Poynting theorem [3, 20].

Furthermore, we give a general time domain definition of dissipation and we show several sufficient characterizations of dissipation, especially it is shown that the commonly used fixed-frequency characterization, *i.e.*, that the imaginary part of the permittivity is negative [4], together with a symmetric, positive definite optical response offers a sufficient condition for dissipation in the time domain.

Finally, notice that there are occasions when it is advantageous to model electromagnetic phenomena with a diffusion equation which has an infinite speed, *i.e.*, highly conducting materials [15]. There are also several applications where it is also necessary to incorporate the effects of the electromagnetic fields on the material, *i.e.*, the mechanical structure of the material have to be considered [8, 9, 20].

2 Basic Equations

Electromagnetic phenomena are modeled by the Maxwell equations

$$\begin{cases} \partial_t \boldsymbol{D} = \nabla \times \boldsymbol{H} - \boldsymbol{J} \\ \partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E} - \boldsymbol{M} \end{cases}$$

where the fields $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}$ and $\boldsymbol{B}$ are unknown and the impressed sources $\boldsymbol{J}$ and $\boldsymbol{M}$ are assumed to be known. To simplify the notation we use a dimensional-free scaling of all quantities. The scaling is similar to the rationalized Gaussian units and all the fields are measured in an energy unit, *i.e.*, $\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{D}$ and $\boldsymbol{B}$ have the unit $(\text{Energy/Volume})^{1/2}$. The transformation to SI units is made by the substitution

$$\begin{gathered} \boldsymbol{E} \mapsto \sqrt{\epsilon_0}\boldsymbol{E},\ \boldsymbol{D} \mapsto \epsilon_0^{-1/2}\boldsymbol{D},\ \boldsymbol{M} \mapsto \sqrt{\epsilon_0}\boldsymbol{M}, \\ \boldsymbol{H} \mapsto \sqrt{\mu_0}\boldsymbol{H},\ \boldsymbol{B} \mapsto \mu_0^{-1/2}\boldsymbol{B},\ \boldsymbol{J} \mapsto \sqrt{\mu_0}\boldsymbol{J}, \\ t \mapsto c_0 t = (\epsilon_0\mu_0)^{-1/2} t \end{gathered}$$

throughout this paper. The electromagnetic interaction with the medium is modeled with a constitutive relation[1] $(\boldsymbol{D}, \boldsymbol{B}) = \boldsymbol{\varepsilon}(\boldsymbol{E}, \boldsymbol{H})$. The structure of the map $\boldsymbol{\varepsilon}$ depends on the medium under consideration. However, most media behave linearly for small amplitudes/energies and the spatial dependence is usually pointwise[2]. Here we restrict ourselves to linear models, and the most general linear, causal and bounded map that is pointwise in space can be written [22]

$$\begin{cases} \boldsymbol{D} = \boldsymbol{\epsilon}\boldsymbol{E} + \boldsymbol{\xi}\boldsymbol{H} + \boldsymbol{\chi}_{\text{ee}} * \boldsymbol{E} + \boldsymbol{\chi}_{\text{em}} * \boldsymbol{H} \\ \boldsymbol{B} = \boldsymbol{\zeta}\boldsymbol{E} + \boldsymbol{\mu}\boldsymbol{H} + \boldsymbol{\chi}_{\text{me}} * \boldsymbol{E} + \boldsymbol{\chi}_{\text{mm}} * \boldsymbol{H} \end{cases}$$

where $\boldsymbol{\epsilon}, \boldsymbol{\mu}, \boldsymbol{\xi}, \boldsymbol{\zeta}$, and $\boldsymbol{\chi}_{kk'},\ k, k' = \text{e}, \text{m}$, are real-valued tensors depending on $(\boldsymbol{x}, t)$, and the operator $*$ denotes the generalized temporal convolution, *i.e.*,

$$\boldsymbol{\chi}(\boldsymbol{x}, t, \cdot) * \boldsymbol{v}(\boldsymbol{x}, \cdot) = \int_{-\infty}^{t} \boldsymbol{\chi}(\boldsymbol{x}, t, t-\tau)\boldsymbol{v}(\boldsymbol{x}, \tau)\, \mathrm{d}\tau.$$

The tensors $\boldsymbol{\epsilon}, \boldsymbol{\mu}, \boldsymbol{\xi}, \boldsymbol{\zeta}$ and integral operators $\boldsymbol{\chi}_{kk'}*,\ k, k' = \text{e}, \text{m}$ are all dimension-free.

The material is classified as non-stationary if at least one of the parameters $\boldsymbol{\epsilon}, \boldsymbol{\mu}, \boldsymbol{\xi}, \boldsymbol{\zeta}$, and $\boldsymbol{\chi}_{kk'}\ k, k' = \text{e}, \text{m}$, depends explicitly on time t. Furthermore, it is anisotropic if at least one of the parameters is not proportional to the identity operator, *i.e.*, a scalar, and bi-isotropic if all the parameters are scalar and at least one of $\boldsymbol{\xi}, \boldsymbol{\zeta}, \boldsymbol{\chi}_{\text{em}}, \boldsymbol{\chi}_{\text{me}}$ is

[1] In this paper the field strengths $\boldsymbol{E}, \boldsymbol{H}$ are the primary fields and the flux densities $\boldsymbol{D}, \boldsymbol{B}$ are defined through the constitutive relation. Observe that there are several other possible formulations of constitutive relations, see Ref. 13 for a discussion of different forms and the relation between them.

[2] With a pointwise map we mean a map that only depends on the value at a point and not on some neighborhood about the point, *e.g.*, differentiation and convolution are not pointwise operators.

non-zero [1, 22].
To simplify the analysis, we introduce the 6 vectors

$$\boldsymbol{e} = \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}, \quad \boldsymbol{d} = \begin{pmatrix} \boldsymbol{D} \\ \boldsymbol{B} \end{pmatrix}, \quad \boldsymbol{j} = \begin{pmatrix} \boldsymbol{J} \\ \boldsymbol{M} \end{pmatrix}$$

and the Maxwell equations and the constitutive relations can be written in a short-hand notation

$$\begin{cases} \partial_t \boldsymbol{d} = \mathsf{M}(\nabla)\boldsymbol{e} - \boldsymbol{j} \\ \boldsymbol{d} = \boldsymbol{\varepsilon}\boldsymbol{e} = \boldsymbol{\varepsilon}_\infty \boldsymbol{e} + \boldsymbol{\chi} * \boldsymbol{e} \end{cases} \tag{2.1}$$

where

$$\boldsymbol{\varepsilon}_\infty = \begin{pmatrix} \boldsymbol{\epsilon} & \boldsymbol{\xi} \\ \boldsymbol{\zeta} & \boldsymbol{\mu} \end{pmatrix}, \quad \boldsymbol{\chi} = \begin{pmatrix} \boldsymbol{\chi}_{\mathrm{ee}} & \boldsymbol{\chi}_{\mathrm{em}} \\ \boldsymbol{\chi}_{\mathrm{me}} & \boldsymbol{\chi}_{\mathrm{mm}} \end{pmatrix}, \quad \mathsf{M}(\nabla) = \begin{pmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{pmatrix}.$$

In this notation, $\boldsymbol{\varepsilon}$ is the linear operator relating $\boldsymbol{d}$ and $\boldsymbol{e}$. We assume that the operator $\boldsymbol{\varepsilon}$ can be divided into an optical response, $\boldsymbol{\varepsilon}_\infty$, and a dispersive part, $\boldsymbol{\chi}*$. The optical part models the fast/instantaneous response of the material, and the dispersive part models the slower memory effects [22]. The optical response $\boldsymbol{\varepsilon}_\infty$ is in general a 6×6 matrix depending on both space and time, *i.e.*, $\boldsymbol{\varepsilon}_\infty = \boldsymbol{\varepsilon}_\infty(\boldsymbol{x}, t)$, and the dispersive part $\boldsymbol{\chi}(\boldsymbol{x}, t, \cdot)*$ is a 6×6 integral operator also depending on space and time.
Fourier transformation gives the more well known time harmonic results[3]. In the time harmonic case, the constitutive relation is only a matrix multiplication $\hat{\boldsymbol{d}} = \hat{\boldsymbol{\varepsilon}}\hat{\boldsymbol{e}}$ (for notation see Appendix A). The same division as above can be made here, where the optical response now is the high frequency component of $\hat{\boldsymbol{\varepsilon}}$, *i.e.*, $\boldsymbol{\varepsilon}_\infty = \lim_{\omega\to\infty} \hat{\boldsymbol{\varepsilon}}$ and the dispersive part is the rest $\hat{\boldsymbol{\chi}} = \hat{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}_\infty$.
Through out this paper, we assume that the optical response is symmetric and positive definite, see also Section 3, *i.e.*,

$$\boldsymbol{\varepsilon}_\infty^{\mathrm{T}} = \boldsymbol{\varepsilon}_\infty, \quad \boldsymbol{e}^{\mathrm{T}}\boldsymbol{\varepsilon}_\infty \boldsymbol{e} \geq \delta \boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}, \quad \text{for some } \delta > 0.$$

This implies that all eigenvalues are positive and that the smallest eigenvalue is greater then or equal to δ. Furthermore, we assume that the convolution kernel $\boldsymbol{\chi}(\boldsymbol{x}, t, \tau)$ is a smooth function in $\boldsymbol{x}, t$ and that $\int_0^\infty |\boldsymbol{\chi}| \,\mathrm{d}\tau, \int_0^\infty |\partial_\tau \boldsymbol{\chi}| \,\mathrm{d}\tau < \infty$. The smoothness requirement is not fulfilled for all pertinent susceptibility kernels, *e.g.*, fractional derivatives that are of frequent use in mechanics. The bound of the susceptibility kernel can be interpreted as a stability requirement on the model, *i.e.*, $\boldsymbol{\chi} * \boldsymbol{e} \to 0$ as $t \to \infty$ if $\boldsymbol{e} \in \mathrm{L}^2$. In the Fourier domain, the smoothness of $\boldsymbol{\chi}$ is connected to the asymptotic behavior of $\hat{\boldsymbol{\chi}}$ for large ω. The assumption made in this paper is essentially that $|\omega\hat{\boldsymbol{\chi}}|$ is bounded as $\omega \to \infty$. The stability corresponds to analyticity of $\hat{\boldsymbol{\varepsilon}}$ in the right complex half plane, see also Kramers-Kronig relations [20].

[3] Whenever time harmonic analysis is discussed, we restrict the analysis to the case of stationary material. The time convention is $\mathrm{e}^{\mathrm{i}\omega t}$.

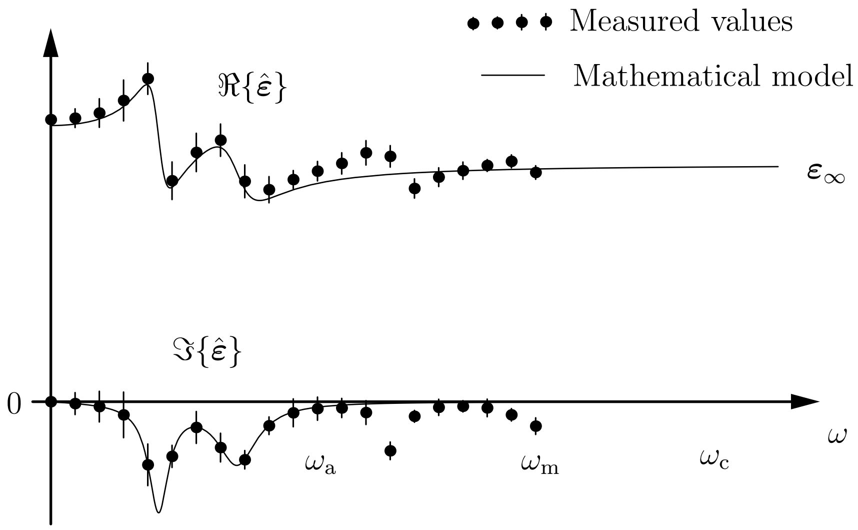

Figure 1: Illustration of the range of validity of the constitutive relations. The measured values of the permittivity $\hat{\boldsymbol{\varepsilon}}$ are known up to frequency ω_{m}, the mathematical model is used for frequencies up to ω_{a} and ω_{c} is the upper limit for the use of a continuum model of the medium, respectively.

3 The optical response

To motivate the assumption of a symmetric, positive definite optical response, $\boldsymbol{\varepsilon}_\infty$, we show that it is always possible to choose an optical response that is symmetric, positive definite, and, further more that the Maxwell equations are not in general well-posed if we relax this requirement.
The non-uniqueness of the optical response is best understood from a fixed-frequency point of view. A typical application, *e.g.*, microwaves or optics, is restricted to a frequency interval, here $0 \leq \omega \leq \omega_{\mathrm{a}}$. The constitutive relations are naturally restricted to the same frequency interval, and the extension of the constitutive relations outside this interval is arbitrary, specially the optical response ($\omega \to \infty$) is arbitrary. Further more, all macroscopic phenomena are restricted to a frequency interval with an upper frequency limit given by the microscopic properties of the material, *i.e.*, the atomic structure, that are better described by the microscopic Maxwell equation or QED (quantum electrodynamics) [10, 11, 20]. We illustrate this in Figure 1, where the real and imaginary part of the permittivity for a typical isotropic, stationary medium is plotted. The parameter values are known (through measurements) in an interval, $0 \leq \omega \leq \omega_{\mathrm{m}}$, and interpolation/extrapolation is used inside/outside the interval. The application is restricted to the frequency interval, $0 \leq \omega \leq \omega_{\mathrm{a}}$, and ω_c is the upper limit for the use of a continuum model of the material.

The optical part, $\omega \to \infty$, of the constitutive relation is hence arbitrary, and it is alway possible to choose a symmetric, positive definite optical response. Observe that the same argument shows that the susceptibility kernel is arbitrary smooth, specially we can assume that $\boldsymbol{\chi}' \in \mathrm{L}^1$. The same argument can also be used to restrict the optical response to the identity operator and the conductivity to zero.

However, it is numerically advantageous to use the more complex models described here.

To illustrate the difficulties with a non symmetric positive definite optical response, we study the bi-isotropic model. In Appendix B we notice that the model is well-behaved only if the optical response is symmetric and definite. The case of a negative definite optical response is ruled out by considering plane waves that impinge normally on a stratified half space, see Appendix C.

This shows that it is reasonable to restrict the class of constitutive relations to a symmetric, positive definite optical response. In the following, we show that this is also a sufficient condition to get mathematical well-posedness and to satisfy the physical limitations. Also observe that symmetry and positive definiteness are connected to dissipation, see Section 7 and Ref. 13.

4 Energy

To derive the necessary estimates of the amplitudes, we start with a short discussion of the electromagnetic energy[4]. As an example we consider a homogeneous isotropic source-free Lorentz medium [20]. The constitutive relation is described by an ordinary differential equation for the polarization $\boldsymbol{P}$.

$$\begin{cases} \boldsymbol{D} = \epsilon_\infty \boldsymbol{E} + \boldsymbol{P} \\ \partial_t^2 \boldsymbol{P} + \nu \partial_t \boldsymbol{P} + \omega_0^2 \boldsymbol{P} = \alpha \boldsymbol{E} \end{cases} \tag{4.1}$$

The material parameter ν is the collision frequency, ω_0 is the harmonic frequency, and α is proportional to the density of the inclusions. Multiply the second equation with the time derivative of the polarization $\dot{\boldsymbol{P}} = \partial_t \boldsymbol{P}$.

$$\frac{1}{2}\partial_t |\dot{\boldsymbol{P}}|^2 + \nu |\dot{\boldsymbol{P}}|^2 + \frac{1}{2}\omega_0^2 \partial_t |\boldsymbol{P}|^2 = \alpha \dot{\boldsymbol{P}} \cdot \boldsymbol{E}.$$

From the first equation in (4.1) we get the power density

$$\begin{aligned} \boldsymbol{e}^{\mathrm{T}} \partial_t \boldsymbol{d} &= \frac{1}{2}\partial_t \left(\epsilon_\infty |\boldsymbol{E}|^2 + |\boldsymbol{H}|^2 \right) + \boldsymbol{E} \cdot \dot{\boldsymbol{P}} \\ &= \frac{1}{2}\partial_t \left(\epsilon_\infty |\boldsymbol{E}|^2 + |\boldsymbol{H}|^2 + \alpha^{-1}\omega_0^2 |\boldsymbol{P}|^2 + \alpha^{-1} |\dot{\boldsymbol{P}}|^2 \right) + \alpha^{-1}\nu |\dot{\boldsymbol{P}}|^2. \end{aligned}$$

The energy in a region Ω is (see (4.2) below)

$$\begin{aligned} \mathcal{E}(t) &= \mathcal{E}(0) + \frac{1}{2}\int_\Omega \epsilon_\infty |\boldsymbol{E}|^2 + |\boldsymbol{H}|^2 + \alpha^{-1}\omega_0^2 |\boldsymbol{P}|^2 + \alpha^{-1} |\dot{\boldsymbol{P}}|^2 \,\mathrm{d}v \Big|_0^t \\ &+ \int_0^t \int_\Omega \alpha^{-1}\nu |\dot{\boldsymbol{P}}|^2 \,\mathrm{d}v \,\mathrm{d}\tau. \end{aligned}$$

[4]Observe that the energy is not uniquely defined [11], the definition given here is the usually accepted one.

Here it is natural to make a separation of the energy density in a rapid optical part, $\epsilon_\infty|\boldsymbol{E}|^2 + |\boldsymbol{H}|^2$, a slower material part, $\alpha^{-1}\omega_0^2|\boldsymbol{P}|^2 + \alpha^{-1}|\dot{\boldsymbol{P}}|^2$, and one part due to losses, $\int_0^t \alpha^{-1}\nu|\dot{\boldsymbol{P}}|^2\,\mathrm{d}\tau$. The loss part constitutes the energy change from electromagnetic energy to heat, whereas the other material part merely represent the electromagnetic energy stored in the material. Observe that we are only interested in the energy difference $\mathcal{E}(t) - \mathcal{E}(0)$ and not in the total value of the energy.

After this introductory example, we proceed with the general case. For the general constitutive relation, we first make a division into two parts, the optical part, $\mathcal{E}_{opt}$, and a slower dispersive part, $\mathcal{E}_{disp}$, where the optical part is most important for the mathematical well-posedness. The slower dispersive part is, however, interesting from the physical perspective of dissipation, see Section 7.

Define the energy $\mathcal{E}(t)$ in a region Ω at time $t > 0$ as

$$\mathcal{E}(t) = \int_0^t \int_\Omega \boldsymbol{e}^{\mathrm{T}} \partial_\tau \boldsymbol{d}\,\mathrm{d}v\,\mathrm{d}\tau + \mathcal{E}(0) \tag{4.2}$$

where $\mathcal{E}(0)$ is the energy at time $t = 0$. Divide the energy in its optical, $\mathcal{E}_{opt}(t)$, and the dispersive part, $\mathcal{E}_{disp}(t)$, *i.e.*, $\mathcal{E}(t) = \mathcal{E}_{opt}(t) + \mathcal{E}_{disp}(t)$ where

$$\begin{cases} \mathcal{E}_{opt}(t) = \dfrac{1}{2}\displaystyle\int_\Omega \boldsymbol{e}^{\mathrm{T}} \boldsymbol{\varepsilon}_\infty \boldsymbol{e}\,\mathrm{d}v \\ \mathcal{E}_{disp}(t) = \displaystyle\int_0^t \int_\Omega \boldsymbol{e}^{\mathrm{T}} (\boldsymbol{\sigma} + \boldsymbol{\chi}'*)\boldsymbol{e}\,\mathrm{d}v\,\mathrm{d}\tau + \mathcal{E}_{disp}(0) \end{cases}$$

respectively.
The conductivity $\boldsymbol{\sigma}$, and the dispersive derivative $\boldsymbol{\chi}'$ are defined by

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = \frac{1}{2}\partial_t \boldsymbol{\varepsilon}_\infty(\boldsymbol{x}, t) + \boldsymbol{\chi}(\boldsymbol{x}, t, 0)$$

and

$$\boldsymbol{\chi}'(\boldsymbol{x}, t, \tau) = \partial_t \boldsymbol{\chi}(\boldsymbol{x}, t, \tau) + \partial_\tau \boldsymbol{\chi}(\boldsymbol{x}, t, \tau)$$

respectively.
Use the Poynting theorem[5]

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) + \boldsymbol{e}^{\mathrm{T}} \partial_t \boldsymbol{d} + \boldsymbol{e}^{\mathrm{T}} \boldsymbol{j} = 0$$

to connect the energy in the region with the energy flow through the boundary and the energy produced by the sources

$$\mathcal{E}(t) = \mathcal{E}(0) - \int_0^t \int_\Gamma \boldsymbol{E} \times \boldsymbol{H} \cdot \boldsymbol{n}\,\mathrm{d}S\,\mathrm{d}\tau - \int_0^t \int_\Omega \boldsymbol{e}^{\mathrm{T}} \boldsymbol{j}\,\mathrm{d}v\,\mathrm{d}\tau \tag{4.3}$$

where $\boldsymbol{n}$ is the outward pointing unit normal to the region Ω. The energy flow can be divided into its in-coming and out-going components[6]. Use the simple (polarization)

[5] Multiply the Maxwell equations (2.1) with the fields, $\boldsymbol{e}$, and integrate over space.
[6] Similar to the wave splitting concept in one space dimension [5, 6].

equality $4\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}+\boldsymbol{b}|^2 - |\boldsymbol{a}-\boldsymbol{b}|^2$ on $(\boldsymbol{E}\times\boldsymbol{H})\cdot\boldsymbol{n} = -[(\boldsymbol{n}\times\boldsymbol{E})\times\boldsymbol{n}]\cdot[\boldsymbol{n}\times\boldsymbol{H}] = |\boldsymbol{e}^-|^2 - |\boldsymbol{e}^+|^2$ *i.e.*,

$$\boldsymbol{e}^{\pm} = \frac{(\boldsymbol{n}\times\boldsymbol{E})\times\boldsymbol{n}\pm\boldsymbol{n}\times\boldsymbol{H}}{2} \tag{4.4}$$

to get the energy balance

$$\mathcal{E}(t) + \int_0^t \|\boldsymbol{e}^-\|_\Gamma^2\,\mathrm{d}\tau = \mathcal{E}(0) + \int_0^t \|\boldsymbol{e}^+\|_\Gamma^2\,\mathrm{d}\tau - \int_0^t\int_\Omega \boldsymbol{e}^{\mathrm{T}}\boldsymbol{j}\,\mathrm{d}v\,\mathrm{d}\tau. \tag{4.5}$$

The physical interpretation of the terms in this expression is: $\mathcal{E}$ is the total energy in the region, $\|\boldsymbol{e}^\pm\|_G$ is the power flowing in/out through the boundary, Γ, and $-\boldsymbol{e}^{\mathrm{T}}\boldsymbol{j}$ is the power density produced by the sources in the region Ω.

5 Energy Estimates

In the mathematical analysis as well as in the numerical treatment of the equations, it is not the physical energy that is of primary interest. What is of importance are estimates of the amplitudes of the fields. We can connect the energy and the field amplitudes by noting that the optical part of the energy is of the same size as the amplitudes and that it is always possible to bound the dispersive energy by the integral of the field amplitudes (the dispersive energy can hence be made arbitrary small by considering a sufficiently small time interval).
The assumption of a positive definite optical response gives the existence of positive constants c, C such that (the constant c should not be mixed up with the phase velocity in Section 6 and Appendix B)

$$c|\boldsymbol{e}|^2 \leq \frac{1}{2}\boldsymbol{e}^{\mathrm{T}}\boldsymbol{\varepsilon}_\infty(\boldsymbol{x},t)\boldsymbol{e} \leq C|\boldsymbol{e}|^2, \quad \text{for all } \boldsymbol{e},\boldsymbol{x},t$$

and by integration over space we get an estimate of the optical energy (notation, see Appendix A)

$$c\|\boldsymbol{e}(\cdot,t)\|^2 \leq \mathcal{E}_{opt}(t) \leq C\|\boldsymbol{e}(\cdot,t)\|^2. \tag{5.1}$$

To derive an estimate of the dispersive energy we start with the conductive part

$$\int_0^t\int_\Omega \boldsymbol{e}^{\mathrm{T}}\boldsymbol{\sigma}\boldsymbol{e}\,\mathrm{d}v\,\mathrm{d}\tau \leq \|\boldsymbol{\sigma}\|_\infty \int_0^t \|\boldsymbol{e}(\cdot,\tau)\|^2\,\mathrm{d}\tau$$

where $\|\boldsymbol{\sigma}\|_\infty = \sup_{\boldsymbol{x},t}|\boldsymbol{\sigma}(\boldsymbol{x},t)|$. The convolution part is more involved, since all previous values of the fields are used. Divide the convolution into two parts

$$\boldsymbol{\chi}' * \boldsymbol{e} = \int_0^t \boldsymbol{\chi}'(\boldsymbol{x},t,t-\tau)\boldsymbol{e}(\boldsymbol{x},\tau)\,\mathrm{d}\tau + \int_{-\infty}^0 \boldsymbol{\chi}'(\boldsymbol{x},t,t-\tau)\boldsymbol{e}(\boldsymbol{x},\tau)\,\mathrm{d}\tau.$$

The first convolution part, only depending on the times $t > 0$, is estimated by the Schwartz and the Young's inequality[7].

$$\int_0^t \int_\Omega \boldsymbol{e}^{\mathrm{T}} \boldsymbol{\chi}' * \boldsymbol{e} \, \mathrm{d}v \, \mathrm{d}\tau = \int_\Omega \int_0^t \boldsymbol{e}^{\mathrm{T}}(\boldsymbol{x}, \tau) \int_0^\tau \boldsymbol{\chi}'(\boldsymbol{x}, \tau, \tau - \tau_1) \boldsymbol{e}(\boldsymbol{x}, \tau_1) \, \mathrm{d}\tau_1 \, \mathrm{d}\tau \, \mathrm{d}v$$

$$\leq \int_\Omega \left(\int_0^t |\boldsymbol{e}(\boldsymbol{x}, \tau)|^2 \, \mathrm{d}\tau \right)^{\frac{1}{2}} \left(\int_0^t \left| \int_0^\tau \boldsymbol{\chi}'(\boldsymbol{x}, \tau, \tau - \tau_1) \boldsymbol{e}(\boldsymbol{x}, \tau_1) \, \mathrm{d}\tau_1 \right|^2 \mathrm{d}\tau \right)^{\frac{1}{2}} \mathrm{d}v$$

$$\leq \int_\Omega \int_0^t |\boldsymbol{e}(\boldsymbol{x}, \tau)|^2 \, \mathrm{d}\tau \sup_{t \in [0,T]} \int_0^T |\boldsymbol{\chi}'(x, t, \tau)| \, \mathrm{d}\tau \, \mathrm{d}v \leq C_1 \int_0^t \|\boldsymbol{e}(\cdot, \tau)\|^2 \, \mathrm{d}\tau$$

where $C_1 = \sup_{t \in [0,T], \boldsymbol{x} \in \Omega} \int_0^T |\boldsymbol{\chi}'(\boldsymbol{x}, t, \tau)| \, \mathrm{d}\tau$. Notice that this essentially implies a bound on the time derivative on the susceptibility kernel $\partial_t \boldsymbol{\chi} \in \mathrm{L}^1$.
On the second integral we use the Schwartz inequality

$$\int_\Omega \int_0^t \boldsymbol{e}^{\mathrm{T}}(\boldsymbol{x}, \tau) \int_{-\infty}^0 \boldsymbol{\chi}'(\boldsymbol{x}, \tau, \tau - \tau_1) \boldsymbol{e}(\boldsymbol{x}, \tau_1) \, \mathrm{d}\tau_1 \, \mathrm{d}\tau \, \mathrm{d}v \leq \frac{1}{2} \int_0^t \|\boldsymbol{e}(\cdot, \tau)\|^2 + \|\tilde{\boldsymbol{p}}\|^2 \, \mathrm{d}\tau$$

where

$$\|\tilde{\boldsymbol{p}}\| = \| \int_{-\infty}^0 \boldsymbol{\chi}'(t, t - \tau) \boldsymbol{e}(\tau) \, \mathrm{d}\tau \| \leq \sup_{t \in [-\infty, 0]} \|\boldsymbol{e}(\cdot, t)\| \sup_{\boldsymbol{x} \in \Omega, t \in [-\infty, 0]} \int_0^\infty |\boldsymbol{\chi}'(\boldsymbol{x}, t, \tau)| \, \mathrm{d}\tau.$$

Finally, we get the dispersive energy estimate

$$\mathcal{E}_{disp}(t) - \mathcal{E}_{disp}(0) \leq C_T \int_0^t \|\boldsymbol{e}(\cdot, \tau)\|^2 + \|\tilde{\boldsymbol{p}}\|^2 \, \mathrm{d}\tau \tag{5.2}$$

for some constant C_T. The source term is estimated as

$$\mathcal{E}_s(t) = \int_0^t \int_\Omega \boldsymbol{e}^{\mathrm{T}} \boldsymbol{j} \, \mathrm{d}\tau \, \mathrm{d}v \leq \frac{1}{2} \int_0^t \|\boldsymbol{e}\|^2 \, \mathrm{d}\tau + \frac{1}{2} \int_0^t \|\boldsymbol{j}\|^2 \, \mathrm{d}\tau. \tag{5.3}$$

5.1 Local constitutive relations

We also give a simplified proof for the special, but useful, case of temporally local constitutive relations[8]. Extra variables $\boldsymbol{p}$, here called the state of the material are introduced in this method. The general linear constitutive relations that are local in time have the form

$$\begin{cases} \boldsymbol{d} = \boldsymbol{\varepsilon}_\infty \boldsymbol{e} + \boldsymbol{\varphi} \boldsymbol{p} & \text{in } \Omega \times [0, T] \\ \partial_t \boldsymbol{p} = \boldsymbol{\phi} \boldsymbol{p} + \boldsymbol{\psi} \boldsymbol{e} & \text{in } \Omega \times [0, T] \\ \boldsymbol{p}(\boldsymbol{x}, 0) = \boldsymbol{p}_0(\boldsymbol{x}) & \text{at } t = 0 \end{cases} \tag{5.4}$$

[7] Young's inequality is [12]:

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p, \quad p \geq 1$$

Here we use $p = 2$.

[8] Compare with the auxiliary differential equation method [30].

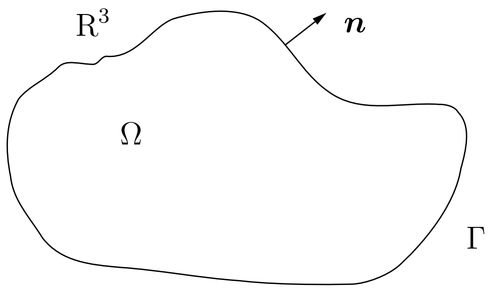

Figure 2: Geometry for the initial-boundary value problem.

where the optical response $\boldsymbol{\varepsilon}_\infty$ is symmetric and positive definite, $\boldsymbol{\varphi} = \boldsymbol{\varphi}(x, t)$ is a $6 \times m$ matrix, $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{x}, t)$ a $m \times 1$ vector, $\boldsymbol{\phi} = \boldsymbol{\phi}(\boldsymbol{x}, t)$ is a $m \times m$ matrix, and $\boldsymbol{\psi} = \boldsymbol{\psi}(\boldsymbol{x}, t)$ is a $m \times 6$ matrix. The matrix $\boldsymbol{\varepsilon}_\infty$ models the optical response and the vector $\boldsymbol{p}$ describes the state of the material. In [30] the Debye and the Lorentz models, the vector $\boldsymbol{p}$ is the polarization of the material.
We start by deriving a bound on the state $\boldsymbol{p}$. Multiply the second equation of (5.4) with $\boldsymbol{p}$, integrate by parts and estimate the products

$$\frac{1}{2}\partial_t|\boldsymbol{p}|^2 = \boldsymbol{p}^{\mathrm{T}}\boldsymbol{\phi}\boldsymbol{p} + \boldsymbol{p}^{\mathrm{T}}\boldsymbol{\psi}\boldsymbol{e} \leq c_1\left(|\boldsymbol{p}|^2 + |\boldsymbol{e}|^2\right)$$

for some constant c_1. Using the Grönwall lemma [24], we get a bound on the states

$$|\boldsymbol{p}|^2 \leq \mathrm{e}^{c_1 t}\left\{|\boldsymbol{p}_0|^2 + \int_0^t |\boldsymbol{e}(\boldsymbol{x}, \tau)|^2 \,\mathrm{d}\tau\right\}$$

and the estimate on the dispersive energy follows:

$$\mathcal{E}_{disp}(t) - \mathcal{E}_{disp}(0) = \int_0^t \int_\Omega \boldsymbol{e}^{\mathrm{T}}\partial_t[\boldsymbol{\varphi}\boldsymbol{p}] \,\mathrm{d}v \,\mathrm{d}\tau$$
$$= \int_0^t \int_\Omega \boldsymbol{e}^{\mathrm{T}}\dot{\boldsymbol{\varphi}}\boldsymbol{p} + \boldsymbol{e}^{\mathrm{T}}\boldsymbol{\varphi}\boldsymbol{\phi}\boldsymbol{p} + \boldsymbol{e}^{\mathrm{T}}\boldsymbol{\varphi}\boldsymbol{\psi}\boldsymbol{e} \,\mathrm{d}v \,\mathrm{d}\tau \leq C_T \int_0^t \left\{\|\boldsymbol{p}_0\|^2 + \|\boldsymbol{e}(\cdot, \tau)\|^2 \,\mathrm{d}\tau\right\}$$

for some constant C_T. This result is similar to (5.2) except that the previous history of the electromagnetic fields are included in $\|\boldsymbol{p}_0\|$.

5.2 Initial-boundary value problem

A typical electromagnetic problem consists of specifying the field values in the region Ω at a time $t = 0$, one of the tangential field components or a linear combination of them at the boundary, Γ, and the impressed sources in the region, see Figure 2. Mathematically, we formulate this as an initial-boundary value problem.

$$\begin{cases} \partial_t[\boldsymbol{\varepsilon}\boldsymbol{e}] = \mathsf{M}(\nabla)\boldsymbol{e} - \boldsymbol{j} & \text{in } \Omega \times [0, T] \\ \boldsymbol{e}(\boldsymbol{x}, 0) = \boldsymbol{f}(\boldsymbol{x}) & \text{in } \Omega \\ \boldsymbol{e}^+ = \boldsymbol{g} + \mathcal{R}\boldsymbol{e}^- & \text{at } \Gamma \times [0, T] \end{cases} \tag{5.5}$$

Here the source $\boldsymbol{j}$, the initial value $\boldsymbol{f}$, and the boundary value $\boldsymbol{g}$ are known. The operator $\mathcal{R}$ denotes a mapping $\mathrm{L}^2(\Gamma\times[0,T]) \mapsto \mathrm{L}^2(\Gamma\times[0,T])$. The split fields $\boldsymbol{e}^{\pm}$ are defined in (4.4). We interpret $\mathcal{R}$ as the reflection operator for the exterior region, $\mathbb{R}^3 \setminus \bar{\Omega}$, *i.e.*, $\mathcal{R}$ models the electromagnetic properties of the region outside Ω, see also Section 8. To solve the problem, we also need to specify the initial values on the states, *i.e.*,

$$\boldsymbol{p}(\boldsymbol{x},0) = \boldsymbol{p}(\boldsymbol{x})$$

in the case of local constitutive relations, see Section 5.1, or all values of the fields for negative times in the case of integral relations[9], *i.e.*,

$$\boldsymbol{e}(\boldsymbol{x},t) = \boldsymbol{e}_{<}(\boldsymbol{x},t) \quad \text{for all } t<0$$

where we assume that $\|\boldsymbol{e}(\cdot,t)\|$ is bounded on $[-\infty,0]$. To show the well-posedness of the problem, we need to estimate the field values at time $t=T$ with their initial values, boundary values, and source terms.
We start to estimate the boundary condition with the triangle inequality and the generalized Cauchy inequality.

$$\|\boldsymbol{e}^+\|_\Gamma^2 \le \|\boldsymbol{g}\|_\Gamma^2 + 2\|\boldsymbol{g}\|_\Gamma\|\mathcal{R}\boldsymbol{e}^-\|_\Gamma + \|\mathcal{R}\boldsymbol{e}^-\|_\Gamma^2 \le (1+\delta^{-1})\|\boldsymbol{g}\|_\Gamma^2 + (1+\delta)\|\mathcal{R}\boldsymbol{e}^-\|_\Gamma^2$$

for all $\delta>0$. Integrate in time and use the induced operator norm on the reflection operator, *i.e.*,

$$\|\mathcal{R}\|^2 = \sup \int_0^T \|\mathcal{R}\boldsymbol{e}^-\|_\Gamma^2\,\mathrm{d}t \quad \text{for all } \boldsymbol{e}^- \text{ such that } \int_0^T \|\boldsymbol{e}^-\|_\Gamma^2\,\mathrm{d}t = 1$$

to get the boundary estimate

$$\int_0^t \|\boldsymbol{e}^+\|_\Gamma^2\,\mathrm{d}\tau \le (1+\delta^{-1})\int_0^t \|\boldsymbol{g}\|_\Gamma^2\,\mathrm{d}\tau + (1+\delta)\|\mathcal{R}\|^2\int_0^t \|\boldsymbol{e}^-\|_\Gamma^2\,\mathrm{d}\tau. \tag{5.6}$$

We proceed by adding the term $\delta_2\int_0^t \|\boldsymbol{e}^+\|_\Gamma^2\,\mathrm{d}\tau$, $\delta_2>0$ on both sides of the energy balance (4.5), and then use the boundary estimate (5.6) to obtain the following estimate:

$$\mathcal{E}(t) + \gamma_1\int_0^t \|\boldsymbol{e}^{\pm}(\cdot,\tau)\|_\Gamma^2\,\mathrm{d}\tau \le \mathcal{E}(0) + \gamma_2\int_0^t \|\boldsymbol{g}(\cdot,\tau)\|_\Gamma^2\,\mathrm{d}\tau + |\mathcal{E}_s(t)|$$

where $\gamma_1 = \min(1-(1+\delta_2)(1+\delta)\|\mathcal{R}\|^2,\delta_2)$ and $\gamma_2=(1+\delta_2)(1+\delta^{-1})$. To be able to bound the boundary terms with a $\gamma_1>0$, we require $\|\mathcal{R}\|<1$.
To proceed, use the definition of the energy $\mathcal{E}=\mathcal{E}_{opt}+\mathcal{E}_{disp}$. We get

$$\mathcal{E}_{opt}(t) + \gamma_1\int_0^t \|\boldsymbol{e}^{\pm}\|_\Gamma^2\,\mathrm{d}\tau \le \mathcal{E}_{opt}(0) + \mathcal{E}_{disp}(0) - \mathcal{E}_{disp}(t) + \gamma_2\int_0^t \|\boldsymbol{g}\|_\Gamma^2\,\mathrm{d}\tau + |\mathcal{E}_s(t)|.$$

[9]Observe that the assumptions made here are slightly more general then the commonly adopted case of quiescent fields, *i.e.*, the fields are assumed to be zero before a finite time, typically $t=0$. The assumption of quiescent fields are too restricted both from a physical and a computational point of view, *e.g.*, problems involving permanent magnetism such as the earth magnetic field.

Finally, we get using the estimates (5.1), (5.2) and (5.3)

$$\|\boldsymbol{e}(\cdot,t)\|^2 + \int_0^t \|\boldsymbol{e}^{\pm}\|_\Gamma^2 \,\mathrm{d}\tau \le C_T \left\{ \|\boldsymbol{f}\|^2 + \int_0^t \|\boldsymbol{e}\|^2 + \|\boldsymbol{g}\|_\Gamma^2 + \|\boldsymbol{j}\|^2 + \|\tilde{\boldsymbol{p}}\|^2 \,\mathrm{d}\tau \right\}$$

for some constant C_T. Grönwall lemma [24] then implies that there is a constant C'_T such that

$$\|\boldsymbol{e}(\cdot,t)\|^2 + \int_0^t \|\boldsymbol{e}^{\pm}(\cdot,\tau)\|_\Gamma^2 \,\mathrm{d}\tau \le C'_T \left\{ \|\boldsymbol{f}\|^2 + \int_0^t \|\boldsymbol{g}(\cdot,\tau)\|_\Gamma^2 + \|\boldsymbol{j}(\cdot,\tau)\|^2 + \|\tilde{\boldsymbol{p}}\|^2 \,\mathrm{d}\tau \right\}.$$

This is the estimate for well-posedness [24]. We can also derive similar estimates on the derivatives of the fields by differentiation of the system, see Refs. 17, 24 for details on this technique. To get well-posedness, we also need to show that there exists a solution to the equation. A constructive method is to use finite difference approximations and fixed-point iterations, see Refs. 17, 24 for a discussion of the technique.

6 Finite Speed of Propagation

The requirement of a finite speed of propagation is natural in most applications of electromagnetics. Here, we consider some of the limitations this requirement implies on the optical response of the material. The finite speed of propagation is also closely connected to the causality condition.

We start by considering plane wave solutions. An ansatz $\boldsymbol{e}(\boldsymbol{x},t) = \boldsymbol{e}_0 f(k(\hat{\boldsymbol{k}}\cdot\boldsymbol{x} - ct))$ in (2.1) gives the equation

$$-ck\boldsymbol{\varepsilon}_\infty \boldsymbol{e}_0 f' = k\mathsf{M}(\hat{\boldsymbol{k}})\boldsymbol{e}_0 f' - \boldsymbol{\chi}' * \boldsymbol{e}_0 f - \boldsymbol{\sigma}\boldsymbol{e}_0 f - \frac{1}{2}[\partial_t \boldsymbol{\varepsilon}_\infty]\boldsymbol{e}_0 f.$$

For large values of k, the two last terms are bounded and we get the following eigenvalue problem[10]

$$-c\boldsymbol{\varepsilon}_\infty \boldsymbol{e}_0 = \mathsf{M}(\hat{\boldsymbol{k}})\boldsymbol{e}_0.$$

By searching for the largest c over all directions $\hat{\boldsymbol{k}}$ we find the speed. Alternatively, we can find an upper limit on the speed by multiplying with $\boldsymbol{e}_0$, and estimating the product $\boldsymbol{e}_0^{\mathrm{T}}\mathsf{M}(\hat{\boldsymbol{k}})\boldsymbol{e}_0$ as

$$\boldsymbol{e}_0^{\mathrm{T}}\mathsf{M}(\hat{\boldsymbol{k}})\boldsymbol{e}_0 = -2\hat{\boldsymbol{k}}\cdot\boldsymbol{E}\times\boldsymbol{H} \le 2|\boldsymbol{E}||\boldsymbol{H}| \le |\boldsymbol{E}|^2 + |\boldsymbol{H}|^2 = \boldsymbol{e}_0^{\mathrm{T}}\boldsymbol{e}_0.$$

This gives the estimate

$$c \le \sup \frac{\boldsymbol{e}_0^{\mathrm{T}}\boldsymbol{e}_0}{\boldsymbol{e}_0^{\mathrm{T}}\boldsymbol{\varepsilon}_\infty\boldsymbol{e}_0} \quad \text{over all } \boldsymbol{e}_0 \in \mathbb{R}^6$$

[10] This is essentially the fact that the principal part of the equation governs the speed.

which is directly related to the positive definiteness of $\boldsymbol{\varepsilon}_\infty$. The positive definiteness is easy to calculate (the smallest eigenvalue of $\boldsymbol{\varepsilon}_\infty$) but the estimate is not very accurate.
Compare the result with the special case of a bi-isotropic material in Appendix B. The smallest eigenvalue of $\boldsymbol{\varepsilon}_\infty$ gives the bound

$$c \leq \frac{1}{\frac{\epsilon+\mu}{2} - \sqrt{\left(\frac{\epsilon-\mu}{2}\right)^2 + \kappa^2}}.$$

The true speed is the geometrical mean value of the eigenvalues.

7 Characterization of Dissipation

A material is called dissipative if the net energy flux through the boundary is non-positive, *i.e.*,

$$\int_0^t \int_\Gamma \boldsymbol{E} \times \boldsymbol{H} \cdot \boldsymbol{n} \,\mathrm{d}S \,\mathrm{d}\tau \leq 0 \quad \text{for all } t \geq 0 \tag{D}$$

for all possible fields $\boldsymbol{E}, \boldsymbol{H}$ at the boundary that are quiescent before time $t = 0$, and all sufficiently smooth regions Ω with boundary, Γ, contained in the material. Using the source-free Poynting theorem, (4.3), we see that this is equivalent to the energy condition $\mathcal{E}(t) \geq 0$ for all possible fields at the boundary and all regions. Stated differently, the medium does not produce energy.
In this section, we give three different characterizations of dissipation for linear, homogeneous, stationary materials, $\boldsymbol{d} = \boldsymbol{\varepsilon}_\infty \boldsymbol{e} + \boldsymbol{\chi} * \boldsymbol{e}$, such that $|\boldsymbol{\chi}(t)| < C\mathrm{e}^{-\eta t}$ for some constants $C, \eta > 0$. We also show that they are equivalent in the case of constitutive relations with a symmetric, positive definite optical response, and that they imply that the material is dissipative in the sense above. The three classes are:
The time domain type

$$\int_0^t \boldsymbol{e}^{\mathrm{T}} \partial_\tau [\boldsymbol{\varepsilon} \boldsymbol{e}] \,\mathrm{d}\tau \geq 0 \quad \text{for all } t \geq 0 \tag{T}$$

for all[11] $\boldsymbol{e} \in \mathrm{C}^1[-\infty, t]$ such that $\boldsymbol{e}(t) = 0$ for $t \leq 0$.
The Laplace type

$$\begin{gathered} \int_0^\infty \mathrm{e}^{-2\eta\tau} \boldsymbol{e}^{\mathrm{T}} \partial_\tau [\boldsymbol{\varepsilon} \boldsymbol{e}] \,\mathrm{d}\tau \geq 0 \quad \text{for all } \eta \geq 0 \\ \text{or} \\ \mathrm{Re}\{s\hat{\boldsymbol{\varepsilon}}(s)\} \geq 0, \quad \text{for all complex } s \text{ such that } \mathrm{Re}\{s\} \geq 0 \end{gathered} \tag{L}$$

[11] It is enough to assume $\boldsymbol{e} \in \mathrm{H}^{1/2+\delta}$ for some $\delta > 0$, *i.e.*, Sobolev space with at least half a derivative.

for all $\boldsymbol{e} \in \mathrm{C}^1[-\infty, \infty]$ such that $\boldsymbol{e}(t) = 0$ for $t \leq 0$. Here, the hat, $\hat{\cdot}$ denotes the Laplace transform.
The Fourier type[12]

$$\begin{gathered} \int_{-\infty}^{\infty} \boldsymbol{e}^{\mathrm{T}} \partial_\tau [\boldsymbol{\varepsilon} \boldsymbol{e}] \, \mathrm{d}\tau \geq 0 \\ \text{or} \\ \mathrm{Re}\{\mathrm{i}\omega \hat{\boldsymbol{\varepsilon}}(\mathrm{i}\omega)\} \geq 0 \quad \text{for all real } \omega \end{gathered} \tag{F}$$

for all $\boldsymbol{e} \in \mathrm{C}_0^1[-\infty, \infty]$. Here $\hat{\boldsymbol{\varepsilon}}(\mathrm{i}\omega)$ denotes the Fourier transform[13] of $\boldsymbol{\varepsilon}$.
The reason for using three different types of dissipation is that they occur naturally in different applications. The time domain, (T), is most natural from a physical point of view, but in this case it is hard to derive sufficient conditions for an arbitrary constitutive relation, see Ref 22. The Laplace transform method is powerful for solving initial-boundary value problems [24]. Finally, the Fourier characterization is widely used in the time harmonic case, and it also provides an easy analytical characterization whether a given constitutive relation is dissipative or not. We show the following implications:

$$\begin{tikzcd}
\mathrm{T} \arrow[rr, "\text{positive definite } \boldsymbol{\varepsilon}_\infty"] \arrow[d, leftrightarrow] & & \mathrm{D} \\
\mathrm{L} \arrow[rr, shift right] & & \mathrm{F} \arrow[ll, shift right, "\text{semi positive definite } \boldsymbol{\varepsilon}_\infty"']
\end{tikzcd}$$

Observe that all conditions imply dissipation (D) for the constitutive relations considered in this paper.

The implications (T)↔(L)→(F) are first proved. In these proofs no additional assumptions on the optical response are made. The equivalence is completed by proving (F)→(T) and (T)→(D). However, for this part of the proof, we need to assume that the optical response is semi positive definite and positive definite, respectively. Notice that the assumption of a semi positive definite optical response is slightly more general then the previously used assumptions on $\boldsymbol{\varepsilon}_\infty$. However, this generalization is in practice vacuous, since in general no existence of a solution can be guaranteed.

We start with the equivalence of the different statements in (L) and (F). The (F) statement follows from the Plancherel relation [12] for real-valued functions

$$\int_{-\infty}^{\infty} f(\tau) g(\tau) \, \mathrm{d}\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Re}\{\hat{f}^*(\mathrm{i}\omega) \hat{g}(\mathrm{i}\omega)\} \, \mathrm{d}\omega.$$

[12]Time convention is $e^{\mathrm{i}\omega t}$.
[13]Observe that we use $\mathrm{i}\omega$ as argument in the Fourier plane.

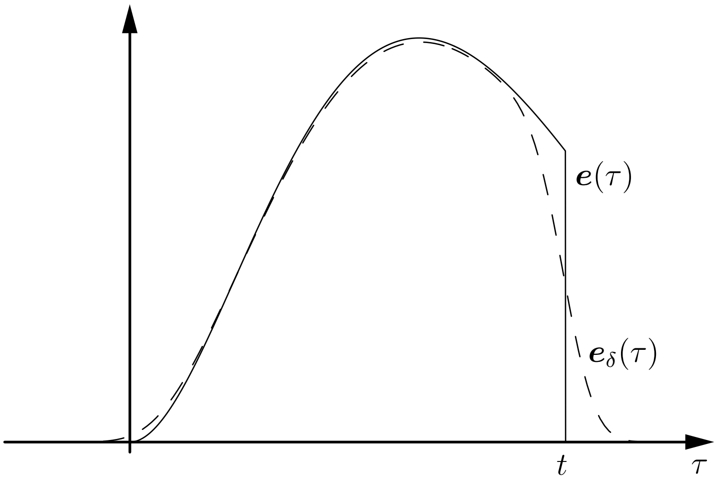

Figure 4: Smoothed function in the proof of F→T.

We get

$$\int_{-\infty}^{\infty} \boldsymbol{e}^{\mathrm{T}} \partial_\tau[\boldsymbol{\varepsilon}\boldsymbol{e}] \,\mathrm{d}\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\boldsymbol{e}}^{\mathrm{H}}(i\omega) \operatorname{Re}\{\mathrm{i}\omega\hat{\boldsymbol{\varepsilon}}(\mathrm{i}\omega)\}\hat{\boldsymbol{e}}(\mathrm{i}\omega) \,\mathrm{d}\omega$$

and hence the medium is dissipative if and only if $\operatorname{Re}\{\mathrm{i}\omega\hat{\boldsymbol{\varepsilon}}(\mathrm{i}\omega)\} = -\omega \operatorname{Im}\{\hat{\boldsymbol{\varepsilon}}(\mathrm{i}\omega)\} \geq 0$ for all real ω. The (L) statement follows from the Plancherel relation and the shift property of the Fourier transform.

$$\int_0^{\infty} \mathrm{e}^{-2\eta t} \boldsymbol{e}^{\mathrm{T}} \partial_t[\boldsymbol{\varepsilon}\boldsymbol{e}] \,\mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\boldsymbol{e}}^{\mathrm{H}}(\eta+\mathrm{i}\omega) \operatorname{Re}\{(\eta+\mathrm{i}\omega)\hat{\boldsymbol{\varepsilon}}(\eta+\mathrm{i}\omega)\}\hat{\boldsymbol{e}}(\eta+\mathrm{i}\omega) \,\mathrm{d}\omega$$

for all real-valued $\boldsymbol{e}$. Hence the characterization $\operatorname{Re}\{s\hat{\boldsymbol{\varepsilon}}(s)\} \geq 0$ holds for all s with $\operatorname{Re}\{s\} \geq 0$. Observe that since $\boldsymbol{\chi}(t)$ is causal, *i.e.*, $\boldsymbol{\chi}(t) = 0$ for $t < 0$, its Laplace transform restricted to the imaginary axis coincide with the Fourier transform.

To show the implication (T)→(L) we use integration by parts

$$\int_0^t \mathrm{e}^{-2\eta\tau} \boldsymbol{e}^{\mathrm{T}} \partial_\tau[\boldsymbol{\varepsilon}\boldsymbol{e}] \,\mathrm{d}\tau = \mathrm{e}^{-2\eta t} \int_0^t \boldsymbol{e}^{\mathrm{T}} \partial_\tau[\boldsymbol{\varepsilon}\boldsymbol{e}] \,\mathrm{d}\tau + \int_0^t \left\{ 2\eta \mathrm{e}^{-2\eta\tau} \int_0^\tau \boldsymbol{e}^{\mathrm{T}} \partial_{\tau_1}[\boldsymbol{\varepsilon}\boldsymbol{e}] \,\mathrm{d}\tau_1 \right\} \mathrm{d}\tau$$

which is positive if $\int_0^t [\boldsymbol{e}^{\mathrm{T}} \partial_t[\boldsymbol{\varepsilon}\boldsymbol{e}]](\tau) \,\mathrm{d}\tau \geq 0$. The implication (L)→(F) is obtained if we use the compact support of the fields in (F) and shift the time scale, or consider the restriction of the Laplace characterization to the imaginary axis.

To show the implication (F)→(T), we start by showing that the optical response is symmetric. We then use that $\hat{\boldsymbol{\varepsilon}}(\mathrm{i}\omega) \sim \boldsymbol{\varepsilon}_\infty$ for large ω and observe that $-\omega \operatorname{Im}\{\boldsymbol{\varepsilon}_\infty\}$ changes sign with ω and hence $\boldsymbol{\varepsilon}_\infty$ is symmetric. Now use the (semi) positive definiteness of $\boldsymbol{\varepsilon}_\infty$ and choose an arbitrary field $\boldsymbol{e} \in \mathrm{C}^1[0,t]$ such that $\boldsymbol{e}(0) = 0$ and extend $\boldsymbol{e}$ smoothly to zero outside $[0,t]$, *i.e.*, set $\boldsymbol{e}_\delta = \boldsymbol{e} * \Phi_\delta$ where $\Phi_\delta(t) = \Phi(t/\delta)$

and Φ is a positive smooth function with unit integral and compact support centered around[14] $t = 0$, see Figure 4. The optical part vanishes due to non contributing limits and in the limit $\delta \to 0$ we get the time domain characterization (T)

$$\int_{-\infty}^{\infty} \boldsymbol{e}_\delta^{\mathrm{T}} \partial_t[\boldsymbol{\varepsilon}\boldsymbol{e}_\delta] \,\mathrm{d}\tau = \int_{-\infty}^{\infty} \boldsymbol{e}_\delta^{\mathrm{T}} \partial_t[\boldsymbol{\chi} * \boldsymbol{e}_\delta] \,\mathrm{d}\tau \overset{\delta\to 0}{\longrightarrow} \int_0^t \boldsymbol{e}^{\mathrm{T}} \partial_\tau[\boldsymbol{\chi} * \boldsymbol{e}] \,\mathrm{d}\tau \le \int_0^t \boldsymbol{e}^{\mathrm{T}} \partial_\tau[\boldsymbol{\varepsilon}\boldsymbol{e}] \,\mathrm{d}\tau$$

These calculations are made under the smoothing assumption that,*e.g.*, $\boldsymbol{\chi}' \in \mathrm{L}^1$ or $\boldsymbol{\chi}$ is piecewise C^1.
(L)$\to$(T) follows from the fact that the Laplace characterization L implies that the optical response is positive semi definite, *viz.* choose a real $s \to \infty$ in (L) to get the result $\mathrm{Re}\{\boldsymbol{\varepsilon}_\infty\} \ge 0$ and hence that positive semi definiteness of the optical response is a necessary condition for Laplace type dissipation [13].
Finally, we observe that to get dissipation (D) it is necessary that there exists a sufficiently smooth solution to the Maxwell equations [17, 24]. From the observation that a symmetric, positive definite optical response gives existence we notice that all characterizations (T),(L) and (F) together with a positive definite optical response imply dissipation (D). This gives the sufficient conditions for dissipation

$$\begin{aligned} &\omega \,\mathrm{Im}\{\hat{\boldsymbol{\varepsilon}}(\mathrm{i}\omega)\} = \omega \frac{\hat{\boldsymbol{\varepsilon}}(\mathrm{i}\omega) - \hat{\boldsymbol{\varepsilon}}^{\mathrm{H}}(\mathrm{i}\omega)}{2\mathrm{i}} \le 0 \quad \text{all } \ \omega \\ &\lim_{\omega\to\pm\infty} \hat{\boldsymbol{\varepsilon}}(\mathrm{i}\omega) = \boldsymbol{\varepsilon}_\infty = \boldsymbol{\varepsilon}_\infty^{\mathrm{T}} > 0 \end{aligned}$$

These conditions are rather easy to check for a specific constitutive relation, both on their integral form, $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_\infty + \boldsymbol{\chi}*$, and on a local form of Equation (5.4).

7.1 Time limited knowledge of the susceptibility kernel

In a time domain determination of the susceptibility kernel it is natural to measure the reflection operator and to calculate the susceptibility kernel by solving an inverse problem [14, 19]. The typical result is a sampling of the susceptibility kernel in an interval $[0, t]$, and, hence, it is not straightforward to use the frequency characterization above. Instead, we use the time domain characterization of dissipation (T) directly.

$$\int_0^t \boldsymbol{e}^{\mathrm{T}} \partial_\tau[\boldsymbol{\varepsilon}\boldsymbol{e}] \,\mathrm{d}\tau = \int_0^t \boldsymbol{e}^{\mathrm{T}} \partial_\tau[(\boldsymbol{\varepsilon}_\infty + \boldsymbol{\chi}*)\boldsymbol{e}] \,\mathrm{d}\tau = \mathcal{E}_{opt}(t) + \int_0^t \boldsymbol{e}^{\mathrm{T}} (\boldsymbol{\sigma} + \boldsymbol{\chi}'*)\boldsymbol{e} \,\mathrm{d}\tau \ge 0.$$

Observe that $\mathcal{E}_{opt}(0) = 0$ since $\boldsymbol{e}(\boldsymbol{x}, 0) = 0$. A sufficient condition for a medium with dissipation is that both $\boldsymbol{\varepsilon}_\infty$ and $\boldsymbol{\sigma} + \boldsymbol{\chi}'*$ are semi positive definite operators on $[0, T]$. Discretize the convolution integral with a stepsize h and set

$$(\tilde{\boldsymbol{e}})_i = \boldsymbol{e}(ih), \quad (\tilde{\boldsymbol{\chi}}')_i = h\boldsymbol{\chi}'(ih), \quad i = 0, 1, 2, \ldots, T/h.$$

[14] Use a mollifier technique [29].

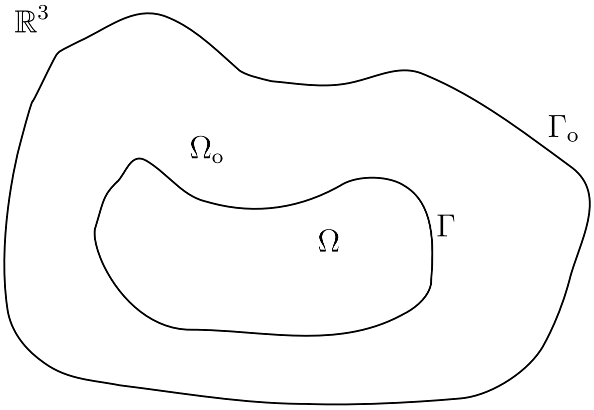

Figure 5: Geometry of the two equivalent problems in Section 8.

Use the trapezoidal rule for the integral, and take the symmetric part of the resulting matrix. For a sufficiently fine mesh we get

$$\tilde{\boldsymbol{e}}^{\mathrm{T}}\boldsymbol{\sigma}\tilde{\boldsymbol{e}} + \frac{1}{2}\tilde{\boldsymbol{e}}^{\mathrm{T}}\mathrm{Toeplitz}(\tilde{\boldsymbol{\chi}}')\tilde{\boldsymbol{e}} \geq 0 \quad \text{for all } \tilde{\boldsymbol{e}} \tag{7.1}$$

where the matrix $\mathrm{Toeplitz}(\boldsymbol{\chi}')$ is the block Toeplitz matrix

$$(\mathrm{Toeplitz}(\tilde{\boldsymbol{\chi}}'))_{i,j} = \tilde{\boldsymbol{\chi}}'_{i-j}$$

and the multiplication is the block multiplication

$$(\mathrm{Toeplitz}(\tilde{\boldsymbol{\chi}}')\tilde{\boldsymbol{e}})_i = \sum_{j=0}^{k} \mathrm{Toeplitz}(\tilde{\boldsymbol{\chi}}')_{i,j}\tilde{\boldsymbol{e}}_j = \sum_{j=0}^{k} \tilde{\boldsymbol{\chi}}'_{i-j}\tilde{\boldsymbol{e}}_j$$

and $\tilde{\boldsymbol{\chi}}'_{i-j}\tilde{\boldsymbol{e}}_j$ is the usual matrix product. This problem of determining the eigenvalues of $\mathrm{Toeplitz}(\tilde{\boldsymbol{\chi}}') + 2\boldsymbol{\sigma}$ is a standard problem in matrix calculus. Such problems are efficiently solved by a consideration of the pivot elements in a Gaussian factorization of the matrix [16]. It is also possible to get some conditions on the susceptibility kernel directly [13].

8 Reflection Operator

The reflection operator for wave propagation in one spatial dimension is thoroughly investigated [25]. This operator is the mapping of the boundary value data of $\boldsymbol{e}^+$ to $\boldsymbol{e}^-$, and efficient numerical algorithms have been developed that solves both the direct and the inverse parameter reconstruction problem efficiently [25].
In this section, we investigate the extension of this reflection operator to three dimensions. Specifically, the reflection operator, $\mathcal{R}$, for a region Ω is a map $\boldsymbol{e}^+ \mapsto \boldsymbol{e}^-$ at the boundary of the region, see Ref 18 for an example of the usefulness of this map in the multidimensional inverse problem.

An explicit representation of the reflection operator is found through the solution of the problem

$$\begin{cases} \partial_t[\boldsymbol{\varepsilon}\boldsymbol{e}] = \mathsf{M}(\nabla)\boldsymbol{e} & \text{in } \Omega \\ \boldsymbol{e}^+ = \boldsymbol{g} & \text{at } \Gamma \\ \boldsymbol{e} = 0 & \text{for } t = 0 \end{cases}$$

The reflection operator contains the tractable information from external measurements of all the electromagnetic properties in the region Ω. To see this, consider the following two problems, see Figure 5:

$$\begin{cases} \partial_t[\boldsymbol{\varepsilon}\boldsymbol{e}] = \mathsf{M}(\nabla)\boldsymbol{e} & \text{in } \Omega_{\mathrm{o}} \\ \boldsymbol{e}^+ = \boldsymbol{g} & \text{at } \Gamma_{\mathrm{o}} \\ \boldsymbol{e} = 0 & \text{for } t = 0 \end{cases}$$

and

$$\begin{cases} \partial_t[\boldsymbol{\varepsilon}\boldsymbol{e}] = \mathsf{M}(\nabla)\boldsymbol{e} & \text{in } \Omega_{\mathrm{o}} \setminus \Omega \\ \boldsymbol{e}^+ = \boldsymbol{g} & \text{at } \Gamma_{\mathrm{o}} \\ \boldsymbol{e}^+ = \mathcal{R}\boldsymbol{e}^- & \text{at } \Gamma \\ \boldsymbol{e} = 0 & \text{for } t = 0 \end{cases}$$

The solution to the later problem is identical to the solution to the former in $\Omega_0 \setminus \Omega$. The reflection operator is useful in numerical implementations since it reduces the computational domain. One realization of this operator is to apply local approximations of the operator, *i.e.*, impedance boundary conditions and absorbing boundary conditions [7, 30]. The reflection operator is bounded $\mathrm{L}^2 \mapsto \mathrm{L}^2$, in contrast to other maps between tangential fields on the boundary that usually do not have this property [27]. The energy estimate is used to prove this statement and also to prove that, in the special case of dissipative materials, it is bounded by unity, *i.e.*, $\|\mathcal{R}\| \leq 1$. Performing the estimates of the derivatives [24] shows that the same types of estimates hold for the derivatives of the reflection operator

$$\|\mathcal{R}\|_{\mathrm{H}^k} \leq C_{kT}$$

Here

$$\|\boldsymbol{e}\|^2_{\mathrm{H}^k} = \sum_{|\alpha|\leq k} \int_0^t \int_\Omega |\partial^\alpha \boldsymbol{e}|^2 \, \mathrm{d}S \, \mathrm{d}t$$

is Sobolev norm, and α is the multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and ∂ denotes the derivative $\partial^\alpha = \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_t$ where x_1, x_2 are in tangential directions of the boundary. The constants C_{kT} depend only on the regularity of the material coefficients, *i.e.*, to get an estimate of k-derivatives all coefficients have to have k bounded derivatives.

The requirement on the bound of $\mathcal{R}$ used in the derivation (5.6) can be relaxed by a division of $\mathcal{R}$ in an optical (fast) part and a slower part, *i.e.*,

$$\mathcal{R}\boldsymbol{e}^- = \mathcal{R}_\infty \boldsymbol{e}^- + \int_0^t \mathcal{R}_s \boldsymbol{e}^- \, d\tau \tag{8.1}$$

where both $\mathcal{R}_\infty, \mathcal{R}_s$ are bounded. Doing this we find the energy estimate

$$\|\boldsymbol{e}(\cdot,t)\|^2 + \int_0^t \|\boldsymbol{e}^\pm(\cdot,t)\|_\Gamma^2 \, d\tau \le C_T \Big\{ \|\boldsymbol{f}\|^2 + \int_0^t \|\boldsymbol{g}(\cdot,t)\|_\Gamma^2 \, d\tau + \int_0^t \|\boldsymbol{j}(\cdot,t)\|^2 \, d\tau + \int_0^t \Big(\|\boldsymbol{e}(\cdot,t)\|^2 + \int_0^\tau \|\boldsymbol{e}^\pm(\cdot,t)\|_\Gamma^2 \, d\tau_1 \Big) \, d\tau \Big\}$$

and with the Grönwall lemma we get a similar estimate, but now we only need a bound $\mathcal{R}_\infty < 1$.

Observe that the common and very important cases of perfectly electric or magnetic conducting materials have $\mathcal{R} = \pm 1$, and that they do not satisfy the energy estimates given here.

9 Conclusion

In this paper the macroscopic Maxwell equations are studied in the time domain. It is observed that the requirements of a symmetric, positive definite optical response offer sufficient conditions for well-posedness of the equations and, further more that the fixed-frequency characterization of dissipation is sufficient for general time domain dissipation. The generalizations of the results to nonlinear constitutive relations are under consideration and will be reported elsewhere.

Acknowledgment

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Appendix A Notation

We use a matrix notation[15] to describe the fields and the operators operating on them, *i.e.*, the vector valued fields are treated as column matrices and the dyadics are

[15] The fields are actually rank 1 tensors and the elements in the constitutive relations are dyadics, *i.e.*, rank 2 tensors. Here we only use the cartesian coordinate system and hence identify the vectors and the dyadics with the matrix representation in a cartesian coordinate system, see Ref. 28 for a discussion of the symmetry properties of the constitutive relations.

quadratic matrices. The dyadic-vector product is the usual matrix product, and with $\mathbf{A}^{\mathrm{T}}$ we denote the transpose of the matrix, *i.e.*, $(\mathbf{A}^{\mathrm{T}})_{i,j} = \mathbf{A}_{j,i}$. We also use the short hand notation for the symmetric and skew part of a matrix, $\mathbf{A} = \mathrm{Re}\{\mathbf{A}\} + \mathrm{i}\,\mathrm{Im}\{\mathbf{A}\}$, where

$$\mathrm{Re}\{\mathbf{A}\} = \frac{\mathbf{A} + \mathbf{A}^{\mathrm{H}}}{2}$$
$$\mathrm{Im}\{\mathbf{A}\} = \frac{\mathbf{A} - \mathbf{A}^{\mathrm{H}}}{2\mathrm{i}}$$

where $\mathbf{A}^{\mathrm{H}}$ denotes the hermite conjugate of $\mathbf{A}$, *i.e.*, $(\mathbf{A}^{\mathrm{H}})_{i,j} = \mathbf{A}^*_{j,i}$ (* means complex conjugate).

All quantities in the time domain formulation are real-valued and defined in a region $\Omega \subset \mathbb{R}^3$ and a time interval $[0, T]$ or $[-\infty, T]$. To simplify the notation we suppress both space, time and frequency arguments where it is not necessary for the understanding.

We denote the pointwise norms with $|\cdot|$, *i.e.*, $|\boldsymbol{e}|^2 = \boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}$, for vectors $\boldsymbol{e}$. The norms depending on space and/or time are denoted $\|\cdot\|$, *i.e.*, $\|\boldsymbol{e}\|^2 = \int_\Omega |\boldsymbol{e}|^2 \,\mathrm{d}v$ and $\|\boldsymbol{e}\|_\infty = \sup_{\boldsymbol{x}\in\Omega, t\in[0,T]} |\boldsymbol{e}|$. For the boundary terms we also use the L^2 norm over the boundary, $\|\boldsymbol{e}^+\|^2_\Gamma = \int_\Gamma |\boldsymbol{e}^+|^2 \,\mathrm{d}S$. For operators, we use the induced operator norm $|\boldsymbol{\varepsilon}_\infty| = \sup_{|\boldsymbol{e}|=1} |\boldsymbol{\varepsilon}_\infty \boldsymbol{e}|$ and similarly for $\|\cdot\|$.

In Section 7 we also use the set of k times continuous functions on an interval $[0, T]$, *i.e.*, $\mathrm{C}^k[0, T]$, and the subset of compactly supported functions C^k_0.

With a hat $\hat{\cdot}$, we denote the Laplace transform, *i.e.*, $\hat{f}(s) = \int_{0^-}^\infty f(t)e^{-st}\,\mathrm{d}t$, and Fourier transform (the time harmonic case) is also denoted with a hat, $\hat{f}(\mathrm{i}\omega) = \int_{-\infty}^\infty f(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t$. Note that we use the argument $\mathrm{i}\omega$ in contrast to the standard ω, *i.e.*, we write $\hat{\boldsymbol{\varepsilon}}(\mathrm{i}\omega)$ instead of the usual $\boldsymbol{\varepsilon}(\omega)$. Note also that the Fourier and Laplace transform coincide on the imaginary axis for causal functions (functions that are supported on the positive real axis).

Appendix B Bi-Isotropic material

To illustrate the problems with an arbitrary optical response, we start by analyzing the homogeneous, non dispersive, stationary, bi-isotropic model in detail, see also Section 6. The bi-isotropic model is the most general linear, isotropic model and its electromagnetic properties in a source-free region are described by the following equations

$$\begin{cases} \epsilon\partial_t \boldsymbol{E} + \xi\partial_t \boldsymbol{H} = \nabla\times\boldsymbol{H} \\ \zeta\partial_t \boldsymbol{E} + \mu\partial_t \boldsymbol{H} = -\nabla\times\boldsymbol{E} \end{cases}$$

For our purpose, it is enough to consider plane wave solutions. A plane wave ansatz, $\boldsymbol{e}(t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$, gives the equation[16]

$$\partial_t \boldsymbol{e} = \mathrm{i}|\boldsymbol{k}|\boldsymbol{\varepsilon}_\infty^{-1}\mathsf{M}(\hat{\boldsymbol{k}})\boldsymbol{e}$$

[16] $\mathrm{i}|\boldsymbol{k}|\boldsymbol{\varepsilon}_\infty^{-1}\mathsf{M}(\hat{\boldsymbol{k}})$ is the symbol of the equation.

where the inverse, $\boldsymbol{\varepsilon}_\infty^{-1}$, has the explicit representation

$$\boldsymbol{\varepsilon}_\infty^{-1} = \frac{1}{\epsilon\mu - \xi\zeta}\begin{pmatrix} \mu & 0 & 0 & -\xi & 0 & 0 \\ 0 & \mu & 0 & 0 & -\xi & 0 \\ 0 & 0 & \mu & 0 & 0 & -\xi \\ -\zeta & 0 & 0 & \epsilon & 0 & 0 \\ 0 & -\zeta & 0 & 0 & \epsilon & 0 \\ 0 & 0 & -\zeta & 0 & 0 & \epsilon \end{pmatrix}$$

Since all directions are equivalent, it is enough to consider $\hat{\boldsymbol{k}} = \hat{\boldsymbol{z}}$. The equation then becomes

$$\partial_t \begin{pmatrix} E_x \\ E_y \\ H_x \\ H_y \end{pmatrix} = \frac{\mathrm{i}k_z}{\epsilon\mu - \xi\zeta}\begin{pmatrix} 0 & -\xi & 0 & -\mu \\ \xi & 0 & \mu & 0 \\ 0 & \epsilon & 0 & \zeta \\ -\epsilon & 0 & -\zeta & 0 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ H_x \\ H_y \end{pmatrix}$$

This system of ordinary differential equation is solved with eigenvalue techniques. The matrix has the eigenvalues

$$\begin{aligned}\lambda_m &= \pm\frac{1}{\epsilon\mu - \xi\zeta}\sqrt{\epsilon\mu - \frac{\zeta^2+\xi^2}{2} \pm \sqrt{\frac{(\zeta-\xi)^2(\zeta+\xi)^2}{4} - \epsilon\mu(\zeta-\xi)^2}} \\ &= \pm\frac{\sqrt{\epsilon\mu - \kappa^2 \pm 2|\chi|\sqrt{\epsilon\mu-\kappa^2+\chi^2}}}{\epsilon\mu-\kappa^2-\chi^2} = \pm\frac{\sqrt{\epsilon\mu-\kappa^2}\pm\chi}{\epsilon\mu-\kappa^2-\chi^2} = \frac{\pm 1}{\sqrt{\epsilon\mu-\kappa^2}\mp\chi}\end{aligned}$$

where χ is the skew part $\chi = (\zeta-\xi)/2\mathrm{i}$ and κ is the symmetric part $\kappa = (\zeta+\xi)/2$. The symmetric part κ measures the reciprocity of the material and the skew part χ, the chirality factor, measures the optical activity of the material [26]. The plane wave solution is

$$\boldsymbol{e}(\boldsymbol{x}, t) = \sum_m \boldsymbol{e}_m \mathrm{e}^{\mathrm{i}k_z(\lambda_m t + z)}$$

where $\boldsymbol{e}_m$ is the eigenvector corresponding to eigenvalue λ_m (we assume that these eigenvectors span $\mathbb{R}^4$). The solution is unbounded if not all eigenvalues are purely real, *i.e.*, $\mathrm{Im}\{\lambda_m\} = 0$, and hence the corresponding problem is not well-posed [24] (the solution grows arbitrary fast for large values of k_z). Since the components of $\boldsymbol{\varepsilon}_\infty$ are real, the requirement is $\chi = 0$ and $\epsilon\mu > \kappa^2$. The eigenvalues then simplify to

$$\lambda_m = \frac{\pm 1}{\sqrt{\epsilon\mu - \kappa^2}}, \quad \epsilon\mu > \kappa^2$$

and from the solution we note that the speed of the plane waves is

$$c = \frac{1}{\sqrt{\epsilon\mu-\kappa^2}}, \quad \epsilon\mu > \kappa^2.$$

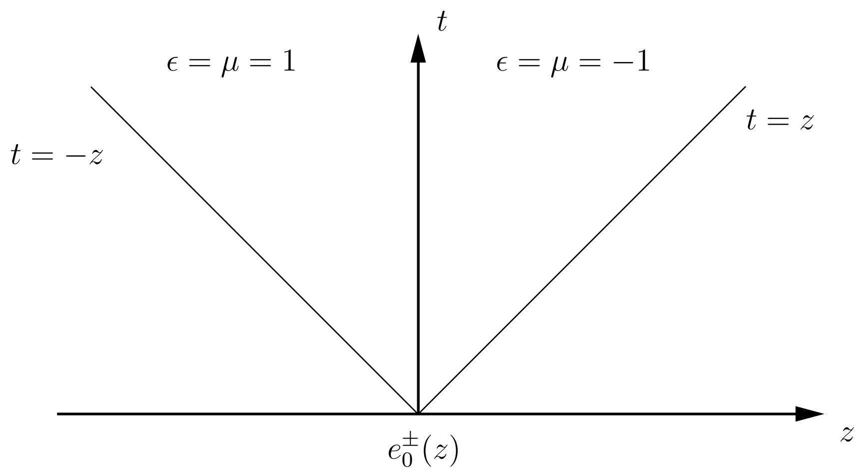

Figure 6: Space-time diagram of the stratified medium problem in Appendix C.

We also notice that the assumption of bounded solutions and a finite speed of propagation give a symmetry and a definiteness requirement on the optical response $\boldsymbol{\varepsilon}_\infty$, *i.e.*,

$$
\begin{aligned}
1 \leq \epsilon\mu - \kappa^2 &= \left(\frac{\epsilon+\mu}{2}\right)^2 - \left(\frac{\epsilon-\mu}{2}\right)^2 - \kappa^2 \\
&= \left(\frac{\epsilon+\mu}{2} + \sqrt{\left(\frac{\epsilon-\mu}{2}\right)^2 + \kappa^2}\right)\left(\frac{\epsilon+\mu}{2} - \sqrt{\left(\frac{\epsilon-\mu}{2}\right)^2 + \kappa^2}\right)
\end{aligned}
$$

and hence all the eigenvalues of $\boldsymbol{\varepsilon}_\infty$ have the same sign and thus $\boldsymbol{\varepsilon}_\infty$ is either positive or negative definite.

To exclude the pathological case of a negative definite optical response $\boldsymbol{\varepsilon}_\infty$, we can consider a wave propagation problem in a stratified space with different signs on $\boldsymbol{\varepsilon}_\infty$, see Appendix C for details. This problem is in general not well-posed. Using the fact that vacuum correspond to $\epsilon = \mu = 1$ and $\kappa = 0$, it is reasonable to choose a positive definite optical response. Physically, we can understand the problem by making an energy interpretation. In this case, the quantity $\boldsymbol{e}^T\boldsymbol{\varepsilon}_\infty\boldsymbol{e}$ is negative which implies that the energy flows in the opposite direction to the propagation of the wave. Finally, we observe that a singular optical response would make it impossible to update the field values in time.

Appendix C Stratified media with different sign in the optical response

In this appendix we present an argument that rules out the case of an negative definite optical response $\boldsymbol{\varepsilon}_\infty$ (isotropic, non-magnetic case).

Consider a plane wave generated in vacuum ($\epsilon = \mu = 1$). This wave impinges on a half-space $z > 0$ that is assumed to have a permittivity $\epsilon = \mu = -1$, see Figure 6.

The electric and magnetic fields $\boldsymbol{E} = E\hat{\boldsymbol{x}}$ and $\boldsymbol{H} = H\hat{\boldsymbol{y}}$ satisfy.

$$\begin{cases} \partial_t E = -\partial_z H \\ \partial_t H = -\partial_z E \end{cases} \quad z < 0$$

$$\begin{cases} \partial_t E = \partial_z H \\ \partial_t H = \partial_z E \end{cases} \quad z > 0$$

The vacuum split fields $e^{\pm} = E \pm H$ diagonalize these sets of equations [25]

$$\begin{cases} \partial_t e^{\pm} = \mp\partial_z e^{\pm} & z < 0 \\ \partial_t e^{\pm} = \pm\partial_z e^{\pm} & z > 0 \end{cases}$$

Assume that all space is source free at times $t \geq 0$, and that the values of the split fields at time $t = 0$ are $e_0^{\pm}(z)$. The solution to this problem is

$$\begin{aligned} e^+(z,t) &= e_0^+(z + \operatorname{sign}(z)t) \\ e^-(z,t) &= \begin{cases} e_0^-(z - \operatorname{sign}(z)t), & |z| \geq t \\ g(z - \operatorname{sign}(z)t), & |z| \leq t \end{cases} \end{aligned}$$

where g is an arbitrary odd function. Observe that the fields have to satisfy the boundary conditions of continuous tangential E, H fields or equivalently that $e^{\pm}$ are continuous. This gives the restriction

$$e_0^+(-t) = e_0^+(t)$$

on the initial conditions. The general initial-boundary value problem is hence not well-posed.

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