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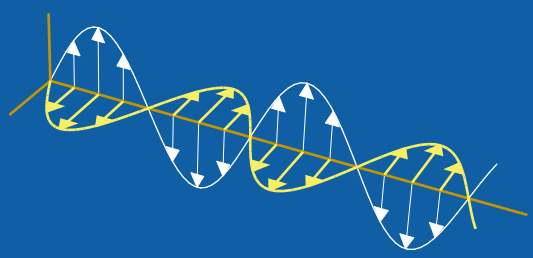

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Abstract

Transient wave propagation in a waveguide filled with an inhomogeneous dispersive medium is analyzed. The waveguide is inhomogeneous in the longitudinal direction, but is homogeneously filled in the transverse directions. The analysis is performed in the time domain and is based upon a wave splitting technique and the method of propagators. The propagator maps the exciting field from one position in the waveguide to the field at another position. This mapping is represented as time convolution integrals. The theory is exemplified by numerical examples where it is shown how wave trains of different shapes are propagating in a waveguide filled with a homogeneous dispersive medium.

1 Introduction

Transient wave propagation in waveguides has been analyzed in a number of papers, see, *e.g.* Refs 4–6, 18, 20–22. It is also described in the classic reference by Collin [2]. The similar transmission line problem solved in the time domain is analyzed by Lindell et al. [16]. A recent analysis of this problem in a homogeneous waveguide using the time domain technique is found in Ref. 13. In that paper the wave propagation in an empty waveguide is treated by a wave splitting technique in combination with a method referred to as the Green functions technique. The purpose with this paper is to extend these results to inhomogeneous waveguides. These extended results might have potential use in the solution of the inverse scattering problem.

The important results in Ref. 13 is the closed form expression for the wave splitting operator and the closed form expressions for the Green operator which is the operator that maps the field at one position to the field in a position further down in the waveguide. When these tools are available, direct and inverse scattering problems can be treated in a very systematic way. The approach has been used in a number of papers on direct and inverse scattering problems in the time-domain, see, *e.g.* Refs 3, 7, 12, 15, 17.

In the present paper, the method of propagators is applied to the problem of wave propagation in a waveguide filled with an inhomogeneous dispersive medium. The waveguide is inhomogeneous in the longitudinal direction, but is homogeneously filled in the transverse directions. The method of propagators is a generalization of the Green functions approach and have been used in earlier papers, see Refs 10, 11. Both the direct and inverse scattering problem can be treated by the technique presented, but in the current work only the direct problem is considered. The present paper indicates that the one-dimensional direct and inverse problems that have earlier been studied by the invariant imbedding method and the Green functions approach can also be solved in the waveguide case. The major difference between the purely one-dimensional case and the waveguide case is that the wave splitting is more complicated in the latter case. Otherwise the structure of the analysis is similar in the two cases.

The outline of the paper is as follows: In Section 2 the wave equation, the time-domain constitutive relations, and the decomposition of the field in longitudinal and

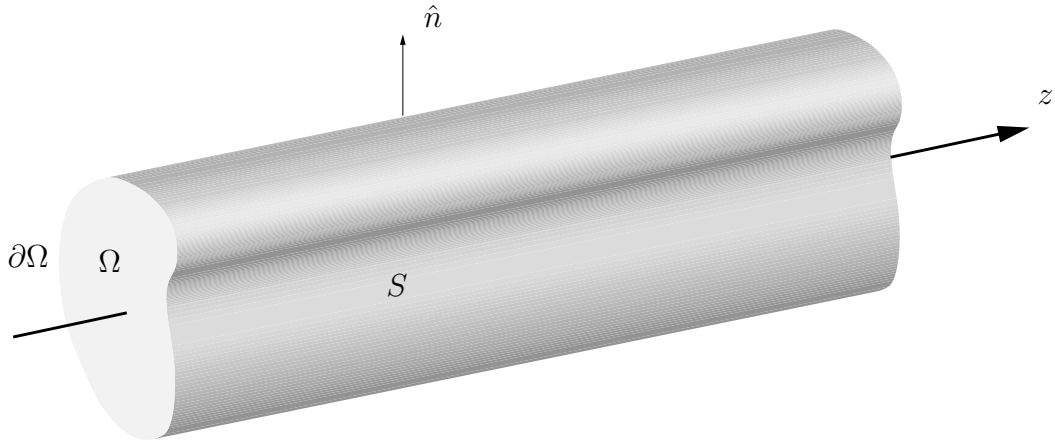

Figure 1: Geometry of the waveguide.

transverse components are presented. The problem is reduced to a one-dimensional problem by separation of variables in Section 3. The wave splitting and the dynamic equations for the split wave components are presented in Section 4. In Section 5 the propagators are defined, and the equations for the corresponding kernels are presented. A special case of the propagators is the reflection operator. The corresponding kernel, the reflection kernel, satisfies an imbedding equation, which is also given in this section. In the case of a homogeneous dispersive medium the equations for the propagators, the propagator kernels, and the reflection kernel simplify. These simplifications are derived in Section 6. This section also contains a note on the precursor field in a waveguide filled with a dispersive material. A numerical example is presented in Section 7 and some conclusions are given in Section 8.

2 Basic equations

The basic equations for the analysis of the propagation of transient waves is the Maxwell equations.

$$\begin{cases} \nabla \times \boldsymbol{E}(\boldsymbol{r},t) = -\dfrac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t} \\ \nabla \times \boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{J}(\boldsymbol{r},t) + \dfrac{\partial \boldsymbol{D}(\boldsymbol{r},t)}{\partial t} \end{cases} \tag{2.1}$$

All fields in this paper are assumed to be quiescent before a fixed time. This property guarantees that all fields vanish at $t \to -\infty$.

The waveguide is assumed to be filled with a dispersive medium that can be stratified wrt z, which is the coordinate along the waveguide, see Figure 1. The appropriate constitutive relations used in this paper are therefore

$$\begin{cases} \boldsymbol{D}(\boldsymbol{r},t) = \epsilon_0 \Big\{ \epsilon(z)\boldsymbol{E}(\boldsymbol{r},t) + \left(\chi(z,\cdot) * \boldsymbol{E}(\boldsymbol{r},\cdot)\right)(t) \Big\} \\ \boldsymbol{B}(\boldsymbol{r},t) = \mu_0 \boldsymbol{H}(\boldsymbol{r},t) \end{cases} \tag{2.2}$$

where the convolution in the time variable is denoted by $*$, *i.e.*,

$$(\chi(z,\cdot) * \boldsymbol{E}(\boldsymbol{r},\cdot))(t) = \int_{-\infty}^{t} \chi(z,t-t')\boldsymbol{E}(\boldsymbol{r},t')\,dt'$$

The permittivity and the permeability of the medium are denoted $\epsilon_0\epsilon(z)$ and μ_0, respectively. For simplicity, $\epsilon(z)$ and $\chi(z,t)$ are assumed to be continuously differentiable functions of z everywhere. Furthermore, $\chi(z,t)$ is assumed to be continuously differentiable as a function of $t \geq 0$. It is convenient to extend the domain of definition of $\chi(z,t)$ to the region $t < 0$ by letting $\chi(z,t) = 0$, when $t < 0$. The phase velocity $c(z)$ and the wave impedance $\eta(z)$ are

$$c(z) = \frac{1}{\sqrt{\epsilon_0\epsilon(z)\mu_0}}, \qquad \eta(z) = \sqrt{\frac{\mu_0}{\epsilon_0\epsilon(z)}}$$

respectively.

The curl of the Maxwell equations, (2.1), and the constitutive relations, (2.2), imply in a source-free region

$$\begin{cases} \nabla^2\boldsymbol{E}(\boldsymbol{r},t) - \nabla(\nabla\cdot\boldsymbol{E}(\boldsymbol{r},t)) = -\nabla\times(\nabla\times\boldsymbol{E}(\boldsymbol{r},t)) \\ \qquad\qquad = \dfrac{1}{c_0^2}\dfrac{\partial^2}{\partial t^2}\Big\{\epsilon(z)\boldsymbol{E}(\boldsymbol{r},t) + (\chi(z,\cdot) * \boldsymbol{E}(\boldsymbol{r},\cdot))(t)\Big\} \\ \nabla^2\boldsymbol{H}(\boldsymbol{r},t) = -\nabla\times(\nabla\times\boldsymbol{H}(\boldsymbol{r},t)) = -\dfrac{\partial}{\partial t}\nabla\times\boldsymbol{D}(\boldsymbol{r},t) \end{cases} \tag{2.3}$$

Furthermore, the divergence of the first equation in (2.3) implies

$$\begin{aligned} \epsilon(z)\nabla\cdot\boldsymbol{E}(\boldsymbol{r},t) &+ (\chi(z,\cdot) * \nabla\cdot\boldsymbol{E}(\boldsymbol{r},\cdot))(t) \\ &= -\epsilon'(z)E_z(\boldsymbol{r},t) - (\partial_z\chi(z,\cdot) * E_z(\boldsymbol{r},\cdot))(t) \end{aligned}$$

This is a Volterra equation of the second kind in $\nabla\cdot\boldsymbol{E}$, with solution

$$\begin{aligned} \nabla\cdot\boldsymbol{E}(\boldsymbol{r},t) = &-\frac{\epsilon'(z)}{\epsilon(z)}E_z(\boldsymbol{r},t) - \frac{1}{\epsilon(z)}(\partial_z\chi(z,\cdot) * E_z(\boldsymbol{r},\cdot))(t) \\ &- \frac{\epsilon'(z)}{\epsilon(z)}(\Psi(z,\cdot) * E_z(\boldsymbol{r},\cdot))(t) - \frac{1}{\epsilon(z)}[\Psi(z,\cdot) * (\partial_z\chi(z,\cdot) * E_z(\boldsymbol{r},\cdot))(\cdot)](t) \end{aligned}$$

where $\Psi(z,t)$ is the resolvent kernel of the kernel $\chi(z,t)$ defined by the resolvent equation.

$$\epsilon(z)\Psi(z,t) + \chi(z,t) + \int_0^t \chi(z,t-t')\Psi(z,t')\,dt' = 0 \tag{2.4}$$

The divergence of the electric field can therefore be expressed in terms of the z-component of the electric field E_z.

2.1 The z-component

The z-component of the first equation in (2.3) can now readily be expressed as an equation in just the z-component of the electric field E_z. A similar treatment can be performed on the z-component of the magnetic field H_z. The results are

$$\begin{cases} \nabla^2 E_z(\boldsymbol{r},t) - \dfrac{1}{c^2(z)}\dfrac{\partial^2}{\partial t^2}E_z(\boldsymbol{r},t) + A(z)\dfrac{\partial}{\partial z}E_z(\boldsymbol{r},t) + B(z)\dfrac{\partial}{\partial t}E_z(\boldsymbol{r},t) \\ \qquad + C(z)E_z(\boldsymbol{r},t) + \left(D(z,\cdot) * E_z(\boldsymbol{r},\cdot)\right)(t) + \left(F(z,\cdot) * \partial_z E_z(\boldsymbol{r},\cdot)\right)(t) = 0 \\ \nabla^2 H_z(\boldsymbol{r},t) - \dfrac{1}{c^2(z)}\dfrac{\partial^2}{\partial t^2}H_z(\boldsymbol{r},t) + B(z)\dfrac{\partial}{\partial t}H_z(\boldsymbol{r},t) \\ \qquad + M(z)H_z(\boldsymbol{r},t) + \left(N(z,\cdot) * H_z(\boldsymbol{r},\cdot)\right)(t) = 0 \end{cases} \tag{2.5}$$

where the coefficients in these PDE's are

$$\begin{cases} A(z) = \dfrac{\epsilon'(z)}{\epsilon(z)} \\ B(z) = -\dfrac{1}{c_0^2}\chi(z,0^+) \\ C(z) = \dfrac{\mathrm{d}}{\mathrm{d}z}A(z) - \dfrac{1}{c_0^2}\dfrac{\partial}{\partial t}\chi(z,0^+) \\ D(z,t) = -\dfrac{1}{c_0^2}\dfrac{\partial^2}{\partial t^2}\chi(z,t) + \dfrac{\partial}{\partial z}F(z,t) \\ F(z,t) = A(z)\Psi(z,t) + \dfrac{\partial_z\chi(z,t)}{\epsilon(z)} + \dfrac{1}{\epsilon(z)}\left(\Psi(z,\cdot) * \partial_z\chi(z,\cdot)\right)(t) \end{cases}$$

and

$$\begin{cases} M(z) = -\dfrac{1}{c_0^2}\dfrac{\partial}{\partial t}\chi(z,0^+) \\ N(z,t) = -\dfrac{1}{c_0^2}\dfrac{\partial^2}{\partial t^2}\chi(z,t) \end{cases}$$

2.2 The transverse components

Decompose the vector fields into their longitudinal and transverse components, see, *e.g.* Ref. 13, and apply to the source-free Maxwell equations (2.1). The result for the z-component is

$$\begin{cases} \hat{z}\cdot\left(\nabla_T \times \boldsymbol{E}_T(\boldsymbol{r},t)\right) = -\mu_0\dfrac{\partial}{\partial t}H_z(\boldsymbol{r},t) \\ \hat{z}\cdot\left(\nabla_T \times \boldsymbol{H}_T(\boldsymbol{r},t)\right) = \epsilon_0\dfrac{\partial}{\partial t}\left\{\epsilon(z)E_z(\boldsymbol{r},t) + \left(\chi(z,\cdot) * E_z(\boldsymbol{r},\cdot)\right)(t)\right\} \end{cases} \tag{2.6}$$

and for the transverse components the result is

$$\begin{cases} \mu_0 \dfrac{\partial}{\partial t}\boldsymbol{H}_T(\boldsymbol{r},t) + \hat{z} \times \dfrac{\partial}{\partial z}\boldsymbol{E}_T(\boldsymbol{r},t) = \hat{z} \times \nabla_T E_z(\boldsymbol{r},t) \\ \epsilon_0 \dfrac{\partial}{\partial t}\Big\{\epsilon(z)\boldsymbol{E}_T(\boldsymbol{r},t) \\ \qquad + \left(\chi(z,\cdot) * \boldsymbol{E}_T(\boldsymbol{r},\cdot)\right)(t)\Big\} - \hat{z} \times \dfrac{\partial}{\partial z}\boldsymbol{H}_T(\boldsymbol{r},t) = -\hat{z} \times \nabla_T H_z(\boldsymbol{r},t) \end{cases} \tag{2.7}$$

This is a general decomposition of the Maxwell equations in terms of the transverse components of the electric and the magnetic fields. From these equations it is seen that the z-components of the electric and the magnetic fields act as sources for the transverse parts of the fields.

Elimination of the transverse magnetic field $\boldsymbol{H}_T$ and the transverse electric field $\boldsymbol{E}_T$, respectively, gives the following second order PDE:s:

$$\begin{cases} \dfrac{\partial^2 \boldsymbol{E}_T(\boldsymbol{r},t)}{\partial z^2} - \dfrac{1}{c_0^2}\dfrac{\partial^2}{\partial t^2}\left[\epsilon(z)\boldsymbol{E}_T(\boldsymbol{r},t) + \left(\chi(z,\cdot) * \boldsymbol{E}_T(\boldsymbol{r},\cdot)\right)(t)\right] \\ \qquad = \dfrac{\partial}{\partial t}\left(\hat{z} \times \nabla_T \mu_0 H_z(\boldsymbol{r},t)\right) + \dfrac{\partial}{\partial z}\nabla_T E_z(\boldsymbol{r},t) \\ \dfrac{\partial^2 \boldsymbol{H}_T(\boldsymbol{r},t)}{\partial z^2} - \dfrac{\epsilon(z)}{c_0^2}\dfrac{\partial^2 \boldsymbol{H}_T(\boldsymbol{r},t)}{\partial t^2} - A(z)\dfrac{\partial \boldsymbol{H}_T(\boldsymbol{r},t)}{\partial z} \\ \qquad + \epsilon(z)\dfrac{\partial}{\partial z}\left(\dfrac{(\Psi(z,\cdot)}{\epsilon(z)} * \partial_z \boldsymbol{H}_T(\boldsymbol{r},\cdot)\right)(t) \\ \qquad = -\epsilon(z)\dfrac{\partial}{\partial t}\left(\hat{z} \times \nabla_T \epsilon_0 E_z(\boldsymbol{r},t)\right) + \epsilon(z)\dfrac{\partial}{\partial z}\left(\dfrac{1}{\epsilon(z)}\nabla_T H_z(\boldsymbol{r},t)\right) \\ \qquad + \epsilon(z)\dfrac{\partial}{\partial z}\left(\dfrac{(\Psi(z,\cdot)}{\epsilon(z)} * \nabla_T H_z(\boldsymbol{r},\cdot)\right)(t) \end{cases}$$

3 Separation of variables

The boundary conditions on the perfectly conducting wall of the waveguide are

$$\begin{cases} \hat{n} \times \boldsymbol{E} = \boldsymbol{0}, \\ \hat{n} \cdot \boldsymbol{H} = 0, \end{cases} \qquad \boldsymbol{r} \in S$$

Since $\hat{n}$ is independent of z, these boundary conditions are equivalent to

$$\begin{cases} E_z = 0, \\ \dfrac{\partial H_z}{\partial n} = 0, \end{cases} \qquad \boldsymbol{r} \in S$$

For a waveguide with perfectly conducting walls, the wave propagation phenomena in the waveguide separate into the well-known two different classes—the TE- and the TM-modes. This is completely analogous to the fixed frequency case.

The TEM-modes are excluded in this presentation due to the absence of an inner conductor in the waveguide.

The ansatz for z-components in the TM- and the TE-cases follows the standard separation of variable procedure.

$$\begin{cases} E_z(\boldsymbol{r},t) = v(\boldsymbol{\rho})a(z,t) \\ H_z(\boldsymbol{r},t) = 0 \end{cases} \qquad \text{(TM-case)}$$

and

$$\begin{cases} E_z(\boldsymbol{r},t) = 0 \\ H_z(\boldsymbol{r},t) = w(\boldsymbol{\rho})b(z,t) \end{cases} \qquad \text{(TE-case)}$$

where the components $E_z(\boldsymbol{r},t)$ and $H_z(\boldsymbol{r},t)$ satisfy (2.5). The functions v and w determine the transverse behavior of the fields and satisfy an eigenvalue problem. Separation of variables determines these eigenvalue problems, which are

$$\begin{cases} \nabla_T^2 v(\boldsymbol{\rho}) + \lambda^2 v(\boldsymbol{\rho}) = 0, & \boldsymbol{\rho} \in \Omega \\ v(\boldsymbol{\rho}) = 0, & \boldsymbol{\rho} \in \partial\Omega \end{cases} \tag{3.1}$$

and

$$\begin{cases} \nabla_T^2 w(\boldsymbol{\rho}) + \lambda^2 w(\boldsymbol{\rho}) = 0, & \boldsymbol{\rho} \in \Omega \\ \hat{n} \cdot \nabla_T w(\boldsymbol{\rho}) = 0, & \boldsymbol{\rho} \in \partial\Omega \end{cases} \tag{3.2}$$

The positive real constant λ is here the eigenvalue for the waveguide listed with due regard to multiplicity.

$$\lambda = \lambda_n, \qquad n = 1, 2, 3, \ldots$$

The eigenfunctions are orthonormal when integrated over the cross section. All fields depend on the index n, but to avoid complicated notation this index is often omitted in this paper. Furthermore, the same notation for the eigenvalue of the Dirichlet (TM) and the Neumann problem (TE) is used. From the context it is always obvious what problem λ refers to. The TEM-case corresponds to the eigenvalue $\lambda = 0$. In this paper it is assumed that no such mode exists and thus the eigenfunctions form a complete orthogonal set in Ω [19, p. 138].

The functions $a(z,t)$ and $b(z,t)$ determine the wave propagation along the waveguide and satisfy a generalized one-dimensional Klein-Gordon equation.

$$\begin{cases} \dfrac{\partial^2 a(z,t)}{\partial z^2} - \dfrac{1}{c^2(z)}\dfrac{\partial^2 a(z,t)}{\partial t^2} + A(z)\dfrac{\partial a(z,t)}{\partial z} + B(z)\dfrac{\partial a(z,t)}{\partial t} \\ \qquad + \left(C(z) - \lambda^2\right) a(z,t) + \left(D(z,\cdot) * a(z,\cdot)\right)(t) + \left(F(z,\cdot) * \partial_z a(z,\cdot)\right)(t) = 0 \\ \dfrac{\partial^2 b(z,t)}{\partial z^2} - \dfrac{1}{c^2(z)}\dfrac{\partial^2 b(z,t)}{\partial t^2} + B(z)\dfrac{\partial b(z,t)}{\partial t} \\ \qquad + \left(M(z) - \lambda^2\right) b(z,t) + \left(N(z,\cdot) * b(z,\cdot)\right)(t) = 0 \end{cases} \tag{3.3}$$

The transverse components have the form, see (2.6) and (2.7)

$$\begin{cases} \boldsymbol{E}_T(\boldsymbol{r},t) = \nabla_T v(\boldsymbol{\rho})\psi_1(z,t) \\ \eta_0 \boldsymbol{H}_T(\boldsymbol{r},t) = [\hat{z} \times \nabla_T v(\boldsymbol{\rho})]\, \phi_1(z,t) \end{cases} \quad \text{(TM-case)}$$
$$\begin{cases} \boldsymbol{E}_T(\boldsymbol{r},t) = -\,[\hat{z} \times \nabla_T w(\boldsymbol{\rho})]\, \phi_2(z,t) \\ \eta_0 \boldsymbol{H}_T(\boldsymbol{r},t) = \nabla_T w(\boldsymbol{\rho})\psi_2(z,t) \end{cases} \quad \text{(TE-case)} \tag{3.4}$$

where η_0 is the wave impedance of vacuum. The functions $\phi_i(z,t)$ and $\psi_i(z,t)$, $i = 1,2$ are related to the functions $a(z,t)$ and $b(z,t)$, which determine the propagation properties of the z-components of the fields. Insert the expressions of the transverse fields into the equations (2.6) and (2.7). The results after some algebra and the use of (3.1) and (3.2) are

$$\begin{cases} \lambda^2 \phi_1(z,t) = -\dfrac{1}{c_0}\left\{\epsilon(z)\dfrac{\partial a(z,t)}{\partial t} + \chi(z,0^+)a(z,t) + \left(\dfrac{\partial \chi(z,\cdot)}{\partial t} * a(z,\cdot)\right)(t)\right\} \\ \lambda^2 \left\{\epsilon(z)\psi_1(z,t) + (\chi(z,\cdot) * \psi_1(z,\cdot))\,(t)\right\} = \dfrac{\partial}{\partial z}\left\{\epsilon(z)a(z,t) + (\chi(z,\cdot) * a(z,\cdot))\,(t)\right\} \end{cases}$$

and

$$\begin{cases} \lambda^2 \phi_2(z,t) = -\dfrac{1}{c_0}\dfrac{\partial b(z,t)}{\partial t} \\ \lambda^2 \psi_2(z,t) = \dfrac{\partial}{\partial z} b(z,t) \end{cases}$$

The propagation properties of the transverse components, $\phi_i(z,t)$ and $\psi_i(z,t)$, $i = 1,2$, are therefore determined by the corresponding properties of the longitudinal components, $a(z,t)$ and $b(z,t)$. Note that the function $\psi_1(z,t)$ can be explicitly written down in terms of the resolvent $\Psi(z,t)$ of the susceptibility kernel $\chi(z,t)$, see (2.4).

The power flux can now easily be expressed in the functions $\phi_i(z,t)$ and $\psi_i(z,t)$, $i = 1,2$. Straightforward calculations give

$$\boldsymbol{S} \cdot \hat{z} = (\boldsymbol{E} \times \boldsymbol{H}) \cdot \hat{z} = (\boldsymbol{E}_T \times \boldsymbol{H}_T) \cdot \hat{z} = \begin{cases} \frac{1}{\eta_0}|\nabla_T v|^2 \psi_1 \phi_1 & \text{(TM-case)} \\ \frac{1}{\eta_0}|\nabla_T w|^2 \psi_2 \phi_2 & \text{(TE-case)} \end{cases}$$

Integration over the cross section and use of the boundary conditions of v or w give

$$P(z,t) = \iint_\Omega \boldsymbol{S} \cdot \hat{z}\, dxdy = \begin{cases} \frac{\lambda^2}{\eta_0}\psi_1(z,t)\phi_1(z,t) & \text{(TM-case)} \\ \frac{\lambda^2}{\eta_0}\psi_2(z,t)\phi_2(z,t) & \text{(TE-case)} \end{cases}$$

The sign power is therefore determined by the sign of the product $\psi_i(z,t)\phi_i(z,t)$, $i = 1,2$.

4 Wave splitting

The starting point in this method is the generalized one-dimensional Klein-Gordon equation (3.3).

$$\frac{\partial^2 a(z,t)}{\partial z^2} - \frac{1}{c^2(z)}\frac{\partial^2 a(z,t)}{\partial t^2} + A(z)\frac{\partial a(z,t)}{\partial z} + B(z)\frac{\partial a(z,t)}{\partial t} \\ + \left(C(z) - \lambda^2\right) a(z,t) + \left(D(z,\cdot) * a(z,\cdot)\right)(t) + \left(F(z,\cdot) * \partial_z a(z,\cdot)\right)(t) = 0$$

It suffices to analyze the TM-case, since the TE-case satisfies a simpler equation and formally the analysis follows the same arguments. The wave equation is conveniently rewritten as a system of first order equations

$$\frac{\partial}{\partial z}\begin{pmatrix} a \\ \partial_z a \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{c^2(z)}\partial_t^2 - B(z)\partial_t - C(z) + \lambda^2 - D(z,\cdot)* & -A(z) - F(z,\cdot)* \end{pmatrix}\begin{pmatrix} a \\ \partial_z a \end{pmatrix} \\ = \mathcal{T}\begin{pmatrix} a \\ \partial_z a \end{pmatrix} \tag{4.1}$$

The matrix-valued operator $\mathcal{T}$ is equivalent to the generalized one-dimensional Klein-Gordon equation (3.3).

The wave splitting transformation used in this paper is defined by [8]

$$a^{\pm}(z,t) = \frac{1}{2}\left[a(z,t) \mp c(z)(\mathcal{K}\partial_z a(z,\cdot))(t)\right]$$

where the operator $\mathcal{K}$, which in general depends on z, has the integral representation

$$(\mathcal{K}f)(t) = \int_{-\infty}^{t} K(z,t-t')f(t')\,dt'$$

where the kernel $K(z,t)$ is

$$K(z,t) = H(t)J_0(c(z)\lambda t)$$

and $H(t)$ is Heaviside's step function. Formally, the splitting can be written in matrix notation as

$$\begin{pmatrix} a^+ \\ a^- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -c(z)\mathcal{K} \\ 1 & c(z)\mathcal{K} \end{pmatrix}\begin{pmatrix} a \\ \partial_z a \end{pmatrix} = \mathcal{S}\begin{pmatrix} a \\ \partial_z a \end{pmatrix} \tag{4.2}$$

where the matrix-valued operator $\mathcal{S}$ denotes the wave splitting.

The operator $\mathcal{K}$ has an inverse $\mathcal{K}^{-1}$, with

$$\mathcal{K}\mathcal{K}^{-1}f = f, \qquad \mathcal{K}^{-1}\mathcal{K}f = f$$

The fields a and $\partial_z a$ are therefore expressed in $a^{\pm}$ as

$$\begin{pmatrix} a \\ \partial_z a \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\frac{1}{c(z)}\mathcal{K}^{-1} & \frac{1}{c(z)}\mathcal{K}^{-1} \end{pmatrix}\begin{pmatrix} a^+ \\ a^- \end{pmatrix} = \mathcal{S}^{-1}\begin{pmatrix} a^+ \\ a^- \end{pmatrix} \tag{4.3}$$

The inverse operator $\mathcal{K}^{-1}$ has an explicit integral representation, see Ref. 13.

$$(\mathcal{K}^{-1}f)(t) = \frac{\partial f}{\partial t} + \left(L(\cdot) * f(\cdot)\right)(t) \tag{4.4}$$

where the kernel $L(z,t)$ is

$$L(z,t) = H(t)\frac{c(z)\lambda J_1(c(z)\lambda t)}{t} \tag{4.5}$$

Another useful identity is [13]

$$\mathcal{K}\frac{\partial f^2}{\partial t^2} = \mathcal{K}^{-1}f - c^2(z)\lambda^2\mathcal{K}f \tag{4.6}$$

The new fields $a^{\pm}(z,t)$ satisfy a system of first order equations, which is obtained from (4.2), (4.3) and (4.1).

$$\frac{\partial}{\partial z}\begin{pmatrix} a^+ \\ a^- \end{pmatrix} = \left\{\mathcal{S}\mathcal{T}\mathcal{S}^{-1} + \frac{\mathrm{d}\mathcal{S}}{\mathrm{d}z}\mathcal{S}^{-1}\right\}\begin{pmatrix} a^+ \\ a^- \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}\begin{pmatrix} a^+ \\ a^- \end{pmatrix} \tag{4.7}$$

The operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are

$$\begin{cases} \mathcal{A} = -\dfrac{1}{c(z)}\dfrac{\partial}{\partial t} - \dfrac{3}{4}A(z) + \dfrac{c(z)}{2}B(z) + \alpha(z,\cdot)* \\ \mathcal{B} = \dfrac{3}{4}A(z) + \dfrac{c(z)}{2}B(z) + \beta(z,\cdot)* \\ \mathcal{C} = \dfrac{3}{4}A(z) - \dfrac{c(z)}{2}B(z) + \gamma(z,\cdot)* \\ \mathcal{D} = \dfrac{1}{c(z)}\dfrac{\partial}{\partial t} - \dfrac{3}{4}A(z) - \dfrac{c(z)}{2}B(z) + \delta(z,\cdot)* \end{cases}$$

where the explicit form of the kernels $\alpha(z,t)$, $\beta(z,t)$, $\gamma(z,t)$, and $\delta(z,t)$ are $(t>0)$

$$\begin{aligned} \alpha(z,t) = &-\frac{1}{c(z)}L(z,t) + \frac{c(z)B(z)}{2}\frac{\partial}{\partial t}K(z,t) + \frac{c(z)C(z)}{2}K(z,t) \\ &+ \frac{c(z)}{2}\left(K(z,\cdot) * D(z,\cdot)\right)(t) - \frac{1}{2}F(z,t) \\ &+ \frac{c^2(z)\lambda^2 A(z)}{4}\left(K(z,\cdot) * K(z,\cdot)\right)(t) \end{aligned}$$

$$\begin{aligned} \beta(z,t) = &\frac{c(z)B(z)}{2}\frac{\partial}{\partial t}K(z,t) + \frac{c(z)C(z)}{2}K(z,t) \\ &+ \frac{c(z)}{2}\left(K(z,\cdot) * D(z,\cdot)\right)(t) + \frac{1}{2}F(z,t) \\ &- \frac{c^2(z)\lambda^2 A(z)}{4}\left(K(z,\cdot) * K(z,\cdot)\right)(t) \end{aligned}$$

$$
\begin{aligned}
\gamma(z,t) = &-\frac{c(z)B(z)}{2}\frac{\partial}{\partial t}K(z,t) - \frac{c(z)C(z)}{2}K(z,t) \\
&-\frac{c(z)}{2}\left(K(z,\cdot) * D(z,\cdot)\right)(t) + \frac{1}{2}F(z,t) \\
&-\frac{c^2(z)\lambda^2 A(z)}{4}\left(K(z,\cdot) * K(z,\cdot)\right)(t) \\
\delta(z,t) = &\frac{1}{c(z)}L(z,t) - \frac{c(z)B(z)}{2}\frac{\partial}{\partial t}K(z,t) - \frac{c(z)C(z)}{2}K(z,t) \\
&-\frac{c(z)}{2}\left(K(z,\cdot) * D(z,\cdot)\right)(t) - \frac{1}{2}F(z,t) \\
&+\frac{c^2(z)\lambda^2 A(z)}{4}\left(K(z,\cdot) * K(z,\cdot)\right)(t)
\end{aligned}
$$

5 The propagators

The propagators map the field $a^+(z,t)$ from a point z_0 to a point z. When all sources are located to the left of a point $z_p < \min(z_0, z)$ the propagators are defined by

$$
\begin{cases}
a^+(z, t+\tau(z,z_0)) = \mathcal{P}^+(z,z_0)a^+(z_0,t) \\
a^-(z, t+\tau(z,z_0)) = \mathcal{P}^-(z,z_0)a^+(z_0,t)
\end{cases}
\tag{5.1}
$$

where the time delay factor is

$$
\tau(z,z_0) = \int_{z_0}^{z} \frac{dz'}{c(z')}
$$

The propagators used in this paper are a generalization of the Green operators, $\mathcal{G}^\pm(z)$ used in Ref. 13. The connection between the Green operators and the propagators is $\mathcal{G}^\pm(z) = \mathcal{P}^\pm(z,0)$. Normally, it is assumed that $z_0 < z$, so that the field is propagated forward in both space and time, but the definition of the propagators and the equations that follow are valid also for the case $z < z_0$. In the definition of $\mathcal{P}^\pm$ wave front time is used. In wave front time at a point z, the time is zero, $t = 0$, when the wavefront passes that point.

From the definition it follows that the propagators are independent of the properties in the region $(-\infty, \min(z_0, z))$. It is also seen that they satisfy the following properties

$$
\begin{cases}
\mathcal{P}^+(z,z_0) = \mathcal{P}^+(z,z_1)\mathcal{P}^+(z_1,z_0) \\
\mathcal{P}^-(z,z_0) = \mathcal{P}^-(z,z_1)\mathcal{P}^+(z_1,z_0)
\end{cases}
$$

for all z_1. Since $\mathcal{P}^+(z,z)$ is the identity operator, it follows that the operator $\mathcal{P}^+(z_0,z)$ is the inverse of the propagator $\mathcal{P}^+(z,z_0)$.

It is convenient to introduce the reflection operator as $\mathcal{R}(z) = \mathcal{P}^-(z,z)$, *i.e.*,

$$
a^-(z,t) = \mathcal{R}(z)a^+(z,t)
$$

The explicit integral representations for the propagators read

$$a^+(z,t+\tau(z,z_0)) = \Gamma(z,z_0)a^+(z_0,t) + \left(P^+(z,z_0,\cdot) * a^+(z_0,\cdot)\right)(t) \tag{5.2}$$
$$a^-(z,t+\tau(z,z_0)) = \left(P^-(z,z_0,\cdot) * a^+(z_0,\cdot)\right)(t)$$
$$a^-(z,t) = \left(R(z,\cdot) * a^+(z,\cdot)\right)(t) \tag{5.3}$$

where the wave front factor $\Gamma(z,z_0)$ is defined as

$$\Gamma(z,z_0) = \exp\left\{\int_{z_0}^{z}\left(-\frac{3}{4}A(z') + \frac{c(z')}{2}B(z')\right)dz'\right\}$$

The boundary condition for the kernel $P^+(z,z_0,t)$ is

$$P^+(z_0,z_0,t) = 0$$

and the equations for the propagator kernels $P^+(z,z_0,t)$ are

$$\begin{cases} \dfrac{\partial P^+(z,z_0,t)}{\partial z} = \Gamma(z,z_0)\alpha(z,t) \\ \quad + \left(\alpha(z,\cdot) * P^+(z,z_0,\cdot)\right)(t) + \left(\beta(z,\cdot) * P^-(z,z_0,\cdot)\right)(t) \\ \quad - \left(\dfrac{3A(z)}{4} - \dfrac{c(z)B(z)}{2}\right)P^+(z,z_0,t) + \left(\dfrac{3A(z)}{4} + \dfrac{c(z)B(z)}{2}\right)P^-(z,z_0,t) \\ \dfrac{\partial P^-(z,z_0,t)}{\partial z} - \dfrac{2}{c(z)}\dfrac{\partial P^-(z,z_0,t)}{\partial t} = \Gamma(z,z_0)\gamma(z,t) \\ \quad + \left(\gamma(z,\cdot) * P^+(z,z_0,\cdot)\right)(t) + \left(\delta(z,\cdot) * P^-(z,z_0,\cdot)\right)(t) \\ \quad + \left(\dfrac{3A(z)}{4} - \dfrac{c(z)B(z)}{2}\right)P^+(z,z_0,t) - \left(\dfrac{3A(z)}{4} + \dfrac{c(z)B(z)}{2}\right)P^-(z,z_0,t) \end{cases} \tag{5.4}$$

The initial condition for $P^-(z,z_0,t)$ is

$$P^-(z,z_0,0) = -\Gamma(z,z_0)\left(\frac{3c(z)A(z)}{8} - \frac{c^2(z)B(z)}{4}\right)$$

By differentiation wrt z_0 another complete set of equations for the kernels are obtained. These equations are related to the imbedding equations for the reflection and transmission kernels in the imbedding theory, see Ref. 14.

Formal equations for the propagators $\mathcal{P}^\pm$ are obtained from the dynamic equation, (4.7), and the definition of the propagators, (5.1)

$$\frac{\partial}{\partial z_1}\begin{pmatrix}\mathcal{P}^+\\ \mathcal{P}^-\end{pmatrix}(z_1,z_0) = \begin{pmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{pmatrix}(z_1)\begin{pmatrix}\mathcal{P}^+\\ \mathcal{P}^-\end{pmatrix}(z_1,z_0) + \frac{1}{c(z_1)}\begin{pmatrix}\frac{\partial}{\partial t} & 0\\ 0 & \frac{\partial}{\partial t}\end{pmatrix}\begin{pmatrix}\mathcal{P}^+\\ \mathcal{P}^-\end{pmatrix}(z_1,z_0)$$

Since $\mathcal{P}^-(z_1,z_0) = \mathcal{R}(z_1)\mathcal{P}^+(z_1,z_0)$ the equation for $\mathcal{P}^+$ read

$$\frac{\partial \mathcal{P}^+(z_1,z_0)}{\partial z_1} = \left(\mathcal{A}(z_1) + \frac{1}{c(z_1)}\frac{\partial}{\partial t} + \mathcal{B}(z_1)\mathcal{R}(z_1)\right)\mathcal{P}^+(z_1,z_0)$$

The formal solution to this operator equation is

$$\mathcal{P}^{+}(z_1, z_0) = \exp\left(\int_{z_0}^{z_1} \mathcal{A}(z) + \frac{1}{c(z_1)}\frac{\partial}{\partial t} + \mathcal{B}(z)\mathcal{R}(z)\, dz\right) \tag{5.5}$$

where the exponential of the operators is defined by its power series expansion. Numerically it is more efficient to solve Eqs. (5.4) than to use the power series expansion of the formal solution. The reason for this is that the equation for the reflection kernel (5.6) is more time consuming to solve than the equations for the propagator kernels (5.4).

5.1 Imbedding equation

For completeness, in this subsection the imbedding equation of the reflection kernel $R(z,t) = P^{-}(z,z,t)$ is given. The derivation of this equation follows from a similar analysis as the one presented in Ref. 13. The result in this case is

$$\begin{aligned}\frac{\partial R(z,t)}{\partial z} - \frac{2}{c(z)}\frac{\partial R(z,t)}{\partial t} &= \gamma(z,t) + \left((\delta(z,\cdot) - \alpha(z,\cdot)) * R(z,\cdot)\right)(t) \\ &- c(z)B(z)R(z,t) - \left(\beta(z,\cdot) * R(z,\cdot) * R(z,\cdot)\right)(t) \\ &- \left(\frac{3}{4}A(z) + \frac{c(z)}{2}B(z)\right)\left(R(z,\cdot) * R(z,\cdot)\right)(t)\end{aligned} \tag{5.6}$$

The augmented initial condition for $R(z,t)$ is

$$R(z,0^{+}) = -\frac{3c(z)}{8}A(z) + \frac{c^2(z)}{4}B(z)$$

6 Waveguide with homogeneous medium

The special case of a waveguide filled with a homogeneous medium is treated in this section. For a homogeneous medium $\chi(z,t) = \chi(t)$ and $c(z) = c$ and the coefficients in the PDE, (2.5), simplify to

$$\begin{cases} A(z) = F(z,t) = 0 \\ B(z) = B = -\dfrac{1}{c_0^2}\chi(0^{+}) \\ C(z) = M(z) = -\dfrac{1}{c_0^2}\dfrac{\partial}{\partial t}\chi(0^{+}) \\ D(z,t) = N(z,t) = -\dfrac{1}{c_0^2}\dfrac{\partial^2}{\partial t^2}\chi(t) \end{cases}$$

In Ref. 13, $B = C = D = M = N = 0$. Similarly, the explicit form of the kernels $\alpha(z,t)$, $\beta(z,t)$, $\gamma(z,t)$, and $\delta(z,t)$ simplify to

$$\begin{cases} \alpha(t) = -\,\delta(t) = -\dfrac{1}{c}L(t) - \dfrac{c}{2c_0^2}\dfrac{\partial^2}{\partial t^2}\left(K * \chi\right)(t) \\ \beta(t) = -\,\gamma(t) = -\dfrac{c}{2c_0^2}\dfrac{\partial^2}{\partial t^2}\left(K * \chi\right)(t) \end{cases}$$

Note that the functions α, β, γ, K, L and χ now are only functions of time t.

6.1 The reflection kernel

The reflection kernel in a homogeneous waveguide is independent of the imbedding parameter z. Thus $R(z,t) = R(t)$ and the imbedding equation, (5.6), simplifies to

$$-\frac{2}{c}\frac{\partial R(t)}{\partial t} = \gamma(t) - \left(2\alpha * R\right)(t) - \left(\beta * R * R\right)(t) - \frac{c}{2}B\left(R * R\right)(t) - cBR(t) \quad (6.1)$$

and the initial condition for $R(t)$ is

$$R(0^+) = -\frac{c^2}{4c_0^2}\chi(0^+) = -\frac{1}{4\epsilon}\chi(0^+) \quad (6.2)$$

Integration of equation (6.1) in t from 0 to t results in

$$\begin{aligned} &\frac{4c_0^2}{c^2}R(t) + \frac{\partial}{\partial t}\left(K * \chi\right)(t) + \frac{4c_0^2}{c^2}\left(P * R\right)(t) \\ &\quad + 2\chi * \left[\frac{\partial}{\partial t}\left(K * R\right)\right](t) + \chi * \left[\frac{\partial}{\partial t}\left(K * R * R\right)\right](t) = 0 \end{aligned} \quad (6.3)$$

where

$$P(t) = \int_0^t L(t')\,dt'$$

6.2 Propagator operators

Since the homogeneous medium is invariant under translation in z, the propagators $\mathcal{P}^{\pm}(z_1, z_0)$ are only dependent on the distance $\zeta = z_1 - z_0$. The definition of the propagators then simplifies to

$$a^{\pm}(z+\zeta, t+\tau(\zeta)) = \left(\mathcal{P}^{\pm}(\zeta)a^{+}(z,\cdot)\right)(t)$$

where

$$\tau(\zeta) = \frac{\zeta}{c}$$

The representations for the propagators of a homogeneous waveguide, see (5.2), are

$$\begin{cases} a^{+}(z+\zeta, t+\tau(\zeta)) = \Gamma(\zeta)a^{+}(z,t) + \left(P^{+}(\zeta,\cdot) * a^{+}(z,\cdot)\right)(t) \\ a^{-}(z+\zeta, t+\tau(\zeta)) = \left(P^{-}(\zeta,\cdot) * a^{+}(z,\cdot)\right)(t) \end{cases} \quad (6.4)$$

with boundary value on $P^+(\zeta,t)$.

$$P^+(0,t)=0$$

The following relation holds for the product of two propagators which commute:

$$\mathcal{P}^+(\zeta_1+\zeta_2)=\mathcal{P}^+(\zeta_1)\mathcal{P}^+(\zeta_2) \tag{6.5}$$

and

$$\mathcal{P}^-(\zeta)=\mathcal{P}^-(0)\mathcal{P}^+(\zeta)=\mathcal{R}\mathcal{P}^+(\zeta)$$

From these relations, it follows that the kernels satisfy

$$P^+(\zeta_1+\zeta_2,t)=\Gamma(\zeta_2)P^+(\zeta_1,t)+\Gamma(\zeta_1)P^+(\zeta_2,t)+\left(P^+(\zeta_1,\cdot)*P^+(\zeta_2,\cdot)\right)(t)$$

and

$$P^-(\zeta,t)=\Gamma(\zeta)R(t)+\left(R(\cdot)*P^+(\zeta,\cdot)\right)(t) \tag{6.6}$$

The wave front factor satisfies

$$\Gamma(\zeta_1+\zeta_2)=\Gamma(\zeta_1)\Gamma(\zeta_2)$$

and the explicit representation of this factor is

$$\Gamma(z)=e^{\frac{cBz}{2}}$$

In a homogeneous waveguide, the equations for the propagator kernels reduce to, see (5.4)

$$\begin{cases} \dfrac{\partial P^+(\zeta,t)}{\partial\zeta}=\Gamma(\zeta)\alpha(t)+\dfrac{cB}{2}\left(P^+(\zeta,t)+P^-(\zeta,t)\right) \\ \qquad +\left(\alpha(\cdot)*P^+(\zeta,\cdot)\right)(t)+\left(\beta(\cdot)*P^-(\zeta,\cdot)\right)(t) \\ \dfrac{\partial P^-(\zeta,t)}{\partial\zeta}-\dfrac{2}{c}\dfrac{\partial P^-(\zeta,t)}{\partial t}=\Gamma(\zeta)\gamma(t)-\dfrac{cB}{2}\left(P^+(\zeta,t)+P^-(\zeta,t)\right) \\ \qquad +\left(\gamma(\cdot)*P^+(\zeta,\cdot)\right)(t)+\left(\delta(\cdot)*P^-(\zeta,\cdot)\right)(t) \end{cases} \tag{6.7}$$

It is here possible to eliminate the kernel $P^-(\zeta,t)$ by using (6.6). The kernel $P^+(\zeta,t)$ then satisfies

$$\begin{cases} \dfrac{\partial P^+(\zeta,t)}{\partial\zeta}=\Gamma(\zeta)\eta(t)+\left(\eta(\cdot)*P^+(\zeta,\cdot)\right)(t)+\dfrac{cB}{2}P^+(\zeta,t) \\ P^+(0,t)=0 \end{cases}$$

where

$$\eta(t)=\alpha(t)+\frac{cB}{2}R(t)+\left(\beta(\cdot)*R(\cdot)\right)(t)$$

With Laplace transform techniques this equation can be transformed into an integral equation.

$$P^+(\zeta,t)=\frac{\zeta}{t}\int_0^t(t-t')\eta(t-t')P^+(\zeta,t')\,dt'+\Gamma(\zeta)\zeta\eta(t) \tag{6.8}$$

Thus, in order to obtain the propagator kernels $P^{\pm}$ for a homogeneous medium the two integral equations (6.3) and (6.8) have to be solved. Both of these equations are Volterra equations of the second kind and are straightforward to solve numerically.

Since the medium is invariant under a translation in z, it follows that the propagator $\mathcal{P}^+(\zeta)$ not only is the propagator for the field $a^+(z,t)$ but also for the total field $a(z,t) = a^+(z,t) + a^-(z,t)$ as well as for the field $a^-(z,t)$. This is seen from

$$\begin{cases} a^+(z+\zeta, t+\tau(\zeta)) = \mathcal{P}^+(\zeta)a^+(z,t) \\ a^-(z+\zeta, t+\tau(\zeta)) = \mathcal{P}^-(\zeta)a^+(z,t) = \mathcal{R}\mathcal{P}^+(\zeta)a^+(z,t) \\ a^-(z,t) = \mathcal{R}a^+(z,t) \end{cases}$$

From the representations, (6.4), it is seen that the propagators $\mathcal{R}$ and $\mathcal{P}^+(\zeta)$ commute. From the identity $a(z,t) = a^+(z,t) + a^-(z,t)$ it then follows that

$$\begin{cases} a^-(z+\zeta, t+\tau(\zeta)) = \mathcal{P}^+(\zeta)\mathcal{R}a^+(z,t) = \mathcal{P}^+(\zeta)a^-(z,t) \\ a(z+\zeta, t+\tau(\zeta)) = \mathcal{P}^+(\zeta)a(z,t) \end{cases}$$

The total field $a(z,t)$ is understood to be a field which is generated by an incident wave from the left. Since the propagator $\mathcal{P}^+(\zeta)$ is a propagator for the entire field it must be independent of the splitting.

6.3 The propagator $\mathcal{P}^+$ for a waveguide filled with a homogeneous, non-dispersive material

The wave propagation problem in a waveguide filled with a homogeneous, non-dispersive medium was examined in Ref. 13. The result concerning the propagator kernel P^+ is reviewed in this subsection.

In the case of a waveguide filled with a homogeneous dielectric material, the splitting, (4.2), diagonalizes the system of equations (4.1). This implies that if there are only sources for $z < z_p$, then there is no field $a^-(z,t)$ in the region $z > z_p$, and $P^-(\zeta,t) \equiv 0$ in this region. The kernel $P^+(\zeta,t)$ satisfies the equation

$$\frac{\partial P^+(\zeta,t)}{\partial \zeta} = \alpha(t) + \left(\alpha(\cdot) * P^+(\zeta,\cdot)\right)(t)$$

where in this case

$$\alpha(t) = -\frac{1}{c}L(z,t) = -H(t)\frac{\lambda J_1(c\lambda t)}{t}$$

In Ref. 13 it was found that the solution to this equation has the explicit expression

$$P^+(\zeta,t) = -c\lambda\zeta\frac{J_1\left(\lambda\sqrt{c^2t^2+2\zeta ct}\right)}{\sqrt{c^2t^2+2\zeta ct}}H(t)$$

6.4 The precursor for a waveguide filled with a homogeneous, dispersive material

In the homogeneous case the formal solution, (5.5), simplifies to

$$\mathcal{P}^+(\zeta) = \exp\left(\zeta(\mathcal{A} + \frac{1}{c}\frac{\partial}{\partial t} + \mathcal{BR})\right) \tag{6.9}$$

The power series expansion of this formal solution can be used to obtain the short time behavior of the propagator. This behavior gives the first precursor of the signal [1]. The exponent reads

$$\begin{aligned}\zeta(\mathcal{A} + \tfrac{1}{c}\tfrac{\partial}{\partial t} + \mathcal{BR}) &= -\zeta\frac{c}{2c_o^2}\chi(0^+) - \zeta\left(\frac{1}{c}L(\cdot) + \frac{c}{2c_0^2}\frac{\partial^2}{\partial t^2}(K*\chi)(\cdot)\right.\\ &\left.+\frac{c}{2c_0^2}\left(R*\frac{\partial^2}{\partial t^2}(K*\chi)\right)(\cdot) + \frac{c}{2c_0^2}\chi(0^+)R(\cdot)\right)*\end{aligned}$$

If we consider times much shorter than the typical time scale of the susceptibility kernel $\chi(t)$ and of the splitting kernel $K(t)$ it is enough to keep the first term in a power series expansion of the exponent and thus

$$\zeta(\mathcal{A} + \tfrac{1}{c}\tfrac{\partial}{\partial t} + \mathcal{BR}) \approx -\zeta\frac{c}{2c_o^2}\chi(0^+) + \zeta\left(\frac{c}{8c_0^2\epsilon}\chi(0^+)^2 - \frac{c}{2}\lambda^2 - \frac{c}{2c_0^2}\frac{\partial}{\partial t}\chi(0^+)\right)*$$

The short time behavior of the propagator is then given by

$$\mathcal{P}^+(\zeta) \approx \Gamma(\zeta)\exp\left(\zeta\left(\frac{c}{8c_0^2\epsilon}\chi(0^+)^2 - \frac{c}{2}\lambda^2 - \frac{c}{2c_0^2}\frac{\partial}{\partial t}\chi(0^+)\right)*\right)$$

From equation (6.7) it is seen that

$$\frac{\partial}{\partial z}P^+(0,0) = P_z^+(0,0) = \frac{c}{8c_0^2\epsilon}\chi(0^+)^2 - \frac{c}{2}\lambda^2 - \frac{c}{2c_0^2}\frac{\partial}{\partial t}\chi(0^+)$$

By expanding the exponential in a power series and performing the convolutions the early time behavior of the propagator follows. The result is

$$\mathcal{P}^+(\zeta) = \Gamma(\zeta)\left(\delta(t) + \zeta P_z^+(0,0)\frac{J_1\left(2\sqrt{-t\zeta P_z^+(0,0)}\right)}{\sqrt{-t\zeta P_z^+(0,0)}}H(t)\right) \tag{6.10}$$

where J_1 denotes the Bessel function of order 1. If $P_z^+(0,0)$ is positive, then the argument of the Bessel function is imaginary and $P^+(\zeta,t)$ has an exponentially growing behavior for small t. If $P_z^+(0,0)$ is negative then the argument of the Bessel function is real and $P^+(\zeta,t)$ has an oscillatory behavior for small times. When $\lambda = 0$ then (6.10) is identical with the expression for the precursor in a dispersive media found by the stationary phase method, see Ref. 9.

In the case of a dispersive medium with $\chi(0^+) = 0$, a more accurate expression can be obtained for the precursor. For small enough times, the power series expansion

$$\chi(t) = t\chi_t(0^+) \tag{6.11}$$

is a good approximation. Thus, for small times, and for a homogeneous dispersive medium with $\chi(0^+) = 0$, both of the Klein-Gordon equations, (3.3), simplify to

$$\frac{\partial^2 a(z,t)}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 a(z,t)}{\partial t^2} - \left(\frac{1}{c_0^2}\chi_t(0^+) + \lambda^2\right) a(z,t) = 0$$

This is the same equation as for the homogeneous dielectric waveguide except that the value of λ is now exchanged by the expression $\sqrt{c_0^{-2}\chi_t(0^+) + \lambda^2}$. An expression for the propagator kernel P^+, which is valid under the approximation (6.11), is given by

$$P^+(\zeta,t) = -c\zeta\sqrt{c_0^{-2}\chi_t(0^+) + \lambda^2}\frac{J_1\left(\sqrt{(c_0^{-2}\chi_t(0^+) + \lambda^2)c^2t^2 + 2\zeta ct}\right)}{\sqrt{c^2t^2 + 2\zeta ct}} \tag{6.12}$$

For times $t \ll \zeta/c$ the two expressions (6.10) and (6.12) for the precursor are analogous. The condition $\chi(0^+) = 0$ is satisfied by, *e.g.* Lorentz media [9].

6.5 Boundary conditions at an interface

The entire waveguide has so far been filled with a homogeneous medium, modeled by $\chi(t)$ and ϵ. If different sections of the waveguide are filled with different homogeneous materials, boundary conditions hold at the interfaces between the materials. In this subsection, one interface is assumed at $z = 0$. The medium that occupies the region $z < 0$ is modeled by $\chi_1(t)$ and ϵ, and the region $z > 0$ is modeled by $\chi_2(t)$ and ϵ. Note that both media are assumed to have the same permittivity ϵ. Furthermore, both media are assumed to be non-magnetic, and the interface has no surface charges.

The TM-case is analyzed first. In this case, the continuity of the normal component of the displacement field implies

$$\begin{aligned}&\epsilon\left(a^+(0^-,t) + a^-(0^-,t)\right) + \left(\chi_1(\cdot) * \left(a^+(0^-,\cdot) + a^-(0^-,\cdot)\right)\right)(t)\\ &= \epsilon\left(a^+(0^+,t) + a^-(0^+,t)\right) + \left(\chi_2(\cdot) * \left(a^+(0^+,\cdot) + a^-(0^+,\cdot)\right)\right)(t)\end{aligned} \tag{6.13}$$

In a homogeneous region $\nabla \cdot \boldsymbol{E} = 0$. Take the limiting values on both sides of the interface and use the continuity of the tangential electric field. The result is

$$\frac{\partial a(0^-,t)}{\partial z} = \frac{\partial a(0^+,t)}{\partial z}$$

The splitting (4.3) implies that this equation can be written as

$$a^-(0^-,t) - a^+(0^-,t) = a^-(0^+,t) - a^+(0^+,t) \tag{6.14}$$

Define a reflection kernel $r(t)$ for the interface by

$$a^-(0^-,t) = \left(r(\cdot) * a^+(0^-,\cdot)\right)(t)$$

On the other hand, from (5.3), in the medium $z > 0$ holds

$$a^-(0^+,t) = \left(R(\cdot) * a^+(0^+,\cdot)\right)(t)$$

These latter two equations are used to eliminate the fields $a^-(0^-,t)$ and $a^-(0^+,t)$ in equations (6.13) and (6.14). The remaining fields in (6.13) and (6.14) after this elimination are $a^+(0^-,t)$ and $a^+(0^+,t)$.

Proceed by eliminating the field $a^+(0^+,t)$ from equation (6.14) by the use of the resolvent F of the kernel $-R$, *i.e.*,

$$F - R - R * F = 0$$

The result of this operation is

$$a^+(0^+,t) = a^+(0^-,t) + \left\{\left[F(\cdot) - r(\cdot) - \left(F(\cdot) * r(\cdot)\right)(\cdot)\right] * a^+(0^-,\cdot)\right\}(t)$$

Finally, apply this expression to (6.13). The final result is

$$2\epsilon r + [\chi_1 + \chi_2 + 2\epsilon F + 2\chi_2 * F] * r = 2\epsilon F - \chi_1 + \chi_2 + 2\chi_2 * F$$

Thus, the kernel r satisfies a Volterra equation of the second kind.

The corresponding result for the TE-case is trivial.

$$r = R$$

7 Numerical examples

As a numerical example, propagation of a TM-mode in two different waveguides is considered. The first waveguide is empty and the other is empty in the region $z < 0$ and filled with a homogeneous dispersive material in $z > 0$. In both waveguides the value of λ is 48.1 m^{-1} which corresponds to the TM_{10} mode in a circular waveguide of radius 5 cm. The cutoff frequency for this mode is $f_c = 2.3$ GHz. The excitation at $z = 0$ is

$$a^+(z = 0^-,t) = H(t)H(T-t)\sin\omega_0 t \tag{7.1}$$

where the duration of the excitation is $T = 5 \cdot 10^{-10}$ s. This choice of excitation makes the incident field at $z = 0$ well-defined. The electric field is calculated at the angular frequency $\omega_0 = 4\pi \cdot 10^{10}$ rad/s and the energy is calculated at five different values of ω_0.

The first two figures, see Figures 2 and 3, show the result in an empty waveguide. The total field as a function of z is depicted in Figure 2, and the time trace of the energy that has passed the point $z = 25$ cm is depicted in Figure 3. The energy is normalized with the total incident energy. The "staircase" leading edge behavior

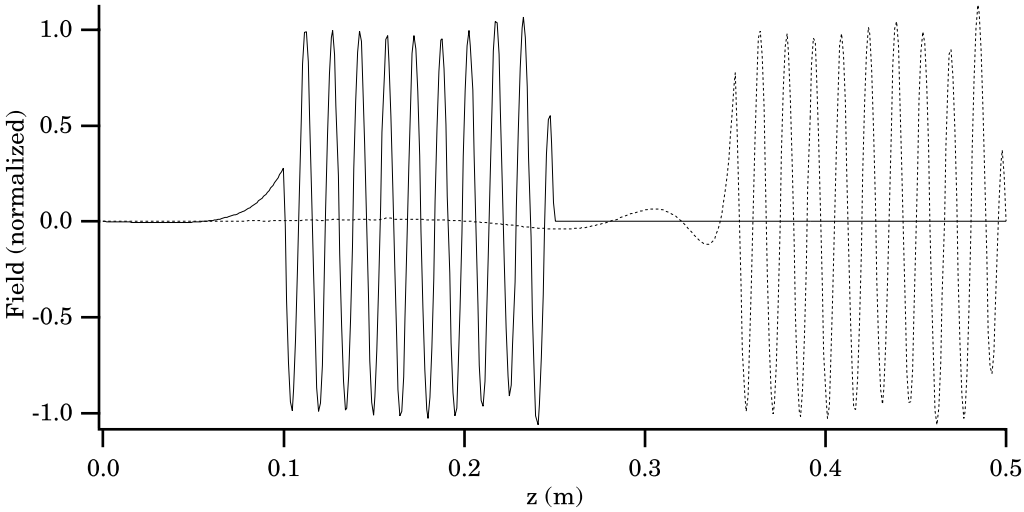

Figure 2: The normalized z-component of the electric field at the two instances, $t = 0.25/c_0$ and $t = 0.5/c_0$, for an empty waveguide. The excitation at $z = 0$ is given in (7.1).

of the curves in Figure 3 is due to the specific excitation of the waveguide, i.e., a sinusoidal excitation. Note that the lowest frequency $f_0 = 2$ GHz is below the cutoff frequency $f_c = 2.3$ GHz.

The next two figures, see Figures 4 and 5, show the result for a waveguide filled with a homogeneous Debye material in the region $z > 0$. The Debye medium is characterized by the constitutive relations

$$\chi(t) = \alpha e^{-t/\tau}$$

where $\alpha = 1 \cdot 10^9$ Hz and $\tau = 1 \cdot 10^{-9}$ s. When the incident wave impinges at the boundary $z = 0$ a part of it will be reflected and the rest of the pulse propagates in the positive z-direction. In Figure 4 the total field as a function of z is depicted, and Figure 5 shows the time trace of the normalized energy as a function of time.

8 Conclusions

In this paper propagation of electromagnetic wave in waveguides of general cross section are analyzed by the use of a time domain method. The waveguide is assumed to be filled with an inhomogeneous (varying with depth), dispersive material. The propagation problem is treated with a wave splitting technique and propagator operators. The propagator kernels satisfy a system of linear PDE's, which together with the appropriate initial and boundary conditions solves the general wave propagation problem. Special emphasize is paid to the homogeneous, dispersive waveguide, where some explicit results are presented. The precursor problem is also addressed, and some numerical computations illustrate the analysis.

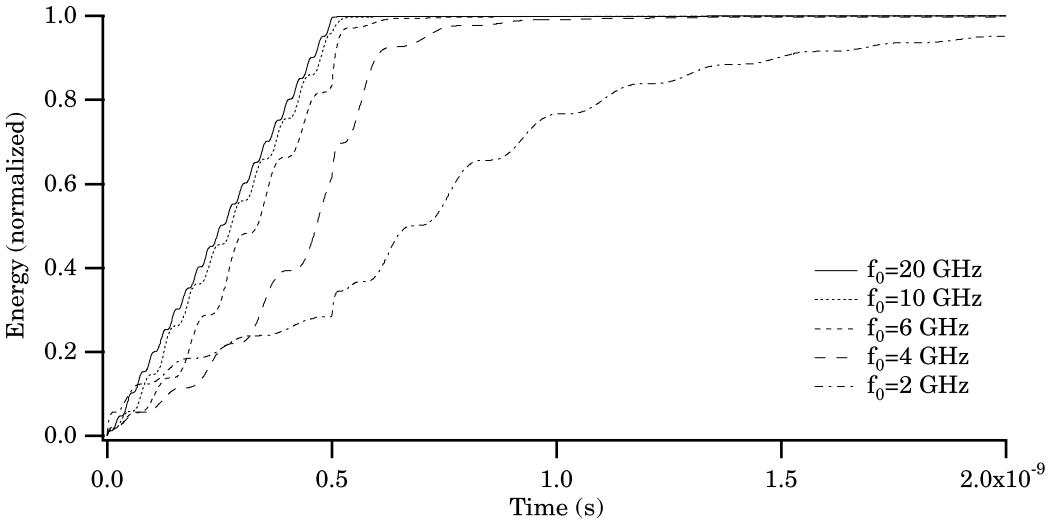

Figure 3: The normalized energy as a function of time at $z = .25$ m for an empty waveguide. The excitation at $z = 0$ is given in (7.1).

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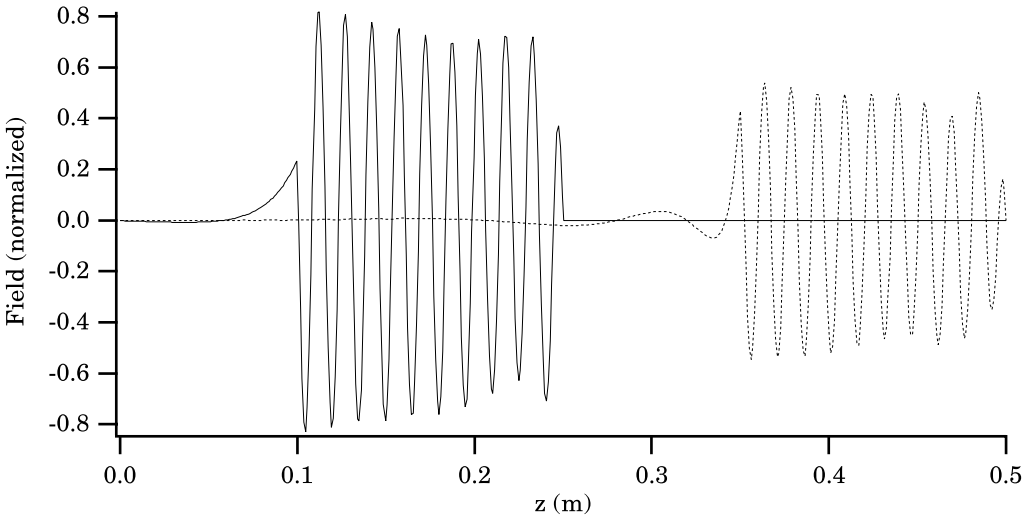

Figure 4: The normalized z-component of the electric field at the two instances, $t = 0.25/c_0$ and $t = 0.5/c_0$, for a waveguide with a Debye medium. The excitation at $z = 0$ is given in (7.1).

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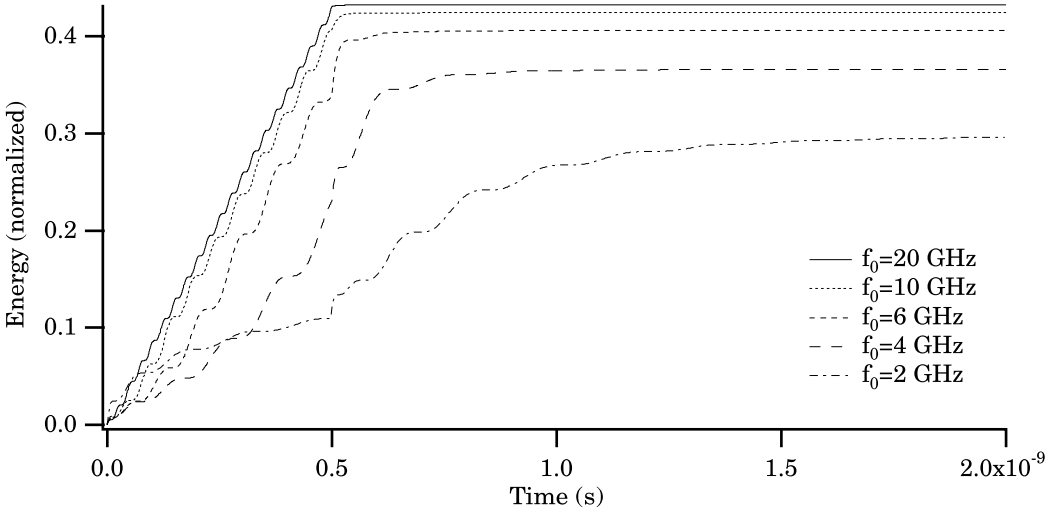

Figure 5: The normalized energy as a function of time at $z = .25$ m for a waveguide with a Debye medium. The excitation at $z = 0$ is given in (7.1).

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