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Eye movements provide an online measure of the strategies students use to solve a problem. Such strategies are difficult to investigate with traditional methods of assessment. In this paper, we present some background and highlights of work in progress, aiming at a larger study where we will investigate how engineering students solve problems in vector calculus.

BASIC CONCEPTS IN VECTOR CALCULUS

Vector calculus is a branch of mathematics that engineering students typically become introduced to during their first or second year at the university. It is used extensively in physics and engineering, especially in topics like electromagnetic fields and fluid mechanics. Vector calculus is usually part of courses in multivariable calculus, and lays the foundation for further studies in mathematics, for example in differential geometry and in studies of partial differential equations. The basic objects in vector calculus are scalar fields and vector fields, and the most basic algebraic operations consist of scalar multiplication, vector addition, dot product, and cross product. These basic operations are usually taught in a prior course in linear algebra. In vector calculus, various differential operators defined on scalar or vector fields are studied, which are typically expressed in terms of the *del operator*, $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$. Two important differential operations in vector calculus that we have used in this pilot study are described in the following table:

Operation	*Notation*	*Description*	*Domain/Range*
Gradient	$\mathrm{grad}(f) = \nabla f$	Measures the rate and direction of change in a scalar field	Maps a scalar field to a vector field
Divergence	$\mathrm{div}(\vec{u}) = \nabla \cdot \vec{u}$	Measures the magnitude of a source or sink at a given point in a vector field	Maps a vector field to a scalar field

The domain for the gradient and divergence differ because the former acts on a scalar field f and the latter is a dot product with $\vec{u}$, where symbols with an arrow denote vector fields. The gradient can for example be used to answer in which direction and at what rate the concentration of a poison increases the most. The divergence can be used to calculate the amount of poison released at a certain place.

Another major topic in vector calculus is integration on curves and (hyper-) surfaces. We have here concentrated on the simplest case of integration, i.e., along curves in a two-dimensional domain. A vector field that is the gradient of a scalar field (i.e., fulfilling $\vec{F} = \nabla V$) is known as a *conservative* field. It is independent of the path (or form) of the curve, but depends only on its start- and end points. Formally the line integral then – see the table below – resembles an expression analogue to the fundamental theorem of calculus for a function of one variable, i.e.

$$\int_a^b V'(x)\mathrm{d}x = V(b) - V(a)$$

Operation	*Notation*	*Description*	*Domain/Range*
Line integration	$\int_\gamma \vec{F} \cdot \overrightarrow{dr}$	Sum up the projection of a vector field on a path	Maps a vector field and a path to a scalar
Line integrtion, conservative field	$\int_\gamma \nabla V \cdot \overrightarrow{dr} = V(\vec{b}) - V(\vec{a})$	Measures the change in potential between two points.	Maps a scalar field and two points to to a scalar

Line integrals are used for example to calculate the work needed to perform a mechanical process. Conservative forces (i.e. vector fields) are common in physics and it is then often intuitively clear why a certain line integral is independent of the path. An example of this is gravitation, illustrated in Figure 1; the work required to lift a stone to the top of the pyramid is independent of the path taken. It only depends on the altitude of the pyramid, i.e., the difference in the gravitational potential at the top of the pyramid compared to at the ground level.

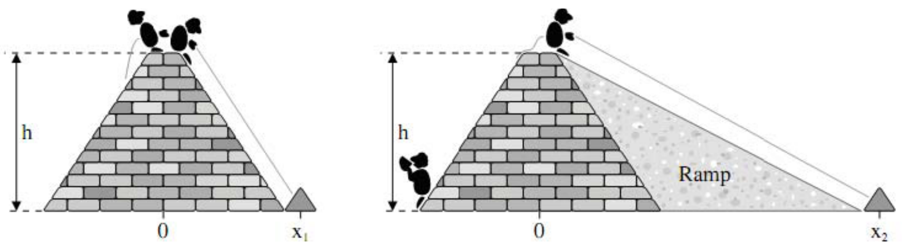

Figure 1. The work required to lift a stone to the top of the pyramid is independent of the path taken.

Vector calculus is often a very visual subject where an abstract mathematical formula can be accompanied with a direct graphical representation. A key to understanding vector calculus is to be able to switch between different representations of a problem, and successfully integrate the information from all representations into a coherent picture. Studies from other domains have shown that integrating relevant information from different sources is an ability overrepresented in high ability students (Hannus & Hyönä, 1999). Tracking how the eyes move during problem solving provides insight into the fine details of such integration.

Eye movements during problem solving

Over the last century, ample evidence has shown that there is a close link between where we look and what we are visually attending (e.g., Deubel & Schneider, 1996). In many conditions, eye movements also provide a valuable link to what is being cognitively processed in the brain, and allow us to monitor such processes closely over time. Studying eye movements thus makes it possible to monitor the entire process of the problem solving, and not only the end product – the proposed solution – which is typically what is evaluated and assessed in today's higher education.

Eye movements are recorded with a device called an eye-tracker, which provides information about where the eyes are directed, i.e., where we look, how the eyes moved to get there, and usually the size of the pupil. When looking at still images, our eyes move with rapid jerks, called *saccades*, interrupted by *fixations* where the eyes remain relatively stable. The duration of fixations, in particular, has been used extensively by researchers interested in problem solving since they have been linked to the complexity of different aspect of the problem; typically, the eye needs to remain still longer when processing more difficult parts of the problem (Rayner, 1998). Along with fixation duration, changes in pupil size have been shown to reflect mental activity (higher activity leads to larger pupil size) during problem solving (Hess & Polt, 1964).

Eye-tracking has previously been used to measure eye movements from people solving various types of problems. Hodgson, Bajwa, Owen, and Kennard (2000) identified three distinct phases that occurred when participants mentally planned to rearrange balls presented on a computer screen, such that the arrangement of balls on one half of the screen (work space) matched the arrangement of balls on the other half of the screen (goal space). This problem is known as the Tower-of-London task. In the three phases identified, there was a bias for participants to first look at the goal space, followed by looks toward the work space, and finally back to the goal space again. Knoblich, Ohlsson, and Raney (2001) investigated how students solved arithmetic matchstick problems and found differences in both fixation duration and allocation of gaze to problem relevant regions over time. Solving geometrical problems, Epelboim and Suppes (2001) illustrated how the eye movements from an expert traced an imagined radius of a circle, which needed to be identified to solve the problem. Eye movement provide valuable information about the problem solving process even when no visual information is present; Yoon and Narayanan (2004), for instance, showed that people re-enact eye movements to positions where a mechanical problem was previously shown – but that now is empty – to solve a new problem where information of the previous problem was helpful.

The above examples show that eye movements provide valuable information of how a problem is approached, and which strategies students use to solve it. While it have been used extensively to study simple problem solving tasks, eye-tracking methodology has only sparingly been used to study more advanced topics in mathematics such as vector calculus. In this paper, we describe the observations and ideas we acquired after conducting a pilot eye-tracking study, where students solved problems dealing with the topics introduced above: line integration of a conservative field, interpretation of the gradient, and finally the concept of divergence in connection to Gauss theorem.

Method

Three topics that are commonly taught in a university course in vector calculus (Ramgard & Larsson, n.d.; Persson & Böiers, 1988; Griffiths & College, 1999) were chosen for this pilot study. They were based on the concepts of line integrals, the gradient and the divergence of a vector field. The topics were selected because they are important components of a basic course in vector calculus and since they represent a subset of problems where a geometric visualization can significantly increase the understanding.

Three participants with good knowledge in vector calculus were being eye-tracked while solving problems from these three topics. Each topic was represented on a two-sided PDF document displayed on a computer screen. On the left side of the document sufficient information to answer the problem was presented (upper half of page) along with a statement the participant was asked to answer (lower half of page). The right hand side of the document contained additional information about the problem in the form of a concrete example, that could help students to better understand the problem. There were three statements for each topic (i.e., nine statements in total).

After a five point calibration the participants were free to inspect the problems for a maximum duration, which was set to four minutes for the first statement in each topic, and one minute each for statements two and three. A keypress (or timeout) ended the trial, whereafter participants where asked whether the statement was correct or incorrect. Finally, information of the participant's confidence in the answer on a scale from 1 (very uncertain) to 7 (very confident) was taken.

Eye movements were recorded at 250 Hz with the RED250 system from SensoMotoric Instruments (Teltow, Germany). The experimental setup is illustrated in Figure 2.

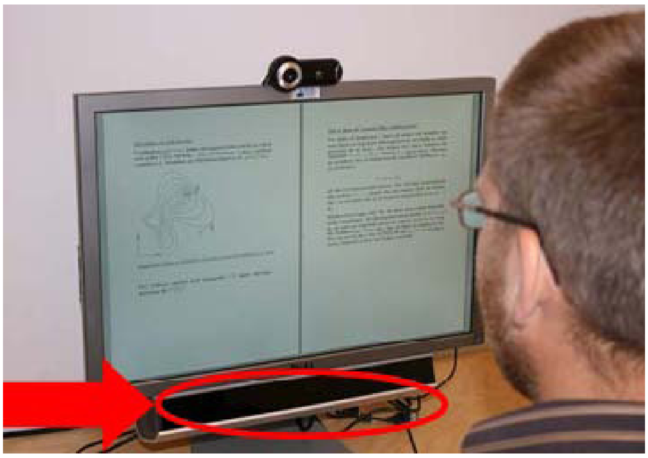

Figure 2. Experimental set-up with the eye-tracker encircled. The webcamera on top of the screen can be used to record speech and facial expressions from the participant.

RESULTS AND DISCUSSION

Figure 3 shows a representation of eye movement data known as a *heatmap*. It represents eye movements collected from one participant and indicates which regions that attracted most attention (warm colors). The heatmap shows that this particular participant used most of the time to study the statement at the bottom of the left page as well as the equations above it. The additional information was used only little, in particular the figure used to concretely illustrate the abstract concept on the left side; it only attracted a few, short glances. One interpretation is that the participant has good previous knowledge in vector calculus, and therefore does not need to use the additional information on the right page. In fact, all three participants seemed to use the additional information mostly when addressing the first statement in a topic, whereas not at all in the second and third statement.

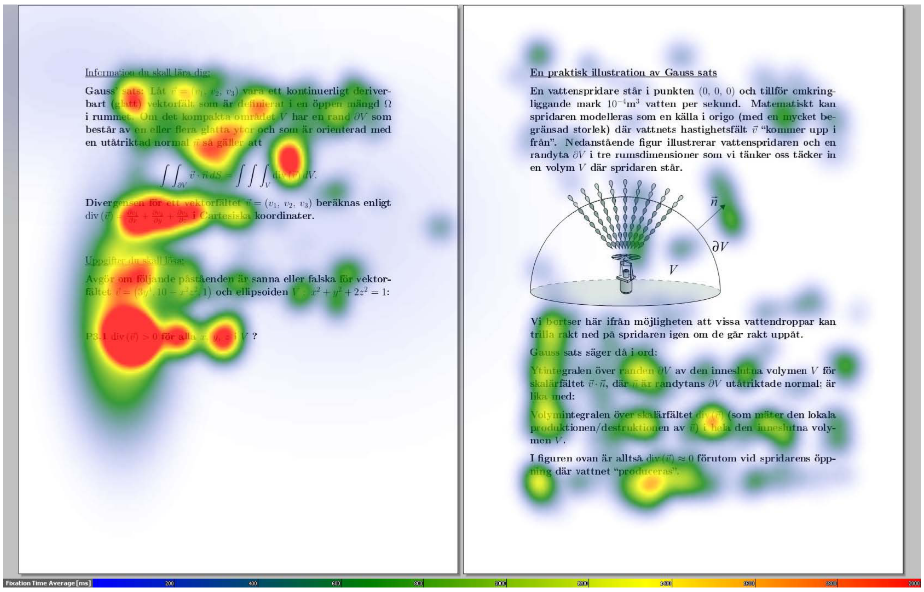

Figure 3. Heatmap showing where one of the three participants focused most of his attention (warmer colors indicate a higher proportion of looks).

Some comments/feedback from the participants in the pilot study:

- ``I did not have time to enter the status of visual thinking in the short time (6 minutes) of each problem"
- ``I chose the strategy to only look on the left hand side due to the limitation in time"
- ``The investigation does not reflect very well the way one usually solves this kind of problems in reality"

It is indeed common to write notes, to perform calculations and draw figures on a piece of scratch paper when working with problems of this type. Moreover, continuously looking at a computer screen when working with one topic may seem unnatural when it, in our experience, is common to look up, down, or even close the eyes when trying to solve a difficult problem.

For future investigations, we plan to reduce the complexity of the problems as well as the amount of text in these problems. To further tap into the participants' thought processes, we will also consider extending the methodology by asking participants to verbalize what they are doing using either concurrent think-aloud or cued retrospection (Holmqvist et al., 2011, Ch 10).

With these changes in the experimental design and with a larger group of participants, we plan to increase the degree of sophistication in our data analysis. Only then it becomes meaningful to provide quantitative results supported by statistical analyses.

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