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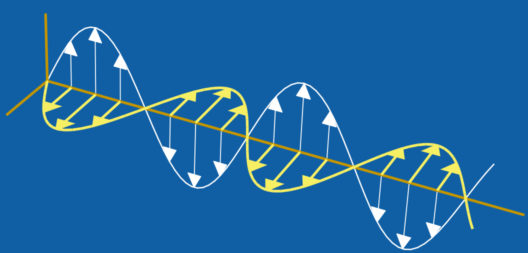

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Abstract

This paper focuses on propagation of transient electromagnetic waves in waveguides of general cross section with perfectly conducting walls. The solution of the transient wave propagation problem relies on a wave splitting technique, which has been frequently used in direct and inverse scattering problems during the last decade. The field in the waveguide is represented as a time convolution of a Green function and the excitation. Some numerical computations illustrate the method. A new way of calculating the first precursor in a Lorentz medium is presented. This method, which is not based upon the classical asymptotic methods, gives an expression of the first precursor at all depths in the medium. The excitation of the waveguide modes for time dependent sources is also addressed.

1 Introduction

Modern information and communication technologies rely on propagation of transient electromagnetic waves, e.g., short pulses. Electromagnetic wave propagation problems in waveguides have traditionally been analyzed using fixed frequency methods. A large body of results has been collected in this field, see, e.g., Ref [2] for a survey of these results. The transient wave propagation phenomena have then been synthesized by Fourier transform techniques [5]. However, modern applications ask for a more effective method, which does not rely on Fourier transform techniques. Specifically, wave front behavior and pulse broadening due to waveguide dispersion are important quantities.

This paper takes a fresh look at the propagation of transient electromagnetic wave in waveguides by using time domain techniques. More explicitly, the transient wave propagation problem is solved using a wave splitting technique. This technique has frequently been used to solve direct and inverse scattering problems in the last decade and was first introduced in one-dimensional scattering problems [3]. In recent years the three-dimensional wave splitting has been developed by Weston [18–21]. For a collection of applications of this theory the reader is referred to Ref [4].

The bounding surface is assumed to be perfectly conducting, but no other assumptions on the cross section of the waveguide have to be made (except mild smoothness requirements). The modal structure, similar to the concept used for fixed frequency, is retained. This concept introduces a set of basis functions in which the transverse field components can be expanded. The propagation of transient electromagnetic waves is systematically treated by means of the wave splitting technique, and for each mode the transient field is expressed as a time convolution of the excitation and a Green function. The TEM-modes are not analyzed in this paper, since these waves propagate as if the medium were free space and they are therefore already covered by previous analyses, see Refs. [10–12].

The first precursor for the propagation of an electromagnetic wave in a slab is also addressed. This result is obtained as a special result in the analysis of the waveguide problem. Explicit expression of the first precursor is given that holds at all depths in the medium. In this respect, it is a generalization of the classical precursor results.

In Section 2 the basic underlying equations of the problem and the decomposition of the fields are outlined. The canonical problems are stated in Section 3 and the wave splitting of the field is presented in Section 4. The Green functions that map an excitation to the response at an interior point are introduced in Section 5, which also contains the exact solution to this problem, the precursor problem, and some numerical illustrations showing the versatility of the method. In Section 6, the power flux of the split waves is analyzed. The appropriate expansion functions are introduced in Section 7 and in Section 8 the excitation of modes from a given source distribution is calculated.

2 Basic equations

In this section, the decomposition of the field and the basic equations are developed. Several of the results are found elsewhere, see, e.g., Refs. [8, 17], but to properly introduce the notation and for the convenience of the reader these results are presented here.

The Maxwell equations are the basic underlying equations that model the fields.

$$\begin{cases} \nabla \times \boldsymbol{E}(\boldsymbol{r},t) = -\dfrac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t} \\ \nabla \times \boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{J}(\boldsymbol{r},t) + \dfrac{\partial \boldsymbol{D}(\boldsymbol{r},t)}{\partial t} \end{cases} \tag{2.1}$$

All fields in this paper are assumed to be quiescent before a fixed time. This property guarantees that all fields vanish at $t \to -\infty$.

The medium in the waveguide is assumed to be non-dispersive and homogeneous, i.e.,

$$\begin{cases} \boldsymbol{D}(\boldsymbol{r},t) = \epsilon_0 \epsilon \boldsymbol{E}(\boldsymbol{r},t) \\ \boldsymbol{B}(\boldsymbol{r},t) = \mu_0 \mu \boldsymbol{H}(\boldsymbol{r},t) \end{cases}$$

The (relative) permittivity and permeability of the medium are denoted ϵ and μ, respectively, and are assumed to be constants. The phase velocity c and the wave impedance η are

$$c = \frac{1}{\sqrt{\epsilon_0 \epsilon \mu_0 \mu}}, \qquad \eta = \sqrt{\frac{\mu_0 \mu}{\epsilon_0 \epsilon}}$$

respectively.

The Maxwell equations (2.1) in a source free region and the constitutive relations imply

$$\nabla^2 \boldsymbol{E}(\boldsymbol{r},t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{0} \tag{2.2}$$

for the electric field $\boldsymbol{E}$ and

$$\nabla^2 \boldsymbol{H}(\boldsymbol{r},t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{0}$$

for the magnetic field $\boldsymbol{H}$.

2.1 Decomposition of the field

The decomposition of the fields in a transverse part and a z-component is, of course, not new. It is a well established technique to handle fixed frequency problems, see, e.g., Ref [8], and even for time domain problems similar decompositions are found elsewhere, see, e.g., Ref [17].

The nabla-operator and a general vector field $\boldsymbol{F}(\boldsymbol{r}, t)$ are decomposed with respect to a fixed direction, here taken as the positive z-direction.

$$\begin{cases} \nabla = \nabla_T + \hat{z}\dfrac{\partial}{\partial z} \\ \boldsymbol{F}(\boldsymbol{r}, t) = \boldsymbol{F}_T(\boldsymbol{r}, t) + \hat{z}F_z(\boldsymbol{r}, t) \end{cases}$$

The space vector $\boldsymbol{r}$ is also decomposed as

$$\boldsymbol{r} = \hat{x}x + \hat{y}y + \hat{z}z = \boldsymbol{\rho} + \hat{z}z$$

For a general vector field $\boldsymbol{F}$ the following identities hold:

$$\begin{cases} \hat{z} \cdot (\nabla \times \boldsymbol{F}(\boldsymbol{r}, t)) = -\nabla_T \cdot (\hat{z} \times \boldsymbol{F}_T(\boldsymbol{r}, t)) = \hat{z} \cdot (\nabla_T \times \boldsymbol{F}_T(\boldsymbol{r}, t)) \\ \nabla \times \boldsymbol{F}(\boldsymbol{r}, t) - \hat{z}\left(\hat{z} \cdot (\nabla \times \boldsymbol{F}(\boldsymbol{r}, t))\right) = \hat{z} \times \dfrac{\partial}{\partial z}\boldsymbol{F}_T(\boldsymbol{r}, t) - \hat{z} \times \nabla_T F_z(\boldsymbol{r}, t) \end{cases}$$

Apply this decomposition to the source-free Maxwell equations (2.1). The result for the z-component is

$$\begin{cases} \hat{z} \cdot (\nabla_T \times \boldsymbol{E}_T(\boldsymbol{r}, t)) = -\dfrac{1}{c}\dfrac{\partial}{\partial t}\eta H_z(\boldsymbol{r}, t) \\ \hat{z} \cdot (\nabla_T \times \eta\boldsymbol{H}_T(\boldsymbol{r}, t)) = \dfrac{1}{c}\dfrac{\partial}{\partial t}E_z(\boldsymbol{r}, t) \end{cases} \tag{2.3}$$

and for the transverse components the result is

$$\begin{cases} \dfrac{1}{c}\dfrac{\partial}{\partial t}\eta\boldsymbol{H}_T(\boldsymbol{r}, t) + \hat{z} \times \dfrac{\partial}{\partial z}\boldsymbol{E}_T(\boldsymbol{r}, t) = \hat{z} \times \nabla_T E_z(\boldsymbol{r}, t) \\ \dfrac{1}{c}\dfrac{\partial}{\partial t}\boldsymbol{E}_T(\boldsymbol{r}, t) - \hat{z} \times \dfrac{\partial}{\partial z}\eta\boldsymbol{H}_T(\boldsymbol{r}, t) = -\hat{z} \times \nabla_T \eta H_z(\boldsymbol{r}, t) \end{cases} \tag{2.4}$$

These equations can also be combined so that the magnetic or the electric transverse fields are eliminated. The transverse fields satisfy the one-dimensional wave equation with source terms.

$$\begin{cases} \left(\dfrac{\partial^2}{\partial z^2} - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\right)\boldsymbol{E}_T(\boldsymbol{r}, t) = \dfrac{1}{c}\dfrac{\partial}{\partial t}\left(\hat{z} \times \nabla_T \eta H_z(\boldsymbol{r}, t)\right) + \dfrac{\partial}{\partial z}\nabla_T E_z(\boldsymbol{r}, t) \\ \left(\dfrac{\partial^2}{\partial z^2} - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\right)\eta\boldsymbol{H}_T(\boldsymbol{r}, t) = -\dfrac{1}{c}\dfrac{\partial}{\partial t}\left(\hat{z} \times \nabla_T E_z(\boldsymbol{r}, t)\right) + \dfrac{\partial}{\partial z}\nabla_T \eta H_z(\boldsymbol{r}, t) \end{cases}$$

The z-components of the electric and the magnetic fields act as sources of the transverse fields. The z-components of the fields therefore generate the corresponding transverse components.

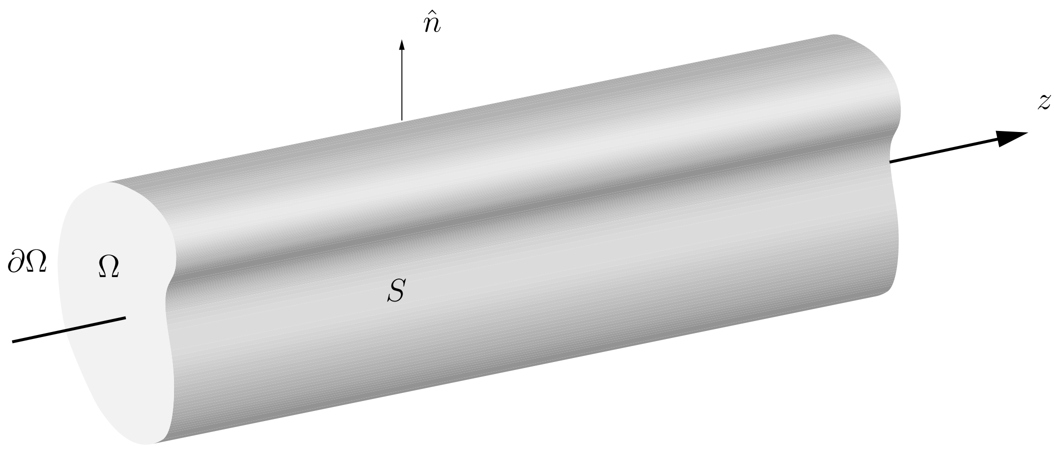

Figure 1: Geometry of the waveguide.

2.2 Boundary conditions

The geometry of the waveguide is depicted in Figure 1. The surface S is the outer bounding surface of the waveguide and $\hat{n}$ is the outward normal to S. The cross section of the waveguide is denoted Ω and $\partial\Omega$ is its bounding curve in the xy-plane. This curve is assumed to be smooth and the domain Ω is simply connected (no interior conductor is present).

The boundary conditions on the perfectly conducting wall of the waveguide are

$$\begin{cases} \hat{n} \times \boldsymbol{E} = \boldsymbol{0}, \\ \hat{n} \cdot \boldsymbol{H} = 0, \end{cases} \qquad \boldsymbol{r} \in S \tag{2.5}$$

Since $\hat{n}$ is independent of z, these boundary conditions are equivalent to

$$\begin{cases} E_z = 0, \\ \dfrac{\partial H_z}{\partial n} = 0, \end{cases} \qquad \boldsymbol{r} \in S$$

For a waveguide with perfectly conducting walls, the wave propagation phenomena in the waveguide separate into two different classes—the TE- and the TM-modes. This is completely analogous to the fixed frequency case. The TEM-modes are excluded in this presentation due to the absence of an inner conductor in the waveguide.

3 The canonical problems

The z-components of the electric and the magnetic field satisfy the three-dimensional wave equation (see (2.2)).

$$\nabla^2 \begin{pmatrix} E_z(\boldsymbol{r},t) \\ \eta H_z(\boldsymbol{r},t) \end{pmatrix} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \begin{pmatrix} E_z(\boldsymbol{r},t) \\ \eta H_z(\boldsymbol{r},t) \end{pmatrix} = 0$$

The method of separation of variables is now applied to this equation. The method is, however, not applied to each coordinate but with respect to the pairs (x, y) and (z, t). The ansatz for z-components in the TM- and the TE-cases is therefore

$$\begin{cases} E_z(\boldsymbol{r}, t) = v(\boldsymbol{\rho})a(z, t) \\ H_z(\boldsymbol{r}, t) = 0 \end{cases} \qquad \text{(TM-case)}$$

$$\begin{cases} E_z(\boldsymbol{r}, t) = 0 \\ H_z(\boldsymbol{r}, t) = w(\boldsymbol{\rho})b(z, t) \end{cases} \qquad \text{(TE-case)}$$

respectively. The functions v and w determine the transverse behavior of the fields and satisfy an eigenvalue problem. These eigenvalue problems are

$$\begin{cases} \nabla_T^2 v(\boldsymbol{\rho}) + \lambda^2 v(\boldsymbol{\rho}) = 0, & \boldsymbol{\rho} \in \Omega \\ v(\boldsymbol{\rho}) = 0, & \boldsymbol{\rho} \in \partial\Omega \end{cases}$$

and

$$\begin{cases} \nabla_T^2 w(\boldsymbol{\rho}) + \lambda^2 w(\boldsymbol{\rho}) = 0, & \boldsymbol{\rho} \in \Omega \\ \hat{n} \cdot \nabla_T w(\boldsymbol{\rho}) = 0, & \boldsymbol{\rho} \in \partial\Omega \end{cases}$$

The positive real constant λ is here the eigenvalue for the waveguide listed with due regard to multiplicity.

$$\lambda = \lambda_n, \qquad n = 1, 2, 3, \ldots$$

All fields depend on the index n, but to avoid complicated notation this index is often omitted in this paper. Furthermore, the same notation for the eigenvalue of the Dirichlet (TM) and the Neumann problem (TE) is used. From the context it is always obvious what problem λ refers to.

The TEM-case corresponds to the eigenvalue $\lambda = 0$. In this paper it is assumed that no such mode exists and thus the eigenfunctions form a complete orthogonal set in Ω [16, p. 138].

The functions $a(z, t)$ and $b(z, t)$ determine the wave propagation along the wave guide and satisfy the one-dimensional Klein-Gordon equation.

$$\frac{\partial^2}{\partial z^2} u(z, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} u(z, t) - \lambda^2 u(z, t) = 0$$

The transverse fields are determined by the z-components of the field. The TM-modes satisfy (see (2.3) and (2.4))

$$\begin{cases} \hat{z} \cdot (\nabla_T \times \boldsymbol{E}_T(\boldsymbol{r}, t)) = 0 \\ \hat{z} \cdot (\nabla_T \times \eta \boldsymbol{H}_T(\boldsymbol{r}, t)) = \dfrac{1}{c} \dfrac{\partial}{\partial t} E_z(\boldsymbol{r}, t) \\ \dfrac{1}{c} \dfrac{\partial}{\partial t} \eta \boldsymbol{H}_T(\boldsymbol{r}, t) + \hat{z} \times \dfrac{\partial}{\partial z} \boldsymbol{E}_T(\boldsymbol{r}, t) = \hat{z} \times \nabla_T E_z(\boldsymbol{r}, t) \\ \dfrac{1}{c} \dfrac{\partial}{\partial t} \boldsymbol{E}_T(\boldsymbol{r}, t) - \hat{z} \times \dfrac{\partial}{\partial z} \eta \boldsymbol{H}_T(\boldsymbol{r}, t) = \boldsymbol{0} \end{cases} \qquad \text{(TM-case)} \qquad (3.1)$$

and the transverse field of the TE-modes satisfies (see (2.3) and (2.4))

$$\begin{cases} \hat{z}\cdot(\nabla_T\times\boldsymbol{E}_T(\boldsymbol{r},t)) = -\dfrac{1}{c}\dfrac{\partial}{\partial t}\eta H_z(\boldsymbol{r},t) \\ \hat{z}\cdot(\nabla_T\times\eta\boldsymbol{H}_T(\boldsymbol{r},t)) = 0 \\ \dfrac{1}{c}\dfrac{\partial}{\partial t}\eta\boldsymbol{H}_T(\boldsymbol{r},t) + \hat{z}\times\dfrac{\partial}{\partial z}\boldsymbol{E}_T(\boldsymbol{r},t) = \boldsymbol{0} \\ \dfrac{1}{c}\dfrac{\partial}{\partial t}\boldsymbol{E}_T(\boldsymbol{r},t) - \hat{z}\times\dfrac{\partial}{\partial z}\eta\boldsymbol{H}_T(\boldsymbol{r},t) = -\hat{z}\times\nabla_T\eta H_z(\boldsymbol{r},t) \end{cases} \qquad \text{(TE-case)} \qquad (3.2)$$

From these equations it is seen that the transverse components have the form

$$\begin{cases} \boldsymbol{E}_T(\boldsymbol{r},t) = \nabla_T v(\boldsymbol{\rho})\psi_1(z,t) \\ \eta\boldsymbol{H}_T(\boldsymbol{r},t) = [\hat{z}\times\nabla_T v(\boldsymbol{\rho})]\,\phi_1(z,t) \end{cases} \qquad \text{(TM-case)}$$
$$\begin{cases} \boldsymbol{E}_T(\boldsymbol{r},t) = -\,[\hat{z}\times\nabla_T w(\boldsymbol{\rho})]\,\phi_2(z,t) \\ \eta\boldsymbol{H}_T(\boldsymbol{r},t) = \nabla_T w(\boldsymbol{\rho})\psi_2(z,t) \end{cases} \qquad \text{(TE-case)} \qquad (3.3)$$

where ψ_i and ϕ_i satisfy (see (3.1) and (3.2))

$$\begin{cases} \lambda^2\phi_1 = -\dfrac{1}{c}\dfrac{\partial a}{\partial t} \\ \dfrac{1}{c}\dfrac{\partial\phi_1}{\partial t} + \dfrac{\partial\psi_1}{\partial z} = a \\ \dfrac{1}{c}\dfrac{\partial\psi_1}{\partial t} + \dfrac{\partial\phi_1}{\partial z} = 0 \end{cases} \quad \text{(TM-case)} \qquad \begin{cases} \lambda^2\phi_2 = -\dfrac{1}{c}\dfrac{\partial b}{\partial t} \\ \dfrac{1}{c}\dfrac{\partial\phi_2}{\partial t} + \dfrac{\partial\psi_2}{\partial z} = b \\ \dfrac{1}{c}\dfrac{\partial\psi_2}{\partial t} + \dfrac{\partial\phi_2}{\partial z} = 0 \end{cases} \quad \text{(TE-case)}$$

These equations imply that the functions ψ_i and ϕ_i can be expressed in terms of the functions a and b as follows, since all fields are assumed to vanish at sufficiently large negative time:

$$\begin{cases} \lambda^2\phi_1 = -\dfrac{1}{c}\dfrac{\partial a}{\partial t} \\ \lambda^2\psi_1 = \dfrac{\partial a}{\partial z} \end{cases} \qquad \begin{cases} \lambda^2\phi_2 = -\dfrac{1}{c}\dfrac{\partial b}{\partial t} \\ \lambda^2\psi_2 = \dfrac{\partial b}{\partial z} \end{cases} \qquad (3.4)$$

The functions a and b can also be eliminated. The result is that the functions ψ_i and ϕ_i satisfy the same kind of one-dimensional Klein-Gordon equations that a and b do, i.e.,

$$\begin{cases} \dfrac{\partial\phi_i^2}{\partial z^2} - \dfrac{1}{c^2}\dfrac{\partial\phi_i^2}{\partial t^2} - \lambda^2\phi_i = 0 \\ \dfrac{\partial\psi_i^2}{\partial z^2} - \dfrac{1}{c^2}\dfrac{\partial\psi_i^2}{\partial t^2} - \lambda^2\psi_i = 0 \end{cases}$$

From this it is seen that the (z,t)-dependence of all field components (a, b, ϕ_i and ψ_i) satisfy the one-dimensional Klein-Gordon equation. Furthermore, the transverse components have (z,t)-dependence that is either the z- or the t-derivative of the corresponding (z,t)-dependence of the z-component of the field, see (3.4). It therefore suffices to study the z-component of the field.

4 Wave splitting

In recent years, a new technique for solving direct and inverse scattering problems in the time domain has been developed. For a collection of results, see Ref [4]. In this section, this technique is adapted to the electromagnetic wave propagation problem in a waveguide.

The one-dimensional Klein-Gordon equation is the starting point for this method.

$$\frac{\partial^2}{\partial z^2}u(z,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}u(z,t) - \lambda^2 u(z,t) = 0 \tag{4.1}$$

This equation is conveniently rewritten as a system of first order equations

$$\frac{\partial}{\partial z}\begin{pmatrix} u \\ \partial_z u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{c^2}\partial_t^2 + \lambda^2 & 0 \end{pmatrix}\begin{pmatrix} u \\ \partial_z u \end{pmatrix} \tag{4.2}$$

The purpose of the wave spitting is to change the dependent variables u and $\partial_z u$ to another set more suited for investigating the propagation problem in the waveguide. The aim is to construct a set of variables, u^+ and u^-, such that the equation for these variables is diagonal. Crudely speaking, the matrix in (4.2) is diagonalized. As is seen below in (4.8), the following change of variables gives a diagonal equation [7]:

$$u^{\pm}(z,t) = \frac{1}{2}\left[u(z,t) \mp c(K\partial_z u(z,\cdot))(t)\right]$$

where the operator K has the integral representation

$$(Kf)(t) = \int_{-\infty}^{t} J_0(c\lambda(t-t'))f(t')\,dt'$$

Formally, the splitting can be written in matrix notation as

$$\begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -cK \\ 1 & cK \end{pmatrix}\begin{pmatrix} u \\ \partial_z u \end{pmatrix} \tag{4.3}$$

This operation defines the wave splitting used in this paper. Even though the TEM-case is not addressed in this paper, it corresponds to $\lambda = 0$ and has the same splitting as in free space, see Ref [3], i.e.,

$$\lim_{\lambda\to 0}(Kf)(t) = \int_{-\infty}^{t} f(t')\,dt'$$

The operator K has a inverse K^{-1}, with

$$KK^{-1}f = f, \qquad K^{-1}Kf = f$$

The fields u and $\partial_z u$ are therefore expressed in $u^{\pm}$ as

$$\begin{pmatrix} u \\ \partial_z u \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\frac{1}{c}K^{-1} & \frac{1}{c}K^{-1} \end{pmatrix}\begin{pmatrix} u^+ \\ u^- \end{pmatrix} \tag{4.4}$$

The inverse operator K^{-1} has the explicit integral representation

$$(K^{-1}f)(t) = \frac{\partial f}{\partial t} + c\lambda \int_{-\infty}^{t} \frac{J_1(c\lambda(t-t'))}{t-t'} f(t')\,dt' = \frac{\partial f}{\partial t} + \left(L(\cdot) * f(\cdot)\right)(t) \tag{4.5}$$

where

$$L(t) = H(t)\frac{c\lambda J_1(c\lambda t)}{t} \tag{4.6}$$

and $H(t)$ is Heaviside's step function. Time convolutions are defined as

$$\left(L(\cdot) * f(\cdot)\right)(t) = \int_{-\infty}^{\infty} L(t-t')f(t')\,dt'$$

The integral representation of the inverse in (4.5) is proved by the identity [6, p. 698]

$$\int_0^x \frac{J_1(x-y)}{x-y} J_0(y)\,dy = J_1(x)$$

Another useful identity, found by integration by parts, is

$$K\frac{\partial f^2}{\partial t^2} = K^{-1}f - c^2\lambda^2 Kf \tag{4.7}$$

The new fields $u^{\pm}(z,t)$ satisfy a system of first order equations, which is obtained from (4.3), (4.4) and (4.2).

$$\begin{aligned}\frac{\partial}{\partial z}\begin{pmatrix} u^+ \\ u^- \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} 1 & -cK \\ 1 & cK \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \frac{1}{c^2}\partial_t^2 + \lambda^2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -\frac{1}{c}K^{-1} & \frac{1}{c}K^{-1} \end{pmatrix}\begin{pmatrix} u^+ \\ u^- \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{c}K^{-1} & 0 \\ 0 & \frac{1}{c}K^{-1} \end{pmatrix}\begin{pmatrix} u^+ \\ u^- \end{pmatrix}\end{aligned} \tag{4.8}$$

The effect of the splitting in (4.3) is now obvious. The transformation K factorizes the Klein-Gordon equation in (4.1), such that no coupling between the two components u^+ and u^- occurs. The field u^+ or u^- expresses the part of the field that propagates power in the $+z$- or $-z$-direction in the waveguide, respectively. Using (4.7), it is clear that u^+- and u^--waves both satisfy the Klein-Gordon equation (4.1).

Finally, the concept of a right- and left-going wave is defined. A field $u(z,t)$ is called right-going if $u(z,t)$ satisfies

$$u(z,t) = -c\left(K\partial_z u(z,\cdot)\right)(t)$$

The field, cf. first line in (4.3),

$$w(z,t) = \frac{1}{2}\left[u(z,t) - c(K\partial_z u(z,\cdot))(t)\right]$$

is right-going since

$$\begin{aligned} w + cK\partial_z w &= \frac{1}{2}\left[u - c^2K^2 u_{zz}\right] = \frac{1}{2}\left[u - c^2K^2(c^{-2}u_{tt} + \lambda^2 u)\right] \\ &= \frac{1}{2}\left[u - KK^{-1}u)\right] = 0 \end{aligned}$$

where (4.7) is used. Similarly, a field $u(z,t)$ is called left-going if $u(z,t)$ satisfies

$$u(z,t) = c\left(K\partial_z u(z,\cdot)\right)(t)$$

To summarize, the condition for right- and left-going fields are

$$u(z,t) = \mp c\left(K\partial_z u(z,\cdot)\right)(t)$$

or by taking the inverse

$$\frac{\partial}{\partial z}u(z,t) = \mp\frac{1}{c}\left(K^{-1}u(z,\cdot)\right)(t) \tag{4.9}$$

5 Green functions

The Green functions represent the mapping of the excitation at the boundary of a section of the waveguide, say $z=0$, to some point $z>0$ in the interior. These Green functions were first introduced by Krueger and Ochs [12] in the non-dispersive case and Kristensson [9] for dispersive media.

The case of propagation in the positive z-direction is first considered. The Green functions $G^{\pm}(z,t)$ for this case are defined as

$$\begin{cases} u^-(z,t+z/c) = \left(G^-(z,\cdot) * u^+(0,\cdot)\right)(t) \\ u^+(z,t+z/c) = u^+(0,t) + \left(G^+(z,\cdot) * u^+(0,\cdot)\right)(t) \end{cases}, \qquad z>0 \tag{5.1}$$

where, as before, time convolutions are defined by a star (*). The field $u^+(z,t)$ at some point $z>0$ consists of two parts—one due to the direct transmission of the incident field $u^+(0,t)$ and one due to scattering effects represented by a time convolution of the incident field $u^+(0,t)$ with the Green function $G^+(z,t)$. In this way, the Green function $G^+(z,t)$ represents the mapping of a right-going field at $z=0$ to a right-going field at some point $z>0$. The other Green function $G^-(z,t)$ represents the corresponding mapping of a right-going field at $z=0$ to a left-going field at $z>0$. Due to causality, $G^{\pm}(z,t)=0$ for $t<0$, and from the definition of $G^+(z,t)$ in (5.1) it follows that

$$G^+(0,t) = 0$$

The Green functions $G^{\pm}(z,t)$ satisfy differential equations, which are found by differentiating (5.1) with respect to z and using (4.8) (see also Ref [7]). The differentiation gives

$$\mp\left(K^{-1}u^{\pm}(z,\cdot)\right)(t+z/c) + \partial_t u^{\pm}(z,t+z/c) = c\left(\partial_z G^{\pm}(z,\cdot) * u^+(0,\cdot)\right)(t)$$

Inserting the explicit integral representation for K^{-1} yields

$$\begin{cases} -\left(L(\cdot) * u^+(z,\cdot)\right)(t+z/c) = c\left(\partial_z G^+(z,\cdot) * u^+(0,\cdot)\right)(t) \\ 2\partial_t u^-(z,t+z/c) + \left(L(\cdot) * u^-(z,\cdot)\right)(t+z/c) = c\left(\partial_z G^-(z,\cdot) * u^+(0,\cdot)\right)(t) \end{cases}$$

where L is defined in (4.6). Apply (5.1) once more. This gives

$$\begin{aligned} &c\left(\partial_z G^+(z,\cdot) * u^+(0,\cdot)\right)(t) \\ &\qquad = -\left(L(\cdot) * u^+(0,\cdot)\right)(t) - \left(L(\cdot) * G^+(z,\cdot) * u^+(0,\cdot)\right)(t) \end{aligned}$$

and

$$\begin{aligned} c\left(\partial_z G^-(z,\cdot) * u^+(0,\cdot)\right)(t) &= 2\left(\partial_t G^-(z,\cdot) * u^+(0,\cdot)\right)(t) \\ &\quad + 2G^-(z,0^+)u^+(0,t) + \left(L(\cdot) * G^-(z,\cdot) * u^+(0,\cdot)\right)(t) \end{aligned}$$

These expressions hold for all excitations $u^+(0,t)$. This implies that

$$\begin{cases} c\partial_z G^+(z,t) = -L(t) - \left(L(\cdot) * G^+(z,\cdot)\right)(t) \\ G^+(0,t) = 0 \end{cases} \tag{5.2}$$

and

$$\begin{cases} c\partial_z G^-(z,t) = 2\partial_t G^-(z,t) + \left(L(\cdot) * G^-(z,\cdot)\right)(t) \\ G^-(z,0^+) = 0 \end{cases} \tag{5.3}$$

The corresponding mapping of an excitation at $z = 0$ to some interior point $z < 0$ is obtained by changing sign of the z-coordinate, i.e., $z \to -z$. This mapping describes the propagation of waves in the negative z-direction.

5.1 Exact solutions

The Green function $G^-(z,t)$ is identically zero in this splitting (unique solubility), as can be seen from (5.3) by rewriting the equation as an initial value problem along the characteristic direction of the equation.

$$G^-(z,t) = 0$$

This fact implies that a right-going wave, the way it is defined by the wave splitting in (4.3), continues to propagate as a right-going wave in the region $z > 0$. Similarly, a left-going wave continues to propagate as a left-going wave in the region $z < 0$.

The other Green function, $G^+(z,t)$, is solved from (5.2) with Laplace transform techniques. The solution is

$$G^+(z,t) = -c\lambda z H(t) \frac{J_1\left(\lambda\sqrt{c^2t^2 + 2zct}\right)}{\sqrt{c^2t^2 + 2zct}}$$

This solution is equivalent to the following complicated identity of Bessel functions:

$$J_1(x) = \frac{\mathrm{d}}{\mathrm{d}z}\left[z\int_0^x \frac{J_1(\sqrt{y^2+2zy})}{\sqrt{y^2+2zy}} J_0(x-y)\,dy\right] + z\int_0^x \frac{J_1(\sqrt{y^2+2zy})}{\sqrt{y^2+2zy}} J_1(x-y)\,dy$$

for all x and z.

The Green function $G^+(z,t)$ represents the propagation of an excitation at $z=0$ in the positive z-direction. The propagation in the negative z-direction is obtained by replacing z by $-z$. It is therefore natural to a define propagator kernel $\mathcal{G}(z,t)$

$$\mathcal{G}(z,t) = -c\lambda|z|H(t)\frac{J_1\left(\lambda\sqrt{c^2t^2+2|z|ct}\right)}{\sqrt{c^2t^2+2|z|ct}} \tag{5.4}$$

which propagates an excitation at $z=0$ in the positive or the negative z-direction depending on the sign of z. This propagator kernel is closely related to the Green function of the Klein-Gordon equation. For further reference, see Refs. [13, 22].

5.2 Fundamental solutions

For any excitation $u^\pm(0,t)$ at $z=0$, the response $u^\pm(z,t)$ is therefore found as

$$u^\pm(z,t\pm z/c) = u^\pm(0,t) + \int_{-\infty}^{t} \mathcal{G}(z,t-t')u^\pm(0,t')\,dt' \tag{5.5}$$

where the plus sign holds in the region $z>0$ and the negative sign in the region $z<0$. Notice that the time parameter t is the time measured from the wave front, which propagates with speed c. An alternative way of writing this expression using real time is

$$u^\pm(z,t) = u^\pm(0,t-|z|/c) - c\lambda|z|\int_{-\infty}^{t-|z|/c} \frac{J_1\left(\lambda\sqrt{c^2(t-t')^2-z^2}\right)}{\sqrt{c^2(t-t')^2-z^2}}u^\pm(0,t')\,dt'$$

By construction, these wave are right-going ($z>0$) or left-going ($z<0$) waves. Note that different modes (different λ) basically is a scaling in the space and time variables. The dimensionless variables, in which the scaling is performed, are

$$\begin{cases} x = \lambda z \\ s = \lambda ct \end{cases}$$

5.3 First precursor of the slab problem

Related to the wave propagation problem in the waveguide, is the propagation of a electromagnetic wave in a homogeneous Lorentz medium. This propagation problem and the related precursor (forerunner) analysis (early arrival of the signal) is analyzed in, e.g., Refs. [8, Sect. 7.11] and [1]. In a Lorentz medium the (transversely polarized) electric field $E(z,t)$ satisfies

$$\frac{\partial^2}{\partial z^2}E(z,t) - \frac{\mu}{c_0^2}\frac{\partial^2}{\partial t^2}\left[\epsilon E(z,t) + \left(\chi(\cdot) * E(z,\cdot)\right)(t)\right] = 0 \tag{5.6}$$

where c_0 is the speed of light in vacuum, ϵ and μ are the permittivity and the permeability of the medium, respectively, and $\chi(t)$ the susceptibility kernel of the medium given by [8]

$$\chi(t) = H(t)\frac{\omega_p^2}{\nu_0}e^{-\nu t/2}\sin\nu_0 t$$

The plasma and the collision frequencies are ω_p and ν, respectively. The frequency ν_0 is defined in terms of the resonance frequency ω_0 as

$$\nu_0 = \sqrt{\omega_0^2 - \nu^2/4}$$

The short time behavior of $\chi(t)$ is

$$\chi(t) \approx \omega_p^2 t, \qquad t \to 0^+$$

If the medium is characterized by several resonances, this expression is modified such that the short time behavior of χ is t times a sum of the squares of the corresponding plasma frequencies.

The approximate behavior of the convolution term in (5.6) evaluated at short times is

$$\frac{\partial^2}{\partial t^2} \left(\chi(\cdot) * E(z,\cdot)\right)(t) \approx \omega_p^2 E(z,t), \qquad t \to 0^+$$

The wave propagating problem in a Lorentz medium at short times after the arrival of the wave front is therefore formally identical to the waveguide propagation problem treated in this paper if λ is identified as

$$\lambda = \frac{\omega_p \sqrt{\mu}}{c_0}$$

The propagator kernel $\mathcal{G}^+(z,t)$ in (5.4) is therefore the appropriate propagator of the electric field in the slab for short times t (from the wave front). The field $E(z,t)$ at a point $z > 0$ in the slab is related to the excitation $E(0,t)$, see (5.5).

$$E(z, t + z/c) = E(0,t) - 2\xi \int_{-\infty}^{t} \frac{J_1\left(\sqrt{\lambda^2 c^2 (t-t')^2 + 4\xi(t-t')}\right)}{\sqrt{\lambda^2 c^2 (t-t')^2 + 4\xi(t-t')}} E(0,t')\, dt' \tag{5.7}$$

where

$$\xi = \frac{c\lambda^2 z}{2} = \frac{\omega_p^2 z}{2c_0}\sqrt{\frac{\mu}{\epsilon}} = \frac{\omega_p^2 z}{2c\epsilon}$$

At large distances z, the integrand in (5.7) can be approximated as ($t > 0$)

$$E(z, t + z/c) = \frac{t^m}{m!} - \frac{2\xi}{m!} \int_0^t \frac{J_1\left(2\sqrt{\xi(t-t')}\right)}{2\sqrt{\xi(t-t')}} t'^m \, dt'$$

where the excitation at $z = 0$ is assumed to have the explicit form

$$E(0,t) = H(t)\frac{t^m}{m!}, \qquad m \geq 0$$

The integral can be evaluated by using [14]

$$\int_0^a J_1(x)\left(a^2 - x^2\right)^m dx = a^{2m} - (2a)^m m! J_m(a)$$

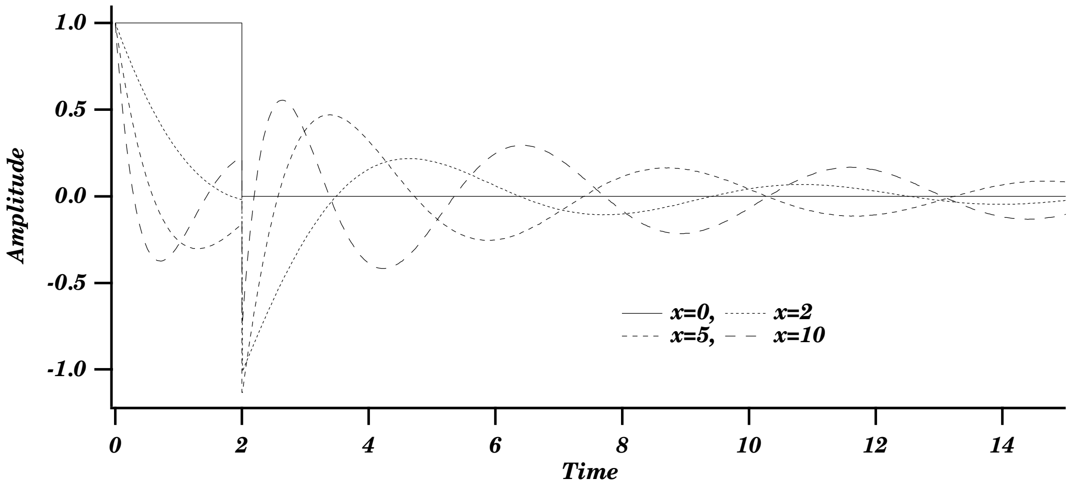

Figure 2: The amplitude of the z-component of the field in the waveguide as a function of the time t, measured from the wave front, i.e., the real time is $t + |z|/c$. The input is given by (5.8), where $T = 2$. The time scale is given in units of $(c\lambda_n)^{-1}$ and the parameter $x = \lambda_n|z|$.

and the field at large distances is ($t > 0$)

$$E(z, t + z/c) = \left(\frac{t}{\xi}\right)^{m/2} J_m\left(2\sqrt{\xi t}\right)$$

which is in agreement with the result in Ref. [8]. However, the solution to the precursor problem, see (5.7), is not restricted to the special excitation which is used here to illustrate the result. Furthermore, it is not restricted to large values of z, but holds for all values of z, provided the time t (from the wave front) is small. In this sense, it is a generalization of the classical precursor results.

5.4 Numerical illustrations

The response of a pulse excitation

$$u^+(0, t) = H(t)H(T - t) \tag{5.8}$$

is depicted in Figure 2. The transient behavior of the field at increasing distance along the waveguide is clearly illustrated for this excitation. At larger distances down the waveguide the pulse gets broadened and more oscillations occur.

In Figures 3 and 4 the shape of the wave is depicted at different positions along the waveguide. In these examples the exciting field is the modulated sine wave.

$$u^+(0, t) = H(t)H(\pi/\omega_m - t)\sin\omega_m t \sin\omega_c t \tag{5.9}$$

In Figures 5, 6 and 7 the transient behavior of a sinusoidal excitation is shown at different positions along the waveguide. The exciting field in these plots is

$$u^+(0, t) = H(t)\sin\omega t$$

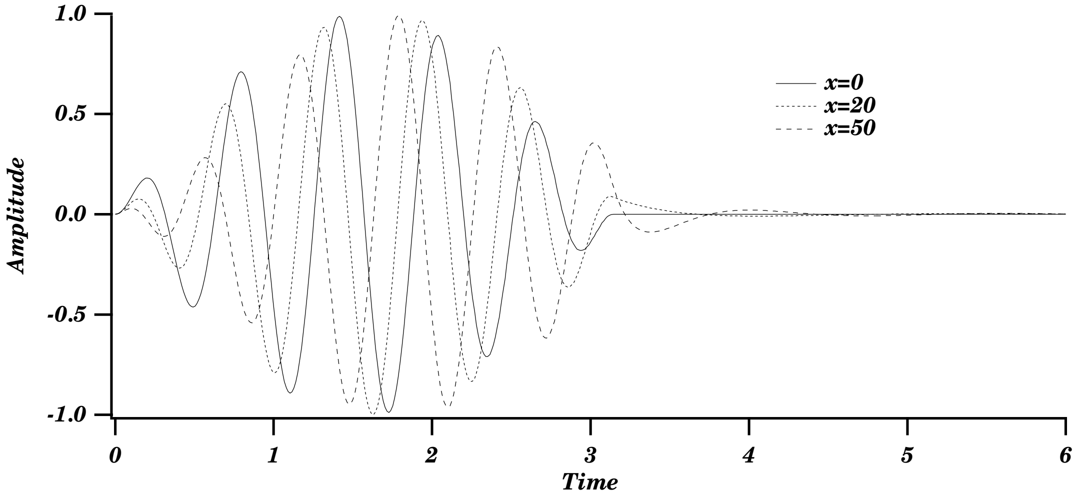

Figure 3: The amplitude of the z-component of the field in the waveguide as a function of the time t, measured from the wave front, i.e., the real time is $t + |z|/c$. The input is given by (5.9), where the carrier frequency is $\omega_c = 10$ and the pulse is modulation frequency $\omega_m = 1$. The time scale is given in units of $(c\lambda_n)^{-1}$, the frequency scale in units of $c\lambda_n$ and the parameter $x = \lambda_n|z|$.

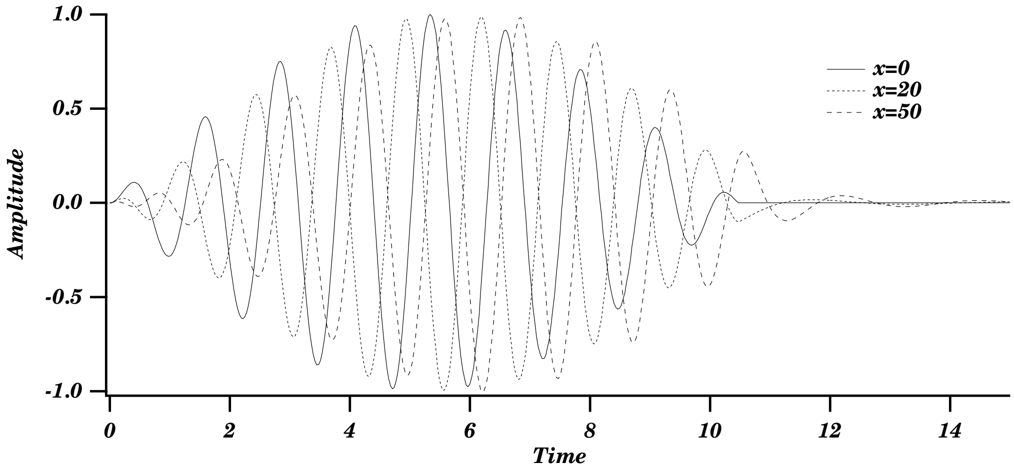

Figure 4: The amplitude of the z-component of the field in the waveguide as a function of the time t, measured from the wave front, i.e., the real time is $t + |z|/c$. The input is given by (5.9), where the carrier frequency is $\omega_c = 5$ and the pulse is modulation frequency $\omega_m = .3$. The time scale is given in units of $(c\lambda_n)^{-1}$, the frequency scale in units of $c\lambda_n$ and the parameter $x = \lambda_n|z|$.

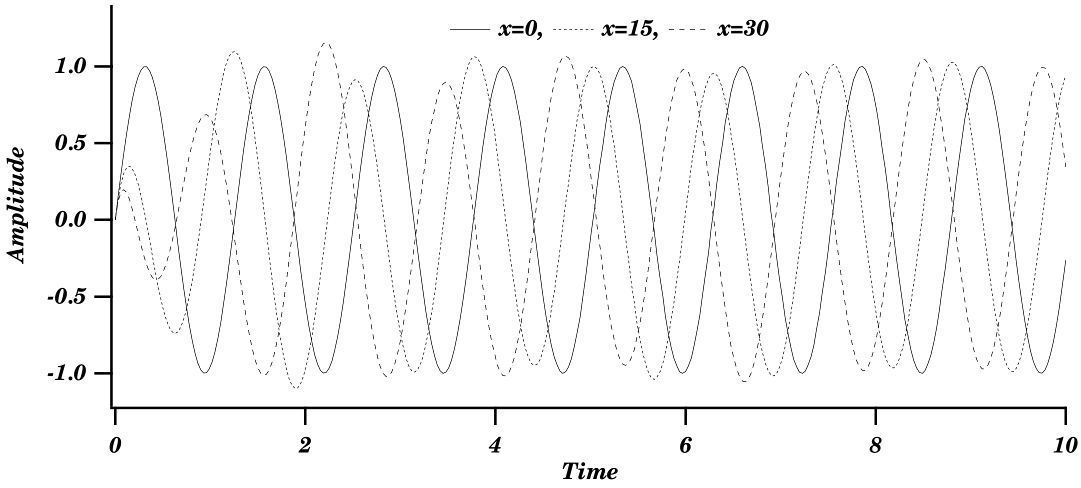

Figure 5: The amplitude of the z-component of the field in the waveguide as a function of the time t, measured from the wave front, i.e., the real time is $t + |z|/c$. The input is given by $u^+(0,t) = H(t)\sin\omega t$, where $\omega = 5$. The time scale is given in units of $(c\lambda_n)^{-1}$, the frequency scale in units of $c\lambda_n$ and the parameter $x = \lambda_n|z|$.

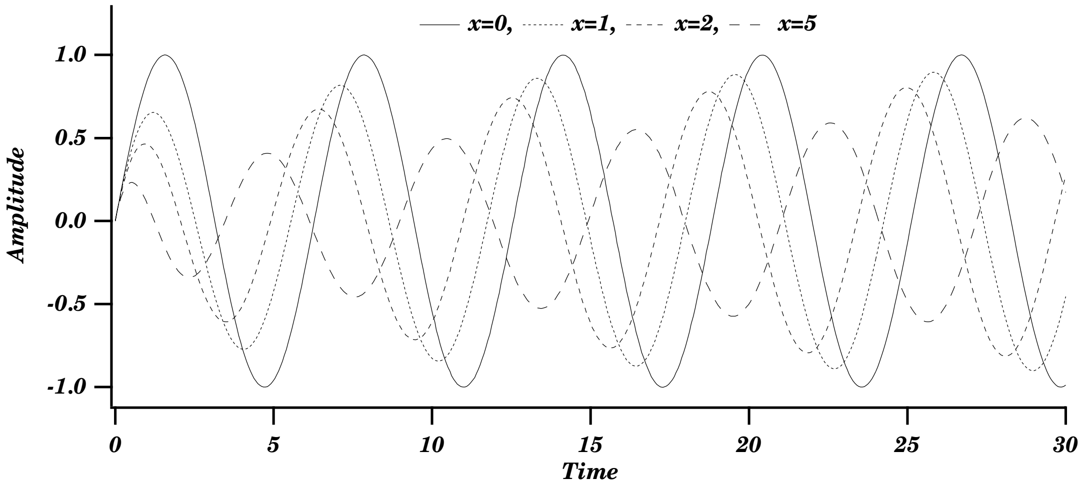

Figure 6: The amplitude of the z-component of the field in the waveguide as a function of the time t, measured from the wave front, i.e., the real time is $t + |z|/c$. The input is given by $u^+(0,t) = H(t)\sin\omega t$, where $\omega = 1$. The time scale is given in units of $(c\lambda_n)^{-1}$, the frequency scale in units of $c\lambda_n$ and the parameter $x = \lambda_n|z|$.

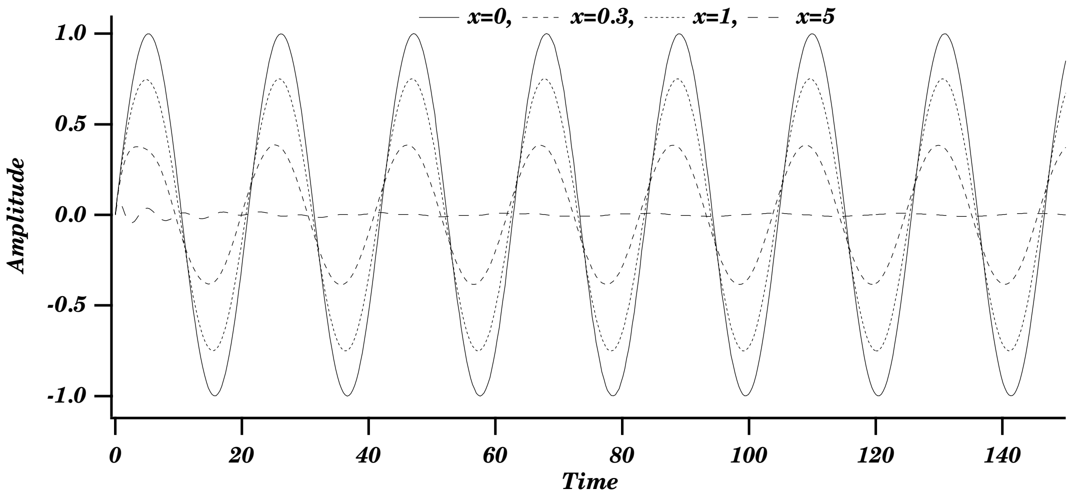

Figure 7: The amplitude of the z-component of the field in the waveguide as a function of the time t, measured from the wave front, i.e., the real time is $t + |z|/c$. The input is given by $u^+(0,t) = H(t) \sin \omega t$, where $\omega = .3$. The time scale is given in units of $(c\lambda_n)^{-1}$, the frequency scale in units of $c\lambda_n$ and the parameter $x = \lambda_n |z|$.

The cutoff frequency for the stationary wave in these examples is $\omega = 1$. Figure 5 shows the transient behavior for a propagating wave ($\omega = 5$). Figure 6 shows the effects for a wave at cutoff ($\omega = 1$) and Figure 7 for a wave below cutoff ($\omega = .3$). Below cutoff the wave is rapidly attenuating.

6 Power flux

From the definition of the split fields u^+ and u^-, it is not obvious that these waves give energy transport in the positive and the negative z-direction, respectively. In this section, the energy flow of each such wave is analyzed.

The power flux is

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} = (\boldsymbol{E}_T + \hat{z}E_z) \times (\boldsymbol{H}_T + \hat{z}H_z) = \boldsymbol{E}_T \times \boldsymbol{H}_T + E_z\hat{z} \times \boldsymbol{H}_T + \boldsymbol{E}_T \times \hat{z}H_z$$

The longitudinal power flux $\boldsymbol{S} \cdot \hat{z}$ is

$$\boldsymbol{S} \cdot \hat{z} = (\boldsymbol{E} \times \boldsymbol{H}) \cdot \hat{z} = (\boldsymbol{E}_T \times \boldsymbol{H}_T) \cdot \hat{z} = \begin{cases} \frac{1}{\eta}|\nabla_T v|^2 \psi_1 \phi_1 & \text{(TM-case)} \\ \frac{1}{\eta}|\nabla_T w|^2 \psi_2 \phi_2 & \text{(TE-case)} \end{cases}$$

where equation (3.3) has been used. The direction of the power flow is determined by the factor $\psi_i \phi_i$. From (3.4) and the wave splitting, it is easy to use (4.5) to obtain (u is either a or b depending upon the mode)

$$\lambda^4 \psi_i \phi_i = -\frac{1}{c}\frac{\partial u}{\partial t}\frac{\partial u}{\partial z} = \pm\frac{1}{c^2}\frac{\partial u}{\partial t} K^{-1} u = \pm\frac{1}{c^2}\frac{\partial u}{\partial t}\left\{\frac{\partial u}{\partial t} + (L(\cdot) * u(\cdot))\,(t)\right\}$$

where the upper (lower) sign holds for a right-(left-)going field. With the convenient change of variable, $x = c\lambda t$, the previous expression is equivalent to analyzing

$$F(x) = \pm f'(x) \left\{ f'(x) + \int_{-\infty}^{x} \frac{J_1(x-y)}{x-y} f(y)\, dy \right\}$$

This is, in general, not a function of a definite sign (compare with a superposition of a propagating and a non-propagating mode for fixed frequency).

The total energy up to time τ is

$$\int_{-\infty}^{\tau} F(x)\, dx = \pm \int_{-\infty}^{\tau} \left(f'(x)\right)^2\, dx \pm \int_{-\infty}^{\tau} f'(x) \int_{-\infty}^{x} \frac{J_1(x-y)}{x-y} f(y)\, dy\, dx$$

Define an even function $g(x)$

$$g(x) = \int_0^{|x|} \frac{J_1(y)}{y}\, dy$$

Then $\lim_{x\to\infty} g(x) = 1$ (see [6, p. 684]) and

$$\begin{aligned} \int_{-\infty}^{\tau} F(x)\, dx &= \pm \int_{-\infty}^{\tau} \left(f'(x)\right)^2\, dx \pm \int_{-\infty}^{\tau} f'(x) \int_{-\infty}^{x} g(x-y) f'(y)\, dy\, dx \\ &= \pm \int_{-\infty}^{\tau} \int_{-\infty}^{\tau} f'(x) G(x-y) f'(y)\, dy\, dx \end{aligned}$$

where

$$G(x) = \delta(x) + \frac{1}{2} \int_0^{|x|} \frac{J_1(y)}{y}\, dy$$

which is even in x.

It is straightforward to prove that

$$\int_{-\infty}^{\infty} G(x) \cos \xi x\, dx = \pi\delta(\xi) + H(|\xi| - 1) \frac{\sqrt{\xi^2 - 1}}{|\xi|}$$

which is an even, positive function (distribution) of ξ. This implies that $G(x)$ is a function (distribution) of positive type (Bochner-Schwartz's theorem, see [15, p. 14]) and therefore

$$\int_{-\infty}^{\tau} \int_{-\infty}^{\tau} f'(x) G(x-y) f'(y)\, dy\, dx \geq 0$$

for all functions $f(x)$.

This inequality implies that the total energy

$$\int_{-\infty}^{\tau} \boldsymbol{S}(\boldsymbol{r}, t) \cdot \hat{z}\, dt$$

in the waveguide up to time τ is positive for a right-going wave and negative for a left-going wave, irrespective of the excitation and the time τ.

7 Expansion functions

To tackle the excitation of the field inside the waveguide and to determine the amplitude of each mode, expansion functions are needed. In this section, the appropriate expansion functions for transient waves are developed.

7.1 Orthogonal modes for transverse fields

In this section, the index n is used on the eigenfunctions v and w. An additional index ν is appended to indicate which mode it is.

Define for ν=TM

$$\begin{cases} \boldsymbol{e}_{n\nu}(\boldsymbol{\rho}) = \nabla_T v_n(\boldsymbol{\rho}) \\ \boldsymbol{h}_{n\nu}(\boldsymbol{\rho}) = \hat{z} \times \nabla_T v_n(\boldsymbol{\rho}) = \hat{z} \times \boldsymbol{e}_{n\nu}(\boldsymbol{\rho}) \end{cases} \qquad \text{(TM-case)}$$

and for ν=TE

$$\begin{cases} \boldsymbol{e}_{n\nu}(\boldsymbol{\rho}) = -\hat{z} \times \nabla_T w_n(\boldsymbol{\rho}) \\ \boldsymbol{h}_{n\nu}(\boldsymbol{\rho}) = \nabla_T w_n(\boldsymbol{\rho}) = \hat{z} \times \boldsymbol{e}_{n\nu}(\boldsymbol{\rho}) \end{cases} \qquad \text{(TE-case)}$$

Notice that $\boldsymbol{e}_{n\nu}(\boldsymbol{\rho})$ and $\boldsymbol{h}_{n\nu}(\boldsymbol{\rho})$ both have the same units. The functions v_n and w_n satisfy

$$\begin{cases} \nabla_T^2 v_n(\boldsymbol{\rho}) + \lambda_n^2 v_n(\boldsymbol{\rho}) = 0, & \boldsymbol{\rho} \in \Omega \\ v_n(\boldsymbol{\rho}) = 0, & \boldsymbol{\rho} \in \partial\Omega \end{cases}$$

and

$$\begin{cases} \nabla_T^2 w_n(\boldsymbol{\rho}) + \lambda_n^2 w_n(\boldsymbol{\rho}) = 0, & \boldsymbol{\rho} \in \Omega \\ \hat{n} \cdot \nabla_T w_n(\boldsymbol{\rho}) = 0, & \boldsymbol{\rho} \in \partial\Omega \end{cases}$$

respectively.

The functions v_n and w_n can always be normalized such that ($\lambda_n \neq 0$)

$$\iint_\Omega v_n(\boldsymbol{\rho}) v_m(\boldsymbol{\rho})\, dS = \iint_\Omega w_n(\boldsymbol{\rho}) w_m(\boldsymbol{\rho})\, dS = \lambda_n^{-2} \delta_{n,m}$$

By doing so, it is easy to see that the vector functions $\boldsymbol{e}_{n\nu}$ and $\boldsymbol{h}_{n\nu}$ are orthonormal.

$$\iint_\Omega \boldsymbol{e}_{n\nu}(\boldsymbol{\rho}) \cdot \boldsymbol{e}_{n'\nu'}(\boldsymbol{\rho})\, dS = \iint_\Omega \boldsymbol{h}_{n\nu}(\boldsymbol{\rho}) \cdot \boldsymbol{h}_{n'\nu'}(\boldsymbol{\rho})\, dS = \delta_{n,n'} \delta_{\nu,\nu'}$$

and

$$\iint_\Omega [\boldsymbol{e}_{n\nu}(\boldsymbol{\rho}) \times \boldsymbol{h}_{n'\nu'}(\boldsymbol{\rho})] \cdot \hat{z}\, dS = \delta_{n,n'} \delta_{\nu,\nu'} \tag{7.1}$$

The sequences $\{\boldsymbol{e}_{n\nu}\}_{n=1}^{\infty}$ and $\{\boldsymbol{h}_{n\nu}\}_{n=1}^{\infty}$, ν=TM, TE form a complete orthonormal set of vector-valued functions in the plane.

7.2 Expansion functions

Let $f_{n\nu}(z,t)$ be any function satisfying

$$\frac{\partial^2}{\partial z^2} f_{n\nu}(z,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} f_{n\nu}(z,t) - \lambda_n^2 f_{n\nu}(z,t) = 0$$

and define for ν=TM

$$\begin{cases} \boldsymbol{E}_{n\nu}(\boldsymbol{r},t) = \boldsymbol{e}_{n\nu}(\boldsymbol{\rho})\lambda_n^{-2}\dfrac{\partial}{\partial z} f_{n\nu}(z,t) + \hat{z} v_n(\boldsymbol{\rho}) f_{n\nu}(z,t) \\ \boldsymbol{H}_{n\nu}(\boldsymbol{r},t) = -\dfrac{1}{c}\boldsymbol{h}_{n\nu}(\boldsymbol{\rho})\lambda_n^{-2}\dfrac{\partial}{\partial t} f_{n\nu}(z,t) \end{cases} \qquad \text{(TM-case)}$$

and for ν=TE

$$\begin{cases} \boldsymbol{E}_{n\nu}(\boldsymbol{r},t) = -\dfrac{1}{c}\boldsymbol{e}_{n\nu}(\boldsymbol{\rho})\lambda_n^{-2}\dfrac{\partial}{\partial t} f_{n\nu}(z,t) \\ \boldsymbol{H}_{n\nu}(\boldsymbol{r},t) = \boldsymbol{h}_{n\nu}(\boldsymbol{\rho})\lambda_n^{-2}\dfrac{\partial}{\partial z} f_{n\nu}(z,t) + \hat{z} w_n(\boldsymbol{\rho}) f_{n\nu}(z,t) \end{cases} \qquad \text{(TE-case)}$$

Then these vector-valued functions satisfy

$$\begin{cases} \nabla \times \boldsymbol{E}_{n\nu}(\boldsymbol{r},t) = -\dfrac{1}{c}\dfrac{\partial}{\partial t}\boldsymbol{H}_{n\nu}(\boldsymbol{r},t) \\ \nabla \times \boldsymbol{H}_{n\nu}(\boldsymbol{r},t) = \dfrac{1}{c}\dfrac{\partial}{\partial t}\boldsymbol{E}_{n\nu}(\boldsymbol{r},t) \end{cases} \tag{7.2}$$

and

$$\begin{cases} \nabla \cdot \boldsymbol{E}_{n\nu}(\boldsymbol{r},t) = 0 \\ \nabla \cdot \boldsymbol{H}_{n\nu}(\boldsymbol{r},t) = 0 \end{cases}$$

and the boundary condition on S

$$\hat{n} \times \boldsymbol{E}_{n\nu}(\boldsymbol{r},t) = \boldsymbol{0}, \qquad \boldsymbol{r} \in S \tag{7.3}$$

Notice also that both fields $\boldsymbol{E}_{n\nu}(\boldsymbol{r},t)$ and $\boldsymbol{H}_{n\nu}(\boldsymbol{r},t)$ have the same units.

7.3 Right- and left-going basis functions

The eigenfunctions $\boldsymbol{E}_{n\nu}$ and $\boldsymbol{H}_{n\nu}$ defined in Section 7.2 are, in general, superpositions of both right- and left-going waves. The right- and left-going eigenmodes $\boldsymbol{E}^{\pm}_{n\nu}(\boldsymbol{\rho},z,t)$ and $\boldsymbol{H}^{\pm}_{n\nu}(\boldsymbol{\rho},z,t)$ are obtained by taking an appropriate choice of the function $f_{n\nu}(z,t)$. The obvious choice here is $u(z,t)$ defined in (5.5). From the definition of right- and left-going waves, see (4.9), the spatial differentiation with respect to z is replaced by the K^{-1}-operator, with the appropriate sign depending on whether the wave is right- or left-going. This operation guarantees that right- and left-going wave are constructed. The definition of $\boldsymbol{E}^{\pm}_{n\nu}(\boldsymbol{\rho},z,t)$ and $\boldsymbol{H}^{\pm}_{n\nu}(\boldsymbol{\rho},z,t)$ for ν=TM is therefore

$$\begin{cases} \boldsymbol{E}^{\pm}_{n\nu}(\boldsymbol{\rho},z,t) = \mp\dfrac{1}{c}\boldsymbol{e}_{n\nu}(\boldsymbol{\rho})\lambda_n^{-2}\left[K^{-1}\Delta_n(z,\cdot)\right](t-|z|/c) + \hat{z} v_n(\boldsymbol{\rho})\Delta_n(z,t-|z|/c) \\ \boldsymbol{H}^{\pm}_{n\nu}(\boldsymbol{\rho},z,t) = -\dfrac{1}{c}\boldsymbol{h}_{n\nu}(\boldsymbol{\rho})\lambda_n^{-2}\dfrac{\partial}{\partial t}\Delta_n(z,t-|z|/c) \end{cases}$$

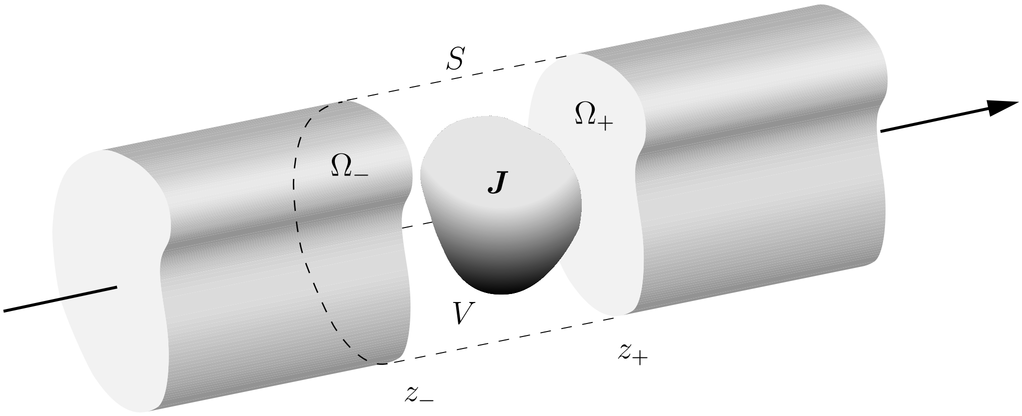

Figure 8: Geometry of the source region.

and for ν=TE

$$\begin{cases} \boldsymbol{E}^{\pm}_{n\nu}(\boldsymbol{\rho},z,t) = \pm\dfrac{1}{c}\boldsymbol{e}_{n\nu}(\boldsymbol{\rho})\lambda_n^{-2}\dfrac{\partial}{\partial t}\Delta_n(z,t-|z|/c) \\ \boldsymbol{H}^{\pm}_{n\nu}(\boldsymbol{\rho},z,t) = \dfrac{1}{c}\boldsymbol{h}_{n\nu}(\boldsymbol{\rho})\lambda_n^{-2}\left[K^{-1}\Delta_n(z,\cdot)\right](t-|z|/c) \mp \hat{z}w_n(\boldsymbol{\rho})\Delta_n(z,t-|z|/c) \end{cases}$$

in the regions $z>0$ (plus sign) and $z<0$ (minus sign), respectively, and where the operator

$$\Delta_n(z,t) = \delta(t) + \mathcal{G}_n(z,t)$$

and $\mathcal{G}(z,t)$ is defined in (5.4). In the region $z>0$, $\boldsymbol{E}^{+}_{n\nu}(\boldsymbol{\rho},z,t)$ and $\boldsymbol{H}^{+}_{n\nu}(\boldsymbol{\rho},z,t)$ are right-going fields, and in the region $z<0$, $\boldsymbol{E}^{-}_{n\nu}(\boldsymbol{\rho},z,t)$ and $\boldsymbol{H}^{-}_{n\nu}(\boldsymbol{\rho},z,t)$ are left-going fields. The action on a function $f(t)$ is always as a time convolution

$$\begin{aligned} \left(\Delta_n(z,\cdot)f(\cdot)\right)(t) &= f(t) + \int_{-\infty}^{t} \mathcal{G}_n(z,t-t')f(t')\,dt' \\ &= f(t) - c\lambda_n|z|\int_{-\infty}^{t} \frac{J_1\left(\lambda_n\sqrt{c^2(t-t')^2+2|z|c(t-t')}\right)}{\sqrt{c^2(t-t')^2+2|z|c(t-t')}} f(t')\,dt' \end{aligned}$$

The action of Δ_n on a function $f(t)$ therefore gives a field $\left(\Delta_n(z,\cdot)f(\cdot)\right)(t)$ that is right- or left-going field, respectively, depending on the sign of z.

8 The source problem

So far, source-free regions in the waveguide have been considered. In this section, the excitation of the different modes are addressed.

The sources of the fields are assumed to be located in a finite section of the wave guide in the interval $[z_-, z_+]$, as in Figure 8. Straightforward calculations using (2.1) and (7.2) yield

$$\int_{-\infty}^{\infty} \nabla \cdot \left[\boldsymbol{E}(\boldsymbol{r}, t-t') \times \boldsymbol{H}_{n\nu}^{\pm}(\boldsymbol{\rho}, z - z_{\mp}, t') + \eta \boldsymbol{H}(\boldsymbol{r}, t-t') \times \boldsymbol{E}_{n\nu}^{\pm}(\boldsymbol{\rho}, z - z_{\mp}, t') \right] dt'$$
$$= \eta \int_{-\infty}^{\infty} \boldsymbol{J}(\boldsymbol{r}, t-t') \cdot \boldsymbol{E}_{n\nu}^{\pm}(\boldsymbol{\rho}, z - z_{\mp}, t')\, dt', \qquad z \in [z_-, z_+]$$

since all fields are assumed to vanish as $t \to -\infty$. Integrate this expression over the volume V bounded by the surfaces S, Ω_- and Ω_+, see Figure 8, and use Gauss' theorem. Due to the boundary conditions (2.5) and (7.3), there is no contribution from S.

$$\begin{aligned}
\eta \int_{-\infty}^{\infty} \iiint_V & \boldsymbol{J}(\boldsymbol{r}, t-t') \cdot \boldsymbol{E}_{n\nu}^{\pm}(\boldsymbol{\rho}, z - z_{\mp}, t')\, dv\, dt' \\
&= \int_{-\infty}^{\infty} \iint_{\Omega_+} \left[\boldsymbol{E}(\boldsymbol{r}, t-t') \times \boldsymbol{H}_{n\nu}^{\pm}(\boldsymbol{\rho}, z - z_{\mp}, t') \right. \\
&\qquad\qquad \left. + \eta \boldsymbol{H}(\boldsymbol{r}, t-t') \times \boldsymbol{E}_{n\nu}^{\pm}(\boldsymbol{\rho}, z - z_{\mp}, t') \right] \cdot \hat{z}\, dS\, dt' \\
&- \int_{-\infty}^{\infty} \iint_{\Omega_-} \left[\boldsymbol{E}(\boldsymbol{r}, t-t') \times \boldsymbol{H}_{n\nu}^{\pm}(\boldsymbol{\rho}, z - z_{\mp}, t') \right. \\
&\qquad\qquad \left. + \eta \boldsymbol{H}(\boldsymbol{r}, t-t') \times \boldsymbol{E}_{n\nu}^{\pm}(\boldsymbol{\rho}, z - z_{\mp}, t') \right] \cdot \hat{z}\, dS\, dt'
\end{aligned}$$

In a region to the left of the sources, $z < z_-$, the fields are assumed to be a superposition of left-going waves.

$$\begin{cases} \boldsymbol{E}(\boldsymbol{r}, t) = \displaystyle\sum_{n\nu} \left(F_{n\nu}^{-}(\cdot) * \boldsymbol{E}_{n\nu}^{-}(\boldsymbol{\rho}, z - z_-, \cdot) \right)(t), \\ \eta \boldsymbol{H}(\boldsymbol{r}, t) = \displaystyle\sum_{n\nu} \left(F_{n\nu}^{-}(\cdot) * \boldsymbol{H}_{n\nu}^{-}(\boldsymbol{\rho}, z - z_-, \cdot) \right)(t), \end{cases} \qquad z < z_-$$

and the fields in the region to the right of the sources, $z > z_+$, are right-going waves

$$\begin{cases} \boldsymbol{E}(\boldsymbol{r}, t) = \displaystyle\sum_{n\nu} \left(F_{n\nu}^{+}(\cdot) * \boldsymbol{E}_{n\nu}^{+}(\boldsymbol{\rho}, z - z_+, \cdot) \right)(t), \\ \eta \boldsymbol{H}(\boldsymbol{r}, t) = \displaystyle\sum_{n\nu} \left(F_{n\nu}^{+}(\cdot) * \boldsymbol{H}_{n\nu}^{+}(\boldsymbol{\rho}, z - z_+, \cdot) \right)(t), \end{cases} \qquad z > z_+$$

where the functions $F_{n\nu}^{\pm}(t)$ are unknown expansion functions depending only on time. The plus and the minus signs correspond to right- and left-going fields, respectively.

The explicit expansions in the region $z > z_+$ for the ν=TM mode are

$$\begin{cases} \boldsymbol{E}(\boldsymbol{r},t) = -\dfrac{1}{c}\displaystyle\sum_{n\nu} \boldsymbol{e}_{n\nu}(\boldsymbol{\rho})\lambda_n^{-2}\left[K^{-1}F_{n\nu}^{+}\right](t-(z-z_+)/c) \\ \qquad\qquad -\dfrac{1}{c}\displaystyle\sum_{n\nu} \boldsymbol{e}_{n\nu}(\boldsymbol{\rho})\lambda_n^{-2}\left[K^{-1}\left(\mathcal{G}(z-z_+,\cdot)*F_{n\nu}^{+}(\cdot)\right)\right](t-(z-z_+)/c) \\ \qquad\qquad +\hat{z}\displaystyle\sum_{n\nu} v_n(\boldsymbol{\rho})\Big[F_{n\nu}^{+}(t-(z-z_+)/c) \\ \qquad\qquad\quad +\left(\mathcal{G}(z-z_+,\cdot)*F_{n\nu}^{+}(\cdot)\right)(t-(z-z_+)/c)\Big] \\ \boldsymbol{H}(\boldsymbol{r},t) = -\dfrac{1}{c}\displaystyle\sum_{n\nu} \boldsymbol{h}_{n\nu}(\boldsymbol{\rho})\lambda_n^{-2}\dfrac{\partial}{\partial t}\Big[F_{n\nu}^{+}(t-(z-z_+)/c) \\ \qquad\qquad\quad +\left(\mathcal{G}(z-z_+,\cdot)*F_{n\nu}^{+}(\cdot)\right)(t-(z-z_+)/c)\Big] \end{cases}$$

Orthogonality implies (see (7.1)) that

$$\int_{-\infty}^{\infty}\iint_{\Omega^{\pm}}\big[\boldsymbol{E}(\boldsymbol{r},t-t')\times\boldsymbol{H}_{n\nu}^{\pm}(\boldsymbol{\rho},z-z_{\mp},t') \\ +\eta\boldsymbol{H}(\boldsymbol{r},t-t')\times\boldsymbol{E}_{n\nu}^{\pm}(\boldsymbol{\rho},z-z_{\mp},t')\big]\cdot\hat{z}\,dS\,dt' = 0$$

and

$$\begin{aligned}&\int_{-\infty}^{\infty}\iint_{\Omega^{\mp}}\big[\boldsymbol{E}(\boldsymbol{r},t-t')\times\boldsymbol{H}_{n\nu}^{\pm}(\boldsymbol{\rho},z-z_{\mp},t') \\ &\quad+\eta\boldsymbol{H}(\boldsymbol{r},t-t')\times\boldsymbol{E}_{n\nu}^{\pm}(\boldsymbol{\rho},z-z_{\mp},t')\big]\cdot\hat{z}\,dS\,dt' \\ &\quad=\mp\frac{2}{c^2\lambda_n^4}\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial t}F_{n\nu}^{\mp}(t)+\left(L(\cdot)*F_{n\nu}^{\mp}(\cdot)\right)(t)\right\}=\mp\frac{2}{c^2\lambda_n^4}\frac{\partial}{\partial t}\left(K^{-1}F_{n\nu}^{\mp}\right)(t)\end{aligned}$$

where the upper (lower) sign of the expansion functions should be read with upper (lower) sign of the integration domain. The expansion functions $F_n^{\pm}(t)$ therefore are

$$\begin{aligned}F_{n\nu}^{\pm}(t) =&\frac{\eta c^2\lambda_n^4}{2}K\partial_t^{-1}\int_{-\infty}^{\infty}\iiint_V \boldsymbol{J}(\boldsymbol{r},t-t')\cdot\boldsymbol{E}_{n\nu}^{\mp}(\boldsymbol{\rho},z-z_{\pm},t')\,dv\,dt' \\ =&\frac{\eta c^2\lambda_n^4}{2}\int_{-\infty}^{t}\int_{-\infty}^{t'} J_0\left(c\lambda(t'-t'')\right)\int_{-\infty}^{\infty} \\ &\iiint_V \boldsymbol{J}(\boldsymbol{r},t''-t''')\cdot\boldsymbol{E}_{n\nu}^{\mp}(\boldsymbol{\rho},z-z_{\pm},t''')\,dv\,dt'''\,dt''\,dt'\end{aligned}$$

which are explicit expressions of the unknown functions $F_{n\nu}^{\pm}$. From these expressions the unknown functions $F_{n\nu}^{\pm}(t)$ and the expansion of the fields in the regions outside the source are determined if the current density $\boldsymbol{J}$ is known.

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