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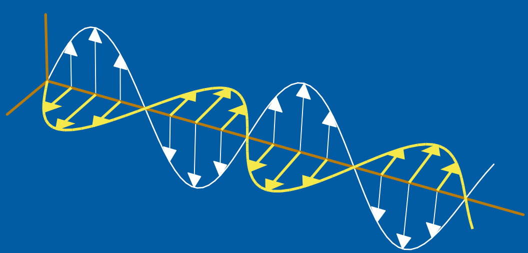

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Abstract

In this paper a systematic analysis that solves the wave propagation problem in a general bianisotropic, stratified media is presented. The method utilizes the concept of propagators, and the representation of these operators is simplified by introducing the Cayley-Hamilton theorem. The propagators propagate the total tangential electric and magnetic fields in the slab and only outside the slab the up/down-going parts of the fields need to be identified. This procedure makes the physical interpretation of the theory intuitive. The reflection and the transmission dyadics for a general bianisotropic medium with an isotropic (vacuum) half space on both sides of the slab are presented in a coordinate-independent dyadic notation, as well as the reflection dyadic for a bianisotropic slab with perfectly electric backing (PEC). In the latter case the current on the metal backing is also given. Some numerical computations that illustrate the algorithm are presented.

1 Introduction

The reflection and the transmission properties of a stratified slab, whose layers are bianisotropic, have been a subject of continued scientific interest over the last decades. The literature on this subject is large, see, e.g., textbooks [4, 12, 14, 21], and recent journal articles [2, 16, 22] and references given therein. The reason for a new investigation on this subject must be that a more systematic approach to solve the problem is available or that new insight is obtained in the numerical treatment or implementation of the problem. The latter reason is the motivation of the recent paper by Yang [22] in which the problem is treated with a spectral recursive transformation method. The motivation behind the present paper is mainly due to the first reason and to the fact that the analysis is presented in coordinate-independent dyadic notation and that it is physically intuitive.

The area of applications that apply to reflection and transmission of plane waves in stratified slabs is vast. We do not intend to give a complete exposition of this field here in this introduction, but refer to the excellent textbooks cited above that contain long lists of applications. Of particular interest and motivation behind the present analysis are the propagation of radar waves through radome walls. For a comprehensive treatment of radome-enclosed antennas, we refer to Ref. 13.

In this paper, the main tool to solve the scattering properties of a stratified slab is the notion of propagators. These operators propagate the *total* field from one position in the slab to another. This is in contrast to the more common approach of propagating the eigenmodes (up- and down-going fields), respectively, of the slab. Moreover, the reflection and transmission problems are treated in a concise way using a coordinate-free dyadic notation. The Cayley-Hamilton theorem simplifies the evaluation of the propagators. The results are then very easy to implement in *e.g.*, MATLAB or any other language that supports matrix manipulations.

The outline of this paper is as follows: In Section 2 the time-harmonic constitutive relations of a general linear, plane-stratified, and bianisotropic medium are presented. In Section 3 the fundamental equation for time-harmonic, plane-wave propagation in layered bianisotropic structures is given. This equation forms the basis for the present discussion. The wave propagator for a piecewise homogeneous, complex structure is derived in Section 4, and a wave splitting is presented in Section 5. Reflection and transmission are discussed in Section 6. In Section 7, we develop the theory for some particularly common and important classes of materials, e.g., isotropic and magnetic dielectrics, biisotropic or isotropic chiral media, and nonmagnetic uniaxial materials. Finally, in Section 8 some numerical computations are presented. A series of appendices contains the technical details of the analysis.

The results presented in this paper are given in a dyadic notion [15]. Scalars are typed in italic letters, vectors in italic boldface, and dyadics in roman boldface. The radius vector is denoted by $\boldsymbol{r} = \hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y + \hat{\boldsymbol{z}}z$, where $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ are the Cartesian basis vectors. Similarly, we denote the radius vector in the x-y-plane as $\boldsymbol{\rho} = \hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y$.

2 Basic equations

2.1 The Maxwell equations

The Maxwell equations model the dynamics of the fields in macroscopic media. The time-dependence of the electric and magnetic fields, $\boldsymbol{E}(\boldsymbol{r},\omega)$ and $\boldsymbol{H}(\boldsymbol{r},\omega)$, and the flux densities, $\boldsymbol{D}(\boldsymbol{r},\omega)$ and $\boldsymbol{B}(\boldsymbol{r},\omega)$, is assumed to be $e^{-i\omega t}$. In source-free regions, the time-harmonic Maxwell field equations are

$$\begin{cases} \nabla \times \boldsymbol{E} = ik_0\,(c_0\boldsymbol{B}) \\ \nabla \times (\eta_0\boldsymbol{H}) = -ik_0\,(c_0\eta_0\boldsymbol{D}) \end{cases} \tag{2.1}$$

where $\eta_0 = \sqrt{\mu_0/\epsilon_0}$ is the intrinsic impedance of vacuum, $c_0 = 1/\sqrt{\epsilon_0\mu_0}$ the speed of light in vacuum, and $k_0 = \omega/c_0$ is the wave number in vacuum.

2.2 The constitutive relations—bianisotropic case

The bianisotropic medium is the most general linear complex medium comprising at most 36 different scalar constitutive parameters (functions). The time-harmonic constitutive relations of a general bianisotropic medium are [14]

$$\begin{cases} \boldsymbol{D} = \epsilon_0\,\{\boldsymbol{\epsilon}\cdot\boldsymbol{E} + \eta_0\boldsymbol{\xi}\cdot\boldsymbol{H}\} \\ \boldsymbol{B} = \dfrac{1}{c_0}\,\{\boldsymbol{\zeta}\cdot\boldsymbol{E} + \eta_0\boldsymbol{\mu}\cdot\boldsymbol{H}\} \end{cases} \tag{2.2}$$

The dyadics $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ in (2.2) are the permittivity and the permeability dyadics, respectively, which for anisotropic materials are general, that is, comprising nine parameters each. For an isotropic medium, $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ are proportional to the identity dyadic[1] $\mathbf{I}_3$. In a biisotropic medium, which is the simplest complex material involving

[1]The identity dyadic in three dimensions is denoted $\mathbf{I}_3$ and in two dimensions (the x-y-plane) it is denoted $\mathbf{I}_2$.

the cross-coupling terms $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$, all the constitutive dyadics are proportional to the identity dyadic $\mathbf{I}_3$.

The four dyadics $\boldsymbol{\epsilon}$, $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$, and $\boldsymbol{\mu}$ depend in general of the spatial variables (x, y, z). For the case the material dyadics depend on the spatial variable z only, the medium is said to be plane-stratified (or simply stratified) in the z direction. For a homogeneous material, the constitutive dyadics are independent of (x, y, z). Notice that $\boldsymbol{\epsilon}$, $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$, and $\boldsymbol{\mu}$ generally are functions of the angular frequency ω owing to (material) temporal dispersion. Although dispersion is assumed to be anomalous in certain frequency intervals (absorption bands), the angular frequency ω is a fixed parameter in this paper. However, when the absorption bands and the frequency range of interest intersect, these effects must be taken into consideration. In these highly dispersive cases, time domain techniques are usually more effective [7, 8, 19].

2.3 Decomposition of dyadics

For the purpose of studying wave propagation in layered bianisotropic structures, it is appropriate to decompose each constitutive dyadic, *i.e.*, a three dimensional dyadic $\mathbf{A}$ is decomposed as

$$\mathbf{A} = \mathbf{A}_{\perp\perp} + \hat{\boldsymbol{z}}\boldsymbol{A}_z + \boldsymbol{A}_\perp\hat{\boldsymbol{z}} + \hat{\boldsymbol{z}}A_{zz}\hat{\boldsymbol{z}}$$

where

$$\begin{cases} \mathbf{A}_{\perp\perp} = \mathbf{I}_2 \cdot \mathbf{A} \cdot \mathbf{I}_2 \\ A_{zz} = \hat{\boldsymbol{z}} \cdot \mathbf{A} \cdot \hat{\boldsymbol{z}} \end{cases} \qquad \begin{cases} \boldsymbol{A}_z = \hat{\boldsymbol{z}} \cdot \mathbf{A} \cdot \mathbf{I}_2 \\ \boldsymbol{A}_\perp = \mathbf{I}_2 \cdot \mathbf{A} \cdot \hat{\boldsymbol{z}} \end{cases}$$

The dyadic $\mathbf{A}_{\perp\perp}$ is a two-dimensional dyadic in the x-y plane, and the vectors $\boldsymbol{A}_z$ and $\boldsymbol{A}_\perp$ are two two-dimensional vectors in this plane. A_{zz} is a scalar.

3 The fundamental equation

In this section, the scattering problem for a bianisotropic structure that is plane-stratified in the z direction is formulated. Stated differently, the medium in which the waves propagate has no variation in the coordinates x and y, *i.e.*, the medium is laterally homogeneous. Furthermore, it is assumed that the electromagnetic sources are located to the vacuum region $z < z_0$, see Figure 1.

In a geometry where the medium is laterally homogeneous in the variables x and y, it is natural to decompose the electromagnetic field in a spectrum of plane waves [5]. The plane wave decomposition amounts to a Fourier transformation of the electric and magnetic fields and flux densities with respect to the lateral variables x and y. The Fourier transform of a time-harmonic field $\boldsymbol{E}(\boldsymbol{r}, \omega)$ is denoted by

$$\boldsymbol{E}(z, \boldsymbol{k}_t, \omega) = \iint_{-\infty}^{\infty} \boldsymbol{E}(\boldsymbol{r}, \omega) e^{-i\boldsymbol{k}_t\cdot\boldsymbol{\rho}}\, dxdy \qquad z > z_0$$

where

$$\boldsymbol{k}_t = \hat{\boldsymbol{x}}k_x + \hat{\boldsymbol{y}}k_y = k_t\hat{\boldsymbol{e}}_\parallel$$

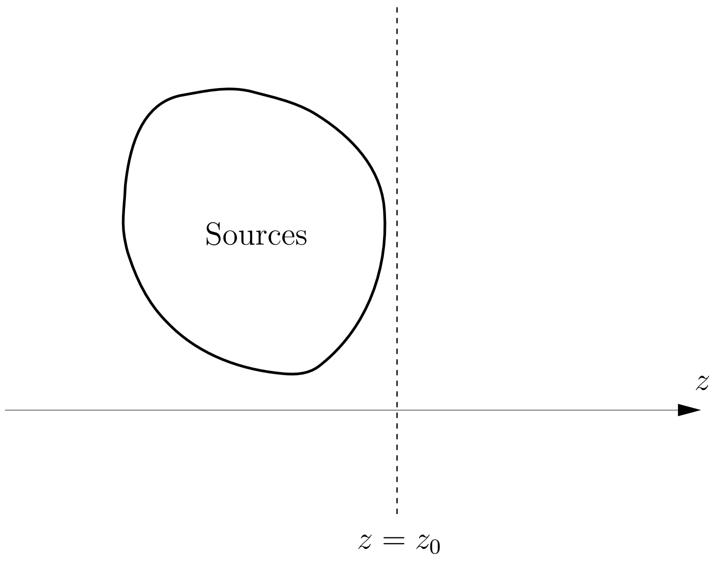


Figure 1: The source region and the plane, $z = z_0$, that limits its extent. Fourier transformation of the fields on any plane to the right of the source region, $z > z_0$, is well defined.

is the tangential wave vector and

$$k_t = \sqrt{k_x^2 + k_y^2}$$

the tangential wave number. The inverse Fourier transform is defined by

$$\boldsymbol{E}(\boldsymbol{r}, \omega) = \frac{1}{4\pi^2} \iint\limits_{-\infty}^{\infty} \boldsymbol{E}(z, \boldsymbol{k}_t, \omega) e^{i\boldsymbol{k}_t \cdot \boldsymbol{\rho}} \, dk_x dk_y \qquad z > z_0 \tag{3.1}$$

Notice that the same letter is used to denote the Fourier transform of the field and the field itself. The argument of the field shows what field is intended.

From now on, the tangential wave vector, $\boldsymbol{k}_t$, is fixed but arbitrary. Substituting the operator identity $\nabla = i\boldsymbol{k}_t + \hat{\boldsymbol{z}}\partial_z$ into the Maxwell field equations (2.1) gives a system of linear, coupled ordinary differential equations (ODEs). This is possible due to the fact that the medium is laterally homogeneous, and the reduction to a set of ODEs is, of course, not possible for a medium with variations in x or y. In vacuum regions, the solutions are either homogeneous, obliquely propagating plane waves or inhomogeneous (evanescent) plane waves depending on whether the tangential wave number, k_t, is less or greater than the wave number in vacuum, $k_0 = \omega/c_0$. It is appropriate to introduce an angle of incidence, θ_i, which is real for homogeneous plane waves, and a normal wave number, k_z, defined by

$$k_z = k_0 \cos\theta_i = \left(k_0^2 - k_t^2\right)^{1/2} = \begin{cases} \sqrt{k_0^2 - k_t^2} \text{ for } k_t < k_0 \\ i\sqrt{k_t^2 - k_0^2} \text{ for } k_t > k_0 \end{cases} \tag{3.2}$$

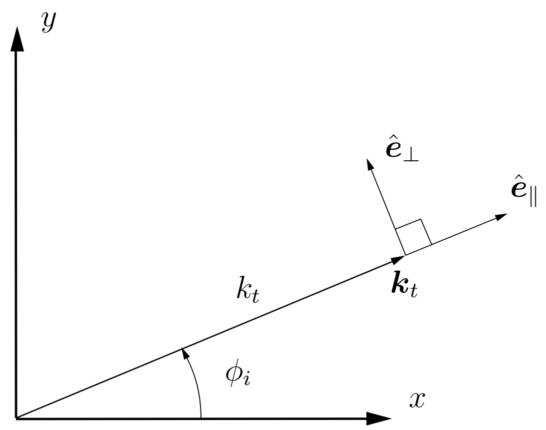


Figure 2: The Fourier variable $\boldsymbol{k}_t$ and the unit vectors $\hat{\boldsymbol{e}}_\parallel$ and $\hat{\boldsymbol{e}}_\perp$.

By this definition, k_z applies to up-going waves and $-k_z$ to down-going waves, see also wave splitting in Section 5. Furthermore, it is convenient to introduce a set of coordinate independent orthonormal basis vectors in the x-y plane by

$$\begin{cases} \hat{\boldsymbol{e}}_\parallel = \boldsymbol{k}_t/k_t = \hat{\boldsymbol{x}}\cos\phi_i + \hat{\boldsymbol{y}}\sin\phi_i \\ \hat{\boldsymbol{e}}_\perp = \hat{\boldsymbol{z}} \times \hat{\boldsymbol{e}}_\parallel = -\hat{\boldsymbol{x}}\sin\phi_i + \hat{\boldsymbol{y}}\cos\phi_i \end{cases}$$

where the azimuth angle of incidence, ϕ_i, is defined in Figure 2. The basis vectors $\{\hat{\boldsymbol{e}}_\parallel, \hat{\boldsymbol{e}}_\perp, \hat{\boldsymbol{z}}\}$ form a positively oriented ON-system. At normal incidence, this does not apply, and we define, *e.g.*, $\hat{\boldsymbol{e}}_\parallel = \hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{e}}_\perp = \hat{\boldsymbol{y}}$.

The Fourier components of the electric and magnetic fields can be decomposed in their tangential and normal components as

$$\begin{cases} \boldsymbol{E}(z, \boldsymbol{k}_t, \omega) = \boldsymbol{E}_{xy}(z) + \hat{\boldsymbol{z}}E_z(z) \\ \boldsymbol{H}(z, \boldsymbol{k}_t, \omega) = \boldsymbol{H}_{xy}(z) + \hat{\boldsymbol{z}}H_z(z) \end{cases}$$

Substituting the constitutive relations into the Maxwell field equations gives a system of ODEs in the tangential components of the electric and magnetic fields only. The fundamental equation for one-dimensional wave propagation becomes

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix} \boldsymbol{E}_{xy}(z) \\ \eta_0\mathbf{J}\cdot\boldsymbol{H}_{xy}(z) \end{pmatrix} = ik_0\mathbf{M}(z)\cdot\begin{pmatrix} \boldsymbol{E}_{xy}(z) \\ \eta_0\mathbf{J}\cdot\boldsymbol{H}_{xy}(z) \end{pmatrix} \tag{3.3}$$

where $\mathbf{J} = \hat{\boldsymbol{z}} \times \mathbf{I}_3 = \hat{\boldsymbol{z}} \times \mathbf{I}_2$ is a two-dimensional rotation dyadic (rotation of $\pi/2$ in the x-y-plane) and

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{pmatrix}$$

is the fundamental dyadic of the bianisotropic medium. Equation (3.3) is the general equation for wave propagation in general linear, laterally homogeneous, media. From the solution of this equation, all pertinent electromagnetic properties can be computed. The fundamental dyadic depends on the tangential wave vector, $\boldsymbol{k}_t$, and the constitutive dyadics, which may or may not depend on the depth z. The explicit expression for the fundamental dyadic is given in Appendix C. For homogeneous materials, $\mathbf{M}$ is independent of z. The fundamental dyadic in vacuum is

$$\mathbf{M}_0 = \begin{pmatrix} \mathbf{0} & -\mathbf{I}_2 + \frac{1}{k_0^2}\boldsymbol{k}_t\boldsymbol{k}_t \\ -\mathbf{I}_2 - \frac{1}{k_0^2}\mathbf{J}\cdot\boldsymbol{k}_t\boldsymbol{k}_t\cdot\mathbf{J} & \mathbf{0} \end{pmatrix} \tag{3.4}$$

The fundamental equation is derived in detail in Appendix C.

4 Propagation of fields

In this section, the wave propagator for a layered bianisotropic structure is introduced. The propagator maps the tangential electric and magnetic fields at the front surface of the structure to the tangential electric and magnetic fields at the rear surface of the structure. We first investigate the form of propagator in a single layer, then we apply this result to several layers.

4.1 Single layer

The propagator of a single layer, (z_1, z), is investigated first. This amounts to solving the fundamental equation (3.3) in the interval (z_1, z). The formal solution can be written in the form

$$\begin{pmatrix} \boldsymbol{E}_{xy}(z) \\ \eta_0\mathbf{J}\cdot\boldsymbol{H}_{xy}(z) \end{pmatrix} = \mathcal{S}\exp\left\{ik_0\int_{z_1}^{z}\mathbf{M}(z')\,dz'\right\}\cdot\begin{pmatrix} \boldsymbol{E}_{xy}(z_1) \\ \eta_0\mathbf{J}\cdot\boldsymbol{H}_{xy}(z_1) \end{pmatrix}$$

where $\mathcal{S}$ is the spatial ordering operator [8, 9]. This operator corresponds to the time ordering operator which appears in quantum mechanics [3]. Naturally, the propagator

$$\mathbf{P}(z, z_1) = \mathcal{S}\exp\left\{ik_0\int_{z_1}^{z}\mathbf{M}(z')\,dz'\right\}, \qquad \mathbf{P}(z_1, z_1) = \mathbf{I}_4$$

can be calculated numerically using standard ODE solvers. For a homogeneous material, an explicit solution can be obtained:

$$\begin{pmatrix} \boldsymbol{E}_{xy}(z) \\ \eta_0\mathbf{J}\cdot\boldsymbol{H}_{xy}(z) \end{pmatrix} = \mathbf{P}(z, z_1)\cdot\begin{pmatrix} \boldsymbol{E}_{xy}(z_1) \\ \eta_0\mathbf{J}\cdot\boldsymbol{H}_{xy}(z_1) \end{pmatrix} \tag{4.1}$$

where the propagator is

$$\mathbf{P}(z, z_1) = e^{ik_0(z-z_1)\mathbf{M}}$$

The exponential of a square dyadic is defined in term of the Taylor series of the exponential, *i.e.*,

$$\exp\mathbf{A} = \sum_{n=0}^{\infty}\frac{1}{n!}\mathbf{A}^n$$

This series converges for all matrices since the exponential is an entire function.

Notice the very simple structure of the propagator in (4.1). The fundamental dyadic $\mathbf{M}$ contains all the wave propagation properties of the slab and then the exponential function propagates the field in the correct way from one position z_1 to another position z.

There are several ways to compute the propagator in the homogeneous case and one such method is accounted for below in Section 4.2. Notice also that the exponential of a square dyadic is a standard routine in *e.g.*, MATLAB. However, as pointed out by Yang, caution should always be exercised when strongly evanescent waves occur [22].

4.2 Homogeneous layer—distinct eigenvalues

A general result for the propagator of a single homogeneous layer can be obtained using the Cayley-Hamilton theorem, see Appendix A, provided the eigenvalues of the fundamental dyadic, $\mathbf{M}$, are distinct. Since the exponential is an entire analytic function, the Cayley-Hamilton theorem gives, see Appendix A ($d = z - z_1$)

$$e^{ik_0 d\mathbf{M}} = q_0(k_0 d)\mathbf{I}_4 + q_2(k_0 d)\mathbf{M}\cdot\mathbf{M} + (q_1(k_0 d)\mathbf{I}_4 + q_3(k_0 d)\mathbf{M}\cdot\mathbf{M})\cdot\mathbf{M}$$

The coefficients, $q_l(k_0 d)$, $l = 1, 2, 3, 4$, are given by the system of linear equations

$$e^{ik_0 d\lambda_l} = q_0(k_0 d) + q_2(k_0 d)\lambda_l^2 + \left(q_1(k_0 d) + q_3(k_0 d)\lambda_l^2\right)\lambda_l, \qquad l = 1, 2, 3, 4 \tag{4.2}$$

provided the eigenvalues, λ_l, $l = 1, 2, 3, 4$, of the fundamental dyadic, $\mathbf{M}$, are distinct. This can generally be assumed unless the medium is isotropic or Tellegen. In the isotropic case, the propagator can be obtained as a limit of the results obtained below, see Section 7.

Typically, for non-pathological materials, two eigenvalues, say λ_1 and λ_2, have positive real parts and two eigenvalues, λ_3 and λ_4, have negative real parts. These eigenvalues correspond to up-going and down-going waves, respectively, see also the wave splitting in Section 5.

In terms of the Vandermonde matrix [10]

$$V = \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 \\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^3 \\ 1 & \lambda_3 & \lambda_3^2 & \lambda_3^3 \\ 1 & \lambda_4 & \lambda_4^2 & \lambda_4^3 \end{pmatrix} = \begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{pmatrix}$$

the system of equations (4.2) can be written as

$$e = V\cdot q, \qquad q = V^{-1}\cdot e$$

where

$$e = \begin{pmatrix} e^{ik_0 d\lambda_1} \\ e^{ik_0 d\lambda_2} \\ e^{ik_0 d\lambda_3} \\ e^{ik_0 d\lambda_4} \end{pmatrix}, \qquad q = \begin{pmatrix} q_0(k_0 d) \\ q_1(k_0 d) \\ q_2(k_0 d) \\ q_3(k_0 d) \end{pmatrix}$$

The inverse of the matrix V is given by

$$V^{-1} = \frac{1}{\Delta}\begin{pmatrix} V_{11} & V_{21} & V_{31} & V_{41} \\ V_{12} & V_{22} & V_{32} & V_{42} \\ V_{13} & V_{23} & V_{33} & V_{43} \\ V_{14} & V_{24} & V_{34} & V_{44} \end{pmatrix}$$

where

$$\Delta = (\lambda_4 - \lambda_3)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)$$

is *Vandermonde*'s determinant and $V_{ij} = (-1)^{i+j} D_{ij}$ is the algebraic complement of the matrix element v_{ij}. Here the determinants D_{ij} are

$$D_{11} = \det\begin{pmatrix} \lambda_2 & \lambda_2^2 & \lambda_2^3 \\ \lambda_3 & \lambda_3^2 & \lambda_3^3 \\ \lambda_4 & \lambda_4^2 & \lambda_4^3 \end{pmatrix} \qquad D_{21} = \det\begin{pmatrix} \lambda_1 & \lambda_1^2 & \lambda_1^3 \\ \lambda_3 & \lambda_3^2 & \lambda_3^3 \\ \lambda_4 & \lambda_4^2 & \lambda_4^3 \end{pmatrix}$$

$$D_{12} = \det\begin{pmatrix} 1 & \lambda_2^2 & \lambda_2^3 \\ 1 & \lambda_3^2 & \lambda_3^3 \\ 1 & \lambda_4^2 & \lambda_4^3 \end{pmatrix} \qquad D_{22} = \det\begin{pmatrix} 1 & \lambda_1^2 & \lambda_1^3 \\ 1 & \lambda_3^2 & \lambda_3^3 \\ 1 & \lambda_4^2 & \lambda_4^3 \end{pmatrix}$$

$$D_{13} = \det\begin{pmatrix} 1 & \lambda_2 & \lambda_2^3 \\ 1 & \lambda_3 & \lambda_3^3 \\ 1 & \lambda_4 & \lambda_4^3 \end{pmatrix} \qquad D_{23} = \det\begin{pmatrix} 1 & \lambda_1 & \lambda_1^3 \\ 1 & \lambda_3 & \lambda_3^3 \\ 1 & \lambda_4 & \lambda_4^3 \end{pmatrix}$$

$$D_{14} = \det\begin{pmatrix} 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \\ 1 & \lambda_4 & \lambda_4^2 \end{pmatrix} \qquad D_{24} = \det\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_3 & \lambda_3^2 \\ 1 & \lambda_4 & \lambda_4^2 \end{pmatrix}$$

$$D_{31} = \det\begin{pmatrix} \lambda_1 & \lambda_1^2 & \lambda_1^3 \\ \lambda_2 & \lambda_2^2 & \lambda_2^3 \\ \lambda_4 & \lambda_4^2 & \lambda_4^3 \end{pmatrix} \qquad D_{41} = \det\begin{pmatrix} \lambda_1 & \lambda_1^2 & \lambda_1^3 \\ \lambda_2 & \lambda_2^2 & \lambda_2^3 \\ \lambda_3 & \lambda_3^2 & \lambda_3^3 \end{pmatrix}$$

$$D_{32} = \det\begin{pmatrix} 1 & \lambda_1^2 & \lambda_1^3 \\ 1 & \lambda_2^2 & \lambda_2^3 \\ 1 & \lambda_4^2 & \lambda_4^3 \end{pmatrix} \qquad D_{42} = \det\begin{pmatrix} 1 & \lambda_1^2 & \lambda_1^3 \\ 1 & \lambda_2^2 & \lambda_2^3 \\ 1 & \lambda_3^2 & \lambda_3^3 \end{pmatrix}$$

$$D_{33} = \det\begin{pmatrix} 1 & \lambda_1 & \lambda_1^3 \\ 1 & \lambda_2 & \lambda_2^3 \\ 1 & \lambda_4 & \lambda_4^3 \end{pmatrix} \qquad D_{43} = \det\begin{pmatrix} 1 & \lambda_1 & \lambda_1^3 \\ 1 & \lambda_2 & \lambda_2^3 \\ 1 & \lambda_3 & \lambda_3^3 \end{pmatrix}$$

$$D_{34} = \det\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_4 & \lambda_4^2 \end{pmatrix} \qquad D_{44} = \det\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix}$$

For the important special case when $(\lambda_+^2 \neq \lambda_-^2)$

$$\lambda_1 = -\lambda_4 = \lambda_+, \qquad \lambda_2 = -\lambda_3 = \lambda_- \tag{4.3}$$

one gets

$$\begin{aligned} e^{ik_0 d\mathbf{M}} = & \frac{1}{\lambda_-^2 - \lambda_+^2}\left(\mathbf{I}_4\lambda_-^2 - \mathbf{M}\cdot\mathbf{M}\right)\cdot\left(\mathbf{I}_4\cos(k_0 d\lambda_+) + \frac{i}{\lambda_+}\mathbf{M}\sin(k_0 d\lambda_+)\right) \\ & - \frac{1}{\lambda_-^2 - \lambda_+^2}\left(\mathbf{I}_4\lambda_+^2 - \mathbf{M}\cdot\mathbf{M}\right)\cdot\left(\mathbf{I}_4\cos(k_0 d\lambda_-) + \frac{i}{\lambda_-}\mathbf{M}\sin(k_0 d\lambda_-)\right) \end{aligned} \tag{4.4}$$

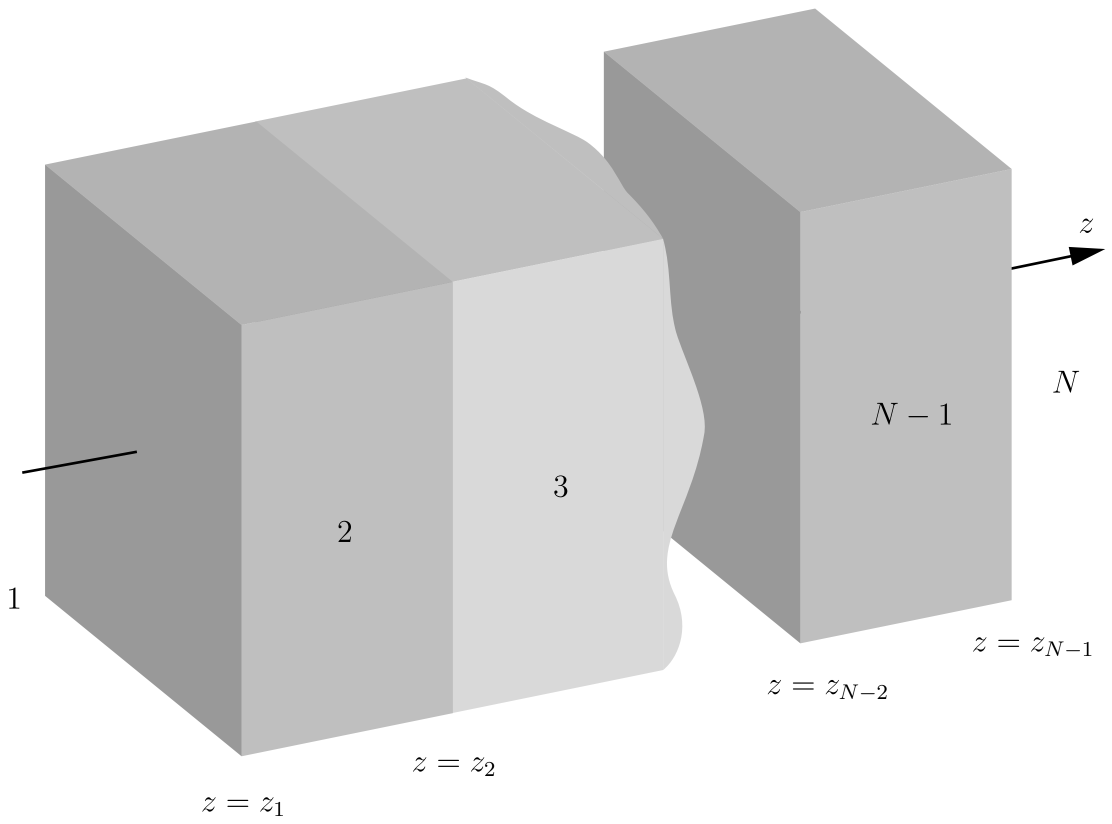


Figure 3: The geometry of a stratified structure of N bianisotropic layers. Regions 1 and N are immaterial (vacuous).

For the case equation (4.3) applies, λ_+^2 and λ_-^2 are eigenvalues of $\mathbf{M} \cdot \mathbf{M}$.

4.3 Several layers

Let z_j, $j = 1, \ldots, N-1$, be the location of $N-1$ parallel interfaces, see Figure 3, and let $\mathbf{M}_j$, $j = 1, \ldots, N$, be the fundamental dyadics of the corresponding regions, respectively. It is assumed that all slabs are homogeneous and that regions $j = 1$ and $j = N$ are immaterial, $\mathbf{M}_1 = \mathbf{M}_N = \mathbf{M}_0$, see (3.4), *i.e.*, $\boldsymbol{\epsilon} = \boldsymbol{\mu} = \mathbf{I}_3$ and $\boldsymbol{\xi} = \boldsymbol{\zeta} = \mathbf{0}$ in these half spaces.

Since the tangential electric and magnetic fields are continuous at the boundaries, a cascade coupling technique can be applied. Using (4.1) repeatedly gives

$$\begin{pmatrix} \boldsymbol{E}_{xy}(z_{N-1}) \\ \eta_0 \mathbf{J} \cdot \boldsymbol{H}_{xy}(z_{N-1}) \end{pmatrix} = \mathbf{P}(z_{N-1}, z_1) \cdot \begin{pmatrix} \boldsymbol{E}_{xy}(z_1) \\ \eta_0 \mathbf{J} \cdot \boldsymbol{H}_{xy}(z_1) \end{pmatrix} \tag{4.5}$$

where the propagator for the layered bianisotropic structure is

$$\mathbf{P}(z_{N-1}, z_1) = e^{ik_0(z_{N-1}-z_{N-2})\mathbf{M}_{N-1}} \cdot \ldots \cdot e^{ik_0(z_3-z_2)\mathbf{M}_3} \cdot e^{ik_0(z_2-z_1)\mathbf{M}_2}$$

Provided all fundamental dyadics $\mathbf{M}_j$, $j = 2, \ldots, N-1$, commute, the total propagator of the slab can be written as one single exponential of the sum of the fundamental

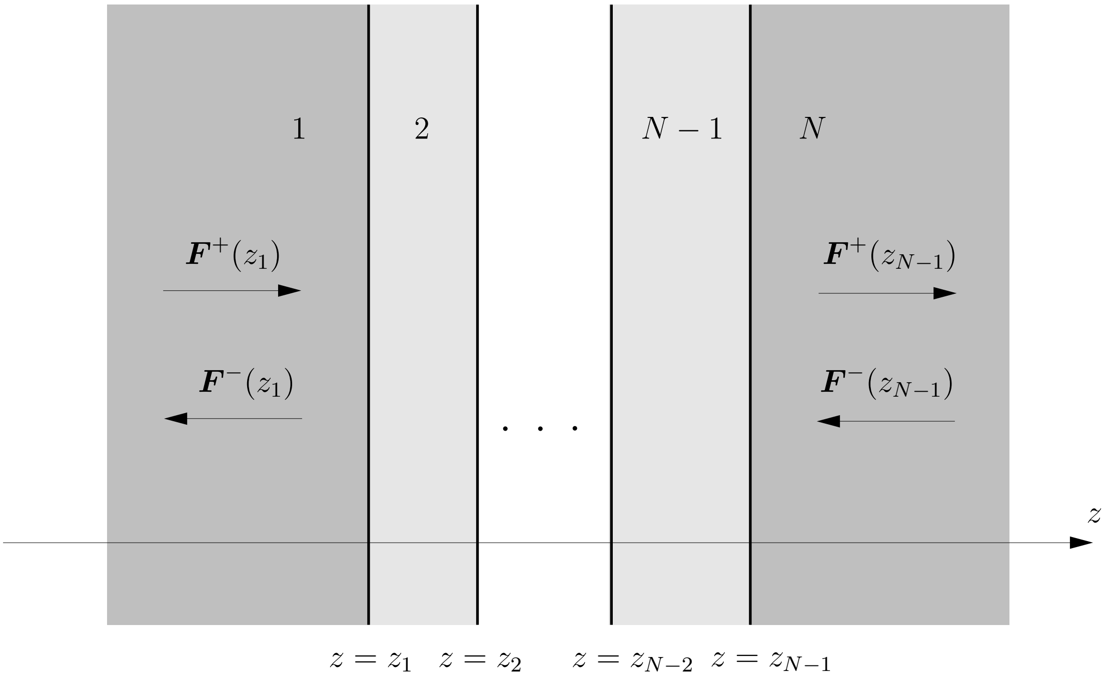


Figure 4: A symbolic representation of the wave splitting. The arrows represent the sign of the z-component of the power flow of the electromagnetic field.

dyadics of each subslab:

$$\mathbf{P}(z_{N-1}, z_1) = e^{ik_0 \sum_{j=2}^{N-1}(z_j - z_{j-1})\mathbf{M}_j} \tag{4.6}$$

This is a very rare case. However, by referring to the Campbell-Hausdorff series, it can be argued that equation (4.6) holds as an approximation when all the layers are thin, see Appendix B.

5 Wave splitting

One way to organize efficiently the input to and the output from the bianisotropic scatterer is to introduce a wave splitting. A wave splitting is a one-to-one correspondence between the dependent vector field variables, *i.e.*, the tangential electric field and the tangential magnetic field, and two new so called split vector field variables, commonly denoted by $\boldsymbol{F}^{+}$ and $\boldsymbol{F}^{-}$, that represent the up-going waves and the down-going waves, respectively. Usually, $\boldsymbol{F}^{+}$ and $\boldsymbol{F}^{-}$ are taken to be the up-going and down-going tangential electric fields. Although the wave splitting applies to all layers, it is adopted in the vacuum regions only [17]. A symbolic representation of the wave splitting is given in Figure 4.

The simplest way to find the wave splitting is, perhaps, to consider the fundamental equation in *vacua*. Combining equations (3.3) and (3.4) implies than the up-going and down-going tangential electromagnetic fields (eigen-modes), $(\boldsymbol{E}_{xy}^{+}, \boldsymbol{H}_{xy}^{+})$

and $(\boldsymbol{E}^-_{xy}, \boldsymbol{H}^-_{xy})$, respectively, satisfy

$$\eta_0 \mathbf{J}\cdot \boldsymbol{H}^{\pm}_{xy}(z) = \mp \frac{k_0}{k_z}\left(\mathbf{I}_2 + \frac{1}{k_0^2}\mathbf{J}\cdot \boldsymbol{k}_t\boldsymbol{k}_t\cdot \mathbf{J}\right)\cdot \boldsymbol{E}^{\pm}_{xy}(z)$$

where the normal wave number for up-going waves, k_z, is defined by equation (3.2). In view of this, the split field vectors, $\boldsymbol{F}^{\pm}$, are defined by

$$\begin{cases} \boldsymbol{E}_{xy}(z) = \boldsymbol{F}^+(z) + \boldsymbol{F}^-(z) \\ \eta_0\mathbf{J}\cdot \boldsymbol{H}_{xy}(z) = -\mathbf{W}^{-1}\cdot \boldsymbol{F}^+(z) + \mathbf{W}^{-1}\cdot \boldsymbol{F}^-(z) \end{cases}$$

where the operator $\mathbf{W}^{-1}$ is defined by[2]

$$\mathbf{W}^{-1} = \frac{k_0}{k_z}\left(\mathbf{I}_2 + \frac{1}{k_0^2}\boldsymbol{k}_t\times(\boldsymbol{k}_t\times\mathbf{I}_2)\right) = \hat{\boldsymbol{e}}_{\parallel}\hat{\boldsymbol{e}}_{\parallel}\frac{1}{\cos\theta_i} + \hat{\boldsymbol{e}}_{\perp}\hat{\boldsymbol{e}}_{\perp}\cos\theta_i$$

and $\boldsymbol{k}_t\times(\boldsymbol{k}_t\times\mathbf{I}_2) = \mathbf{J}\cdot\boldsymbol{k}_t\boldsymbol{k}_t\cdot\mathbf{J} = -k_t^2\hat{\boldsymbol{e}}_{\perp}\hat{\boldsymbol{e}}_{\perp}$. Equivalently,

$$\boldsymbol{F}^{\pm}(z) = \frac{1}{2}\left(\boldsymbol{E}_{xy}(z) \mp \mathbf{W}\cdot\eta_0\mathbf{J}\cdot\boldsymbol{H}_{xy}(z)\right)$$

where

$$\mathbf{W} = \frac{k_z}{k_0}\left(\mathbf{I}_2 - \frac{1}{k_z^2}\boldsymbol{k}_t\times(\boldsymbol{k}_t\times\mathbf{I}_2)\right) = \hat{\boldsymbol{e}}_{\parallel}\hat{\boldsymbol{e}}_{\parallel}\cos\theta_i + \hat{\boldsymbol{e}}_{\perp}\hat{\boldsymbol{e}}_{\perp}\frac{1}{\cos\theta_i}$$

In matrix notation the wave splitting becomes

$$\begin{pmatrix}\boldsymbol{F}^+(z)\\ \boldsymbol{F}^-(z)\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\mathbf{I}_2 & -\mathbf{W}\\ \mathbf{I}_2 & \mathbf{W}\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{E}_{xy}(z)\\ \eta_0\mathbf{J}\cdot\boldsymbol{H}_{xy}(z)\end{pmatrix} \tag{5.1}$$

with inverse

$$\begin{pmatrix}\boldsymbol{E}_{xy}(z)\\ \eta_0\mathbf{J}\cdot\boldsymbol{H}_{xy}(z)\end{pmatrix} = \begin{pmatrix}\mathbf{I}_2 & \mathbf{I}_2\\ -\mathbf{W}^{-1} & \mathbf{W}^{-1}\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{F}^+(z)\\ \boldsymbol{F}^-(z)\end{pmatrix} \tag{5.2}$$

At normal incidence, $\mathbf{W} = \mathbf{W}^{-1} = \mathbf{I}_2$.

Since the normal parts of the electric and magnetic fields can be expressed in terms of the tangential parts as, see (C.2)

$$\begin{pmatrix}E_z(z)\\ \eta_0 H_z(z)\end{pmatrix} = \frac{1}{k_0}\begin{pmatrix}\mathbf{0} & \boldsymbol{k}_t\\ \mathbf{J}\cdot\boldsymbol{k}_t & \mathbf{0}\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{E}_{xy}(z)\\ \eta_0\mathbf{J}\cdot\boldsymbol{H}_{xy}(z)\end{pmatrix}$$

the total electric and magnetic fields are

$$\begin{cases} \boldsymbol{E} = \left(\mathbf{I}_2 - \frac{1}{k_z}\hat{\boldsymbol{z}}\boldsymbol{k}_t\right)\cdot\boldsymbol{F}^+ + \left(\mathbf{I}_2 + \frac{1}{k_z}\hat{\boldsymbol{z}}\boldsymbol{k}_t\right)\cdot\boldsymbol{F}^- \\ \eta_0\boldsymbol{H} = \frac{1}{k_0}\boldsymbol{k}^+\times\left(\mathbf{I}_2 - \frac{1}{k_z}\hat{\boldsymbol{z}}\boldsymbol{k}_t\right)\cdot\boldsymbol{F}^+ + \frac{1}{k_0}\boldsymbol{k}^-\times\left(\mathbf{I}_2 + \frac{1}{k_z}\hat{\boldsymbol{z}}\boldsymbol{k}_t\right)\cdot\boldsymbol{F}^- \end{cases} \tag{5.3}$$

[2]This dyadic is related to the admittance dyadic $\mathbf{Y}(\boldsymbol{k}_t)$.

$$\mathbf{Y}(\boldsymbol{k}_t) = \frac{1}{k_0k_z}\left\{k_0^2\hat{\boldsymbol{e}}_{\perp}\hat{\boldsymbol{e}}_{\parallel} - k_z^2\hat{\boldsymbol{e}}_{\parallel}\hat{\boldsymbol{e}}_{\perp}\right\} = \mathbf{J}\cdot\mathbf{W}^{-1}(\boldsymbol{k}_t)$$

where

$$\boldsymbol{k}^{\pm} = \boldsymbol{k}_t \pm \hat{\boldsymbol{z}} k_z$$

The relations

$$\boldsymbol{k}^{\pm} \cdot \left(\mathbf{I}_2 \mp \frac{1}{k_z} \hat{\boldsymbol{z}} \boldsymbol{k}_t \right) \cdot \boldsymbol{F}^{\pm} = 0$$

hold also.

Straightforward calculations show that the split fields satisfy the ODEs

$$\frac{\mathrm{d}}{\mathrm{d}z} \boldsymbol{F}^{\pm}(z) = \pm i k_z \boldsymbol{F}^{\pm}(z)$$

which give

$$\begin{cases} \boldsymbol{F}^{\pm}(z) = \boldsymbol{F}^{\pm}(z_1) e^{\pm i k_z (z - z_1)}, & z < z_1 \\ \boldsymbol{F}^{\pm}(z) = \boldsymbol{F}^{\pm}(z_{N-1}) e^{\pm i k_z (z - z_{N-1})}, & z > z_{N-1} \end{cases} \tag{5.4}$$

Notice that $\boldsymbol{F}^{\pm}(z)$ are damped in the $\pm z$ directions, respectively, for the case the plane waves are inhomogeneous (evanescent).

5.1 Power flow

For the case of homogeneous plane waves, the Poynting vector

$$\boldsymbol{S} = \frac{1}{4} \left(\boldsymbol{E} \times \boldsymbol{H}^* + \boldsymbol{E}^* \times \boldsymbol{H} \right)$$

becomes

$$\begin{aligned} \boldsymbol{S} = & \frac{1}{4\eta_0 k_0} \left(\boldsymbol{k}^{\pm} + (\boldsymbol{k}^{\pm})^* \right) \left| \left(\mathbf{I}_2 \mp \frac{1}{k_z} \hat{\boldsymbol{z}} \boldsymbol{k}_t \right) \cdot \boldsymbol{F}^{\pm} \right|^2 = \\ = & \frac{1}{2\eta_0 k_0} \boldsymbol{k}^{\pm} \left| \hat{\boldsymbol{e}}_{\parallel} F_{\parallel}^{\pm} + \hat{\boldsymbol{e}}_{\perp} F_{\perp}^{\pm} \mp \hat{\boldsymbol{z}} F_{\parallel}^{\pm} \frac{k_t}{k_z} \right|^2 = \frac{1}{2\eta_0 k_0} \boldsymbol{k}^{\pm} \left(|F_{\parallel}^{\pm}|^2 \frac{k_0^2}{k_z^2} + |F_{\perp}^{\pm}|^2 \right) \end{aligned} \tag{5.5}$$

for up-going and down-going waves, respectively, where the projections $F_{\perp}^{\pm}$ and $F_{\parallel}^{\pm}$ are defined by $F_{\perp}^{\pm} = \hat{\boldsymbol{e}}_{\perp} \cdot \boldsymbol{F}^{\pm}$ and $F_{\parallel}^{\pm} = \hat{\boldsymbol{e}}_{\parallel} \cdot \boldsymbol{F}^{\pm}$.

6 Reflection and transmission

In this section, the reflection and transmission dyadics for the plane-stratified bian-isotropic structure are computed. These dyadics are easy to obtain using the wave splitting, (5.1)–(5.2), on the solution of the propagator problem, see (4.5). Recall that all the generating sources are located in the vacuous half-space, $z < z_0 < z_1$, and that the half-space, $z > z_{N-1}$, is either vacuous or perfectly conducting. In the latter case, transmission is, of course, zero. Recall also that $\boldsymbol{F}^{+}(z_1)$ and $\boldsymbol{F}^{-}(z_1)$ are the incident and reflected tangential electric fields at $z = z_1$, respectively, and that $\boldsymbol{F}^{+}(z_{N-1})$ is the transmitted tangential electric field at $z = z_{N-1}$. All these fields are associated with the transverse wave vector $\boldsymbol{k}_t$. Specifically, the total incident field $\boldsymbol{E}^i(\boldsymbol{r},\omega)$ at $z \le z_1$ (but to the right of the sources, *i.e.*, $z_0 \le z \le z_1$) in terms of $\boldsymbol{F}^+(z_1)$ is

$$\boldsymbol{E}^i(\boldsymbol{r},\omega) = \frac{1}{4\pi^2}\iint\limits_{-\infty}^{\infty}\left(\mathbf{I}_2 - \frac{1}{k_z}\hat{\boldsymbol{z}}\boldsymbol{k}_t\right)\cdot\boldsymbol{F}^+(z_1,\boldsymbol{k}_t,\omega)e^{i\boldsymbol{k}_t\cdot\boldsymbol{\rho}+ik_z(z-z_1)}\,dk_xdk_y \quad z_0 \le z \le z_1$$

by the use of (3.1), (5.3), and (5.4). Notice that all components of the field, not just the tangential ones, are given.

In a direct scattering problem, the incident fields are given. In our case, we have specified sources to the left and none to the right of the slab, *i.e.*, the given fields are

$$\begin{cases}\boldsymbol{F}^+(z_1) = \text{ given} = -\hat{\boldsymbol{z}}\times\left(\hat{\boldsymbol{z}}\times\boldsymbol{E}^i(z_1)\right)\\ \boldsymbol{F}^-(z_{N-1}) = \mathbf{0}\end{cases}$$

where $\boldsymbol{E}^i(z_1)$ is the incident electric field (Fourier transformed field) at $z = z_1$ associated with the transverse wave vector $\boldsymbol{k}_t$. The double cross product projects the field to the x-y-plane, since only that part is relevant in the propagation problem as seen from the previous analysis.

Writing the solution to the propagation problem, (4.5), in block-matrix form as

$$\begin{pmatrix}\boldsymbol{E}_{xy}(z_{N-1})\\ \eta_0\mathbf{J}\cdot\boldsymbol{H}_{xy}(z_{N-1})\end{pmatrix} = \begin{pmatrix}\mathbf{P}_{11} & \mathbf{P}_{12}\\ \mathbf{P}_{21} & \mathbf{P}_{22}\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{E}_{xy}(z_1)\\ \eta_0\mathbf{J}\cdot\boldsymbol{H}_{xy}(z_1)\end{pmatrix} \tag{6.1}$$

and combining it with the wave splitting, (5.1)–(5.2), gives the scattering relation

$$\begin{aligned}\begin{pmatrix}\boldsymbol{F}^+(z_{N-1})\\ \boldsymbol{F}^-(z_{N-1})\end{pmatrix} &= \frac{1}{2}\begin{pmatrix}\mathbf{I}_2 & -\mathbf{W}\\ \mathbf{I}_2 & \mathbf{W}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{P}_{11} & \mathbf{P}_{12}\\ \mathbf{P}_{21} & \mathbf{P}_{22}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{I}_2 & \mathbf{I}_2\\ -\mathbf{W}^{-1} & \mathbf{W}^{-1}\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{F}^+(z_1)\\ \boldsymbol{F}^-(z_1)\end{pmatrix}\\ &= \begin{pmatrix}\mathbf{T}_{11} & \mathbf{T}_{12}\\ \mathbf{T}_{21} & \mathbf{T}_{22}\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{F}^+(z_1)\\ \boldsymbol{F}^-(z_1)\end{pmatrix} = \mathbf{T}\cdot\begin{pmatrix}\boldsymbol{F}^+(z_1)\\ \boldsymbol{F}^-(z_1)\end{pmatrix}\end{aligned}$$

where

$$\begin{cases}2\mathbf{T}_{11} = \mathbf{P}_{11} - \mathbf{P}_{12}\cdot\mathbf{W}^{-1} - \mathbf{W}\cdot\mathbf{P}_{21} + \mathbf{W}\cdot\mathbf{P}_{22}\cdot\mathbf{W}^{-1}\\ 2\mathbf{T}_{12} = \mathbf{P}_{11} + \mathbf{P}_{12}\cdot\mathbf{W}^{-1} - \mathbf{W}\cdot\mathbf{P}_{21} - \mathbf{W}\cdot\mathbf{P}_{22}\cdot\mathbf{W}^{-1}\\ 2\mathbf{T}_{21} = \mathbf{P}_{11} - \mathbf{P}_{12}\cdot\mathbf{W}^{-1} + \mathbf{W}\cdot\mathbf{P}_{21} - \mathbf{W}\cdot\mathbf{P}_{22}\cdot\mathbf{W}^{-1}\\ 2\mathbf{T}_{22} = \mathbf{P}_{11} + \mathbf{P}_{12}\cdot\mathbf{W}^{-1} + \mathbf{W}\cdot\mathbf{P}_{21} + \mathbf{W}\cdot\mathbf{P}_{22}\cdot\mathbf{W}^{-1}\end{cases}$$

We manipulate this set of equations to

$$\begin{cases}\boldsymbol{F}^-(z_1) = \mathbf{r}\cdot\boldsymbol{F}^+(z_1)\\ \boldsymbol{F}^+(z_{N-1}) = \mathbf{t}\cdot\boldsymbol{F}^+(z_1)\end{cases} \tag{6.2}$$

where the reflection and transmission dyadics for the tangential electric field, $\mathbf{r}$ and $\mathbf{t}$, respectively, are defined by

$$\begin{cases}\mathbf{r} = -\mathbf{T}_{22}^{-1}\cdot\mathbf{T}_{21}\\ \mathbf{t} = \mathbf{T}_{11} + \mathbf{T}_{12}\cdot\mathbf{r}\end{cases}$$

The reflected and transmitted fields $\boldsymbol{E}^r(\boldsymbol{r},\omega)$ and $\boldsymbol{E}^t(\boldsymbol{r},\omega)$ for $z \le z_1$ and $z \ge z_N$, respectively, are then given by

$$\begin{cases} \boldsymbol{E}^r(\boldsymbol{r},\omega) = \dfrac{1}{4\pi^2} \displaystyle\iint\limits_{-\infty}^{\infty} \left(\mathbf{I}_2 + \dfrac{\hat{\boldsymbol{z}}\boldsymbol{k}_t}{k_z}\right) \cdot \mathbf{r} \cdot \boldsymbol{F}^+(z_1, \boldsymbol{k}_t, \omega) e^{i\boldsymbol{k}_t\cdot\boldsymbol{\rho} + ik_z(z_1 - z)}\, dk_x dk_y \quad z \le z_1 \\ \boldsymbol{E}^t(\boldsymbol{r},\omega) = \dfrac{1}{4\pi^2} \displaystyle\iint\limits_{-\infty}^{\infty} \left(\mathbf{I}_2 - \dfrac{\hat{\boldsymbol{z}}\boldsymbol{k}_t}{k_z}\right) \cdot \mathbf{t} \cdot \boldsymbol{F}^+(z_1, \boldsymbol{k}_t, \omega) e^{i\boldsymbol{k}_t\cdot\boldsymbol{\rho} + ik_z(z - z_N)}\, dk_x dk_y \quad z \ge z_N \end{cases}$$

An alternative way of computing the reflection and transmission properties of a slab is to apply the Redheffer $*$-product [18] to a composition of the slab into discrete subslabs. This procedure is outlined in Appendix D.

6.1 PEC-backing

Finally, we consider the case when medium N is a perfect electric conductor (PEC), which is of great technical interest. For this case, the boundary conditions at $z = z_N$ give the appropriate constraints

$$\begin{pmatrix} \mathbf{0} \\ -\eta_0 \boldsymbol{J}_S \end{pmatrix} = \begin{pmatrix} \boldsymbol{E}_{xy}(z_{N-1}) \\ \eta_0 \mathbf{J}\cdot\boldsymbol{H}_{xy}(z_{N-1}) \end{pmatrix} = \begin{pmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{I}_2 & \mathbf{I}_2 \\ -\mathbf{W}^{-1} & \mathbf{W}^{-1} \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{F}^+(z_1) \\ \mathbf{r}\cdot\boldsymbol{F}^+(z_1) \end{pmatrix}$$

where $\boldsymbol{J}_S$ is the surface current density at $z = z_{N-1}$, and where we have used (6.1), (5.2), and (6.2). The upper equation gives the reflection dyadic in this case. The result is

$$\mathbf{r} = -\left(\mathbf{P}_{11} + \mathbf{P}_{12}\cdot\mathbf{W}^{-1}\right)^{-1} \cdot \left(\mathbf{P}_{11} - \mathbf{P}_{12}\cdot\mathbf{W}^{-1}\right) \tag{6.3}$$

and the lower the current density

$$\boldsymbol{J}_S = \frac{1}{\eta_0}\left(\mathbf{P}_{22}\cdot\mathbf{W}^{-1}\cdot(\mathbf{I}_2 - \mathbf{r}) - \mathbf{P}_{21}\cdot(\mathbf{I}_2 + \mathbf{r})\right)\cdot\boldsymbol{F}^+(z_1)$$

This latter equation can be formulated as a dyadic relation, $\boldsymbol{J}_S = \mathbf{C}\cdot\boldsymbol{F}^+(z_1)$, where the surface current dyadic $\mathbf{C}$ is

$$\mathbf{C} = \frac{1}{\eta_0}\left(\mathbf{P}_{22}\cdot\mathbf{W}^{-1}\cdot(\mathbf{I}_2 - \mathbf{r}) - \mathbf{P}_{21}\cdot(\mathbf{I}_2 + \mathbf{r})\right)$$

7 Examples

The fundamental dyadic for a general, stratified, bianisotropic material is given in Appendix C. In this section, explicit expressions for the single-slab propagator are given for some important classes of linear and homogeneous materials. These results, which are of independent interest, can be used to check numerical codes.

7.1 Isotropic media

The results for simple media are well known. For a homogeneous isotropic slab ($\boldsymbol{\xi} = \boldsymbol{\zeta} = \mathbf{0}$, $\boldsymbol{\epsilon} = \epsilon\mathbf{I}_3$, $\boldsymbol{\mu} = \mu\mathbf{I}_3$), the fundamental dyadic is

$$\mathbf{M} = \begin{pmatrix} \mathbf{0} & -\mu\mathbf{I}_2 + \frac{1}{\epsilon k_0^2}\boldsymbol{k}_t\boldsymbol{k}_t \\ -\epsilon\mathbf{I}_2 - \frac{1}{\mu k_0^2}\mathbf{J}\cdot\boldsymbol{k}_t\boldsymbol{k}_t\cdot\mathbf{J} & \mathbf{0} \end{pmatrix}$$

For this case, the eigenvalues are given by equation (4.3) with

$$\lambda_+^2 = \lambda_-^2 = \lambda^2 = \epsilon\mu - k_t^2/k_0^2$$

Equation (4.4) for the single-slab propagator reduces to ($d = z - z_1$)

$$\mathbf{P} = e^{ik_0 d\mathbf{M}} = \mathbf{I}_4 \cos(k_0 d\lambda) + \frac{i}{\lambda}\mathbf{M}\sin(k_0 d\lambda) \tag{7.1}$$

This result can either be obtained directly from the Cayley-Hamilton theorem or by a limit process of the results in Section 4.2.

It is easy to see that two different fundamental dyadics of do not commute unless the materials are impedance matched. In fact,

$$\mathbf{M}_n\cdot\mathbf{M}_m = \begin{pmatrix} \epsilon_m\mu_n\left(\mathbf{I}_2 - \tau_m\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp - \tau_n\hat{\boldsymbol{e}}_\parallel\hat{\boldsymbol{e}}_\parallel\right) & \mathbf{0} \\ \mathbf{0} & \epsilon_n\mu_m\left(\mathbf{I}_2 - \tau_m\hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp - \tau_n\hat{\boldsymbol{e}}_\parallel\hat{\boldsymbol{e}}_\parallel\right) \end{pmatrix}$$

where $\hat{\boldsymbol{e}}_\parallel = \boldsymbol{k}_t/k_t$, $\hat{\boldsymbol{e}}_\perp = \mathbf{J}\cdot\hat{\boldsymbol{e}}_\parallel$, and $\tau_m = k_t^2/(\epsilon_m\mu_m k_0^2)$.

7.1.1 Single isotropic layer embedded in vacuum

The propagator for a single layer is given by (7.1). Explicitly, in the $\left\{\hat{\boldsymbol{e}}_\parallel, \hat{\boldsymbol{e}}_\perp\right\}$-system, it is

$$\begin{aligned} \mathbf{P} &= e^{ik_0 d\mathbf{M}} \\ &= \begin{pmatrix} \left(\hat{\boldsymbol{e}}_\parallel\hat{\boldsymbol{e}}_\parallel + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\right)\cos(k_0 d\lambda) & -i\left(\hat{\boldsymbol{e}}_\parallel\hat{\boldsymbol{e}}_\parallel\frac{\lambda}{\epsilon_2} + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\frac{\mu_2}{\lambda}\right)\sin(k_0 d\lambda) \\ -i\left(\hat{\boldsymbol{e}}_\parallel\hat{\boldsymbol{e}}_\parallel\frac{\epsilon_2}{\lambda} + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\frac{\lambda}{\mu_2}\right)\sin(k_0 d\lambda) & \left(\hat{\boldsymbol{e}}_\parallel\hat{\boldsymbol{e}}_\parallel + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\right)\cos(k_0 d\lambda) \end{pmatrix} \end{aligned}$$

where $\lambda^2 = \epsilon_2\mu_2 - k_t^2/k_0^2$. Straightforward calculations show that principal blocks of the scattering dyadic are

$$\begin{cases} 2\mathbf{T}_{11} = \hat{\boldsymbol{e}}_\parallel\hat{\boldsymbol{e}}_\parallel\left(2\cos(k_0 d\lambda) + i\left(\dfrac{\lambda}{\epsilon_2\cos\theta_i} + \dfrac{\epsilon_2\cos\theta_i}{\lambda}\right)\sin(k_0 d\lambda)\right) + \\ \qquad + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\left(2\cos(k_0 d\lambda) + i\left(\dfrac{\lambda}{\mu_2\cos\theta_i} + \dfrac{\mu_2\cos\theta_i}{\lambda}\right)\sin(k_0 d\lambda)\right) \\ 2\mathbf{T}_{22} = \hat{\boldsymbol{e}}_\parallel\hat{\boldsymbol{e}}_\parallel\left(2\cos(k_0 d\lambda) - i\left(\dfrac{\lambda}{\epsilon_2\cos\theta_i} + \dfrac{\epsilon_2\cos\theta_i}{\lambda}\right)\sin(k_0 d\lambda)\right) + \\ \qquad + \hat{\boldsymbol{e}}_\perp\hat{\boldsymbol{e}}_\perp\left(2\cos(k_0 d\lambda) - i\left(\dfrac{\lambda}{\mu_2\cos\theta_i} + \dfrac{\mu_2\cos\theta_i}{\lambda}\right)\sin(k_0 d\lambda)\right) \end{cases}$$

and

$$\begin{cases} 2\mathbf{T}_{12} = i\Bigg(\hat{\boldsymbol{e}}_{\parallel}\hat{\boldsymbol{e}}_{\parallel}\left(-\dfrac{\lambda}{\epsilon_2\cos\theta_i} + \dfrac{\epsilon_2\cos\theta_i}{\lambda}\right) \\ \qquad\qquad + \hat{\boldsymbol{e}}_{\perp}\hat{\boldsymbol{e}}_{\perp}\left(\dfrac{\lambda}{\mu_2\cos\theta_i} - \dfrac{\mu_2\cos\theta_i}{\lambda}\right)\Bigg)\sin(k_0 d\lambda) \\ \mathbf{T}_{21} = -\mathbf{T}_{12} \end{cases}$$

where θ_i is the angle of incidence. Consequently, the reflection and transmission dyadics are

$$\begin{cases} \mathbf{r} = \hat{\boldsymbol{e}}_{\parallel}\hat{\boldsymbol{e}}_{\parallel} r_{\parallel\parallel} + \hat{\boldsymbol{e}}_{\perp}\hat{\boldsymbol{e}}_{\perp} r_{\perp\perp} \\ \mathbf{t} = \hat{\boldsymbol{e}}_{\parallel}\hat{\boldsymbol{e}}_{\parallel} t_{\parallel\parallel} + \hat{\boldsymbol{e}}_{\perp}\hat{\boldsymbol{e}}_{\perp} t_{\perp\perp} \end{cases} \tag{7.2}$$

where

$$\begin{cases} r_{\parallel\parallel} = \dfrac{i\left(-\frac{\lambda}{\epsilon_2\cos\theta_i} + \frac{\epsilon_2\cos\theta_i}{\lambda}\right)\sin(k_0 d\lambda)}{2\cos(k_0 d\lambda) - i\left(\frac{\lambda}{\epsilon_2\cos\theta_i} + \frac{\epsilon_2\cos\theta_i}{\lambda}\right)\sin(k_0 d\lambda)} = r_{1\parallel\parallel}\dfrac{1 - e^{2ik_0\lambda d}}{1 - r_{1\parallel\parallel}^2 e^{2ik_0\lambda d}} \\ r_{\perp\perp} = \dfrac{i\left(\frac{\lambda}{\mu_2\cos\theta_i} - \frac{\mu_2\cos\theta_i}{\lambda}\right)\sin(k_0 d\lambda)}{2\cos(k_0 d\lambda) - i\left(\frac{\lambda}{\mu_2\cos\theta_i} + \frac{\mu_2\cos\theta_i}{\lambda}\right)\sin(k_0 d\lambda)} = r_{1\perp\perp}\dfrac{1 - e^{2ik_0\lambda d}}{1 - r_{1\perp\perp}^2 e^{2ik_0\lambda d}} \end{cases}$$

$$\begin{cases} t_{\parallel\parallel} = \dfrac{2}{2\cos(k_0 d\lambda) - i\left(\frac{\lambda}{\epsilon_2\cos\theta_i} + \frac{\epsilon_2\cos\theta_i}{\lambda}\right)\sin(k_0 d\lambda)} = \dfrac{(1 - r_{1\parallel\parallel}^2)e^{ik_0\lambda d}}{1 - r_{1\parallel\parallel}^2 e^{2ik_0\lambda d}} \\ t_{\perp\perp} = \dfrac{2}{2\cos(k_0 d\lambda) - i\left(\frac{\lambda}{\mu_2\cos\theta_i} + \frac{\mu_2\cos\theta_i}{\lambda}\right)\sin(k_0 d\lambda)} = \dfrac{(1 - r_{1\perp\perp}^2)e^{ik_0\lambda d}}{1 - r_{1\perp\perp}^2 e^{2ik_0\lambda d}} \end{cases}$$

and

$$\begin{cases} r_{1\parallel\parallel} = \dfrac{\frac{1}{2}\left(\frac{\lambda}{\epsilon_2\cos\theta_i} - \frac{\epsilon_2\cos\theta_i}{\lambda}\right)}{1 + \frac{1}{2}\left(\frac{\lambda}{\epsilon_2\cos\theta_i} + \frac{\epsilon_2\cos\theta_i}{\lambda}\right)} = \dfrac{\lambda - \epsilon_2\cos\theta_i}{\lambda + \epsilon_2\cos\theta_i} \\ r_{1\perp\perp} = \dfrac{\frac{1}{2}\left(\frac{\mu_2\cos\theta_i}{\lambda} - \frac{\lambda}{\mu_2\cos\theta_i}\right)}{1 + \frac{1}{2}\left(\frac{\mu_2\cos\theta_i}{\lambda} + \frac{\lambda}{\mu_2\cos\theta_i}\right)} = \dfrac{\mu_2\cos\theta_i - \lambda}{\mu_2\cos\theta_i + \lambda} \end{cases}$$

are recognized as Fresnel's equations [11].

For a PEC-backed isotropic slab, equation (6.3) yields

$$\begin{cases} r_{\parallel\parallel} = -\dfrac{\cos(k_0 d\lambda) + i\frac{\lambda}{\epsilon_2\cos\theta_i}\sin(k_0 d\lambda)}{\cos(k_0 d\lambda) - i\frac{\lambda}{\epsilon_2\cos\theta_i}\sin(k_0 d\lambda)} = \dfrac{r_{1\parallel\parallel} - e^{2ik_0\lambda d}}{1 - r_{1\parallel\parallel}e^{2ik_0\lambda d}} \\ r_{\perp\perp} = -\dfrac{\cos(k_0 d\lambda) + i\frac{\mu_2\cos\theta_i}{\lambda}\sin(k_0 d\lambda)}{\cos(k_0 d\lambda) - i\frac{\mu_2\cos\theta_i}{\lambda}\sin(k_0 d\lambda)} = \dfrac{r_{1\perp\perp} - e^{2ik_0\lambda d}}{1 - r_{1\perp\perp}e^{2ik_0\lambda d}} \end{cases}$$

where $r_{1\parallel\parallel}$ and $r_{1\perp\perp}$ are given by Fresnel's equations.

7.2 Biisotropic media

Electromagnetic wave propagation in biisotropic media has received extensive attention during recent years. An excellent review of the area is given in Ref. 14. For a homogeneous biisotropic slab ($\boldsymbol{\epsilon} = \epsilon \mathbf{I}_3$, $\boldsymbol{\mu} = \mu \mathbf{I}_3$, $\boldsymbol{\xi} = \xi \mathbf{I}_3$, $\boldsymbol{\zeta} = \zeta \mathbf{I}_3$), the fundamental dyadic is

$$\mathbf{M} = \begin{pmatrix} -\mathbf{J}\zeta + \frac{a\xi}{k_0^2}\boldsymbol{k}_t\boldsymbol{k}_t \cdot \mathbf{J} & -\mu\mathbf{I}_2 + \frac{a\mu}{k_0^2}\boldsymbol{k}_t\boldsymbol{k}_t \\ -\epsilon\mathbf{I}_2 - \frac{a\epsilon}{k_0^2}\mathbf{J}\cdot\boldsymbol{k}_t\boldsymbol{k}_t\cdot\mathbf{J} & \mathbf{J}\xi - \frac{a\zeta}{k_0^2}\mathbf{J}\cdot\boldsymbol{k}_t\boldsymbol{k}_t \end{pmatrix}$$

where $a^{-1} = \epsilon\mu - \xi\zeta$. Two special cases are of interest, *viz.*, reciprocal, biisotropic medium (isotropic chiral medium or Pasteur medium), which is characterized by $\zeta = -\xi$, and non-reciprocal, achiral, biisotropic medium (Tellegen medium), which is characterized by $\zeta = \xi$.

Straightforward calculations show that the eigenvalues of $\mathbf{M}$ are all distinct unless the medium is Tellegen; specifically, equation (4.3) holds, where the eigenvalues

$$\lambda_\pm^2 = n_\pm^2 - \frac{k_t^2}{k_0^2}, \qquad n_\pm = \sqrt{\mu\epsilon - \left(\frac{\xi+\zeta}{2}\right)^2} \pm i\,\frac{\xi-\zeta}{2}$$

correspond to up-going (down-going waves correspond to the similar negative values) right and left-hand circularly polarized plane waves in the medium, respectively. Unless the medium is Tellegen, the single-slab propagator is given by equation (4.4), where

$$\mathbf{M}\cdot\mathbf{M} = \begin{pmatrix} \mathbf{I}_2\left(\mu\epsilon - \zeta^2 - k_t^2/k_0^2\right) & \mathbf{J}\mu\left(\zeta-\xi\right) \\ \mathbf{J}\epsilon\left(\zeta-\xi\right) & \mathbf{I}_2\left(\mu\epsilon - \xi^2 - k_t^2/k_0^2\right) \end{pmatrix}$$

In particular, for Pasteur media ($\zeta = -\xi$), the propagator is

$$\mathbf{P} = \frac{1}{2}\begin{pmatrix} \mathbf{P}_+ & i\sqrt{\frac{\mu}{\epsilon}}\mathbf{P}_+\cdot\mathbf{J} \\ i\sqrt{\frac{\epsilon}{\mu}}\mathbf{J}\cdot\mathbf{P}_+ & -\mathbf{J}\cdot\mathbf{P}_+\cdot\mathbf{J} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \mathbf{P}_- & -i\sqrt{\frac{\mu}{\epsilon}}\mathbf{P}_-\cdot\mathbf{J} \\ -i\sqrt{\frac{\epsilon}{\mu}}\mathbf{J}\cdot\mathbf{P}_- & -\mathbf{J}\cdot\mathbf{P}_-\cdot\mathbf{J} \end{pmatrix}$$

where the dyadics

$$\mathbf{P}_\pm = \mathbf{I}_2\cos(k_0 d\lambda_\pm) \pm \frac{1}{\lambda_\pm}\left(\mathbf{I}_2 n_\pm - \frac{1}{n_\pm k_0^2}\boldsymbol{k}_t\boldsymbol{k}_t\right)\cdot\mathbf{J}\sin(k_0 d\lambda_\pm), \quad n_\pm = \sqrt{\mu\epsilon} \pm i\,\xi$$

are the propagators of the tangential components of particular linear combinations of the electric and magnetic fields known as wave fields, namely, $\left(\boldsymbol{E} \mp i\sqrt{\frac{\mu}{\epsilon}}\eta_0\boldsymbol{H}\right)/2$, respectively [14]. For a Tellegen material ($\zeta = \xi$), equation (7.1) applies with $\lambda^2 = \mu\epsilon - \xi^2 - k_t^2/k_0^2$.

Using the technique in presented in Section 6, it is a straightforward matter to obtain the reflection and transmission dyadics for a single biisotropic slab. For results, the reader is referred to Ref. 14 and references given therein.

7.3 Anisotropic media

For anisotropic materials ($\boldsymbol{\xi} = \boldsymbol{\zeta} = \mathbf{0}$), the blocks of the fundamental dyadic are

$$\begin{cases} \mathbf{M}_{11} = -\dfrac{1}{\epsilon_{zz}k_0}\boldsymbol{k}_t\boldsymbol{\epsilon}_z + \dfrac{1}{\mu_{zz}k_0}\mathbf{J}\cdot\boldsymbol{\mu}_\perp\boldsymbol{k}_t\cdot\mathbf{J} \\ \mathbf{M}_{12} = \mathbf{J}\cdot\boldsymbol{\mu}_{\perp\perp}\cdot\mathbf{J} + \dfrac{1}{\epsilon_{zz}k_0^2}\boldsymbol{k}_t\boldsymbol{k}_t - \dfrac{1}{\mu_{zz}}\mathbf{J}\cdot\boldsymbol{\mu}_\perp\boldsymbol{\mu}_z\cdot\mathbf{J} \\ \mathbf{M}_{21} = -\boldsymbol{\epsilon}_{\perp\perp} + \dfrac{1}{\epsilon_{zz}}\boldsymbol{\epsilon}_\perp\boldsymbol{\epsilon}_z - \dfrac{1}{\mu_{zz}k_0^2}\mathbf{J}\cdot\boldsymbol{k}_t\boldsymbol{k}_t\cdot\mathbf{J} \\ \mathbf{M}_{22} = -\dfrac{1}{\epsilon_{zz}k_0}\boldsymbol{\epsilon}_\perp\boldsymbol{k}_t + \dfrac{1}{\mu_{zz}k_0}\mathbf{J}\cdot\boldsymbol{k}_t\boldsymbol{\mu}_z\cdot\mathbf{J} \end{cases}$$

For nonmagnetic anisotropic materials ($\boldsymbol{\xi} = \boldsymbol{\zeta} = \mathbf{0}$, $\boldsymbol{\mu} = \mathbf{I}_3$)

$$\begin{cases} \mathbf{M}_{11} = -\dfrac{1}{\epsilon_{zz}k_0}\boldsymbol{k}_t\boldsymbol{\epsilon}_z \\ \mathbf{M}_{12} = -\mathbf{I}_2 + \dfrac{1}{\epsilon_{zz}k_0^2}\boldsymbol{k}_t\boldsymbol{k}_t \\ \mathbf{M}_{21} = -\boldsymbol{\epsilon}_{\perp\perp} + \dfrac{1}{\epsilon_{zz}}\boldsymbol{\epsilon}_\perp\boldsymbol{\epsilon}_z - \dfrac{1}{k_0^2}\mathbf{J}\cdot\boldsymbol{k}_t\boldsymbol{k}_t\cdot\mathbf{J} \\ \mathbf{M}_{22} = -\dfrac{1}{\epsilon_{zz}k_0}\boldsymbol{\epsilon}_\perp\boldsymbol{k}_t \end{cases}$$

It seems hard to obtain explicit expressions for the single-slab propagator in the general anisotropic case. However, closed-form solutions can be obtained in some special cases which are of interest.

7.3.1 Nonmagnetic uniaxial media

The permittivity dyadic of the uniaxial medium can be written as

$$\boldsymbol{\epsilon} = \epsilon_1\left(\mathbf{I}_3 - \hat{\boldsymbol{u}}\hat{\boldsymbol{u}}\right) + \epsilon_2\hat{\boldsymbol{u}}\hat{\boldsymbol{u}}$$

where the unit vector $\hat{\boldsymbol{u}}$ defines an optical axis in the material. This unit vector has a general orientation in space. Consequently, the decomposition of the permittivity dyadic is, see Section 2.3

$$\begin{cases} \boldsymbol{\epsilon}_{\perp\perp} = \mathbf{I}_2\cdot\boldsymbol{\epsilon}\cdot\mathbf{I}_2 = \epsilon_1\mathbf{I}_2 + (\epsilon_2-\epsilon_1)\boldsymbol{u}_{xy}\boldsymbol{u}_{xy} \\ \boldsymbol{\epsilon}_z = \hat{\boldsymbol{z}}\cdot\boldsymbol{\epsilon}\cdot\mathbf{I}_2 = (\epsilon_2-\epsilon_1)u_z\boldsymbol{u}_{xy} \\ \boldsymbol{\epsilon}_\perp = \mathbf{I}_2\cdot\boldsymbol{\epsilon}\cdot\hat{\boldsymbol{z}} = (\epsilon_2-\epsilon_1)u_z\boldsymbol{u}_{xy} = \boldsymbol{\epsilon}_z \\ \epsilon_{zz} = \hat{\boldsymbol{z}}\cdot\boldsymbol{\epsilon}\cdot\hat{\boldsymbol{z}} = \epsilon_1 + (\epsilon_2-\epsilon_1)u_z^2 \end{cases}$$

where the projection of the unit vector $\hat{\boldsymbol{u}}$ on the x-y plane is $\boldsymbol{u}_{xy} = \mathbf{I}_2\cdot\hat{\boldsymbol{u}}$ and the projection along the z-axis is $u_z = \hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{z}}$. The blocks of the fundamental dyadic become

$$\begin{cases} \mathbf{M}_{11} = -\dfrac{(\epsilon_2-\epsilon_1)u_z}{\epsilon_{zz}k_0}\boldsymbol{k}_t\boldsymbol{u}_{xy} \\ \mathbf{M}_{12} = -\mathbf{I}_2 + \dfrac{1}{\epsilon_{zz}k_0^2}\boldsymbol{k}_t\boldsymbol{k}_t \\ \mathbf{M}_{21} = -\epsilon_1\mathbf{I}_2 - \dfrac{\epsilon_1(\epsilon_2-\epsilon_1)}{\epsilon_{zz}}\boldsymbol{u}_{xy}\boldsymbol{u}_{xy} - \dfrac{1}{k_0^2}\mathbf{J}\cdot\boldsymbol{k}_t\boldsymbol{k}_t\cdot\mathbf{J} \\ \mathbf{M}_{22} = -\dfrac{(\epsilon_2-\epsilon_1)u_z}{\epsilon_{zz}k_0}\boldsymbol{u}_{xy}\boldsymbol{k}_t = \mathbf{M}_{11}^t \end{cases}$$

An appropriate matrix representation of the fundamental dyadic $\mathbf{M}$ in the $\{\hat{\boldsymbol{e}}_\parallel, \hat{\boldsymbol{e}}_\perp\}$-system is

$$\begin{pmatrix} -\frac{(\epsilon_2-\epsilon_1)}{\epsilon_{zz}}\frac{k_t}{k_0}u_z u_\parallel & -\frac{(\epsilon_2-\epsilon_1)}{\epsilon_{zz}}\frac{k_t}{k_0}u_z u_\perp & \frac{1}{\epsilon_{zz}}\frac{k_t^2}{k_0^2}-1 & 0 \\ 0 & 0 & 0 & -1 \\ -\epsilon_1-\frac{\epsilon_1(\epsilon_2-\epsilon_1)}{\epsilon_{zz}}u_\parallel^2 & -\frac{\epsilon_1(\epsilon_2-\epsilon_1)}{\epsilon_{zz}}u_\parallel u_\perp & -\frac{(\epsilon_2-\epsilon_1)}{\epsilon_{zz}}\frac{k_t}{k_0}u_z u_\parallel & 0 \\ -\frac{\epsilon_1(\epsilon_2-\epsilon_1)}{\epsilon_{zz}}u_\parallel u_\perp & -\epsilon_1-\frac{\epsilon_1(\epsilon_2-\epsilon_1)}{\epsilon_{zz}}u_\perp^2+\frac{k_t^2}{k_0^2} & -\frac{(\epsilon_2-\epsilon_1)}{\epsilon_{zz}}\frac{k_t}{k_0}u_z u_\perp & 0 \end{pmatrix}$$

where $u_\parallel = \hat{\boldsymbol{e}}_\parallel \cdot \boldsymbol{u}_{xy}$ and $u_\perp = \hat{\boldsymbol{e}}_\perp \cdot \boldsymbol{u}_{xy}$.

Normal incidence: At normal incidence, $k_t = 0$; hence

$$\mathbf{M} = \begin{pmatrix} \mathbf{0} & -\mathbf{I}_2 \\ -\epsilon_1\mathbf{I}_2 - \frac{\epsilon_1(\epsilon_2-\epsilon_1)}{\epsilon_{zz}}\boldsymbol{u}_{xy}\boldsymbol{u}_{xy} & \mathbf{0} \end{pmatrix}$$

and

$$\mathbf{M}\cdot\mathbf{M} = \begin{pmatrix} \epsilon_1\mathbf{I}_2 + \frac{\epsilon_1(\epsilon_2-\epsilon_1)}{\epsilon_{zz}}\boldsymbol{u}_{xy}\boldsymbol{u}_{xy} & \mathbf{0} \\ \mathbf{0} & \epsilon_1\mathbf{I}_2 + \frac{\epsilon_1(\epsilon_2-\epsilon_1)}{\epsilon_{zz}}\boldsymbol{u}_{xy}\boldsymbol{u}_{xy} \end{pmatrix}$$

The eigenvalues of $\mathbf{M}$ are all distinct; specifically, equation (4.3) holds, where

$$\lambda_+^2 = \epsilon_1, \qquad \lambda_-^2 = \frac{\epsilon_1\epsilon_2}{\epsilon_{zz}} = \frac{\epsilon_1\epsilon_2}{\epsilon_1 + (\epsilon_2-\epsilon_1)u_z^2}$$

These eigenvalues correspond to up-going (down-going waves correspond to the similar negative values) ordinary and extra-ordinary waves in the medium, respectively. The single-slab propagator is given by equation (4.4).

Optical axis in the normal direction: In this case, $\boldsymbol{u}_{xy} = \mathbf{0}$ ($u_z = \pm 1$); consequently,

$$\mathbf{M} = \begin{pmatrix} \mathbf{0} & -\mathbf{I}_2 + \frac{1}{\epsilon_2 k_0^2}\boldsymbol{k}_t\boldsymbol{k}_t \\ -\epsilon_1\mathbf{I}_2 - \frac{1}{k_0^2}\mathbf{J}\cdot\boldsymbol{k}_t\boldsymbol{k}_t\cdot\mathbf{J} & \mathbf{0} \end{pmatrix}$$

and

$$\mathbf{M}\cdot\mathbf{M} = \begin{pmatrix} \epsilon_1\left(\mathbf{I}_2 - \frac{1}{\epsilon_2 k_0^2}\boldsymbol{k}_t\boldsymbol{k}_t\right) + \frac{1}{k_0^2}\mathbf{J}\cdot\boldsymbol{k}_t\boldsymbol{k}_t\cdot\mathbf{J} & \mathbf{0} \\ \mathbf{0} & \epsilon_1\left(\mathbf{I}_2 - \frac{1}{\epsilon_2 k_0^2}\boldsymbol{k}_t\boldsymbol{k}_t\right) + \frac{1}{k_0^2}\mathbf{J}\cdot\boldsymbol{k}_t\boldsymbol{k}_t\cdot\mathbf{J} \end{pmatrix}$$

The eigenvalues of $\mathbf{M}$ are all distinct; specifically, equation (4.3) holds, where

$$\lambda_+^2 = \epsilon_1 - \frac{k_t^2}{k_0^2}, \qquad \lambda_-^2 = \epsilon_1 - \frac{k_t^2}{k_0^2}\frac{\epsilon_1}{\epsilon_2}$$

The single-slab propagator is given by equation (4.4):

$$\begin{aligned}\mathbf{P} = e^{ik_0 d\mathbf{M}} &= \begin{pmatrix} \hat{\boldsymbol{e}}_\parallel \hat{\boldsymbol{e}}_\parallel \cos(k_0 d\lambda_-) & -i\hat{\boldsymbol{e}}_\parallel \hat{\boldsymbol{e}}_\parallel \frac{\lambda_-}{\epsilon_1} \sin(k_0 d\lambda_-) \\ -i\hat{\boldsymbol{e}}_\parallel \hat{\boldsymbol{e}}_\parallel \frac{\epsilon_1}{\lambda_-} \sin(k_0 d\lambda_-) & \hat{\boldsymbol{e}}_\parallel \hat{\boldsymbol{e}}_\parallel \cos(k_0 d\lambda_-) \end{pmatrix} \\ &+ \begin{pmatrix} \hat{\boldsymbol{e}}_\perp \hat{\boldsymbol{e}}_\perp \cos(k_0 d\lambda_+) & -i\hat{\boldsymbol{e}}_\perp \hat{\boldsymbol{e}}_\perp \frac{1}{\lambda_+} \sin(k_0 d\lambda_+) \\ -i\hat{\boldsymbol{e}}_\perp \hat{\boldsymbol{e}}_\perp \lambda_+ \sin(k_0 d\lambda_+) & \hat{\boldsymbol{e}}_\perp \hat{\boldsymbol{e}}_\perp \cos(k_0 d\lambda_+) \end{pmatrix}\end{aligned}$$

Similar to the isotropic case, one can show that the reflection and transmission dyadics for the uniaxial slab can be written in the form (7.2), where

$$\begin{cases} r_{\parallel\parallel} = r_{1\parallel\parallel} \dfrac{1 - e^{2ik_0\lambda_- d}}{1 - r_{1\parallel\parallel}^2 e^{2ik_0\lambda_- d}} \\ r_{\perp\perp} = r_{1\perp\perp} \dfrac{1 - e^{2ik_0\lambda_+ d}}{1 - r_{1\perp\perp}^2 e^{2ik_0\lambda_+ d}} \\ t_{\parallel\parallel} = \dfrac{(1 - r_{1\parallel\parallel}^2) e^{ik_0\lambda_- d}}{1 - r_{1\parallel\parallel}^2 e^{2ik_0\lambda_- d}} \\ t_{\perp\perp} = \dfrac{(1 - r_{1\perp\perp}^2) e^{ik_0\lambda_+ d}}{1 - r_{1\perp\perp}^2 e^{2ik_0\lambda_+ d}} \end{cases}$$

and

$$\begin{cases} r_{1\parallel\parallel} = \dfrac{\lambda_- - \epsilon_1 \cos\theta_i}{\lambda_- + \epsilon_1 \cos\theta_i} \\ r_{1\perp\perp} = \dfrac{\cos\theta_i - \lambda_+}{\cos\theta_i + \lambda_+} \end{cases}$$

For a PEC-backed uniaxial slab similar results holds. Explicitly, the result is

$$\begin{cases} r_{\perp\perp} = \dfrac{r_{1\parallel\parallel} - e^{2ik_0\lambda_- d}}{1 - r_{1\parallel\parallel} e^{2ik_0\lambda_- d}} \\ r_{\perp\perp} = \dfrac{r_{1\perp\perp} - e^{2ik_0\lambda_+ d}}{1 - r_{1\perp\perp} e^{2ik_0\lambda_+ d}} \end{cases}$$

8 Numerical computations

In this section, we illustrate the analysis presented in the previous sections in a series of numerical computations. The programming task is most easily done in a language that supports matrix manipulations, *e.g.*, MATLAB. The numerical implementation is straightforward and causes no problem except for cases where strongly dissipative layers and evanescent waves are present. In these cases special considerations have to made [22].

8.1 Reflectance and transmittance

In view of equation (5.5) for the Poynting vectors associated with the up-going and the down-going fields, the reflectance and the transmittance of the planar bian-isotropic slab are defined by

$$\begin{cases} R = \dfrac{|F_{\|}^{-}(z_1)|^2/\cos^2\theta_i + |F_{\perp}^{-}(z_1)|^2}{|F_{\|}^{+}(z_1)|^2/\cos^2\theta_i + |F_{\perp}^{+}(z_1)|^2} \\ T = \dfrac{|F_{\|}^{+}(z_{N-1})|^2/\cos^2\theta_i + |F_{\perp}^{+}(z_{N-1})|^2}{|F_{\|}^{+}(z_1)|^2/\cos^2\theta_i + |F_{\perp}^{+}(z_1)|^2} \end{cases}$$

respectively. Notice that the reflectance and the transmittance depend on the angles θ_i and ϕ_i (or equivalently on the tangential wave vector $\boldsymbol{k}_t$). These quantities can be expressed in terms of the components of the reflection and transmission dyadics for the electric field

$$\begin{cases} \mathbf{r} = \hat{\boldsymbol{e}}_{\|}\hat{\boldsymbol{e}}_{\|}r_{\|\|} + \hat{\boldsymbol{e}}_{\|}\hat{\boldsymbol{e}}_{\perp}r_{\|\perp} + \hat{\boldsymbol{e}}_{\perp}\hat{\boldsymbol{e}}_{\|}r_{\perp\|} + \hat{\boldsymbol{e}}_{\perp}\hat{\boldsymbol{e}}_{\perp}r_{\perp\perp} \\ \mathbf{t} = \hat{\boldsymbol{e}}_{\|}\hat{\boldsymbol{e}}_{\|}t_{\|\|} + \hat{\boldsymbol{e}}_{\|}\hat{\boldsymbol{e}}_{\perp}t_{\|\perp} + \hat{\boldsymbol{e}}_{\perp}\hat{\boldsymbol{e}}_{\|}t_{\perp\|} + \hat{\boldsymbol{e}}_{\perp}\hat{\boldsymbol{e}}_{\perp}t_{\perp\perp} \end{cases}$$

the angles θ_i and ϕ_i, and the polarization angle, χ, of the incident electric field at $z = z_1$ defined by

$$\boldsymbol{E}^i(z_1) = E_0(z_1)\left(\hat{\boldsymbol{e}}_{\perp}\cos\chi + \hat{\boldsymbol{e}}_{\perp}\times\hat{\boldsymbol{k}}^{+}\sin\chi\right)$$

where $E_0(z_1)$ is a complex number that gives the amplitude and phase of the incident plane polarized wave at the front end of the slab. For a plane polarized incident field the result is

$$\begin{cases} R = |r_{\|\|}\sin\chi + r_{\|\perp}\cos\chi/\cos\theta_i|^2 + |r_{\perp\|}\cos\theta_i\sin\chi + r_{\perp\perp}\cos\chi|^2 \\ T = |t_{\|\|}\sin\chi + t_{\|\perp}\cos\chi/\cos\theta_i|^2 + |t_{\perp\|}\cos\theta_i\sin\chi + t_{\perp\perp}\cos\chi|^2 \end{cases} \tag{8.1}$$

which in the absence of any cross-polarized terms, *e.g.*, for isotropic materials, reduce to

$$\begin{cases} R = |r_{\|\|}|^2\sin^2\chi + |r_{\perp\perp}|^2\cos^2\chi \\ T = |t_{\|\|}|^2\sin^2\chi + |t_{\perp\perp}|^2\cos^2\chi \end{cases}$$

The perpendicular polarization (TE polarization) corresponds to $\chi = 0$ and the parallel polarization (TM polarization) to $\chi = \pi/2$.

8.2 Example—radome

In this first example we illustrate how the transmitted power (transmittance T) is affected by the introduction of one or several uniaxial layers in a radome. The specific parameters of the radome with only isotropic layers and with a uniaxial reinforcement are given in Table 2. The dielectric data of the materials are obtained from Ref. 1, which also provides the pertinent mixing formulas for the glass fiber reinforced Epoxy.

Without reinforcing layer

	ϵ	d (mm)
Epoxy	3.65(1+i0.0320)	0.8
Rohacell	1.10(1+i0.0004)	6.4
Epoxy	3.65(1+i0.0320)	0.8

With reinforcing layer (E-glass, $\epsilon = 6.32(1 + i0.0037)$**) [1]**

	$\epsilon_{xx} = \epsilon_{yy}$	ϵ_{zz}	d (mm)
Epoxy + E-glass (30%)	4.44(1+i0.0218)	4.23(1+i0.0248)	0.8
Rohacell	1.10(1+i0.0004)	1.10(1+i0.0004)	6.4
Epoxy + E-glass (30%)	4.44(1+i0.0218)	4.23(1+i0.0248)	0.8

Table 2: Material parameters for the radome in Example 8.2 which are presented in Figures 5–6.

The transmitted power for a series of incident angles for these materials are presented in Figures 5–6. The perpendicular polarization (TE-case) is depicted in Figure 5 and the parallel polarization (TM-case) is depicted in Figure 6. The perpendicular polarization (TE-case) is not affected by the anisotropy, but the change of the permittivity in the lateral directions, *i.e.*, ϵ_{xx} and ϵ_{yy}, alter the transmission properties. This is due to the special orientation of the uniaxial layers ($\hat{\boldsymbol{u}} = \hat{\boldsymbol{z}}$ for the optical axis of the layers) in this example.

The effect of the uniaxial layer (glass fiber reinforcement) is not negligible. This is especially true at higher frequencies. At these higher frequencies an extra transmission loss of several dB is observed due to the glass fiber reinforcement.

As a second example of the analysis presented in this paper we consider another radome application. The radome is a 13-layer construction of E-glass/resin and polyethen/resin. The 13 layers are periodically repeated as: E-glass/resin 0.20 mm, polyethen/resin 0.40 mm, E-glass/resin 0.40 mm, polyethen/resin 0.40 mm,..., polyethen/resin 0.40 mm, and E-glass/resin 0.20 mm, respectively. The permittivity of the layers are E-glass/resin $4.40(1 + i0.0100)$, and polythene/resin $2.60(1 + i0.0060)$. The transmittance for an angle of incidence of 30° is shown in Figure 7. The figure clearly shows that the radome acts as a homogeneous slab at low frequencies ($\leq$ 90 GHz), *i.e.*, there is no resolution of the 13 layers. The resonance phenomena at $\approx$ 110 GHz gives the desired transmission reduction due to the layered structure.

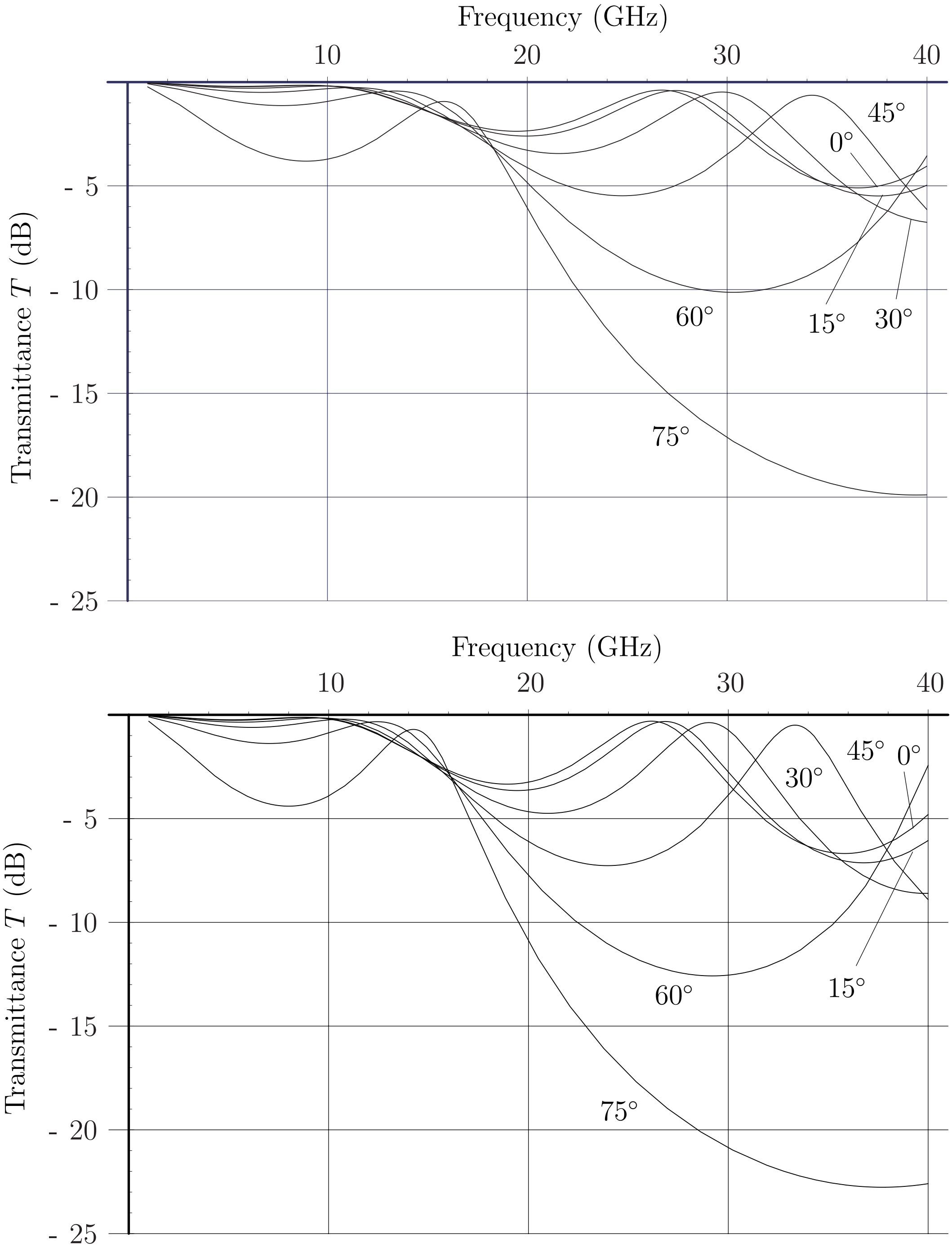
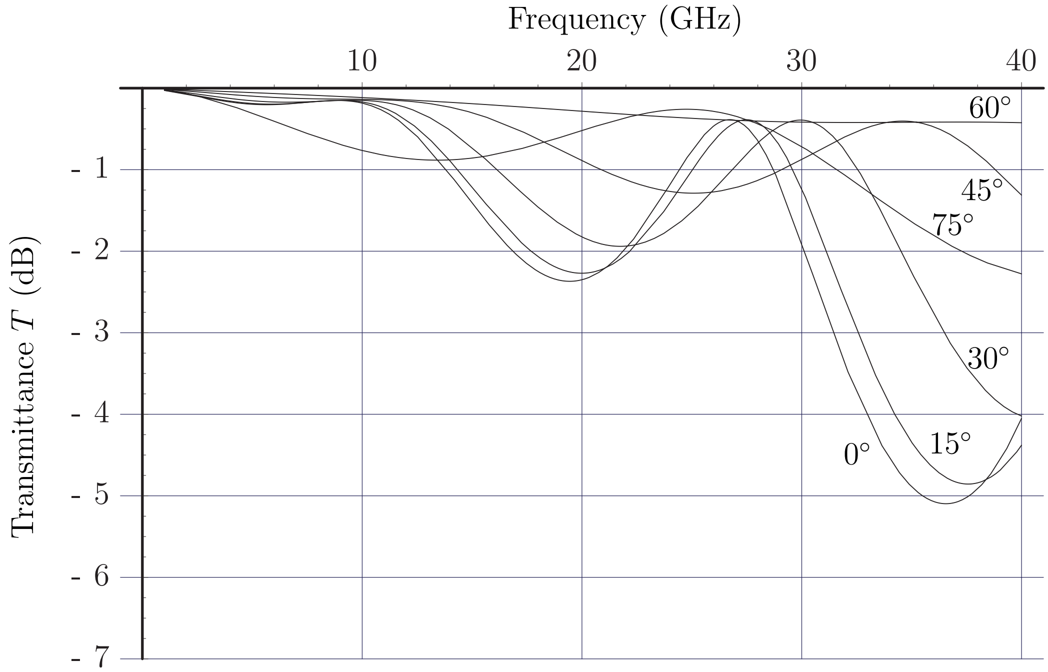


Figure 5: The transmitted power (transmittance) for an isotropic slab (upper) and an uniaxially anisotropic slab (lower) with data given in Table 2 as a function of frequency for different angles of incidence. The polarization of the incident electric field is $\chi = 0$ (TE polarization).



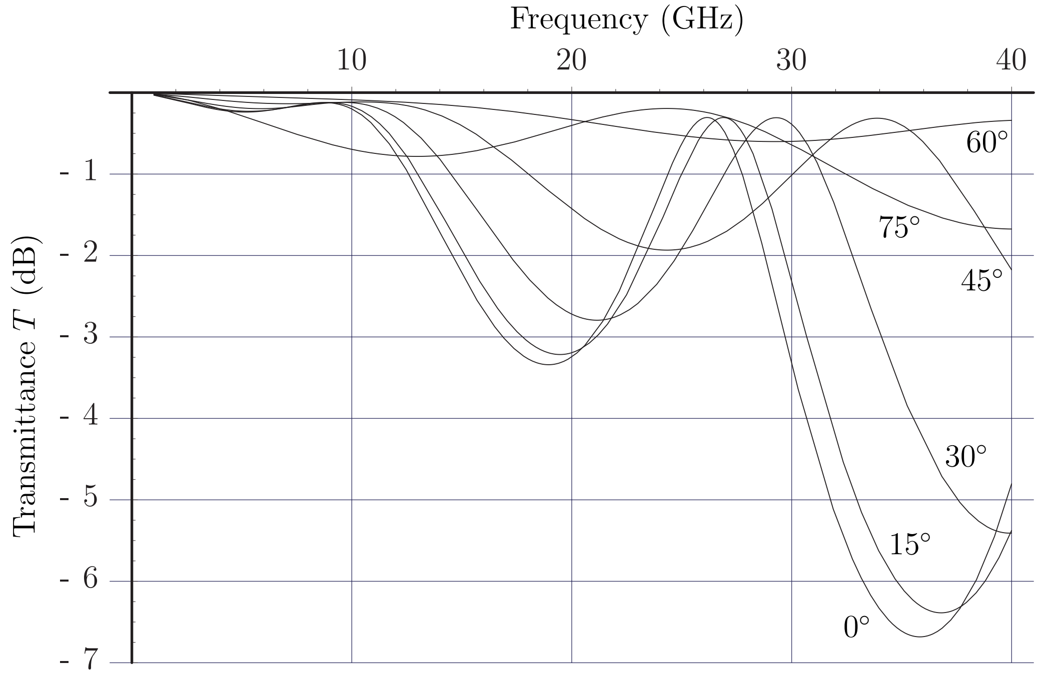


Figure 6: The transmitted power (transmittance) for an isotropic slab (upper) and an uniaxially anisotropic slab (lower) with data given in Table 2 as a function of frequency for different angles of incidence. The polarization of the incident electric field is $\chi = \pi/2$ (TM polarization).

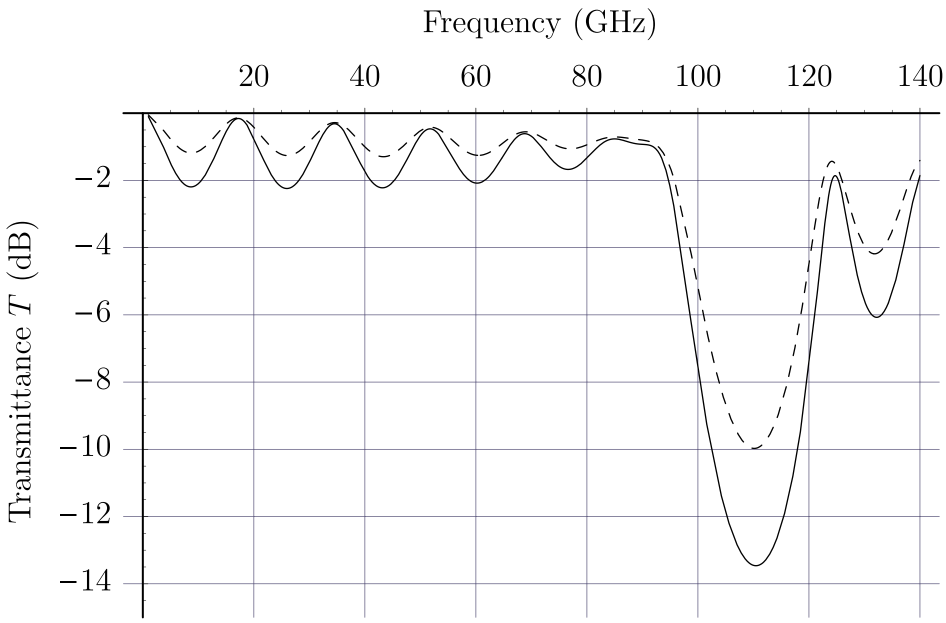


Figure 7: The transmitted power (transmittance) for a 13-layer slab with data given in the text as a function of frequency for an angle of incidence of 30°. The solid line shows TE polarization and the broken line the TM polarization.

$\boldsymbol{\epsilon}$	$\boldsymbol{\mu}$	$\boldsymbol{\xi}$	$\boldsymbol{\zeta}$	k_0d
$\begin{pmatrix}3 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 3\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1.1\end{pmatrix}$	$\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & i0.5\\ 0 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & -i0.5 & 0\end{pmatrix}$	2π

Table 4: Material parameters for the bianisotropic material in Example 8.3 which is presented in Figure 8.

8.3 Example—bianisotropic media

As an illustration of the performance of the algorithm in the general bianisotropic case, we choose to calculate the transmission properties of a complex material. An example of such a material is the Ω-material, which has been investigated intensely during recent years [17, 20].

In Figure 8 we illustrate the transmission properties of a Ω-material. Specifically, we show the transmittance T, see (8.1), of the slab as a function of the two angles θ_i and ϕ_i (or equivalently $\boldsymbol{k}_t$) for the two generic polarization of the incident wave (TE- and TM-cases). The specific data for the material is given in Table 4. This material can be manufactured by putting small Ω-shaped elements in the x-y-plane in a host medium [17].

From these computations we clearly see that the transmittance vary as a function

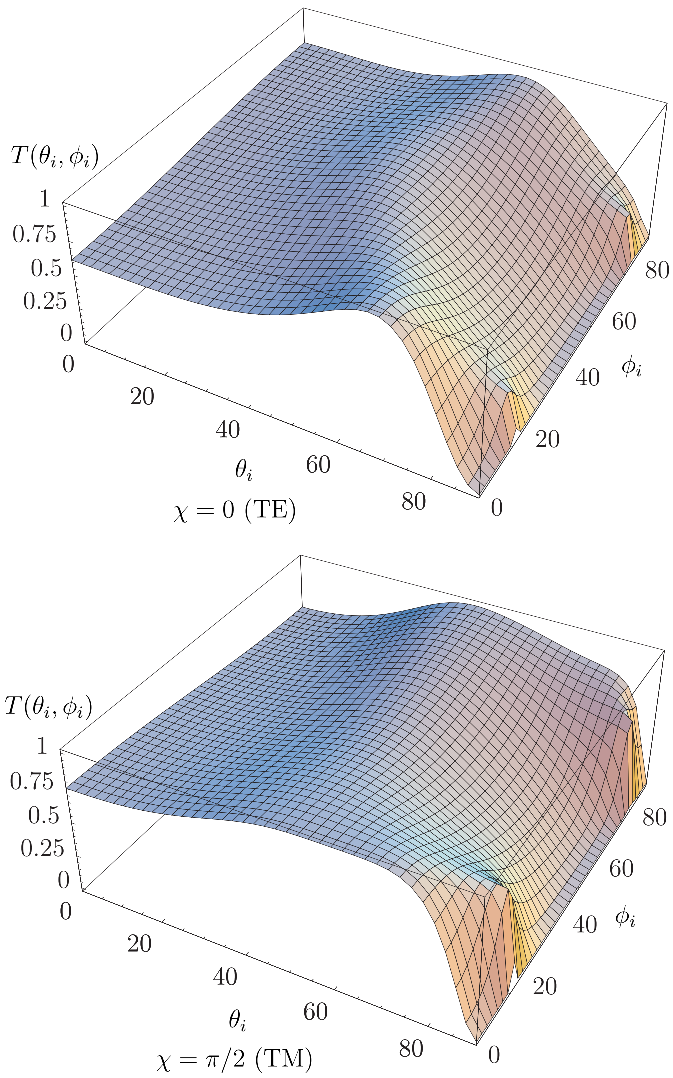


Figure 8: The transmittance for a bianisotropic slab given in Table 4. The two generic polarizations, TE and TM, of the incident wave are depicted.

of the incident direction ϕ_i, which shows that the slab lacks symmetry in the x-y-plane. The difference between the two different plane of polarization is not very striking. Another choice of parameters shows larger differences [17].

9 Conclusions

In this paper we have shown how the wave propagation properties in plane-stratified slab comprised of complex (bianisotropic) media can be analyzed by the notion of propagators. The propagators map the total field at one position, *e.g.*, the left hand side boundary of the slab to another position, *e.g.*, the right hand side of the slab. The Cayley-Hamilton theorem simplifies the evaluation of the propagators.

Wave splitting of the total field then easily gives the reflection and the transmission dyadics of the slab. Several numerical computations show the performance of the analysis.

Acknowledgments

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Appendix A Cayley-Hamilton theorem

The following theorems are of fundamental importance for computing the action of an entire function of a square dyadic [6]

Theorem A.1 (Cayley-Hamilton). *A quadratic dyadic* $\mathbf{A}$ *satisfies its own characteristic equation:*

$$\textit{If } p_{\mathbf{A}}(\lambda) = \det(\lambda \mathbf{I} - \mathbf{A}), \ \textit{then } p_{\mathbf{A}}(\mathbf{A}) = \mathbf{0}$$

From this theorem, one can prove the following important theorem.

Theorem A.2. *Let* $\lambda_1, \ldots, \lambda_m$ *be the different eigenvalues of the* n*-dimensional dyadic* $\mathbf{A}$*, and* $n_1, \ldots, n_m$ *their multiplicity. If* $f(z)$ *is an entire function, then*

$$f(\mathbf{A}) = q(\mathbf{A})$$

where the uniquely defined polynomial q *of degree* $\leq n-1$ *is defined by the following conditions:*

$$\frac{d^j q}{dz^j}(\lambda_k) = \frac{d^j f}{dz^j}(\lambda_k), \qquad j = 0, \ldots, n_k - 1, k = 1, \ldots, m$$

Appendix B Campbell-Hausdorff series

Let $\mathbf{A}_i$, $i = 1, 2$ be two dyadics, and $[\mathbf{A}_1, \mathbf{A}_2] = \mathbf{A}_1 \cdot \mathbf{A}_2 - \mathbf{A}_2 \cdot \mathbf{A}_1$ be the commutator of the dyadics. Then we construct a dyadic $\mathbf{A}$, such that

$$e^{\mathbf{A}} = e^{\mathbf{A}_1} \cdot e^{\mathbf{A}_2}$$

by

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 + \frac{1}{2}[\mathbf{A}_1, \mathbf{A}_2] + \frac{1}{3!}\{[[\mathbf{A}_1, \mathbf{A}_2], \mathbf{A}_2] + [[\mathbf{A}_2, \mathbf{A}_1], \mathbf{A}_1]\} + \ldots$$

In the long-wave limit, the approximation $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$ can be used to homogenize a two-component structure of bianisotropic layers.

Appendix C Derivation of fundamental equation

The Maxwell equations, (2.1), can be decomposed in tangential and normal parts as

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix}\boldsymbol{E}_{xy}(z)\\ \eta_0\mathbf{J}\cdot\boldsymbol{H}_{xy}(z)\end{pmatrix}=i\begin{pmatrix}\boldsymbol{k}_t & \mathbf{0}\\ \mathbf{0} & \mathbf{J}\cdot\boldsymbol{k}_t\end{pmatrix}\begin{pmatrix}E_z(z)\\ \eta_0H_z(z)\end{pmatrix}-ik_0\begin{pmatrix}c_0\mathbf{J}\cdot\boldsymbol{B}_{xy}(z)\\ c_0\eta_0\boldsymbol{D}_{xy}(z)\end{pmatrix}\tag{C.1}$$

and

$$\begin{pmatrix}c_0\eta_0D_z(z)\\ c_0B_z(z)\end{pmatrix}=\frac{1}{k_0}\begin{pmatrix}\mathbf{0} & \boldsymbol{k}_t\\ \mathbf{J}\cdot\boldsymbol{k}_t & \mathbf{0}\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{E}_{xy}(z)\\ \eta_0\mathbf{J}\cdot\boldsymbol{H}_{xy}(z)\end{pmatrix}\tag{C.2}$$

where an appropriate matrix notion has been introduced.

Similarly, the constitutive relations, (2.2), can be decomposed in tangential and normal parts as

$$\begin{aligned}\begin{pmatrix}c_0\mathbf{J}\cdot\boldsymbol{B}_{xy}(z)\\ c_0\eta_0\boldsymbol{D}_{xy}(z)\end{pmatrix}&=\begin{pmatrix}\mathbf{J}\cdot\boldsymbol{\zeta}_{\perp\perp} & -\mathbf{J}\cdot\boldsymbol{\mu}_{\perp\perp}\cdot\mathbf{J}\\ \boldsymbol{\epsilon}_{\perp\perp} & -\boldsymbol{\xi}_{\perp\perp}\cdot\mathbf{J}\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{E}_{xy}(z)\\ \eta_0\mathbf{J}\cdot\boldsymbol{H}_{xy}(z)\end{pmatrix}\\ &+\begin{pmatrix}\mathbf{J}\cdot\boldsymbol{\zeta}_{\perp} & \mathbf{J}\cdot\boldsymbol{\mu}_{\perp}\\ \boldsymbol{\epsilon}_{\perp} & \boldsymbol{\xi}_{\perp}\end{pmatrix}\begin{pmatrix}E_z(z)\\ \eta_0H_z(z)\end{pmatrix}\end{aligned}\tag{C.3}$$

and

$$\begin{pmatrix}\epsilon_{zz} & \xi_{zz}\\ \zeta_{zz} & \mu_{zz}\end{pmatrix}\begin{pmatrix}E_z(z)\\ \eta_0H_z(z)\end{pmatrix}=\begin{pmatrix}c_0\eta_0D_z(z)\\ c_0B_z(z)\end{pmatrix}+\begin{pmatrix}-\boldsymbol{\epsilon}_z & \boldsymbol{\xi}_z\cdot\mathbf{J}\\ -\boldsymbol{\zeta}_z & \boldsymbol{\mu}_z\cdot\mathbf{J}\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{E}_{xy}(z)\\ \eta_0\mathbf{J}\cdot\boldsymbol{H}_{xy}(z)\end{pmatrix}\tag{C.4}$$

where

$$\begin{cases}\boldsymbol{\epsilon}=\boldsymbol{\epsilon}_{\perp\perp}+\hat{\boldsymbol{z}}\boldsymbol{\epsilon}_z+\boldsymbol{\epsilon}_{\perp}\hat{\boldsymbol{z}}+\hat{\boldsymbol{z}}\epsilon_{zz}\hat{\boldsymbol{z}}\\ \boldsymbol{\xi}=\boldsymbol{\xi}_{\perp\perp}+\hat{\boldsymbol{z}}\boldsymbol{\xi}_z+\boldsymbol{\xi}_{\perp}\hat{\boldsymbol{z}}+\hat{\boldsymbol{z}}\xi_{zz}\hat{\boldsymbol{z}}\\ \boldsymbol{\zeta}=\boldsymbol{\zeta}_{\perp\perp}+\hat{\boldsymbol{z}}\boldsymbol{\zeta}_z+\boldsymbol{\zeta}_{\perp}\hat{\boldsymbol{z}}+\hat{\boldsymbol{z}}\zeta_{zz}\hat{\boldsymbol{z}}\\ \boldsymbol{\mu}=\boldsymbol{\mu}_{\perp\perp}+\hat{\boldsymbol{z}}\boldsymbol{\mu}_z+\boldsymbol{\mu}_{\perp}\hat{\boldsymbol{z}}+\hat{\boldsymbol{z}}\mu_{zz}\hat{\boldsymbol{z}}\end{cases}$$

are defined in Section 2.3.

Combining equations (C.1) and (C.3) gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix}\boldsymbol{E}_{xy}(z)\\ \eta_0\mathbf{J}\cdot\boldsymbol{H}_{xy}(z)\end{pmatrix}=&-ik_0\begin{pmatrix}\mathbf{J}\cdot\boldsymbol{\zeta}_{\perp\perp} & -\mathbf{J}\cdot\boldsymbol{\mu}_{\perp\perp}\cdot\mathbf{J}\\ \boldsymbol{\epsilon}_{\perp\perp} & -\boldsymbol{\xi}_{\perp\perp}\cdot\mathbf{J}\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{E}_{xy}(z)\\ \eta_0\mathbf{J}\cdot\boldsymbol{H}_{xy}(z)\end{pmatrix}\\ &-ik_0\begin{pmatrix}-\boldsymbol{k}_t/k_0+\mathbf{J}\cdot\boldsymbol{\zeta}_{\perp} & \mathbf{J}\cdot\boldsymbol{\mu}_{\perp}\\ \boldsymbol{\epsilon}_{\perp} & -\mathbf{J}\cdot\boldsymbol{k}_t/k_0+\boldsymbol{\xi}_{\perp}\end{pmatrix}\begin{pmatrix}E_z(z)\\ \eta_0H_z(z)\end{pmatrix}\end{aligned}\tag{C.5}$$

Similarly, combining equations (C.2) and (C.4) yields

$$\begin{pmatrix}\epsilon_{zz} & \xi_{zz}\\ \zeta_{zz} & \mu_{zz}\end{pmatrix}\begin{pmatrix}E_z(z)\\ \eta_0H_z(z)\end{pmatrix}=\begin{pmatrix}-\boldsymbol{\epsilon}_z & \boldsymbol{k}_t/k_0+\boldsymbol{\xi}_z\cdot\mathbf{J}\\ \mathbf{J}\cdot\boldsymbol{k}_t/k_0-\boldsymbol{\zeta}_z & \boldsymbol{\mu}_z\cdot\mathbf{J}\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{E}_{xy}(z)\\ \eta_0\mathbf{J}\cdot\boldsymbol{H}_{xy}(z)\end{pmatrix}\tag{C.6}$$

which is an expression for the normal components of the electric and magnetic fields in terms of tangential components.

Eliminating the normal field components gives the fundamental equation (3.3). The fundamental dyadic is found to be

$$\mathbf{M} = \begin{pmatrix} -\mathbf{J}\cdot\boldsymbol{\zeta}_{\perp\perp} & \mathbf{J}\cdot\boldsymbol{\mu}_{\perp\perp}\cdot\mathbf{J} \\ -\boldsymbol{\epsilon}_{\perp\perp} & \boldsymbol{\xi}_{\perp\perp}\cdot\mathbf{J} \end{pmatrix} + \begin{pmatrix} \boldsymbol{k}_t/k_0 - \mathbf{J}\cdot\boldsymbol{\zeta}_{\perp} & -\mathbf{J}\cdot\boldsymbol{\mu}_{\perp} \\ -\boldsymbol{\epsilon}_{\perp} & \mathbf{J}\cdot\boldsymbol{k}_t/k_0 - \boldsymbol{\xi}_{\perp} \end{pmatrix}$$
$$\frac{1}{\epsilon_{zz}\mu_{zz} - \xi_{zz}\zeta_{zz}} \begin{pmatrix} -\mu_{zz}\boldsymbol{\epsilon}_z - \xi_{zz}\mathbf{J}\cdot\boldsymbol{k}_t/k_0 + \xi_{zz}\boldsymbol{\zeta}_z & \mu_{zz}\boldsymbol{k}_t/k_0 + \mu_{zz}\boldsymbol{\xi}_z\cdot\mathbf{J} - \xi_{zz}\boldsymbol{\mu}_z\cdot\mathbf{J} \\ \zeta_{zz}\boldsymbol{\epsilon}_z + \epsilon_{zz}\mathbf{J}\cdot\boldsymbol{k}_t/k_0 - \epsilon_{zz}\boldsymbol{\zeta}_z & -\zeta_{zz}\boldsymbol{k}_t/k_0 - \zeta_{zz}\boldsymbol{\xi}_z\cdot\mathbf{J} + \epsilon_{zz}\boldsymbol{\mu}_z\cdot\mathbf{J} \end{pmatrix}$$

Specifically, the block dyadics, $\mathbf{M}_{lm}$, $l, m = 1, 2$, are

$$\begin{cases} \mathbf{M}_{11} = -\mathbf{J}\cdot\boldsymbol{\zeta}_{\perp\perp} + a\left(\boldsymbol{k}_t/k_0 - \mathbf{J}\cdot\boldsymbol{\zeta}_{\perp}\right)\left(-\mu_{zz}\boldsymbol{\epsilon}_z - \xi_{zz}\mathbf{J}\cdot\boldsymbol{k}_t/k_0 + \xi_{zz}\boldsymbol{\zeta}_z\right) \\ \qquad\qquad - a\left(\mathbf{J}\cdot\boldsymbol{\mu}_{\perp}\right)\left(\zeta_{zz}\boldsymbol{\epsilon}_z + \epsilon_{zz}\mathbf{J}\cdot\boldsymbol{k}_t/k_0 - \epsilon_{zz}\boldsymbol{\zeta}_z\right) \\ \mathbf{M}_{12} = \mathbf{J}\cdot\boldsymbol{\mu}_{\perp\perp}\cdot\mathbf{J} + a\left(\boldsymbol{k}_t/k_0 - \mathbf{J}\cdot\boldsymbol{\zeta}_{\perp}\right)\left(\mu_{zz}\boldsymbol{k}_t/k_0 + \mu_{zz}\boldsymbol{\xi}_z\cdot\mathbf{J} - \xi_{zz}\boldsymbol{\mu}_z\cdot\mathbf{J}\right) \\ \qquad\qquad - a\left(\mathbf{J}\cdot\boldsymbol{\mu}_{\perp}\right)\left(-\zeta_{zz}\boldsymbol{k}_t/k_0 - \zeta_{zz}\boldsymbol{\xi}_z\cdot\mathbf{J} + \epsilon_{zz}\boldsymbol{\mu}_z\cdot\mathbf{J}\right) \\ \mathbf{M}_{21} = -\boldsymbol{\epsilon}_{\perp\perp} - a\boldsymbol{\epsilon}_{\perp}\left(-\mu_{zz}\boldsymbol{\epsilon}_z - \xi_{zz}\mathbf{J}\cdot\boldsymbol{k}_t/k_0 + \xi_{zz}\boldsymbol{\zeta}_z\right) \\ \qquad\qquad + a\left(\mathbf{J}\cdot\boldsymbol{k}_t/k_0 - \boldsymbol{\xi}_{\perp}\right)\left(\zeta_{zz}\boldsymbol{\epsilon}_z + \epsilon_{zz}\mathbf{J}\cdot\boldsymbol{k}_t/k_0 - \epsilon_{zz}\boldsymbol{\zeta}_z\right) \\ \mathbf{M}_{22} = \boldsymbol{\xi}_{\perp\perp}\cdot\mathbf{J} - a\boldsymbol{\epsilon}_{\perp}\left(\mu_{zz}\boldsymbol{k}_t/k_0 + \mu_{zz}\boldsymbol{\xi}_z\cdot\mathbf{J} - \xi_{zz}\boldsymbol{\mu}_z\cdot\mathbf{J}\right) \\ \qquad\qquad + a\left(\mathbf{J}\cdot\boldsymbol{k}_t/k_0 - \boldsymbol{\xi}_{\perp}\right)\left(-\zeta_{zz}\boldsymbol{k}_t/k_0 - \zeta_{zz}\boldsymbol{\xi}_z\cdot\mathbf{J} + \epsilon_{zz}\boldsymbol{\mu}_z\cdot\mathbf{J}\right) \end{cases}$$

where $a^{-1} = \epsilon_{zz}\mu_{zz} - \xi_{zz}\zeta_{zz}$. Notice that the result holds for materials that are stratified in the z direction, not only the homogeneous case.

Appendix D Composition of two slabs

In this appendix we derive an algorithm of how the reflection and transmission dyadics of a slab, that is composed of two different subslabs, are related to the individual reflection and transmission dyadics of the subslabs. This problem is closely related to the Redheffer $*$-product [18] or a composition of transmission lines.

Let $\mathbf{r}_i^{\pm}$ and $\mathbf{t}_i^{\pm}$, $i = 1, 2$, denote the reflection and the transmission dyadics for two subslabs. Furthermore, let the split fields at the left boundary of the first slab be $\boldsymbol{F}_1^{\pm}$, on the intermediate boundary be $\boldsymbol{F}^{\pm}$, and on the right boundary be $\boldsymbol{F}_2^{\pm}$, see Figure 9. The relation between the outputs and inputs on each boundary is

$$\begin{cases} \boldsymbol{F}_1^- = \mathbf{r}_1^+\cdot\boldsymbol{F}_1^+ + \mathbf{t}_1^-\cdot\boldsymbol{F}^- \\ \boldsymbol{F}^+ = \mathbf{t}_1^+\cdot\boldsymbol{F}_1^+ + \mathbf{r}_1^-\cdot\boldsymbol{F}^- \end{cases}$$

and

$$\begin{cases} \boldsymbol{F}^- = \mathbf{r}_2^+\cdot\boldsymbol{F}^+ + \mathbf{t}_2^-\cdot\boldsymbol{F}_2^- \\ \boldsymbol{F}_2^+ = \mathbf{t}_2^+\cdot\boldsymbol{F}^+ + \mathbf{r}_2^-\cdot\boldsymbol{F}_2^- \end{cases}$$

Solve for the intermediate fields $\boldsymbol{F}^{\pm}$, and we get

$$\begin{cases} \boldsymbol{F}^+ = \left(\mathbf{I}_2 - \mathbf{r}_1^-\cdot\mathbf{r}_2^+\right)^{-1}\cdot\left(\mathbf{t}_1^+\cdot\boldsymbol{F}_1^+ + \mathbf{r}_1^-\cdot\mathbf{t}_2^-\cdot\boldsymbol{F}_2^-\right) \\ \boldsymbol{F}^- = \mathbf{r}_2^+\cdot\left(\mathbf{I}_2 - \mathbf{r}_1^-\cdot\mathbf{r}_2^+\right)^{-1}\cdot\left(\mathbf{t}_1^+\cdot\boldsymbol{F}_1^+ + \mathbf{r}_1^-\cdot\mathbf{t}_2^-\cdot\boldsymbol{F}_2^-\right) + \mathbf{t}_2^-\cdot\boldsymbol{F}_2^- \end{cases}$$

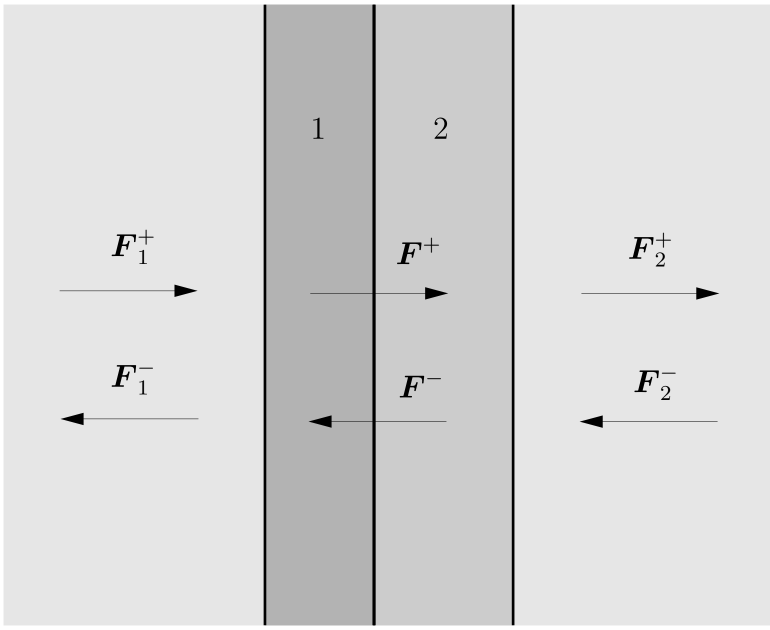

Figure 9: The composite slab and its two subslabs.

The reflection and transmission dyadics of the composite slab $\mathbf{r}^{\pm}$ and $\mathbf{t}^{\pm}$ are

$$\begin{cases} \boldsymbol{F}_1^- = \mathbf{r}^+ \cdot \boldsymbol{F}_1^+ + \mathbf{t}^- \cdot \boldsymbol{F}_2^- \\ \boldsymbol{F}_2^+ = \mathbf{t}^+ \cdot \boldsymbol{F}_1^+ + \mathbf{r}^- \cdot \boldsymbol{F}_2^- \end{cases}$$

where

$$\begin{cases} \mathbf{r}^+ = \mathbf{r}_1^+ + \mathbf{t}_1^- \cdot \mathbf{r}_2^+ \cdot \left(\mathbf{I}_2 - \mathbf{r}_1^- \cdot \mathbf{r}_2^+\right)^{-1} \cdot \mathbf{t}_1^+ \\ \mathbf{t}^- = \mathbf{t}_1^- \cdot \mathbf{r}_2^+ \cdot \left(\mathbf{I}_2 - \mathbf{r}_1^- \cdot \mathbf{r}_2^+\right)^{-1} \cdot \mathbf{r}_1^- \cdot \mathbf{t}_2^- + \mathbf{t}_1^- \cdot \mathbf{t}_2^- \\ \mathbf{r}^- = \mathbf{r}_2^- + \mathbf{t}_2^+ \cdot \left(\mathbf{I}_2 - \mathbf{r}_1^- \cdot \mathbf{r}_2^+\right)^{-1} \cdot \mathbf{r}_1^- \cdot \mathbf{t}_2^- \\ \mathbf{t}^+ = \mathbf{t}_2^+ \cdot \left(\mathbf{I}_2 - \mathbf{r}_1^- \cdot \mathbf{r}_2^+\right)^{-1} \cdot \mathbf{t}_1^+ \end{cases}$$

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