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Rikte, Sten

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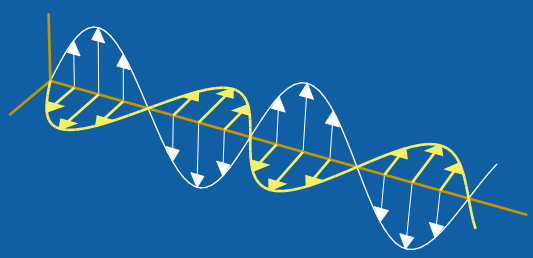
Sten Rikte

Department of Electroscience
Electromagnetic Theory
Lund Institute of Technology
Sweden



Sten Rikte

Department of Electroscience
Electromagnetic Theory
Lund Institute of Technology
P.O. Box 118
SE-221 00 Lund
Sweden

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Abstract

The use of complex, time-varying electromagnetic field vectors in the analysis of pulse propagation problems in linear, homogeneous, bi-isotropic media is proposed. In the most general case, these dispersive materials are characterized by four time-dependent susceptibility kernels. Properly defined, the complex field vector satisfies a first-order, dispersive wave equation. Refractive, admittance, and impedance kernels are identified as solutions of temporal Volterra integral equations of the second kind and these equations are solved numerically for the case of a single-resonance dispersion model. A few standard examples of application are commented upon briefly.

1 Introduction

The use of complex field vectors in order to write the free-space Maxwell equations in a more economical form has been proposed by several authors, see, e.g., Stratton for references [14]. The basic idea is to define a complex field vector such that the real part represents the electric field and the imaginary part the magnetic field, or *vice versa*, and such that the Maxwell equations decouple. The first ideas along these lines seem to be due to Beltrami [1]. In the time-harmonic analysis, these fields or generalizations of these fields are referred to as Beltrami fields [13], wave fields [12], or self-dual fields [12]. Wave fields have been used with success in the analysis of monochromatic wave propagation phenomena in linear bi-isotropic materials, see, e.g., Lindell *et al.* [12]. The bi-isotropic family of materials embraces not only reciprocal, isotropic chiral (or Pasteur) media, but also isotropic, nonreciprocal, achiral (or Tellegen) media, and — in a wide sense — isotropic dielectric and magnetic media [12]. Recently, Beltrami fields have been used analyzing Gaussian pulse propagation in certain weakly dispersive isotropic chiral materials [13].

All material media are dispersive to some extent and bi-isotropic materials are surmised to display anomalous dispersion in certain frequency bands in a similar way as isotropic Lorentz materials [12]. In the time-harmonic analysis, it is natural to suppress or even neglect temporal dispersion (not since these effects can be neglected but since the Fourier transformation of a convolution of two time-dependent functions is the product of the Fourier transforms of the two functions so that the material parameters merely become functions of frequency). At pulse propagation, however, this is not allowed. In the present article, a complex time-dependent field vector for arbitrary temporally dispersive bi-isotropic materials is defined and a first-order, dispersive wave equation for this field derived. Complex time-dependent field vectors would be useful in the analysis of pulse propagation phenomena in bi-isotropic materials, for instance, in chirowaveguide theory. This motivates the present study.

In the present article, attention is focused on the dispersive effects of wave propagation in bi-isotropic media. Results on transient wave propagation in stratified bi-isotropic slabs have been reported before, see, e.g., Refs 9–11. In these articles, the analysis was based on wave splitting and a matrix formalism. From Section 4 in

the present work follows that combining the wave-splitting technique with the complex field vector concept simplifies the analysis of the homogeneous slab problem.

In this introduction, the basic field equations for the bi-isotropic medium are given. In Section 2, the complex electromagnetic field vector is defined. Appropriate intrinsic material operators are introduced and expressed in terms of the susceptibility operators of the bi-isotropic medium. In practise, this amounts to solving temporal Volterra integral equations of the second kind. Numerical results for a single-resonance Pasteur medium are presented in Section 3. Knowledge of the intrinsic operators of the medium is an appropriate starting point for the discussion of pulse propagation phenomena in bi-isotropic materials and three fundamental examples demonstrating this are given in Section 4.

The following notation is used: the radius vector is denoted by $\boldsymbol{r}$, the time by t, the electric and magnetic field vectors at $(\boldsymbol{r}, t)$ by $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{H}(\boldsymbol{r}, t)$, respectively, and the corresponding flux densities by $\boldsymbol{D}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$. Each field vector is written in the form $\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{u}_x E_x(\boldsymbol{r}, t) + \boldsymbol{u}_y E_y(\boldsymbol{r}, t) + \boldsymbol{u}_z E_z(\boldsymbol{r}, t)$, where $\boldsymbol{u}_x$, $\boldsymbol{u}_y$, and $\boldsymbol{u}_z$ are the basis vectors in the Cartesian frame. The dynamics of the fields is modeled by the macroscopic Maxwell equations:

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}, t) = -\partial_t \boldsymbol{B}(\boldsymbol{r}, t), \qquad \nabla \times \boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{J}(\boldsymbol{r}, t) + \partial_t \boldsymbol{D}(\boldsymbol{r}, t),$$

where $\boldsymbol{J}(\boldsymbol{r}, t)$ is the current density at $(\boldsymbol{r}, t)$. For brevity, the independent variables $(\boldsymbol{r}, t)$ are often suppressed.

The constitutive relations of a linear, homogeneous, bi-isotropic material can be written in the form

$$c_0 \eta_0 \boldsymbol{D} = \varepsilon \boldsymbol{E} + \xi \eta_0 \boldsymbol{H}, \qquad c_0 \boldsymbol{B} = \zeta \boldsymbol{E} + \mu \eta_0 \boldsymbol{H}, \tag{1.1}$$

where the *relative* permittivity and permeability operators of the medium are

$$\varepsilon = 1 + \chi^{ee}(t)*, \qquad \mu = 1 + \chi^{mm}(t)*,$$

the *relative* cross-coupling operators

$$\xi = \chi^{em}(t)*, \qquad \zeta = \chi^{me}(t)*,$$

and the star $(*)$ denotes temporal convolution:

$$[\zeta \boldsymbol{E}]\,(\boldsymbol{r}, t) = (\chi^{me} * \boldsymbol{E})(\boldsymbol{r}, t) = \int_{-\infty}^{t} \chi^{me}(t - t') \boldsymbol{E}(\boldsymbol{r}, t')\, dt'.$$

This is one proper way to model the temporal dispersion of the bi-isotropic medium in the absence of a material optical response [6, 12]. Another set of proper constitutive relations is presented in Section 3. The constants c_0 and η_0 are the speed of light in vacuum and the intrinsic impedance of vacuum, respectively. The integral kernels $\chi^{ee}(t)$, $\chi^{em}(t)$, $\chi^{me}(t)$, and $\chi^{mm}(t)$ are the susceptibility kernels of the medium. The susceptibility kernels are measured in s^{-1}, whereas the integral operators ε, ξ, ζ, and μ are dimensionless. Due to causality, the susceptibility kernels

are identically zero for $t < 0$. For $t > 0$, these kernels are assumed to be twice continuously differentiable. Pasteur media satisfy $\chi^{me}(t) = -\chi^{em}(t)$ and Tellegen materials $\chi^{me}(t) = \chi^{em}(t)$. In the short-wave limit, the constitutive relations (1.1) reduce to the ones in vacuum provided that the susceptibility kernels are absolutely integrable. This is deduced from the Riemann-Lebesgue lemma.

Substituting the constitutive relations into the Maxwell equations gives a linear system of first-order hyperbolic integro-differential equations in the electric and magnetic field vectors only:

$$\begin{cases} \nabla \times \boldsymbol{E} = -c_0^{-1}\partial_t \left(\zeta \boldsymbol{E} + \mu\eta_0 \boldsymbol{H}\right), \\ \nabla \times \eta_0 \boldsymbol{H} = \eta_0 \boldsymbol{J} + c_0^{-1}\partial_t \left(\varepsilon \boldsymbol{E} + \xi\eta_0 \boldsymbol{H}\right). \end{cases} \tag{1.2}$$

The aim is to decouple this system of equations by a linear change of variables.

2 The complex electromagnetic field vector

An arbitrary time-dependent electromagnetic field $\boldsymbol{E}(\boldsymbol{r},t)$, $\boldsymbol{H}(\boldsymbol{r},t)$ in a linear, homogeneous, bi-isotropic medium can be represented uniquely by a complex field vector, $\boldsymbol{Q}(\boldsymbol{r},t)$, (and its complex conjugate, $\boldsymbol{Q}^*(\boldsymbol{r},t)$,) as

$$\begin{cases} \boldsymbol{E} = \boldsymbol{Q} + \boldsymbol{Q}^*, \\ \eta_0 \boldsymbol{H} = i\mathcal{Y}\boldsymbol{Q} - i\mathcal{Y}^*\boldsymbol{Q}^*, \end{cases} \tag{2.1}$$

where the complex-valued temporal integral operator

$$\mathcal{Y} = 1 + Y(t) *$$

is the relative intrinsic admittance of the medium and $\mathcal{Y}^*$ its complex conjugate. The imaginary unit is denoted by i. The inverse of the transformation (2.1) is

$$\boldsymbol{Q} = \frac{1}{2}\mathcal{Z}\left(\mathcal{Y}^*\boldsymbol{E} - i\eta_0\boldsymbol{H}\right), \tag{2.2}$$

where the real-valued temporal integral operator

$$\mathcal{Z} = 1 + Z(t) *$$

is a relative intrinsic impedance defined by

$$\left(\mathcal{Y} + \mathcal{Y}^*\right)\mathcal{Z}/2 = 1. \tag{2.3}$$

Obviously, the field vectors $\boldsymbol{Q}(\boldsymbol{r},t)$ and $\boldsymbol{Q}^*(\boldsymbol{r},t)$ can be interpreted as complex electric fields.

The introduction of the complex electromagnetic field vector reduces the system of integro-differential equations (1.2) to the first-order, dispersive wave equation

$$\nabla \times \boldsymbol{Q} = -ic_0^{-1}\partial_t\mathcal{N}\boldsymbol{Q} - i\eta_0\mathcal{Z}\boldsymbol{J}/2, \tag{2.4}$$

where the complex-valued temporal integral operator

$$\mathcal{N} = 1 + N(t) *$$

is referred to as the index of refraction. It is understood that $\mathcal{Y}$, $\mathcal{N}$, and $\mathcal{Z}$ are intrinsic operators of the medium, that is, independent of the field vectors. Below, equation (2.4) is referred to as the wave-field equation.

The decoupling of the Maxwell equations leads to conditions on the relative intrinsic admittance and the index of refraction in terms of the susceptibility operators of the bi-isotropic medium:

$$\mathcal{N} = \mu\mathcal{Y}^* + i\xi, \qquad \mathcal{N}\mathcal{Y}^* = \varepsilon - i\zeta\mathcal{Y}^*.$$

Combining these equations gives

$$(\mathcal{N} - i\xi)(\mathcal{N} + i\zeta) = \mu\varepsilon, \qquad \mu\mathcal{Y} = \mathcal{N}^* + i\xi \tag{2.5}$$

with solutions

$$\begin{cases} \mathcal{N} = i\dfrac{\xi - \zeta}{2} + \sqrt{\mu\varepsilon - \dfrac{(\xi+\zeta)^2}{4}}, \\ \mu\mathcal{Y} = i\dfrac{\xi + \zeta}{2} + \sqrt{\mu\varepsilon - \dfrac{(\xi+\zeta)^2}{4}}, \end{cases}$$

where the positive square-root operator has been chosen:

$$\sqrt{\mu\varepsilon - \frac{(\xi+\zeta)^2}{4}} = 1 + N_{\mathrm{co}}(t) * .$$

Here, the real-valued integral kernel $N_{\mathrm{co}}(t)$ satisfies the nonlinear Volterra integral equation of the second kind

$$2N_{\mathrm{co}}(t) + (N_{\mathrm{co}} * N_{\mathrm{co}})(t) = \chi^{ee}(t) + \chi^{mm}(t) + (\chi^{ee} * \chi^{mm})(t) - (\chi * \chi)(t),$$

where

$$\chi(t) = \chi^{em}(t)/2 + \chi^{me}(t)/2$$

is the nonreciprocity kernel. Volterra integral equations of the second kind are uniquely solvable in the space of of continuous functions in each compact time-interval and the solutions depends continuously on data [7]. Consequently, the kernel $N_{\mathrm{co}}(t)$ inherits causality and smoothness properties from the susceptibility kernels. Straightforward analysis shows that a choice of the negative square-root operator does not add or alter anything of significance for the present discussion.

In terms of the kernel $N_{\mathrm{co}}(t)$ and the chirality kernel

$$\kappa(t) = \chi^{em}(t)/2 - \chi^{me}(t)/2,$$

the complex-valued refractive kernel becomes

$$N(t) = N_{\rm co}(t) + i\kappa(t).$$

Clearly, the refractive kernel of the bi-isotropic medium is real iff the medium is Tellegen, that is, $\kappa(t) = 0$. Similarly, the admittance kernel can be written as

$$Y(t) = Y_{\rm co}(t) + iY_{\rm cross}(t),$$

where the components $Y_{\rm co}(t)$ and $Y_{\rm cross}(t)$ are real-valued functions. The second identity (2.5) implies that the admittance kernel satisfies the linear Volterra integral equation of the second kind

$$Y(t) + (Y * \chi^{mm})(t) = N_{\rm co}(t) - \chi^{mm}(t) + i\chi(t).$$

In particular, the admittance kernel inherits causality and smoothness properties from the susceptibility kernels. Unique solubility gives that the admittance kernel of the bi-isotropic medium is real iff the medium is Pasteur, that is, $\chi(t) = 0$. Finally, the impedance kernel $Z(t)$ satisfies a linear Volterra integral equation of the second kind:

$$Z(t) + (Z * N_{\rm co})(t) = \chi^{mm}(t) - N_{\rm co}(t).$$

This follows from equation (2.3). The impedance kernel inherits causality and smoothness properties from the susceptibility kernels.

In summary, by introducing the complex field vector (2.2), the Maxwell equations for the linear, homogeneous, bi-isotropic medium (1.2) reduce to the first-order dispersive wave equation (2.4). The refractive, admittance, and impedance kernels present in equations (2.1) and (2.4) depend on the susceptibility kernels of the medium only. Obtaining these intrinsic kernels of the medium, of which the first two are complex and the third real, is a well posed problem. Specifically, Volterra integral equations of the second kind are solved. The final result, the wave equation (2.4), is an appropriate starting point for discussing pulse propagation phenomena in bi-isotropic materials and scattering from such materials.

Of course, there is a close connection between the complex field vectors $\boldsymbol{Q}(\boldsymbol{r}, t)$ and $\boldsymbol{Q}^*(\boldsymbol{r}, t)$ used in the present article and the monochromatic wave fields $\boldsymbol{E}_\pm(\boldsymbol{r}, \omega)$ employed by Lindell *et al.* [12]. The relation between these fields is given by the Fourier transformation:

$$\boldsymbol{E}_+(\boldsymbol{r}, \omega) = \int_{-\infty}^{\infty} \exp\left(-i\omega t\right)\boldsymbol{Q}(\boldsymbol{r}, t)\, dt,$$
$$\boldsymbol{E}_-(\boldsymbol{r}, \omega) = \int_{-\infty}^{\infty} \exp\left(-i\omega t\right)\boldsymbol{Q}^*(\boldsymbol{r}, t)\, dt.$$

Observe that the relation $\boldsymbol{E}_-(\boldsymbol{r}, \omega) = \boldsymbol{E}_+^*(\boldsymbol{r}, -\omega)$ holds.

3 Intrinsic kernels for a single-resonance model

In this section, the refractive, admittance, and impedance kernels for a single-resonance model are computed numerically. The results have been obtained by solving the Volterra integral equations of the second kind presented in Section 2. A heuristic time-domain derivation of the single-resonance dispersion model is presented first.

Post's constitutive relations are often used to model optical activity [12]:

$$\begin{cases} c_0\eta_0\boldsymbol{D}(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r},t) + \big(\chi_P^{ee} * \boldsymbol{E}\big)(\boldsymbol{r},t) + \big(\chi_P^{em} * c_0\boldsymbol{B}\big)(\boldsymbol{r},t), \\ \eta_0\boldsymbol{H}(\boldsymbol{r},t) = \big(\chi_P^{me} * \boldsymbol{E}\big)(\boldsymbol{r},t) + c_0\boldsymbol{B}(\boldsymbol{r},t) + \big(\chi_P^{mm} * c_0\boldsymbol{B}\big)(\boldsymbol{r},t), \end{cases}$$

These constitutive relations are equivalent to the Lindell-Sihvola constitutive relations (1.1) and the connection between the susceptibility kernels is given by a linear system of uncoupled Volterra integral equations of the second kind.

Suppose that Post's constitutive parameters are given. Then the Lindell-Sihvola constitutive parameters are obtained by solving the integral equations

$$\begin{cases} \chi^{mm}(t) + \chi_P^{mm}(t) + (\chi^{mm} * \chi_P^{mm}*)(t) = 0, \\ \chi^{me}(t) + \chi_P^{me}(t) + (\chi^{me} * \chi_P^{mm}*)(t) = 0, \\ \chi^{em}(t) - \chi_P^{em}(t) + (\chi^{em} * \chi_P^{mm}*)(t) = 0, \\ \chi^{ee}(t) + (\chi^{ee} * \chi_P^{mm}*)(t) = \chi_P^{ee}(t) + (\chi_P^{ee} * \chi_P^{mm}*)(t) - (\chi_P^{em} * \chi_P^{me}*)(t). \end{cases}$$

For nonmagnetic Pasteur materials,

$$\chi_P^{em}(t) = \chi_P^{me}(t), \qquad \chi_P^{mm}(t) = 0.$$

Unique solubility then gives

$$\chi^{mm}(t) = 0, \qquad \chi^{em}(t) = -\chi^{me}(t) = \kappa(t), \qquad \chi^{ee}(t) = \chi_P^{ee}(t) - (\kappa * \kappa)(t),$$

where $\chi_P^{ee}(t)$ and $\chi_P^{em}(t) = \kappa(t)$ are to be determined.

Denoting the electron density by n, the electric polarization can be written as

$$nq\boldsymbol{r}(t) = \epsilon_0(\chi_P^{ee} * \boldsymbol{E})(t) + \epsilon_0(\chi_P^{em} * c_0\boldsymbol{B})(t),$$

where $\boldsymbol{r}(t)$ is the displacement of an electron with mass m and charge $q = -e$ relative to its position of equilibrium. For an electron describing a helical path, the equation of motion becomes [4]

$$m\partial_t^2\boldsymbol{r}(t) = -m\nu\partial_t\boldsymbol{r}(t) - m\omega_0^2\boldsymbol{r}(t) + q\boldsymbol{E}(t) - qc_0\tau\nabla\times\boldsymbol{E}(t),$$

where $\nu \geq 0$ is the collision frequency, $\omega_0 \geq 0$ the natural frequency, and τ is a positive or negative time-constant that depends on the micro structure of the chiral medium. Substituting the electric polarization and Faraday's law into the equation of motion gives

$$\big(\partial_t^2 + \nu\partial_t + \omega_0^2\big)\big((\chi_P^{ee} * \boldsymbol{E})(t) + (\chi_P^{em} * c_0\boldsymbol{B})(t)\big) = \omega_p^2\big(\boldsymbol{E}(t) + \tau\partial_t c_0\boldsymbol{B}(t)\big).$$

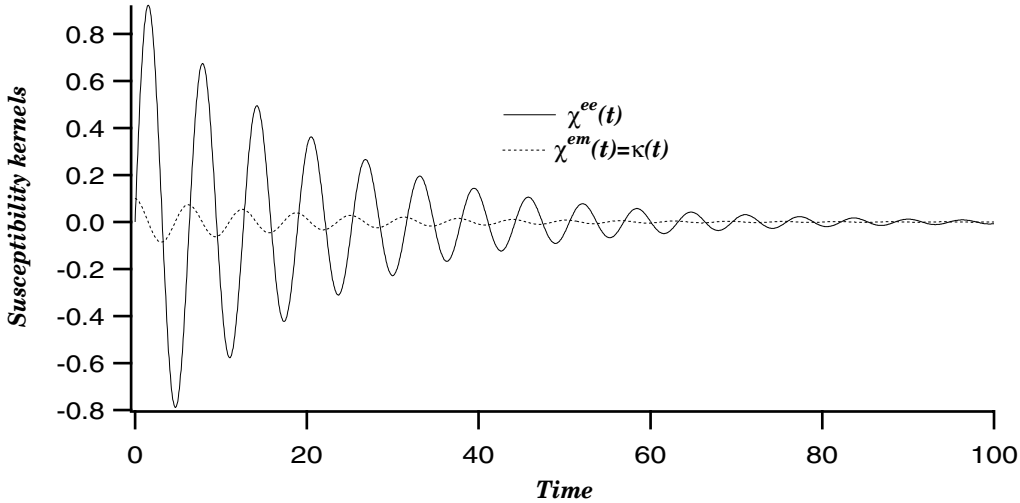

Figure 1: The non-zero susceptibility kernels for the single-resonance Pasteur medium discussed in the example. The kernels are measured in 10^{16} Hz and the time in 10^{-16} s.

Setting $\boldsymbol{B} = \boldsymbol{0}$ shows that the electric susceptibility kernel is given by the Lorentz model; setting $\boldsymbol{E} = \boldsymbol{0}$ shows that the chirality kernel is proportional to the time-derivative of the electric susceptibility kernel; consequently,

$$\chi_P^{ee}(t) = H(t)\frac{\omega_p^2}{\nu_0}\sin(\nu_0 t)\exp\left(-\frac{\nu t}{2}\right), \qquad \chi_P^{em}(t) = \tau\partial_t\chi_P^{ee}(t),$$

where $\omega_p = \sqrt{(nq)^2/(\epsilon_0 m)}$, $\nu_0 = \sqrt{\omega_0^2 - \nu^2/4}$, and $H(t)$ denotes the Heaviside step function. This model is often referred to as Condon's model although the classical arguments presented in this section were known already to Drude [3, 4].

As an example, choose the following parameters in the optical regime:

$$\omega_p = \omega_0 = 10^{16}\text{Hz}, \qquad \nu = .1 \times \omega_0, \qquad \tau = .1/\omega_0.$$

The non-zero susceptibility kernels are depicted in Figure 1. The admittance and impedance kernels for this model are displayed in Figure 2. The co-refractive kernel equals the admittance kernel and the cross-refractive kernel equals the chirality kernel. The impedance kernel is the resolvent kernel of the admittance kernel.

4 Examples of application

In this section, the propagation of plane pulses in bi-isotropic materials is analyzed using complex field vectors and a new wave spitting. Basic results for pulse propagation in chirowaveguides are presented as well. The general idea is to demonstrate the power of the complex field vector concept without going into details.

In the time-harmonic plane-wave analysis [12], the Fourier transform of $\boldsymbol{Q}(\boldsymbol{r}, t)$ represents the right-hand circularly polarized (RCP) waves in the medium, whereas

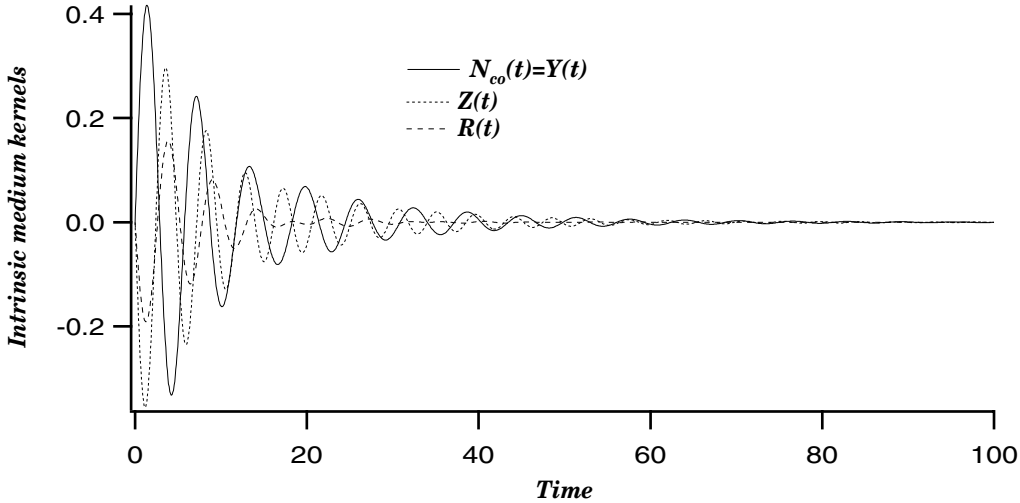

Figure 2: The admittance and impedance kernels for the single-resonance Pasteur medium. The reflection kernel at normal incidence on the Pasteur half-space from vacuum is presented as well. All kernels are measured in 10^{16} Hz and the time in 10^{-16} s.

the Fourier transform of $\boldsymbol{Q}^*(\boldsymbol{r},t)$ represents the left-hand circularly polarized (LCP) waves. In this section, these labels are used for the time-varying fields as well.

4.1 Transverse electric and magnetic (TEM) pulses

In a bi-isotropic slab, a current distribution of the form

$$\boldsymbol{J}(\boldsymbol{r},t) = \boldsymbol{J}_T(\boldsymbol{r},t) = \boldsymbol{u}_x J_x(z,t) + \boldsymbol{u}_y J_y(z,t)$$

supports transverse electromagnetic (TEM) waves:

$$\boldsymbol{Q}(\boldsymbol{r},t) = \boldsymbol{Q}_T(\boldsymbol{r},t) = \boldsymbol{u}_x Q_x(z,t) + \boldsymbol{u}_y Q_y(z,t),$$

The normal incidence problem can be considered as a special case. This Ansatz reduces the wave-field equation (2.4) to

$$\partial_z \boldsymbol{Q} = i c_0^{-1} \partial_t \mathcal{N} \boldsymbol{u}_z \times \boldsymbol{Q} + i\eta_0 \mathcal{Z} \boldsymbol{u}_z \times \boldsymbol{J}/2.$$

The wave splitting

$$\boldsymbol{E}_+^\pm = \frac{1}{2}\left(\boldsymbol{Q} \mp i\,\boldsymbol{u}_z \times \boldsymbol{Q}\right) = \left(\boldsymbol{u}_x \mp i\,\boldsymbol{u}_y\right)\left(Q_x \pm i\,Q_y\right)/2 \equiv \left(\boldsymbol{u}_x \mp i\,\boldsymbol{u}_y\right) E_+^\pm$$

decomposes the total electric RCP field, $\boldsymbol{Q}$, into an up-going RCP field, $\boldsymbol{E}_+^+$, and a down-going RCP field, $\boldsymbol{E}_+^-$: $\boldsymbol{Q} = \boldsymbol{E}_+^+ + \boldsymbol{E}_+^-$. The wave equations for the amplitudes of these waves, $E_+^\pm = (Q_x \pm i\,Q_y)/2$, are

$$\partial_z E_+^\pm = \mp c_0^{-1} \partial_t \mathcal{N} E_+^\pm \mp \eta_0 \mathcal{Z} J_+^\pm/2, \tag{4.1}$$

where $J_+^\pm = (J_x \pm i\, J_y)/2$.

Similarly, the total electric LCP field, $\boldsymbol{Q}^*$, can be split into an up-going LCP field, $\boldsymbol{E}_-^+$, and a down-going LCP field, $\boldsymbol{E}_-^-$: $\boldsymbol{Q}^* = \boldsymbol{E}_-^+ + \boldsymbol{E}_-^-$. These fields are

$$\boldsymbol{E}_-^\pm = \left(\boldsymbol{E}_+^\pm\right)^* = \frac{1}{2}\left(\boldsymbol{Q}^* \pm i\, \boldsymbol{u}_z \times \boldsymbol{Q}^*\right) \equiv \left(\boldsymbol{u}_x \pm i\, \boldsymbol{u}_y\right) E_-^\pm,$$

where the amplitudes of the LCP fields, $E_-^\pm = (E_+^\pm)^* = \left(Q_x^* \mp i\, Q_y^*\right)/2$, satisfy the wave equations $\partial_z E_-^\pm = \mp c_0^{-1} \partial_t \mathcal{N}^* E_-^\pm \mp \eta_0 \mathcal{Z} J_-^\pm/2$, where $J_-^\pm = (J_+^\pm)^* = (J_x \mp i\, J_y)/2$. Consequently, the total electric field in the bi-isotropic medium can be written as the sum of the up-going and the down-going electric RCP and LCP fields:

$$\boldsymbol{E} = \left(\boldsymbol{u}_x - i\, \boldsymbol{u}_y\right) E_+^+ + \left(\boldsymbol{u}_x + i\, \boldsymbol{u}_y\right) E_+^- + \left(\boldsymbol{u}_x + i\, \boldsymbol{u}_y\right) E_-^+ + \left(\boldsymbol{u}_x - i\, \boldsymbol{u}_y\right) E_-^-.$$

The total magnetic field is

$$\begin{aligned} \eta_0 \boldsymbol{H} = {} & i\mathcal{Y}\left(\boldsymbol{u}_x - i\, \boldsymbol{u}_y\right) E_+^+ + i\mathcal{Y}\left(\boldsymbol{u}_x + i\, \boldsymbol{u}_y\right) E_+^- \\ & - i\mathcal{Y}^*\left(\boldsymbol{u}_x + i\, \boldsymbol{u}_y\right) E_-^+ - i\mathcal{Y}^*\left(\boldsymbol{u}_x - i\, \boldsymbol{u}_y\right) E_-^-. \end{aligned}$$

In summary, combining the complex field vector concept with the wave splitting technique decomposes the propagation problem into to subproblems: (i) solve the dispersive wave equations (4.1) and (ii) obtain scattering relations at the plane interfaces using the boundary conditions.

At normal incidence on a bi-isotropic half-space from vacuum, the scattering relation at the interface can be written as

$$\boldsymbol{E}^r = \left(\mathcal{R}_{\text{co}} \mathbf{I} + \mathcal{R}_{\text{cross}} \mathbf{J}\right) \cdot \boldsymbol{E}^i = \mathcal{R} \boldsymbol{E}_+^i + \mathcal{R}^* \boldsymbol{E}_-^i,$$

where the two-dimensional dyadics

$$\mathbf{I} = \boldsymbol{u}_x \boldsymbol{u}_x + \boldsymbol{u}_y \boldsymbol{u}_y, \qquad \mathbf{J} = \boldsymbol{u}_z \times \mathbf{I} = \boldsymbol{u}_y \boldsymbol{u}_x - \boldsymbol{u}_x \boldsymbol{u}_y \tag{4.2}$$

have been introduced. Here, $\boldsymbol{E}^i(t)$ denotes the incident electric field and $\boldsymbol{E}^r(t)$ the reflected electric field. The reflection operator

$$R(t)* \equiv \mathcal{R} = \mathcal{R}_{\text{co}} + i\mathcal{R}_{\text{cross}} \equiv \left(R_{\text{co}}(t) + iR_{\text{cross}}\right)(t)\, *$$

is a complex-valued temporal convolution operator that takes the up-going (incident) electric RCP field, $\boldsymbol{E}_+^i = (\mathbf{I} - i\mathbf{J}) \cdot \boldsymbol{E}^i/2$, to the down-going (reflected) electric LCP field. The reflection operator $\mathcal{R}^*$ takes the up-going electric LCP field, $\boldsymbol{E}_-^i = (\mathbf{I} + i\mathbf{J}) \cdot \boldsymbol{E}^i/2$, to the down-going electric RCP field. There is no coupling between incoming and reflected RCP fields and no coupling between incoming and reflected LCP fields. The transmission operator for the electric RCP field is $1 + R(t)*$. Similarly, the transmission operator for the electric LCP field is found to be $1 + R^*(t)*$.

Straightforward calculations show that

$$\mathcal{R} = (1 + \mathcal{Y})^{-1}(1 - \mathcal{Y}).$$

Consequently, the reflection kernel, $R(t)$, satisfies the resolvent equation

$$R(t) + (R * Y)(t)/2 + Y(t)/2 = 0.$$

Unique solubility shows that the reflection kernel is real iff the bi-isotropic medium is Pasteur. The reflection kernel relevant for the specific Condon model discussed in Section 3 is depicted in Figure 2. The wave equations (4.1) for the amplitudes of the refracted circularly polarized waves can be solved using the method of characteristics.

4.2 Oblique incidence on a bi-isotropic slab

A plane pulse traveling in vacuum impinges on a bi-isotropic slab under the angle of incidence θ. No internal sources are present. For the case the planes of incidence are perpendicular to $\boldsymbol{u}_x$, an appropriate Ansatz is

$$\boldsymbol{Q}(\boldsymbol{r}, t) = \boldsymbol{Q}\left(z, t - c_0^{-1} y \sin\theta\right) = \boldsymbol{Q}_T\left(z, t - c_0^{-1} y \sin\theta\right) + \boldsymbol{u}_z Q_z\left(z, t - c_0^{-1} y \sin\theta\right).$$

This problem has been discussed in detail by Kristensson and Rikte using matrix notation [11]. The results presented in this subsection facilitates the analysis of the problem and would be useful in the inverse scattering problem of determining finite time traces of the susceptibility kernels of a bi-isotropic half-space from a measurement of finite time traces of reflection data at oblique incidence.

Snell's law of refraction for RCP waves can be written as

$$\mathcal{N} \sin\vartheta = \sin\theta,$$

where $\sin\vartheta$ is a complex-valued temporal integral operator of the form

$$\sin\vartheta = \sin\theta + A(t) * .$$

The convolution kernel $A(t)$ is obtained by solving the linear Volterra integral equation of the second kind

$$A(t) + N(t)\sin\theta + (A * N)(t) = 0.$$

At normal incidence, $\theta = 0$, unique solubility shows that $A(t) = 0$; consequently, $\sin\vartheta = 0$. Introduce also the complex-valued temporal integral operator

$$\cos\vartheta = \cos\theta + B(t) *$$

defined by the operator identity

$$\cos^2\vartheta + \sin^2\vartheta = 1.$$

The kernel $B(t)$ satisfies the Volterra integral equation of the second kind

$$2B(t)\cos\theta + (B * B)(t) + 2A(t)\sin\theta + (A * A)(t) = 0.$$

Unique solubility of Volterra integral equation of the second kind implies that the operators $\sin\vartheta$ and $\cos\vartheta$ are real for isotropic media and for Tellegen media. At normal incidence, $B(t) = 0$; consequently, $\cos\vartheta = 1$. In the general case, the kernels $A(t)$ and $B(t)$ inherit causality and smoothness properties from the susceptibility kernels.

Substituting the above Ansatz into the wave equation (2.4) for RCP fields gives

$$Q_z = i \sin\vartheta Q_x \tag{4.3}$$

and

$$\partial_z \boldsymbol{Q}_T = -ic_0^{-1}\partial_t \mathcal{N}(\boldsymbol{u}_x\boldsymbol{u}_y - \boldsymbol{u}_y\boldsymbol{u}_x \cos^2\vartheta) \cdot \boldsymbol{Q}_T.$$

The total electric RCP field can be decomposed into two uncoupled RCP fields using the wave splitting

$$\begin{pmatrix} Q_x \\ Q_y \\ Q_z \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -i\,\cos\vartheta & i\,\cos\vartheta \\ i\,\sin\vartheta & i\,\sin\vartheta \end{pmatrix} \begin{pmatrix} E_+^+ \\ E_+^- \end{pmatrix},$$

where the scalar functions $E_+^\pm$ can be interpreted as the amplitudes of the up-going and down-going RCP electric fields, respectively, and the relation (4.3) has been included. The amplitudes, $E_+^\pm$, satisfy the wave equations

$$\partial_z E_+^\pm = \mp c_0^{-1}\partial_t \mathcal{N} \cos\vartheta E_+^\pm.$$

The complex conjugates of the amplitudes of the RCP electric fields, $E_-^\pm = (E_+^\pm)^*$, are the amplitudes of the up-going and down-going LCP electric fields, respectively.

The total electric field can be written as

$$\begin{aligned}\boldsymbol{E} = {} & (\boldsymbol{u}_x - i\,(\boldsymbol{u}_y\cos\vartheta - \boldsymbol{u}_z\sin\vartheta))\,E_+^+ + (\boldsymbol{u}_x + i\,(\boldsymbol{u}_y\cos\vartheta - \boldsymbol{u}_z\sin\vartheta))\,E_+^- \\ & + (\boldsymbol{u}_x + i\,(\boldsymbol{u}_y\cos\vartheta^* - \boldsymbol{u}_z\sin\vartheta^*))\,E_-^+ + (\boldsymbol{u}_x - i\,(\boldsymbol{u}_y\cos\vartheta^* - \boldsymbol{u}_z\sin\vartheta^*))\,E_-^-.\end{aligned}$$

Notice that the wave fronts of all the circularly polarized waves make the angle θ with the z-axis. The total magnetic field is given by

$$\begin{aligned}\eta_0\boldsymbol{H} = {} & i\mathcal{Y}\,(\boldsymbol{u}_x - i\,(\boldsymbol{u}_y\cos\vartheta - \boldsymbol{u}_z\sin\vartheta))\,E_+^+ + i\mathcal{Y}\,(\boldsymbol{u}_x + i\,(\boldsymbol{u}_y\cos\vartheta - \boldsymbol{u}_z\sin\vartheta))\,E_+^- \\ & - i\mathcal{Y}^*\,(\boldsymbol{u}_x + i\,(\boldsymbol{u}_y\cos\vartheta^* - \boldsymbol{u}_z\sin\vartheta^*))\,E_-^+ - i\mathcal{Y}^*\,(\boldsymbol{u}_x - i\,(\boldsymbol{u}_y\cos\vartheta^* - \boldsymbol{u}_z\sin\vartheta^*))\,E_-^-.\end{aligned}$$

Using these expressions, the oblique incidence problem can be solved much the same way as the normal incidence problem, although the complexity has increased.

4.3 Basic theory of pulse propagation in chirowaveguides

This section concerns pulse propagation in chirowaveguides. A chirowaveguide is a closed (conventional) waveguide that consists of one single, hollow, perfect conductor

with bi-isotropic filling. A time-harmonic theory of chirowaveguides based on the wave-field concept can be found in Ref. 12.

Pulse propagation in empty waveguides has been discussed by Kristensson using time-domain wave splitting [8]. Recently, Bernekorn *et al.* generalized the analysis in Ref. 8 to waveguides with stratified, isotropic fillings using similar techniques [2]. In this section, the basics of the theory of pulse propagation in chirowaveguides is developed using complex field vectors. By specialization, the results in Ref. 8 can be obtained and the theory in Ref. 2 for the homogeneous medium be improved.

Consider a straight, cylindrical chirowaveguide extended in the z-direction. The interior of the waveguide is denoted by V and the smooth boundary by S. The cross-section of the waveguide is denoted by Ω and the boundary of the cross-section by $\partial\Omega$. The reference direction of the normal vector field $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{r}_T)$ along $\partial\Omega$ is outward with respect to the filling. The tangential vector field $\boldsymbol{\tau} = \boldsymbol{\tau}(\boldsymbol{r}_T)$ along $\partial\Omega$ is defined by $\boldsymbol{\tau} = \boldsymbol{n} \times \boldsymbol{u}_z$. The (constant) binormal vector field along $\partial\Omega$ is $\boldsymbol{u}_z$.

As before, the complex electromagnetic field vector is decomposed in its transverse and longitudinal components: $\boldsymbol{Q} = \boldsymbol{Q}_T + \boldsymbol{u}_z Q_z$. Similarly, the nabla operator is written as $\nabla = \nabla_T + \boldsymbol{u}_z \partial_z$ and the Laplacian as $\Delta = \Delta_T + \partial_z^2$. These decompositions are standard in waveguide theory.

In the absence of sources, the longitudinal component of the wave-field equation (2.4) reads

$$\boldsymbol{u}_z \cdot (\nabla_T \times \boldsymbol{Q}_T) = -ic_0^{-1}\partial_t \mathcal{N} Q_z, \tag{4.4}$$

whereas the transverse part becomes

$$\partial_z(\boldsymbol{u}_z \times \boldsymbol{Q}_T) + \nabla_T Q_z \times \boldsymbol{u}_z = -ic_0^{-1}\partial_t \mathcal{N}\, \boldsymbol{Q}_T. \tag{4.5}$$

Taking the divergence of both members of the wave-field equation gives

$$\nabla_T \cdot \boldsymbol{Q}_T = -\partial_z Q_z. \tag{4.6}$$

Taking the curl gives second-order wave equations for the wave-field components:

$$-\Delta_T \boldsymbol{Q}_T = (\partial_z^2 - c_0^{-2}\partial_t^2 \mathcal{N}^2)\boldsymbol{Q}_T, \quad -\Delta_T Q_z = (\partial_z^2 - c_0^{-2}\partial_t^2 \mathcal{N}^2) Q_z.$$

Transverse differentiation of the longitudinal equation (4.4) in combination with the condition (4.6) yields

$$-ic_0^{-1}\partial_t \mathcal{N} \nabla_T Q_z = -\partial_z(\boldsymbol{u}_z \times \nabla_T Q_z) - \Delta_T(\boldsymbol{u}_z \times \boldsymbol{Q}_T). \tag{4.7}$$

Using the two-dimensional dyadics (4.2), equations (4.5) and (4.7) can be written in the compact form

$$\begin{pmatrix} (ic_0^{-1}\partial_t \mathcal{N}\mathbf{I} + \partial_z \mathbf{J}) & -\mathbf{J} \\ -\Delta_T \mathbf{J} & (ic_0^{-1}\partial_t \mathcal{N}\mathbf{I} - \partial_z \mathbf{J}) \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{Q}_T \\ \nabla_T Q_z \end{pmatrix} = \mathbf{0}.$$

Assuming that the inverse negative transverse Laplacian, $(-\Delta_T)^{-1}$, exists, equation (4.7) can be written as

$$\boldsymbol{Q}_T = (-\Delta_T)^{-1}(\partial_z \mathbf{I} + ic_0^{-1}\partial_t \mathcal{N}\mathbf{J}) \cdot \nabla_T Q_z. \tag{4.8}$$

In particular, the tangential and normal wave-field components at the boundary are

$$\begin{cases} Q_\tau = (-\Delta_T)^{-1}\big(\partial_z\partial_\tau Q_z - ic_0^{-1}\partial_t\mathcal{N}\partial_n Q_z\big), \\ Q_n = (-\Delta_T)^{-1}\big(\partial_z\partial_n Q_z + ic_0^{-1}\partial_t\mathcal{N}\partial_\tau Q_z\big), \end{cases}$$

where tangential and normal derivatives are defined by

$$\partial_\tau Q_z(\boldsymbol{r},t) = \boldsymbol{\tau}(\boldsymbol{r}_T)\cdot\nabla_T Q_z(\boldsymbol{r},t), \qquad \partial_n Q_z(\boldsymbol{r},t) = \boldsymbol{n}(\boldsymbol{r}_T)\cdot\nabla_T Q_z(\boldsymbol{r},t).$$

Specification of the complex-valued operator $(-\Delta_T)^{-1}$ reduces the problem to a wave propagation problem for the longitudinal wave-field component only:

$$\begin{cases} -\Delta_T Q_z(\boldsymbol{r},t) = (\partial_z^2 - c_0^{-2}\partial_t^2\mathcal{N}^2)Q_z(\boldsymbol{r},t), & \boldsymbol{r}\in V, \\ \mathrm{Re}\,\{Q_z(\boldsymbol{r},t)\} = 0, & \boldsymbol{r}\in S, \\ \mathrm{Re}\,\big\{(-\Delta_T)^{-1}\big(\partial_z\partial_\tau Q_z(\boldsymbol{r},t) - ic_0^{-1}\partial_t\mathcal{N}\partial_n Q_z(\boldsymbol{r},t)\big)\big\} = 0, & \boldsymbol{r}\in S. \end{cases} \tag{4.9}$$

The first boundary condition (4.9) is the usual Dirichlet condition $E_z(\boldsymbol{r},t) = 0$, $\boldsymbol{r}\in S$ employed at TM mode propagation in conventional waveguides with isotropic filling. The second boundary condition is more complicated. However, in the isotropic case, when the refractive index $\mathcal{N}$ is real, it reduces to the familiar Neumann condition $\partial_n H_z(\boldsymbol{r},t) = 0$, $\boldsymbol{r}\in S$ at TE mode propagation. To see this, notice that the wave equation (4.9) reduces to one wave equation for $E_z(\boldsymbol{r},t)$ and one for $H_z(\boldsymbol{r},t)$ in conventional waveguides, and, furthermore, that the first boundary condition implies that $\partial_\tau\mathrm{Re}\,\{Q_z(\boldsymbol{r},t)\} = 0$, $\boldsymbol{r}\in S$. Since, for each specific eigenmode, the operator $(-\Delta_T)^{-1}$ is real, the second boundary condition (4.9) implies that $\partial_n\mathrm{Im}\,\{Q_z(\boldsymbol{r},t)\} = 0$, $\boldsymbol{r}\in S$, which, by definition, is the Neumann condition. The situation is similar for Tellegen fillings, see below.

The system of equations (4.9) is an appropriate starting point for the study of pulse propagation in chirowaveguides. The wave front travels along the axis of the guide with the speed of light in vacuum, c_0. Therefore, a temporal integral operator representing a longitudinal refractive index for up-going modes would be of the form

$$\mathcal{N}_z = 1 + N_z(t)*,$$

and satisfy the identity

$$\partial_z = -c_0^{-1}\partial_t\mathcal{N}_z. \tag{4.10}$$

The kernel $N_z(t)$ is required to be *real*, causal, and sufficiently smooth. This Ansatz allows the wave to propagate but *not* to rotate along the axis of the guide and leads to z-independent boundary conditions. In transverse directions, the wave does *not* propagate. Consequently, a transverse refractive index satisfying the operator identity

$$\mathcal{N}_T^2 = \mathcal{N}_z^2 - \mathcal{N}^2 \tag{4.11}$$

must be of the form

$$\mathcal{N}_T = N_T(t)*.$$

By definition, $\mathcal{N}_T$ is complex-valued if $\mathcal{N}$ is complex-valued. The operator identity (4.11) is equivalent to the Volterra integral equation of the second kind

$$2N_z(t) + (N_z * N_z)(t) - 2N(t) - (N * N)(t) - (N_T * N_T)(t) = 0.$$

It can be verified that the refractive kernels $N_z(t)$ and $N_T(t)$ inherit causality and smoothness properties from the susceptibility kernels.

The modal operator identity (4.10) leads to a non-unique Ansatz,

$$Q_z(\boldsymbol{r}, t) = \int_{-\infty}^{\infty} u(z, t - t') v(\boldsymbol{r}_T, t')\, dt',$$

where the function can be chosen $u(z, t)$ *real*. This Ansatz decomposes the propagation problem (4.9) into the longitudinal, first-order, dispersive wave equation

$$\partial_z u(z, t) = -c_0^{-1} \partial_t \mathcal{N}_z u(z, t) \tag{4.12}$$

and the transverse, dispersive, boundary-value problem

$$\begin{cases} -\Delta_T v(\boldsymbol{r}_T, t) = c_0^{-2} \partial_t^2 \mathcal{N}_T^2 v(\boldsymbol{r}_T, t), \quad \boldsymbol{r}_T \in \Omega, \\ \mathrm{Re}\,\{v(\boldsymbol{r}_T, t)\} = 0, \quad \boldsymbol{r}_T \in \partial\Omega, \\ \mathrm{Re}\,\big\{(c_0^{-1}\partial_t \mathcal{N}_T)^{-2} c_0^{-1} \partial_t \big(\mathcal{N}_z \partial_\tau v(\boldsymbol{r}_T, t) + i\mathcal{N} \partial_n v(\boldsymbol{r}_T, t)\big)\big\} = 0, \quad \boldsymbol{r}_T \in \partial\Omega. \end{cases} \tag{4.13}$$

The wave equation (4.12) can be solved using the method of characteristics once the longitudinal refractive index for the mode has been obtained. The longitudinal and transverse refractive indices are obtained by solving problem (4.13) subjected to the condition (4.11). It is important to observe that the temporal integral operator $(-\Delta_T)^{-1}$ does not exist unless $N_T(+0) \neq 0$. If $N_T(+0) = 0$, the integral operator $\partial_t \mathcal{N}_T = N_T(+0) + N_T'(t)*$ lacks principal part and cannot be inverted (the prime denotes the classical time-derivative). The theory of the problem (4.13) in the general bi-isotropic case will be developed elsewhere.

In a conventional waveguide with achiral filling, the kernel of the transverse refractive index for each mode can be written in the form $N_T(t) = c_0 \lambda H(t)$, where $H(t)$ is the Heaviside step function and λ is a positive constant. Solutions, $v(\boldsymbol{r}_T, t)$, of the dispersive boundary-value problem (4.13) are then either real and satisfy the Dirichlet boundary-value problem

$$\begin{cases} -\Delta_T v = \lambda^2 v, \quad \boldsymbol{r}_T \in \Omega, \\ v = 0, \quad \boldsymbol{r}_T \in \partial\Omega \end{cases} \tag{4.14}$$

or purely imaginary, $v(\boldsymbol{r}_T, t) = iw(\boldsymbol{r}_T, t)$, and satisfy the Neumann boundary-value problem

$$\begin{cases} -\Delta_T w = \lambda^2 w, \quad \boldsymbol{r}_T \in \Omega, \\ \partial_n w = 0, \quad \boldsymbol{r}_T \in \partial\Omega. \end{cases} \tag{4.15}$$

Obviously, the time-dependence can be separated out and included in $u(z, t)$ so that λ^2 is either an eigenvalue of the Dirichlet boundary-value problem or an eigenvalue

of the Neumann boundary-value problem. The kernel of the longitudinal refractive index of the mode is obtained by solving the Volterra equation of the second kind

$$2N_z(t) + (N_z * N_z)(t) - 2N(t) - (N * N)(t) = c_0^2\lambda^2 t H(t). \tag{4.16}$$

In summary, the following solutions arise in the *achiral* case:

- Dirichlet solutions:

$$\begin{cases} Q_z(\boldsymbol{r},t) = u(z,t)v(\boldsymbol{r}_T), \\ E_z(\boldsymbol{r},t)/2 = u(z,t)v(\boldsymbol{r}_T), \\ \eta_0 H_z(\boldsymbol{r},t)/2 = -\left(\displaystyle\int_{-\infty}^{t} Y_{\text{cross}}(t-t')u(z,t')\,dt'\right) v(\boldsymbol{r}_T), \end{cases}$$

  where the real function $u(z,t)$ satisfies the first-order, dispersive wave equation (4.10) and the real function $v(\boldsymbol{r}_T)$ satisfies the Dirichlet boundary-value problem (4.14). In the reciprocal (isotropic) case, these solutions reduce to the usual TM modes.

- Neumann solutions:

$$\begin{cases} Q_z(\boldsymbol{r},t) = iu(z,t)w(\boldsymbol{r}_T), \\ E_z(\boldsymbol{r},t) = 0, \\ \eta_0 H_z(\boldsymbol{r},t)/2 = -\left(u(z,t) + \displaystyle\int_{-\infty}^{t} Y_{\text{co}}(t-t')u(z,t')\,dt'\right) w(\boldsymbol{r}_T), \end{cases}$$

  where the real function $u(z,t)$ satisfies the first-order, dispersive wave equation (4.10) and the real function $w(\boldsymbol{r}_T)$ satisfies the Neumann boundary-value problem (4.15). These solutions are TE modes.

In both cases, the transverse components are given by equation (4.8):

$$\boldsymbol{Q}_T = \lambda^{-2}c_0^{-1}\partial_t(-\mathcal{N}_z\mathbf{I} + i\mathcal{N}\mathbf{J}) \cdot \nabla_T Q_z.$$

In the empty waveguide, $N(t) = 0$, and the longitudinal index of refraction can be computed exactly using the method of successive approximations. Since $N_T(t) = c_0\lambda H(t)$ for each specific mode,

$$N_z(t) = \sum_{k=1}^{\infty}\binom{\frac{1}{2}}{k}\left\{(N_T*)^{2k-1}N_T\right\}(t) = H(t)\sum_{k=1}^{\infty}\binom{\frac{1}{2}}{k}\frac{(c_0\lambda)^{2k}}{(2k-1)!}t^{2k-1} =$$
$$= H(t)\int_0^t\left\{\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k)!!}\frac{(c_0\lambda)^{2k}}{(2k-2)!!}t^{2k-2}\right\}dt = H(t)\int_0^t\frac{c_0\lambda}{t}J_1(c_0\lambda t)\,dt,$$

where a Bessel function expansion has been used:

$$J_1(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+1)!}\left(\frac{x}{2}\right)^{2k+1}.$$

The longitudinal wave equation (4.12) becomes

$$\partial_z u(z,t) = -c_0^{-1}\partial_t u(z,t) - \lambda \int_{-\infty}^{t} \frac{J_1(c_0\lambda(t-t'))}{t-t'} u(z,t')\,dt'.$$

Assuming the excitation $u(0,t)$ at $z=0$ be given, the solution to this equation is

$$u(z,t) = \int_{-\infty}^{\infty} \mathcal{E}(z;t-t')u(0,t')\,dt',$$

where the fundamental solution

$$\begin{aligned}\mathcal{E}(z;t) =& H(z)\frac{z}{c_0 t}\frac{d}{dt}\left(J_0\left(\omega_p\sqrt{t^2-z^2/c_0^2}\right)H\left(t-\frac{z}{c_0}\right)\right)\\ =& H(z)\delta\left(t-\frac{z}{c_0}\right) - H(z)\frac{\omega_p^2 z}{2c_0}\left(J_0\left(\omega_p\sqrt{t^2-z^2/c_0^2}\right)\right.\\ &\left.+ J_2\left(\omega_p\sqrt{t^2-z^2/c_0^2}\right)\right)H\left(t-\frac{z}{c_0}\right),\end{aligned}$$

oscillates with a characteristic frequency $\omega_p = c_0\lambda$. These results can be found in Kristensson [8]. The fundamental solution $\mathcal{E}(z;t)$ is also relevant for propagation of radio waves in the ionized layers of the atmosphere, where the mean free path of the electrons is exceedingly long, see, for instance, Felsen and Marcuvitz [5, p. 163]. For this case, the quantity ω_p denotes the plasma frequency.

In a conventional waveguide with nonmagnetic, isotropic filling, the refractive equation (4.16) for an arbitrary mode becomes

$$2N_z(t) + (N_z * N_z)(t) = c_0^2\lambda^2 t H(t) + \chi^{ee}(t). \tag{4.17}$$

Solving this equation and the first-order, dispersive wave equation (4.12) would be an effective alternative method to deal with the propagation problem in Ref. 2. In particular, forerunners or precursors in dielectric-filled waveguides can be obtained. Sommerfeld's forerunner (the first precursor) is the leading edge behavior of the propagating field shortly after the arrival of the wave front. In Lorentz materials, this field is highly oscillating. In the absence of the guide, the characteristic frequency of Sommerfeld's forerunner is completely determined by the constant $\partial_t\chi^{ee}(+0)$. In a waveguide, the characteristic frequency depends also on the eigenvalue of the propagating mode or modes. To illustrate this, suppose the filling consists of a single-resonance Lorentz medium with the plasma frequency ω_p. In this case, an approximation to the solution of equation (4.17) for early times is

$$N_z(t) \approx H(t)\int_0^t \frac{\omega}{t} J_1(\omega t)\,dt,\ t \text{ small}, \tag{4.18}$$

where $\omega = \sqrt{c_0^2\lambda^2 + \omega_p^2}$. In analogy with the result for the empty guide, the fundamental solution corresponding to the approximation (4.18) is given by the expression

for $\mathcal{E}(z;t)$ above with ω_p replaced by ω. Thus, Sommerfeld's forerunner is given by

$$H(z)\delta\left(t-\frac{z}{c_0}\right)-H(z)\frac{\omega^2 z}{2c_0}\left(J_0\left(\omega\sqrt{t^2-z^2/c_0^2}\right)\right.$$
$$\left.+J_2\left(\omega\sqrt{t^2-z^2/c_0^2}\right)\right)H\left(t-\frac{z}{c_0}\right),$$

where $\omega=\sqrt{c_0^2\lambda^2+\omega_p^2}$. For the circular waveguide, the eigenvalues λ are reciprocally proportional to the radius a. Therefore, the characteristic frequency of the first precursor is dominated by the contribution from the waveguide mode when the radius is small and by the dispersion of the filling when the radius is large. This is physically reasonable and applies also to other cross sections. Brillouin's forerunner (the second precursor) represents the slowly varying field components and arrives later. This forerunner is harder to analyze than Sommerfeld's forerunner and will not be discussed here. The cases with fillings displaying also magnetic effects and/or Tellegen effects are of the same difficulty as the case with nonmagnetic, isotropic filling: merely replace χ^{ee} in equation (4.17) by $\chi^{ee}+\chi^{mm}+\chi^{ee}*\chi^{mm}-\chi*\chi$.

5 Conclusion

By introducing the complex time-dependent electromagnetic field vector for temporally dispersive bi-isotropic materials, the Maxwell equations can be written economically as a first-order dispersive wave equation. This reduction is accomplished by introducing three intrinsic temporal convolution operators: the refractive index, the relative admittance, and the relative impedance. These operators can be obtained from the temporal susceptibility operators of the medium by solving Volterra integral equations of the second kind. Such equations are numerically stable. The obtained first-order dispersive wave equation seems to be an appropriate starting point for discussing pulse propagation phenomena in bi-isotropic materials and scattering from such materials. Three specific problems illustrating this are discussed briefly. All three problems would be of importance for the inverse problem of finding finite time-traces of the susceptibility kernels of the medium from finite time-traces of reflection and/or transmission data. It is conjectured that an appropriate intermediate step in such an inverse problem would be to find the refractive, admittance, and impedance kernels of the medium. Pulse propagation in chirowaveguides is, perhaps, the most important application. This problem is not yet sufficiently analyzed and can be considered open, although the basic theory is given in the present article.

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