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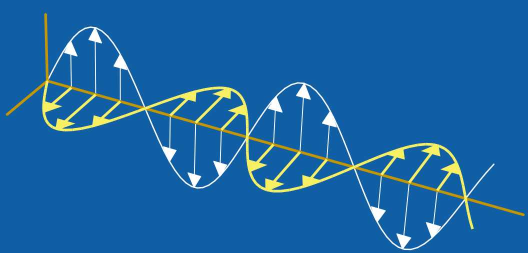

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Abstract

Time-domain Green dyadics for linear, homogeneous, temporally dispersive bi-isotropic media are presented. Complex time-dependent electromagnetic field is introduced. Approximation to the complex field from an electric point dipole in an unbounded bi-isotropic medium with respect to the slowly varying components (second forerunner approximation) is obtained. Numerical examples are presented. Surface integral equations for the tangential components of the electromagnetic fields are derived for two standard scattering problems.

1 Introduction

Green functions and Green dyadics for simple media are well-known notions in time-harmonic field analysis [5, 8, 17]. They are defined as the solutions to the scalar and the dyadic Helmholtz equations, respectively, with impulsive source terms. The knowledge of the Green function (Green dyadic) gives a possibility to obtain the solution of the scalar (vector) Helmholtz equation with an arbitrary source term. Time-harmonic Green functions and Green dyadics are often used to obtain surface integral representations of the electromagnetic fields [16]. These representations together with the boundary conditions lead to integral equations for the tangential components of the electric and magnetic fields on the boundary (equivalent magnetic and electric surface current densities, respectively).

During the last decade, time-harmonic Green functions and Green dyadics for various homogeneous, linear, complex (*e.g.*, bi-isotropic [11] or uniaxial [12]) materials, have been obtained as well. Recently, results for a large class of bi-gyrotropic materials, which involve at most twelve independent parameters, have been presented [13].

The majority of materials are dispersive, *i.e.*, the parameters depend on frequency. This dependence does not play any role in the analysis of time-harmonic fields. However, in order to study pulse propagation, it has to be taken into account. Time-dependent Green functions and Green dyadics offer a natural tool to investigate pulse propagation in dispersive media. They are defined as the solutions of the scalar and the dyadic wave equations (or dispersive wave equations), respectively, with impulsive source terms. The free-space time-dependent Green dyadic is introduced in Ref. 10. In Ref. 4, time-dependent Green dyadics for homogeneous, dispersive, isotropic media are derived. In the present article, the theory is generalized to the case of temporally dispersive, bi-isotropic materials. The complex time-dependent electromagnetic field introduced in Ref. 14 is utilized to simplify the analysis.

The outline of the present paper is as follows. In Section 2, the notation and the basic equations are introduced as well as the constitutive relations relevant to the problem in question. In Section 3, the complex time-dependent electromagnetic field is defined. The Green dyadic for the complex field is derived and given in explicit form in Section 4. In Section 5, the fields from an electric point dipole in an unbounded, temporally dispersive, bi-isotropic medium are obtained and approximated with respect to the slowly varying components (second forerunner approximation), and the numerical results are presented. Surface integral representations of the complex field are obtained in Section 6 and these are used in Section 7 to obtain surface integral equations for the tangential components of the electromagnetic fields. Some conclusions are drawn in Section 8.

2 Basics

In this paper, scalars are typed in italic style, vectors in italic boldface style, and dyadics in Roman boldface style. The three-dimensional identity dyadic is denoted by $\mathbf{I}$, the dyadic differential operators in Cartesian coordinates are given by [8]

$$
\begin{aligned}
\nabla\nabla &= (\partial_x \boldsymbol{u}_x + \partial_y \boldsymbol{u}_y + \partial_z \boldsymbol{u}_z)(\partial_x \boldsymbol{u}_x + \partial_y \boldsymbol{u}_y + \partial_z \boldsymbol{u}_z)\,,\\
\nabla\times\mathbf{I} &= (\partial_x \boldsymbol{u}_x + \partial_y \boldsymbol{u}_y + \partial_z \boldsymbol{u}_z)\times(\boldsymbol{u}_x\boldsymbol{u}_x + \boldsymbol{u}_y\boldsymbol{u}_y + \boldsymbol{u}_z\boldsymbol{u}_z)\,.
\end{aligned}
$$

Standard notation is used for all electromagnetic fields as well as charge and current densities. The speed of light in vacuum is c_0 and the intrinsic impedance of vacuum η_0. The permittivity and permeability of vacuum are denoted by ϵ_0 and μ_0, respectively.

The Maxwell equations,

$$
\begin{cases}
\nabla\times\boldsymbol{E}(\boldsymbol{r},t) = -\partial_t \boldsymbol{B}(\boldsymbol{r},t),\\
\nabla\times\boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{J}(\boldsymbol{r},t) + \partial_t \boldsymbol{D}(\boldsymbol{r},t),
\end{cases}
\tag{2.1}
$$

and the equation of continuity,

$$
\nabla\cdot\boldsymbol{J}(\boldsymbol{r},t) + \partial_t \rho(\boldsymbol{r},t) = 0, \tag{2.2}
$$

describe the dynamics of the electromagnetic fields and charges in macroscopic media. All fields and source terms are assumed to be initially quiescent. This means that all these quantities are zero before a certain time t, say $t=0$.

The constitutive relations of a homogeneous, temporally dispersive, bi-isotropic medium in the absence of an optical response are [6, 14]

$$
\begin{cases}
c_0\eta_0\boldsymbol{D}(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r},t) + (\chi^{ee}*\boldsymbol{E})(\boldsymbol{r},t) + (\chi^{em}*\eta_0\boldsymbol{H})(\boldsymbol{r},t)\\
\qquad\qquad\quad = [\varepsilon\boldsymbol{E}]\,(\boldsymbol{r},t) + [\xi\eta_0\boldsymbol{H}]\,(\boldsymbol{r},t),\\
c_0\boldsymbol{B}(\boldsymbol{r},t) = (\chi^{me}*\boldsymbol{E})(\boldsymbol{r},t) + \eta_0\boldsymbol{H}(\boldsymbol{r},t) + (\chi^{mm}*\eta_0\boldsymbol{H})(\boldsymbol{r},t)\\
\qquad\qquad\quad = [\zeta\boldsymbol{E}]\,(\boldsymbol{r},t) + [\mu\eta_0\boldsymbol{H}]\,(\boldsymbol{r},t),
\end{cases}
$$

where $\chi^{ij}(t)$, $i,j=$ e,m, are the susceptibility kernels of the medium. The relative permittivity and permeability operators, $\varepsilon = (1+\chi^{ee}*)$ and $\mu = (1+\chi^{mm}*)$, as well as the relative cross-coupling terms, $\xi = \chi^{em}*$ and $\zeta = \chi^{me}*$, are temporal integral operators. The asterisk ($*$) denotes temporal convolution:

$$
(\chi*\boldsymbol{E})(\boldsymbol{r},t) = \int \chi(t-t')\boldsymbol{E}(\boldsymbol{r},t')\,dt'.
$$

Here and in the rest of the paper the integration limits $-\infty$ and ∞ are omitted in time integrals, *i.e.*, $\int \ldots dt' := \int_{-\infty}^{\infty} \ldots dt'$. Causality implies that all susceptibility kernels are identically zero for time $t < 0$ which means that the upper limit in the integral above can be set to t. All kernels are supposed to be smooth and bounded for $t > 0$. Bi-isotropic medium is Pasteur if $\chi^{me}(t) = -\chi^{em}(t)$, it is Tellegen if $\chi^{me}(t) = \chi^{em}(t)$ [9].

3 The complex time-dependent electromagnetic field

The complex time-dependent electromagnetic field $\boldsymbol{Q}(\boldsymbol{r}, t)$ is introduced through the relations [14]

$$\begin{cases} \boldsymbol{E} = \boldsymbol{Q} + \boldsymbol{Q}^*, \\ \eta_0 \boldsymbol{H} = i\mathcal{Y}\boldsymbol{Q} - i\mathcal{Y}^*\boldsymbol{Q}^*, \end{cases} \tag{3.1}$$

where i is the imaginary unit and the relative intrinsic admittance $\mathcal{Y} = (1 + Y*)$ is a complex-valued temporal integral operator. $\boldsymbol{Q}^*(\boldsymbol{r}, t)$ and $\mathcal{Y}^*$ denote complex conjugates of $\boldsymbol{Q}(\boldsymbol{r}, t)$ and $\mathcal{Y}$, respectively. Explicitly,

$$\boldsymbol{Q} = \frac{1}{2}\mathcal{Z}(\mathcal{Y}^*\boldsymbol{E} - i\eta_0\boldsymbol{H}),$$

where the relative intrinsic impedance $\mathcal{Z} = (1+Z*)$ is a real-valued temporal integral operator defined by

$$(\mathcal{Y} + \mathcal{Y}^*)\,\mathcal{Z}/2 = 1.$$

Demanding decoupling of the Maxwell equations for the complex fields $\boldsymbol{Q}(\boldsymbol{r}, t)$ and $\boldsymbol{Q}^*(\boldsymbol{r}, t)$ gives the following condition on the operator $\mathcal{Y}$:

$$\mathcal{Y}\zeta - i\varepsilon + i\mathcal{Y}^2\mu + \mathcal{Y}\xi = 0. \tag{3.2}$$

The Maxwell equations then reduce to

$$\nabla \times \boldsymbol{Q} = -\frac{i}{c_0}\partial_t \mathcal{N}\boldsymbol{Q} - \frac{i}{2}\eta_0\mathcal{Z}\boldsymbol{J}, \tag{3.3}$$

where the index of refraction $\mathcal{N} = (1 + N*)$ is a complex-valued temporal integral operator defined by

$$\mathcal{N} = \mu\mathcal{Y}^* + i\xi = \mu\mathcal{Y} - i\zeta. \tag{3.4}$$

Equations (3.2) and (3.4) can be combined to get the following expressions for the operators $\mathcal{N}$ and $\mathcal{Y}$ in terms of the known data:

$$\begin{cases} \mathcal{N} = i\dfrac{\xi - \zeta}{2} + \sqrt{\mu\varepsilon - \dfrac{(\xi + \zeta)^2}{4}}, \\ \mu\mathcal{Y} = i\dfrac{\xi + \zeta}{2} + \sqrt{\mu\varepsilon - \dfrac{(\xi + \zeta)^2}{4}}, \end{cases}$$

where the square-root operator is

$$\sqrt{\mu\varepsilon - \frac{(\xi+\zeta)^2}{4}} = 1 + N_{\mathrm{co}}(t) * .$$

Here, the real-valued integral kernel $N_{\mathrm{co}}(t)$ satisfies the nonlinear Volterra integral equation of the second kind

$$2N_{\mathrm{co}}(t) + (N_{\mathrm{co}} * N_{\mathrm{co}})(t) = \chi^{ee}(t) + \chi^{mm}(t) + (\chi^{ee} * \chi^{mm})(t) - (\chi * \chi)(t),$$

where

$$\chi(t) = \chi^{em}(t)/2 + \chi^{me}(t)/2$$

is the nonreciprocity kernel. For a Pasteur medium $\chi \equiv 0$.

The wave number operator $\mathcal{K}$ is defined by

$$\mathcal{K} = c_0^{-1}\partial_t\mathcal{N} = c_0^{-1}\partial_t(1 + N*).$$

The inverses of all the operators above exist and are well-defined temporal integral operators, *cf.* Ref. 4.

4 Green dyadic for the complex field

The Green dyadic $\mathbf{G}_Q$ for the complex electromagnetic field $\boldsymbol{Q}(\boldsymbol{r},t)$ is defined by [4]

$$\boldsymbol{Q}(\boldsymbol{r},t) = \int_{\mathbb{R}^3}\int \mathbf{G}_Q(\boldsymbol{r}-\boldsymbol{r}';t-t')\cdot\mu_0\partial_{t'}\left[\mu\boldsymbol{J}\right](\boldsymbol{r}',t')\,dt'\,dv'. \tag{4.1}$$

Using (3.3), the following equation for the dyadic $\mathbf{G}_Q$ is obtained

$$\left(\nabla\times\mathbf{I} + i\mathbf{I}\mathcal{K}\right)\mathbf{G}_Q = -\frac{i}{2}\mathbf{I}c_0\partial_t^{-1}\mathcal{Z}\mu^{-1}\left[\delta_{\mathbf{0}}\otimes\delta_0\right], \tag{4.2}$$

where $\delta_{\mathbf{0}} = \delta(\boldsymbol{r})$ and $\delta_0 = \delta(t)$ are the Dirac delta functions in space and in time, respectively. Operating with $\nabla\cdot$ on (4.2) gives

$$\nabla\cdot\mathbf{G}_Q = -\frac{c_0}{2}\nabla\cdot\mathbf{I}\partial_t^{-1}\mathcal{Z}\mathcal{K}^{-1}\mu^{-1}\left[\delta_{\mathbf{0}}\otimes\delta_0\right].$$

Operating with $\nabla\times$ on (4.2) and using the equation above leads to the following dispersive wave equation for the Green dyadic

$$\left[\Delta - \mathcal{K}^2\right]\mathbf{G}_Q = \frac{1}{2}\left[\mathbf{I} - \nabla\nabla\mathcal{K}^{-2} + i\mathcal{K}^{-1}\nabla\times\mathbf{I}\right]\mathcal{Z}\mathcal{N}\mu^{-1}\left[\delta_{\mathbf{0}}\otimes\delta_0\right],$$

where Δ is the Laplace operator. The solution to the equation above can be written in the following form:

$$\mathbf{G}_Q = -\frac{1}{2}\left[\mathbf{I} - \nabla\nabla\mathcal{K}^{-2} + i\mathcal{K}^{-1}\nabla\times\mathbf{I}\right]\mathcal{Z}\mathcal{N}\mu^{-1}\mathcal{E}, \tag{4.3}$$

where $\mathcal{E}=\mathcal{E}(\boldsymbol{r};t)$ is the retarded fundamental solution of the dispersive wave operator $(-\Delta+\mathcal{K}^2)$. The fundamental solution $\mathcal{E}(\boldsymbol{r};t)$ is given by [3, 4]

$$\mathcal{E}(\boldsymbol{r};t)=q(r)\frac{1}{4\pi r}\left(\delta\left(t-\frac{r}{c_0}\right)+P\left(r;t-\frac{r}{c_0}\right)\right), \tag{4.4}$$

where

$$q(r)=\exp\left(-\frac{r}{c_0}N(+0)\right) \tag{4.5}$$

and

$$P(r;t)=\sum_{m=1}^{\infty}\frac{1}{m!}\left(-\frac{r}{c_0}\right)^m((N'*)^{m-1}N')(t). \tag{4.6}$$

The kernel $P(r;t)$ satisfies the Volterra temporal integral equation of the second kind [3, 4]

$$\begin{cases} tP(r;t)=F(r;t)+\big(F(r;\cdot)*P(r;\cdot)\big)(t), \quad F(r;t)=-t\dfrac{r}{c_0}N'(t), \\ P(r;t)=0 \qquad \text{for} \quad t<0. \end{cases} \tag{4.7}$$

An alternative representation of the retarded fundamental solution, which is used below to obtain the second forerunner approximations, is [3, 4]

$$\mathcal{E}(\boldsymbol{r};\cdot)*=\frac{1}{4\pi r}\exp\left(-\frac{r}{c_0}\partial_t\big(\delta(\cdot)+N(\cdot)\big)*\right), \tag{4.8}$$

where the exponential function is understood in terms of its Maclaurin expansion. Notice that (*cf.* Ref. 4)

$$\mathbf{G}_Q(\boldsymbol{r}-\boldsymbol{r}';t-t')=\mathbf{G}_Q^T(\boldsymbol{r}'-\boldsymbol{r};t-t'), \tag{4.9}$$

where $\mathbf{G}_Q^T$ is the transpose of the dyadic $\mathbf{G}_Q$. (Recall that a dyadic $\mathbf{A}^T$ is the transpose of a dyadic $\mathbf{A}$ if $\mathbf{A}\cdot\boldsymbol{F}=\boldsymbol{F}\cdot\mathbf{A}^T$ for all vectors $\boldsymbol{F}$.)

The operator combination $\mathcal{Z}\mathcal{N}\mu^{-1}$ in (4.3) can be expressed in terms of the constitutive operators as

$$\mathcal{Z}\mathcal{N}\mu^{-1}=1+\frac{i}{2}(\xi-\zeta)\left(\mu\varepsilon-\frac{(\xi+\zeta)^2}{4}\right)^{-1/2}.$$

Notice that if the medium is isotropic, then $\mathcal{Z}\mathcal{N}\mu^{-1}=1$ and (4.3) reduces to the result in Ref. 4.

With the help of Schwartz' pseudo-functions [15], the expression (4.3) can be written explicitly as, *cf.* Ref. 4,

$$\begin{aligned} 2\mathbf{G}_Q=&-\frac{1}{3}\mathbf{I}\mathcal{K}^{-2}\mathcal{Z}\mathcal{N}\mu^{-1}\left[\delta_{\mathbf{0}}\otimes\delta_0\right]+(\boldsymbol{u}_r\boldsymbol{u}_r-\mathbf{I})\,\mathcal{Z}\mathcal{N}\mu^{-1}\mathcal{E} \\ &(3\boldsymbol{u}_r\boldsymbol{u}_r-\mathbf{I})\left(Pf.\left(\frac{1}{r^3}\right)\mathcal{K}^{-2}+\frac{1}{r^2}\mathcal{K}^{-1}\right)\mathcal{Z}\mathcal{N}\mu^{-1}r\mathcal{E} \\ &+i\left(\frac{1}{r^2}\mathcal{K}^{-1}+\frac{1}{r}\right)\mathcal{Z}\mathcal{N}\mu^{-1}\left(\boldsymbol{r}\mathcal{E}\times\mathbf{I}\right). \end{aligned} \tag{4.10}$$

5 Example

5.1 Electric point dipole in a dispersive bi-isotropic medium

Suppose that an electric point dipole, placed in an unbounded bi-isotropic medium, flashes on and off at time $t = 0$. Its electric dipole moment is $\boldsymbol{p}\delta(t)$, where $\boldsymbol{p}$ is a constant vector. The charge and current densities of this source are given by

$$\rho = -(\boldsymbol{p}\cdot\nabla)\left(\delta_{\mathbf{0}}\otimes\delta_0\right), \qquad \boldsymbol{J} = \boldsymbol{p}\partial_t\left(\delta_{\mathbf{0}}\otimes\delta_0\right), \tag{5.1}$$

respectively. Using equations (4.1) and (4.10) gives the following expression for the complex time-dependent electromagnetic field generated by an electric point dipole:

$$\begin{aligned} 2\epsilon_0\boldsymbol{Q} = &-\frac{1}{3}\boldsymbol{p}\mathcal{N}^{-1}\mathcal{Z}\left[\delta_{\mathbf{0}}\otimes\delta_0\right] + (\boldsymbol{u}_r\boldsymbol{u}_r-\mathbf{I})\cdot\boldsymbol{p}\,c_0^{-2}\partial_t^2\mathcal{Z}\mathcal{N}\mathcal{E}\\ &+(3\boldsymbol{u}_r\boldsymbol{u}_r-\mathbf{I})\cdot\boldsymbol{p}\left(Pf.\left(\frac{1}{r^3}\right)\mathcal{N}^{-1}+\frac{1}{r^2}c_0^{-1}\partial_t\right)\mathcal{Z}\left(r\mathcal{E}\right)\\ &+i\left(\frac{1}{r^2}+\frac{1}{r}\mathcal{K}\right)c_0^{-1}\partial_t\mathcal{Z}\left(\boldsymbol{r}\mathcal{E}\times\boldsymbol{p}\right). \end{aligned} \tag{5.2}$$

Recall that the operators $\mathcal{N}$ and $\mathcal{K}$ as well as the retarded fundamental solution $\mathcal{E}$ are complex quantities.

The field $\boldsymbol{Q}(\boldsymbol{r},t)$ can be obtained numerically by solving the equation (4.7), substituting (4.4) into (5.2), and performing all the convolutions. This procedure is very time- and memory-consuming.

In analogy with the analysis in Ref. 4, asymptotic methods developed in Refs. 3,7 can be used to get an approximation to the dipole fields with respect to the slowly varying components (the second forerunner approximation). First, write [3]

$$\begin{aligned} \mathcal{E}(\boldsymbol{r};\cdot)* &= \frac{1}{4\pi r}\exp\left(-\frac{r}{c_0}\partial_t\left(1+N(\cdot)*\right)\right)\\ &\approx \frac{1}{4\pi r}\exp\left(-\frac{r}{c_0}\left((1+n_1)\partial_t+n_2\partial_t^2+n_3\partial_t^3\right)\right)=\tilde{\mathcal{E}}(\boldsymbol{r};\cdot)*, \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} &\tilde{\mathcal{E}}(\boldsymbol{r};t) = \frac{1}{4\pi r}\exp\left(\frac{n_2^3}{27n_3^2}\frac{r}{c_0}-\frac{n_2}{3n_3}(t-t_1(r))\right)\frac{\mathrm{Ai}\left(\mathrm{sign}(\mathrm{Re}(n_3))(t-t_1(r))/t_3(r)\right)}{t_3(r)},\\ &n_1=\int_0^\infty N(t)\,dt, \qquad n_2=-\int_0^\infty tN(t)\,dt, \qquad n_3=\frac{1}{2}\int_0^\infty t^2N(t)\,dt,\\ &t_1(r)=\left(n_1+1-\frac{n_2^2}{3n_3}\right)\frac{r}{c_0}, \qquad t_3(r)=\left(\frac{3n_3\mathrm{sign}(\mathrm{Re}(n_3))r}{c_0}\right)^{\frac{1}{3}}. \end{aligned}$$

Now, to get asymptotic expressions for the dipole fields, approximate convolutions in (5.2) by the first three terms in their series representations [3, 7], *e.g.*,

$$\mathcal{Z} = (1+Z*) \approx (1+z_1)+z_2\partial_t+z_3\partial_t^2,$$

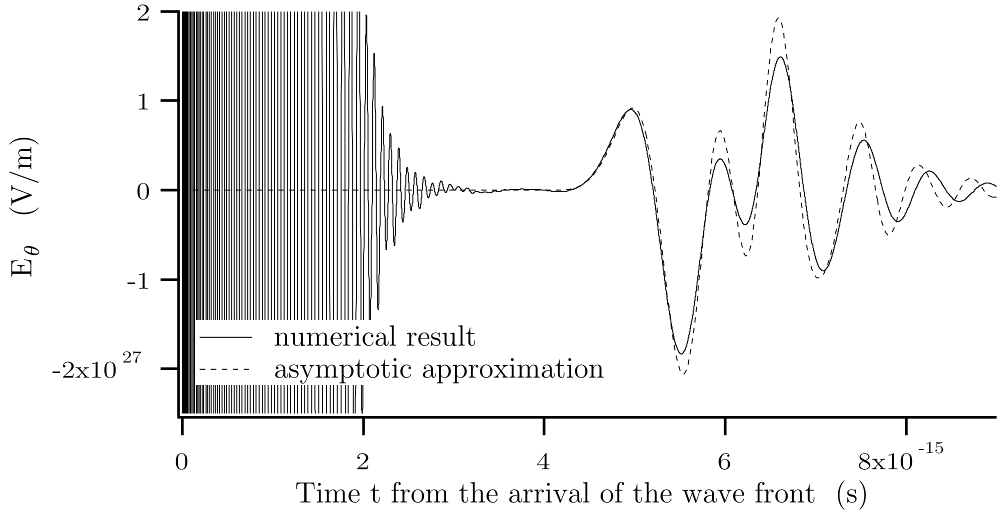

Figure 1: The field E_θ from a dipole $\boldsymbol{u}_z 10^{-17}\delta(t)\,\mathrm{C\,m}$ in a chiral medium at a distance $r = 4\cdot 10^{-6}\,\mathrm{m}$ from the dipole at an angle of observation $\theta = \pi/4$. The medium is characterized by the parameters $\alpha = 3.33\cdot 10^{-18}\,\mathrm{s}$, $\omega_p = \omega_0 = 3\cdot 10^{16}\,\mathrm{s}^{-1}$, $\nu = 6\cdot 10^{15}\,\mathrm{s}^{-1}$.

where the moments are

$$z_1 = \int_0^\infty Z(t)\,dt, \qquad z_2 = -\int_0^\infty tZ(t)\,dt, \qquad z_3 = \frac{1}{2}\int_0^\infty t^2 Z(t)\,dt.$$

Using these approximations and (5.3) in (5.2) give an approximate expression for the complex field $\boldsymbol{Q}(\boldsymbol{r}, t)$ due to an electric point dipole. The main advantage of this method is that the resulting expression contains only algebraic combinations of the exponential function, Airy function and its derivative. No convolutions or other time-consuming operations are involved. An explicit formula for this approximation for the case of an isotropic, nonmagnetic medium is presented in Ref. 4. The general expression is too long to be presented in this paper.

Notice that this technique cannot be used to obtain the wave-front behavior (the first precursor) of the complex field $\boldsymbol{Q}(\boldsymbol{r}, t)$.

5.2 Numerical calculations

In this subsection, the methods described above are used to calculate the dipole fields in a chiral medium. Both the numerical and the asymptotic solutions are obtained and the results are compared with each other.

In terms of the susceptibility kernels, Drude's model for reciprocal, nonmagnetic,

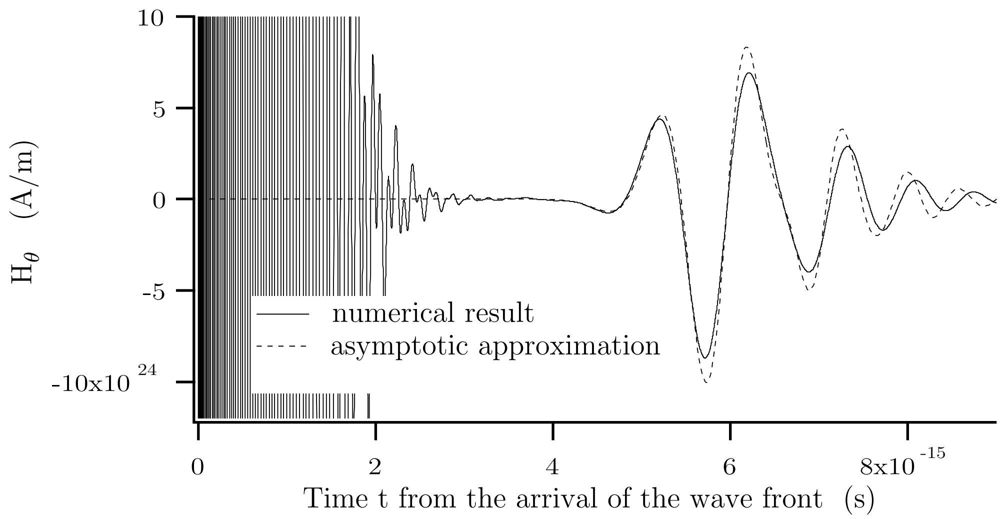

Figure 2: The field H_θ from a dipole in a chiral medium. For details see caption of Figure 1.

isotropic chiral materials (also known as Condon's model) can be described as [3]

$$\begin{cases} G(t) = H(t)\dfrac{\omega_p^2}{\nu_0}\sin(\nu_0 t)\exp\left(-\dfrac{\nu t}{2}\right), \\ \chi^{em}(t) = -\chi^{me}(t) = \kappa(t) = \alpha\partial_t G(t), \\ \chi^{ee}(t) = \chi(t) = G(t) - (\kappa * \kappa)(t), \\ \chi^{mm} = 0, \end{cases} \tag{5.4}$$

where $H(t)$ is the Heaviside step function, ω_0, ω_p, and ν are the harmonic, plasma, and collision frequencies, respectively, $\nu_0 = \sqrt{\omega_0^2 - \nu^2/4}$, and α is a constant depending on the microstructure of the medium. For this model, the susceptibility moments are [3]

$$\begin{cases} \chi_1 = \dfrac{\omega_p^2}{\omega_0^2}, \qquad \chi_2 = -\dfrac{\nu\omega_p^2}{\omega_0^4}, \qquad \chi_3 = -\dfrac{(w_0^2-\nu^2)w_p^2}{w_0^6} - \alpha^2\dfrac{w_p^4}{w_0^4}, \\ n_1 = \sqrt{1+\chi_1} - 1, \qquad n_2 = \dfrac{\chi_2}{2(1+n_1)} + i\alpha\chi_1, \qquad n_3 = \dfrac{\chi_3 - (\mathrm{Re} n_2)^2}{2(1+n_1)} + i\alpha\chi_2, \\ n_{\mathrm{res}1} = -\dfrac{n_1}{1+n_1}, \quad n_{\mathrm{res}2} = -\dfrac{n_2(1+n_{\mathrm{res}1})}{1+n_1}, \quad n_{\mathrm{res}3} = -\dfrac{(1+n_{\mathrm{res}1})n_3 + n_{\mathrm{res}2}n_2}{1+n_1}, \\ z_1 = -\dfrac{n_1}{1+n_1}, \quad z_2 = -\dfrac{\mathrm{Re} n_2(1+z_1)}{1+n_1}, \quad z_3 = -\dfrac{(1+z_1)\mathrm{Re} n_3 + z_2\mathrm{Re} n_2}{1+n_1}. \end{cases}$$

In Figures 1–6, the components of the electric and magnetic fields due to an electric point dipole (5.1) with $\boldsymbol{p} = \boldsymbol{u}_z 10^{-17}\,\mathrm{C\,m\,s}$ in an unbounded chiral medium are presented. The medium is characterized by the parameters $\alpha = 3.33\cdot 10^{-18}\,\mathrm{s}$, $\omega_p = \omega_0 = 3\cdot 10^{16}\,\mathrm{s}^{-1}$, $\nu = 6\cdot 10^{15}\,\mathrm{s}^{-1}$. The distance from the dipole to the observation

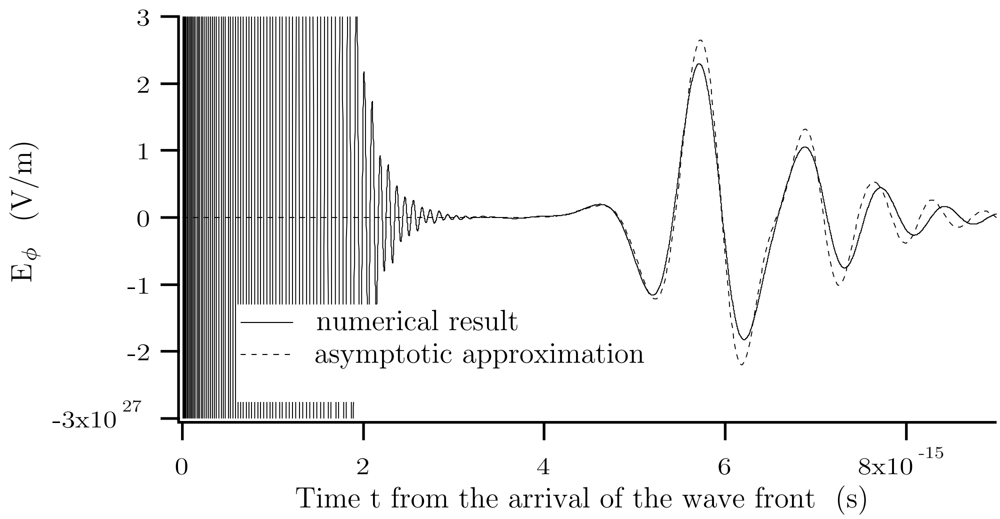

Figure 3: The field E_ϕ from a dipole in a chiral medium. For details see caption of Figure 1.

point is $4 \cdot 10^{-6}$ m and the angle of observation is $\theta = \pi/4$. Both the numerical solution (solid line) and the asymptotic approximation (dashed line) are presented. The agreement between these two solutions is quite good. A quick look at Figures 1 and 7, where the latter presents the θ-component of the electric field at the distance $r = 10^{-6}$ m, reveals that the developed method gives better approximation for larger propagation depths which is intuitively clear from the representation in equation (4.8).

Figure 8 illustrates how the θ-component of the electric field changes with the distance. The left curve represents the field at $r = 4 \cdot 10^{-6}$ m from the dipole while the right one — at a 10 times larger distance. Note that different scales are used for different curves. Only asymptotic results are available for large propagation depths. Other field components change with the distance in a similar way.

Note that t in all figures denotes the time after the arrival of the wave front (*i.e.*, t is "the wave-front time").

6 Surface integral representations

The derivation of the surface integral representations and the surface integral equations for the complex electromagnetic field $\boldsymbol{Q}(\boldsymbol{r}, t)$ in Ref. 4 is now generalized to the case of bi-isotropic materials.

Let V_- and V_+ be two disjoint open domains in $\mathbb{R}^3$ such that $\overline{V_- \cup V_+} = \mathbb{R}^3$. Furthermore, suppose that $S = \mathbb{R}^3 \setminus (V_- \cup V_+)$ is a regular surface. Let $\boldsymbol{u}_n = \boldsymbol{u}_n(\boldsymbol{r})$ denote the outward, with respect to V_-, unit normal vector to S. Furthermore, let

$$\boldsymbol{Q}_\pm(\boldsymbol{r}, t) = \lim_{V_\pm \ni \boldsymbol{r}' \to \boldsymbol{r}} \boldsymbol{Q}(\boldsymbol{r}', t), \qquad \boldsymbol{r} \in S.$$

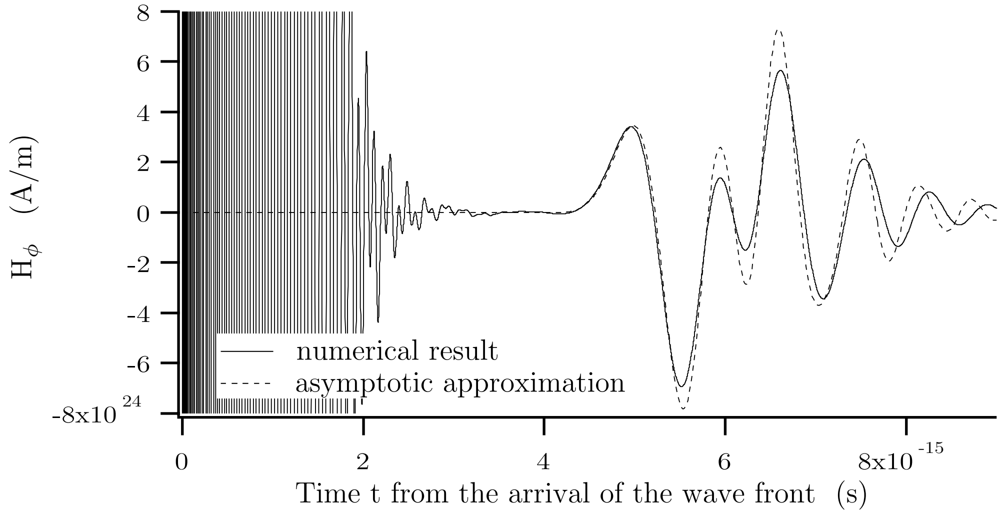

Figure 4: The field H_ϕ from a dipole in a chiral medium. For details see caption of Figure 1.

If the domain V_- is filled with a known temporally dispersive, bi-isotropic medium, it is possible to express the complex field $\boldsymbol{Q}(\boldsymbol{r}, t)$, $\boldsymbol{r} \in V_-$, in terms of its tangential components at the boundary, $\boldsymbol{u}_n \times \boldsymbol{Q}_-(\boldsymbol{r}, t)$, $\boldsymbol{r} \in S$, and the current density $\boldsymbol{J}(\boldsymbol{r}, t)$, $\boldsymbol{r} \in V_-$. No information about the material in the domain V_+ is needed. Using equations (3.3) and (4.2) and the general differentiation rule

$$\begin{aligned}&\nabla' \cdot (\boldsymbol{Q}(\boldsymbol{r}', t') \times \mathbf{G}_Q(\boldsymbol{r}' - \boldsymbol{r}; t - t'))\\&\qquad = (\nabla' \times \boldsymbol{Q}(\boldsymbol{r}', t')) \cdot \mathbf{G}_Q(\boldsymbol{r}' - \boldsymbol{r}; t - t') - \boldsymbol{Q}(\boldsymbol{r}', t') \cdot (\nabla' \times \mathbf{G}_Q(\boldsymbol{r}' - \boldsymbol{r}; t - t'))\end{aligned}$$

give for $\boldsymbol{r} \in V_-$

$$\begin{aligned}\nabla' \cdot (\boldsymbol{Q}(\boldsymbol{r}', t') \times \mathbf{G}_Q(\boldsymbol{r}' - \boldsymbol{r}; t - t')) &= \frac{ic_0}{2}\delta(\boldsymbol{r} - \boldsymbol{r}')\,\delta(t - t')\,\partial_t^{-1}\left[\mathcal{Z}\mu^{-1}\boldsymbol{Q}\right](\boldsymbol{r}, t)\\&- \frac{i\eta_0}{2}\left[\mathcal{Z}\boldsymbol{J}\right](\boldsymbol{r}', t')\cdot\mathbf{G}_Q(\boldsymbol{r}'-\boldsymbol{r}; t-t') - i\left[\mathcal{K}\boldsymbol{Q}\right](\boldsymbol{r}', t')\cdot\mathbf{G}_Q(\boldsymbol{r}'-\boldsymbol{r}; t-t')\\&+ i\boldsymbol{Q}(\boldsymbol{r}', t') \cdot \left[\mathcal{K}\mathbf{G}_Q\right](\boldsymbol{r}' - \boldsymbol{r}; t - t').\end{aligned} \tag{6.1}$$

Note that, due to causality, $\boldsymbol{Q}(\boldsymbol{r}', t') \times \mathbf{G}_Q(\boldsymbol{r}' - \boldsymbol{r}; t - t')$ has bounded support for every fixed $\boldsymbol{r}$ and t. Let $V_{\boldsymbol{r},t}$ be a bounded open domain containing this support and such that the boundary $S_{\boldsymbol{r},t}$ of $V_{\boldsymbol{r},t} \cap V_-$ is regular. Then, using the Gauss theorem for dyadics, identity (4.9), and the equality

$$\left(\boldsymbol{u}_n' \times \boldsymbol{Q}_-(\boldsymbol{r}', t')\right) \cdot \mathbf{G}_Q(\boldsymbol{r}' - \boldsymbol{r}; t - t') = \boldsymbol{u}_n' \cdot \left(\boldsymbol{Q}_-(\boldsymbol{r}', t') \times \mathbf{G}_Q(\boldsymbol{r}' - \boldsymbol{r}; t - t')\right)$$

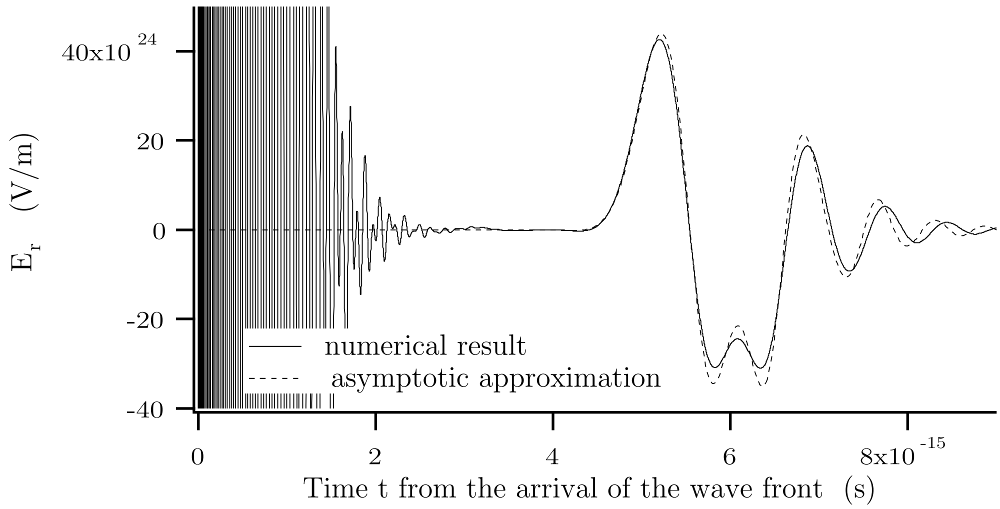

Figure 5: The field E_r from a dipole in a chiral medium. For details see caption of Figure 1.

give

$$\int_{V_-} \nabla' \cdot (\boldsymbol{Q}(\boldsymbol{r}',t') \times \mathbf{G}_Q(\boldsymbol{r}' - \boldsymbol{r}; t-t')) = \int_{V_{\boldsymbol{r},t} \cap V_-} \nabla' \cdot (\boldsymbol{Q}(\boldsymbol{r}',t') \times \mathbf{G}_Q(\boldsymbol{r}' - \boldsymbol{r}; t-t'))$$

$$= \oint_{S_{\boldsymbol{r},t}} \mathbf{G}_Q(\boldsymbol{r}-\boldsymbol{r}'; t-t') \cdot \big(\boldsymbol{u}_n' \times \boldsymbol{Q}_-(\boldsymbol{r}',t')\big) dS' = \int_S \mathbf{G}_Q(\boldsymbol{r}-\boldsymbol{r}'; t-t') \cdot \big(\boldsymbol{u}_n' \times \boldsymbol{Q}_-(\boldsymbol{r}',t')\big) dS', \tag{6.2}$$

where $\boldsymbol{u}_n' = \boldsymbol{u}_n(\boldsymbol{r}')$. Now integrate (6.1) over $(\boldsymbol{r}',t') \in V_- \otimes (-\infty, \infty)$. Integration with respect to t' results in a cancellation of the last two terms on the right-hand side due to the commutative property of temporal convolutions. Finally, integrating with respect to $\boldsymbol{r}'$ gives the following expression, which can be referred to as Huygens' principle:

$$\left.\begin{matrix} \frac{ic_0}{2} \partial_t^{-1} \left[\mathcal{Z}\mu^{-1}\boldsymbol{Q}\right](\boldsymbol{r},t) \\ \mathbf{0} \end{matrix}\right\} = \frac{ic_0}{2} \partial_t^{-1} \left[\mathcal{Z}\mu^{-1}\boldsymbol{Q}_{i-}\right](\boldsymbol{r},t)$$
$$+ \int_S \int \mathbf{G}_Q(\boldsymbol{r}-\boldsymbol{r}'; t-t') \cdot \big(\boldsymbol{u}_n' \times \boldsymbol{Q}_-(\boldsymbol{r}',t')\big) dt' dS', \begin{cases} \boldsymbol{r} \in V_- \\ \boldsymbol{r} \in V_+, \end{cases} \tag{6.3}$$

where the source term is given by

$$\boldsymbol{Q}_{i-}(\boldsymbol{r},t) = \int_{V_-} \int \mathbf{G}_Q(\boldsymbol{r}-\boldsymbol{r}'; t-t') \cdot \mu_0 \partial_{t'} \left[\mu \boldsymbol{J}\right](\boldsymbol{r}',t') \, dt' \, dv'.$$

Obviously, the case when the medium in the domain V_+ is known and the one

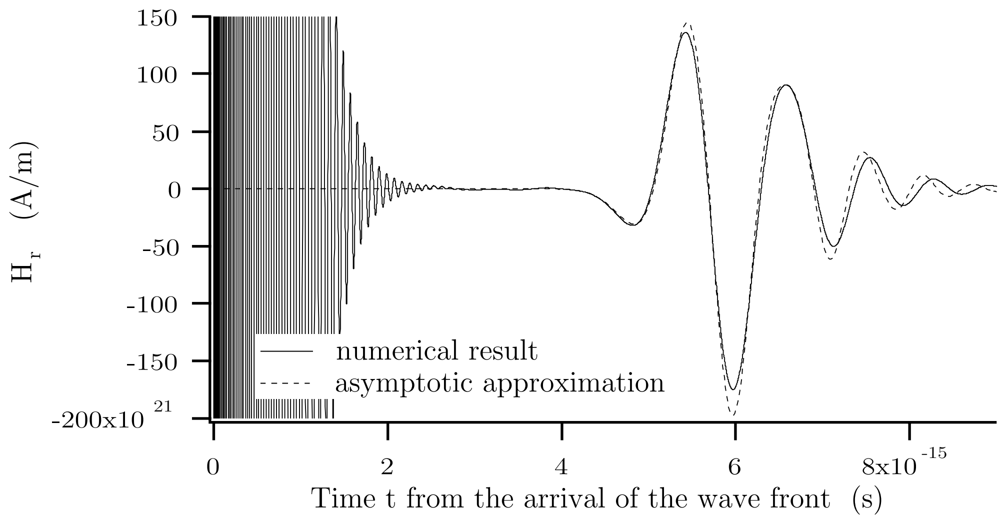

Figure 6: The field H_r from a dipole in a chiral medium. For details see caption of Figure 1.

in V_- is not, can be handled in the same way. The result is

$$\left.\begin{matrix}\dfrac{ic_0}{2}\partial_t^{-1}\left[\mathcal{Z}\mu^{-1}\boldsymbol{Q}\right](\boldsymbol{r},t)\\ \boldsymbol{0}\end{matrix}\right\}=\frac{ic_0}{2}\partial_t^{-1}\left[\mathcal{Z}\mu^{-1}\boldsymbol{Q}_{i+}\right](\boldsymbol{r},t)$$
$$-\int_S\int\mathbf{G}_Q(\boldsymbol{r}-\boldsymbol{r}';t-t')\cdot\big(\boldsymbol{u}_n'\times\boldsymbol{Q}_+(\boldsymbol{r}',t')\big)\,dt'dS',\ \begin{cases}\boldsymbol{r}\in V_+\\ \boldsymbol{r}\in V_-,\end{cases} \tag{6.4}$$

where

$$\boldsymbol{Q}_{i+}(\boldsymbol{r},t)=\int_{V_+}\int\mathbf{G}_Q(\boldsymbol{r}-\boldsymbol{r}';t-t')\cdot\mu_0\partial_{t'}\left[\mu\boldsymbol{J}\right](\boldsymbol{r}',t')\,dt'\,dv'.$$

7 Surface integral equations

In this section, surface integral equations for the tangential components of the field $\boldsymbol{Q}(\boldsymbol{r},t)$ are obtained. The materials in both domains, V_+ and V_-, are supposed to be known. Furthermore, it is assumed that $\boldsymbol{J}(\boldsymbol{r},t)=\boldsymbol{0}$ when $\boldsymbol{r}\in S$. The analysis follows the guidelines of the discussion in Ref. 4 (see also Ref. 16).

From the Gauss surface divergence theorem it follows that

$$\begin{aligned}&\int_S(\nabla\nabla\mathcal{E}(\boldsymbol{r}-\boldsymbol{r}';t-t'))\cdot\big(\boldsymbol{u}_n'\times\boldsymbol{Q}_-(\boldsymbol{r}',t')\big)\,dS'\\&\quad=\int_S(\nabla\nabla_S\mathcal{E}(\boldsymbol{r}-\boldsymbol{r}';t-t'))\cdot\big(\boldsymbol{u}_n'\times\boldsymbol{Q}_-(\boldsymbol{r}',t')\big)\,dS'\\&\quad=\nabla\int_S(\mathcal{E}(\boldsymbol{r}-\boldsymbol{r}';t-t'))\nabla_S'\cdot\big(\boldsymbol{u}_n'\times\boldsymbol{Q}_-(\boldsymbol{r}',t')\big)\,dS',\qquad\boldsymbol{r}\notin S,\end{aligned} \tag{7.1}$$

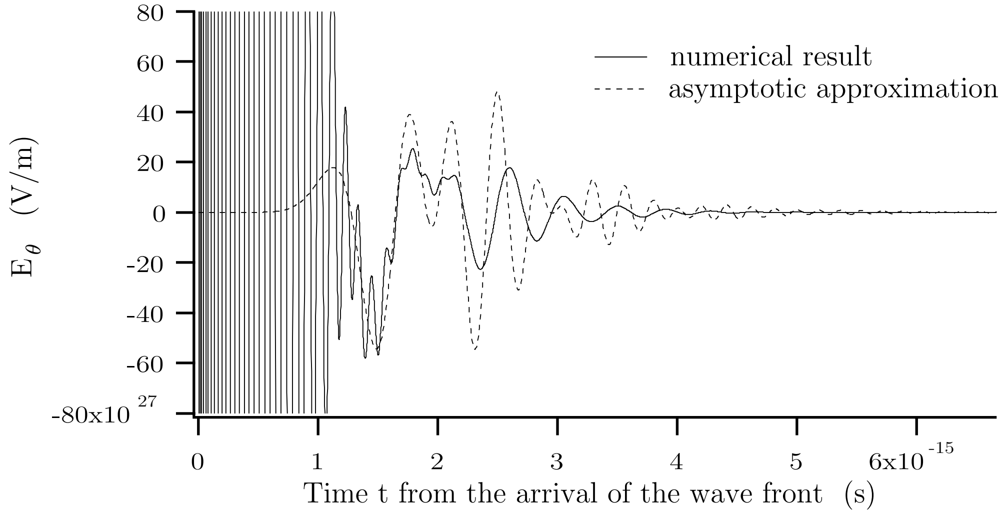

Figure 7: The field E_θ from a dipole in a chiral medium at a distance $r = 10^{-6}$ m from the dipole at the observation angle $\theta = \pi/4$. For material parameters see caption of Figure 1.

where $\nabla_S\cdot$ is the surface divergence [1, 9]. The surface S does not need to be closed due to the bounded support of the integrand. Equation (7.1) together with (4.3) lead to the following form of the surface integral on the right-hand side of (6.3):

$$\begin{aligned}
&\int_S\int \mathbf{G}_Q(\boldsymbol{r}-\boldsymbol{r}';t-t')\cdot\big(\boldsymbol{u}_n'\times\boldsymbol{Q}_-(\boldsymbol{r}',t')\big)dt'dS' \\
&=\mathcal{Z}\mathcal{N}\mu^{-1}\Big\{-\frac{1}{2}\int_S\int \mathcal{E}(\boldsymbol{r}-\boldsymbol{r}';t-t')\boldsymbol{u}_n'\times\boldsymbol{Q}_-(\boldsymbol{r}',t')dt'dS' \\
&\qquad+\frac{1}{2}\nabla\int_S\int\big[\mathcal{K}^{-2}\mathcal{E}\big]\,(\boldsymbol{r}-\boldsymbol{r}';t-t')\nabla_S'\cdot\big(\boldsymbol{u}_n'\times\boldsymbol{Q}_-(\boldsymbol{r}',t')\big)dt'dS' \\
&\qquad-\frac{i}{2}\nabla\times\int_S\int\big[\mathcal{K}^{-1}\mathcal{E}\big]\,(\boldsymbol{r}-\boldsymbol{r}';t-t')\,\big(\boldsymbol{u}_n'\times\boldsymbol{Q}_-(\boldsymbol{r}',t')\big)dt'dS'\Big\},\quad \boldsymbol{r}\notin S.
\end{aligned}\tag{7.2}$$

In the limit $\boldsymbol{r}\to S_\pm$ (*i.e.*, $V_\pm \ni \boldsymbol{r}\to S$), the representation (6.3) transforms into the surface integral *relation* for the complex field $\boldsymbol{Q}_-(\boldsymbol{r},t)$. Using (7.2), (6.3), and the jump relations [4]

$$\begin{aligned}
&\nabla\int_S\int \mathcal{E}(\boldsymbol{r}-\boldsymbol{r}';t-t')f(\boldsymbol{r}',t')dt'\,dS' \\
&\qquad=\int_S\int \nabla\mathcal{E}(\boldsymbol{r}-\boldsymbol{r}';t-t')f(\boldsymbol{r}',t')dt'\,dS'\pm\frac{1}{2}\boldsymbol{u}_n f(\boldsymbol{r},t),\qquad \boldsymbol{r}\to S_\pm, \\
&\nabla\times\int_S\int \mathcal{E}(\boldsymbol{r}-\boldsymbol{r}';t-t')\boldsymbol{F}(\boldsymbol{r}',t')dt'\,dS' \\
&\qquad=\int_S\int (\nabla\mathcal{E}(\boldsymbol{r}-\boldsymbol{r}';t-t'))\times\boldsymbol{F}(\boldsymbol{r}',t')dt'\,dS'\pm\frac{1}{2}\boldsymbol{u}_n\times\boldsymbol{F}(\boldsymbol{r},t),\qquad \boldsymbol{r}\to S_\pm,
\end{aligned}$$

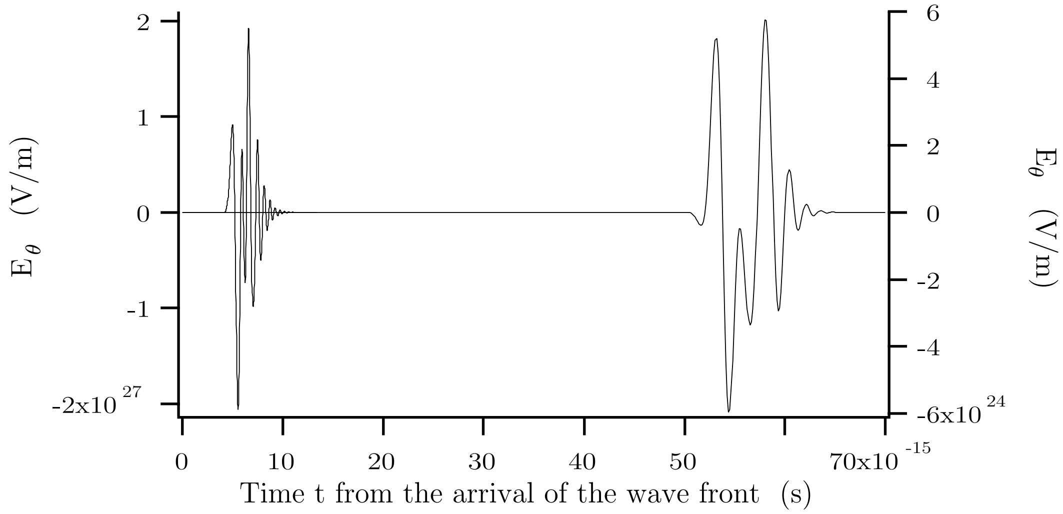

Figure 8: The θ-component of the field $\boldsymbol{E}$ from a dipole $\boldsymbol{u}_z 10^{-17}\delta(t)\,\mathrm{C\,m}$ in a chiral medium at distances $r = 4\cdot 10^{-6}\,\mathrm{m}$ and $r = 4\cdot 10^{-5}\,\mathrm{m}$ from the source. For material parameters see caption of Figure 1.

which are valid for any sufficiently regular scalar field $f(\boldsymbol{r},t)$ and vector field $\boldsymbol{F}(\boldsymbol{r},t)$, give for $\boldsymbol{r}\in S$

$$\left.\begin{matrix}\frac{ic_0}{2}\partial_t^{-1}\left[\mathcal{Z}\mu^{-1}\boldsymbol{Q}_-\right](\boldsymbol{r},t)\\ \mathbf{0}\end{matrix}\right\} = \frac{ic_0}{2}\partial_t^{-1}\left[\mathcal{Z}\mu^{-1}\boldsymbol{Q}_{i-}\right](\boldsymbol{r},t)$$
$$+\int_S\int \mathbf{G}_Q(\boldsymbol{r}-\boldsymbol{r}';t-t')\cdot\big(\boldsymbol{u}_n'\times\boldsymbol{Q}_-(\boldsymbol{r}',t')\big)\,dt'\,dS'$$
$$\pm\,\mathcal{Z}\mathcal{N}\mu^{-1}\Big\{\frac{1}{4}\boldsymbol{u}_n\nabla_S\cdot\left(\boldsymbol{u}_n\times\left[\mathcal{K}^{-2}\boldsymbol{Q}_-\right](\boldsymbol{r},t)\right)-\frac{i}{4}\boldsymbol{u}_n\times\left(\boldsymbol{u}_n\times\left[\mathcal{K}^{-1}\boldsymbol{Q}_-\right](\boldsymbol{r},t)\right)\Big\}. \tag{7.3}$$

The surface integral on the right-hand side of (7.3) is interpreted as

$$\int_S\int \mathbf{G}_Q(\boldsymbol{r}-\boldsymbol{r}';t-t')\cdot\big(\boldsymbol{u}_n'\times\boldsymbol{Q}_-(\boldsymbol{r}',t')\big)dt'dS'$$
$$=\mathcal{Z}\mathcal{N}\mu^{-1}\Big\{-\frac{1}{2}\int_S\int \mathcal{E}(\boldsymbol{r}-\boldsymbol{r}';t-t')\boldsymbol{u}_n'\times\boldsymbol{Q}_-(\boldsymbol{r}',t')dt'dS'$$
$$+\frac{1}{2}\int_S\int \nabla\left(\left[\mathcal{K}^{-2}\mathcal{E}\right](\boldsymbol{r}-\boldsymbol{r}';t-t')\right)\nabla_S'\cdot\big(\boldsymbol{u}_n'\times\boldsymbol{Q}_-(\boldsymbol{r}',t')\big)dt'dS'$$
$$-\frac{i}{2}\int_S\int \nabla\left(\left[\mathcal{K}^{-1}\mathcal{E}\right](\boldsymbol{r}-\boldsymbol{r}';t-t')\right)\times\big(\boldsymbol{u}_n'\times\boldsymbol{Q}_-(\boldsymbol{r}',t')\big)dt'dS'\Big\},\quad \boldsymbol{r}\in S, \tag{7.4}$$

where the integrals exist as principal value integrals. Using the Maxwell equations (3.3) and the fact that $\nabla_S\cdot(\boldsymbol{u}_n\times\boldsymbol{Q}_-) = -\boldsymbol{u}_n\cdot(\nabla\times\boldsymbol{Q}_-)$, both equations (7.3) reduce to

$$
\begin{aligned}
\frac{ic_0}{4}\partial_t^{-1}\left[\mathcal{Z}\mu^{-1}\boldsymbol{Q}_-\right](\boldsymbol{r},t) = & \frac{ic_0}{2}\partial_t^{-1}\left[\mathcal{Z}\mu^{-1}\boldsymbol{Q}_{i-}\right](\boldsymbol{r},t) \\
& + \int_S\int \mathbf{G}_Q(\boldsymbol{r}-\boldsymbol{r}';t-t')\cdot\left(\boldsymbol{u}_n'\times\boldsymbol{Q}_-(\boldsymbol{r}',t')\right)dt'\,dS', \qquad \boldsymbol{r}\in S,
\end{aligned}
\tag{7.5}
$$

where the surface integral term is given by (7.4).

The integral relation based on the equation (6.4) can be derived in the same way. The result is

$$
\begin{aligned}
\frac{ic_0}{4}\partial_t^{-1}\left[\mathcal{Z}\mu^{-1}\boldsymbol{Q}_+\right](\boldsymbol{r},t) = & \frac{ic_0}{2}\partial_t^{-1}\left[\mathcal{Z}\mu^{-1}\boldsymbol{Q}_{i+}\right](\boldsymbol{r},t) \\
& - \int_S\int \mathbf{G}_Q(\boldsymbol{r}-\boldsymbol{r}';t-t')\cdot\left(\boldsymbol{u}_n'\times\boldsymbol{Q}_+(\boldsymbol{r}',t')\right)dt'\,dS', \qquad \boldsymbol{r}\in S.
\end{aligned}
\tag{7.6}
$$

To get further in solving the scattering problem, boundary conditions on the surface S (*i.e.*, the connection between $\boldsymbol{Q}_+(\boldsymbol{r},t)$ and $\boldsymbol{Q}_-(\boldsymbol{r},t)$, for $\boldsymbol{r}\in S$) have to be specified. In the next two subsections, two standard scattering problems are discussed.

7.1 Perfectly conducting scatterer

In this subsection, V_- is a perfect conductor and V_+ a temporally dispersive bi-isotropic medium. The boundary condition on the surface S is $\boldsymbol{u}_n\times\boldsymbol{E}=\boldsymbol{0}$. In terms of the complex field $\boldsymbol{Q}(\boldsymbol{r},t)$, it becomes $\boldsymbol{u}_n\times[\boldsymbol{Q}+\boldsymbol{Q}^*](\boldsymbol{r},t)=\boldsymbol{0}$, $\boldsymbol{r}\in S$. Taking the cross product of both members in (7.6) with $\boldsymbol{u}_n$ and using the boundary condition give the following integral equation for the surface current density $\boldsymbol{J}_S^{\mathrm{e}}(\boldsymbol{r},t):=\boldsymbol{u}_n\times\boldsymbol{H}(\boldsymbol{r},t)=\boldsymbol{u}_n\times[i\mathcal{Y}\boldsymbol{Q}-i\mathcal{Y}^*\boldsymbol{Q}^*](\boldsymbol{r},t)/\eta_0$, $\boldsymbol{r}\in S$:

$$
\boldsymbol{J}_S^{\mathrm{e}}(\boldsymbol{r},t)=\frac{i4}{\eta_0}\mathcal{Z}^{-1}\Big(\boldsymbol{u}_n\times\boldsymbol{Q}_{i+}(\boldsymbol{r},t)+\boldsymbol{u}_n\times\int_S\int\mathbf{G}_Q(\boldsymbol{r}-\boldsymbol{r}';t-t')\cdot\mu_0\partial_{t'}\left[\mu\boldsymbol{J}_S^{\mathrm{e}}\right](\boldsymbol{r}',t')dt'\,dS'\Big).
\tag{7.7}
$$

Note that the equation above has exactly the same form as the one in the isotropic case [4]. The difference is in a more complicated structure of the Green dyadic $\mathbf{G}_Q$. (Recall that the refractive index $\mathcal{N}$, the wave number $\mathcal{K}$, and the fundamental solution $\mathcal{E}$ in (4.3) are all complex.) Separating (7.7) into its real and imaginary parts gives two alternative integral equations for the surface current density, the first being of the second kind and the second of the first kind. Unfortunately, both of them contain the surface divergence $\nabla_S\cdot\boldsymbol{J}_S^{\mathrm{e}}$ of the unknown field $\boldsymbol{J}_S^{\mathrm{e}}$ (*cf.* (7.4)). This makes numerical treatment of these equations unattractive. Moving the derivative from the surface field to the $\nabla\mathcal{E}$-term (integration by parts) does not reduce this inconvenience because then the highly singular second space derivatives of the kernel $\mathcal{E}(\boldsymbol{r}-\boldsymbol{r}';t-t')$ have to be dealt with. However, it is possible to combine these equations to obtain an integral equation which does not possess the mentioned shortcomings. Applying the operator $\mathcal{K}$ to the both sides of (7.7) and using (4.3) gives

$$\begin{aligned}\mathcal{K}\boldsymbol{J}^{\mathrm{e}}_S(\boldsymbol{r},t) =& \frac{i4}{\eta_0}\mathcal{Z}^{-1}\mathcal{K}(\boldsymbol{u}_n\times\boldsymbol{Q}_{i+}(\boldsymbol{r},t)) + i2\boldsymbol{u}_n\times\int_S\int\nabla\mathcal{E}(\boldsymbol{r}-\boldsymbol{r}';t-t')\nabla'\cdot\boldsymbol{J}^{\mathrm{e}}_S(\boldsymbol{r}',t')dt'\,dS' \\ & - i2\boldsymbol{u}_n\times\int_S\int(\mathbf{I}\mathcal{K}^2+i\nabla\times\mathbf{I}\mathcal{K})\mathcal{E}(\boldsymbol{r}-\boldsymbol{r}';t-t')\cdot\boldsymbol{J}^{\mathrm{e}}_S(\boldsymbol{r}',t')dt'\,dS'. \end{aligned} \tag{7.8}$$

Now, taking the real part of the both sides leads to the following integral equation of the second kind for the surface current density:

$$\begin{aligned}\mathcal{K}_{\mathrm{co}}\boldsymbol{J}^{\mathrm{e}}_S(\boldsymbol{r},t) =& -\frac{4}{\eta_0}\mathcal{Z}^{-1}\mathrm{Im}\left\{\mathcal{K}(\boldsymbol{u}_n\times\boldsymbol{Q}_{i+}(\boldsymbol{r},t))\right\} \\ & -2\boldsymbol{u}_n\times\int_S\int\nabla\mathcal{E}_{\mathrm{cr}}(\boldsymbol{r}-\boldsymbol{r}';t-t')\nabla'\cdot\boldsymbol{J}^{\mathrm{e}}_S(\boldsymbol{r}',t')dt'\,dS' \\ & +2\boldsymbol{u}_n\times\int_S\int\mathrm{Im}\left\{(\mathbf{I}\mathcal{K}^2+i\nabla\times\mathbf{I}\mathcal{K})\mathcal{E}(\boldsymbol{r}-\boldsymbol{r}';t-t')\right\}\cdot\boldsymbol{J}^{\mathrm{e}}_S(\boldsymbol{r}',t')dt'\,dS', \end{aligned} \tag{7.9}$$

where $\mathcal{K}=\mathcal{K}_{\mathrm{co}}+i\mathcal{K}_{\mathrm{cr}}$, with the similar notation for the other complex quantities. All integrals in the equation above exist as principle value integrals. From (4.4) it follows that

$$\mathcal{E}_{\mathrm{cr}}(\boldsymbol{r};t)=\frac{1}{4\pi r}\left(q_{\mathrm{cr}}(r)\delta\left(t-\frac{r}{c_0}\right)+\mathrm{Im}\left\{q(r)P\left(r;t-\frac{r}{c_0}\right)\right\}\right).$$

Representation (4.6) and the equality (*cf.* (4.5))

$$q_{\mathrm{cr}}(r)=\exp\left(-\frac{r}{c_0}N_{\mathrm{co}}(0+)\right)\sin\left(-\frac{r}{c_0}N_{\mathrm{cr}}(0+)\right),$$

show that $\mathcal{E}_{\mathrm{cr}}$ has no singularity and $\nabla\nabla\mathcal{E}_{\mathrm{cr}}(\boldsymbol{r};t)$ has at most $1/r$ singularity at the origin. Now, performing integration by parts in the second term on the right-hand side of (7.9) gives

$$\begin{aligned}\mathcal{K}_{\mathrm{co}}\boldsymbol{J}^{\mathrm{e}}_S(\boldsymbol{r},t) =& -\frac{4}{\eta_0}\mathcal{Z}^{-1}\mathrm{Im}\left\{\mathcal{K}(\boldsymbol{u}_n\times\boldsymbol{Q}_{i+}(\boldsymbol{r},t))\right\} \\ & -2\boldsymbol{u}_n\times\int_S\int\nabla\nabla\mathcal{E}_{\mathrm{cr}}(\boldsymbol{r}-\boldsymbol{r}';t-t')\cdot\boldsymbol{J}^{\mathrm{e}}_S(\boldsymbol{r}',t')dt'\,dS' \\ & +2\boldsymbol{u}_n\times\int_S\int\mathrm{Im}\left\{(\mathbf{I}\mathcal{K}^2+i\nabla\times\mathbf{I}\mathcal{K})\mathcal{E}(\boldsymbol{r}-\boldsymbol{r}';t-t')\right\}\cdot\boldsymbol{J}^{\mathrm{e}}_S(\boldsymbol{r}',t')dt'\,dS'. \end{aligned} \tag{7.10}$$

This equation can be used in numerical calculations.

7.2 Permeable scatterer

In this subsection, the surface S is supposed to be an interface between two different temporally dispersive materials. To distinguish the two sets of parameters, the intrinsic integral operators $\mathcal{N}$, $\mathcal{Z}$, ε, and μ as well as the dispersive fundamental solutions $\mathcal{E}(\boldsymbol{r};t)$ and the Green dyadics $\mathbf{G}_Q(\boldsymbol{r};t)$ in the domains $V_\pm$ are endowed with the subscripts $\pm$, respectively. The boundary conditions on the surface S are

$$\begin{aligned}
&\boldsymbol{u}_n\times(\boldsymbol{Q}_+(\boldsymbol{r},t)+\boldsymbol{Q}^*_+(\boldsymbol{r},t)) = \boldsymbol{u}_n\times(\boldsymbol{Q}_-(\boldsymbol{r},t)+\boldsymbol{Q}^*_-(\boldsymbol{r},t)) =: \boldsymbol{J}^m_S(\boldsymbol{r},t)\\
&\boldsymbol{u}_n\times(i\left[\mathcal{Y}_+\boldsymbol{Q}_+\right](\boldsymbol{r},t)-i\left[\mathcal{Y}^*_+(\boldsymbol{r},t)\boldsymbol{Q}^*_+\right](\boldsymbol{r},t))/\eta_0\\
&\qquad = \boldsymbol{u}_n\times(i\left[\mathcal{Y}_-\boldsymbol{Q}_-\right](\boldsymbol{r},t)-i\left[\mathcal{Y}^*_-(\boldsymbol{r},t)\boldsymbol{Q}^*_-\right](\boldsymbol{r},t))/\eta_0 =: \boldsymbol{J}^e_S(\boldsymbol{r},t).
\end{aligned}$$

Taking the cross product of the left- and right-hand sides of equations (7.5)–(7.6) with $\boldsymbol{u}_n$ and using the boundary conditions give the following integral equations for the surface fields

$$\begin{aligned}
&\frac{ic_0}{8}\partial_t^{-1}\left[\mathcal{Z}^2_\pm\mu^{-1}_\pm(\mathcal{Y}^*_\pm\boldsymbol{J}^m_S-i\eta_0\boldsymbol{J}^e_S)\right](\boldsymbol{r},t)=\frac{ic_0}{2}\partial_t^{-1}\left[\mathcal{Z}_\pm\mu^{-1}_\pm(\boldsymbol{u}_n\times\boldsymbol{Q}_{i\pm})\right](\boldsymbol{r},t)\\
&\qquad\mp\frac{1}{2}\boldsymbol{u}_n\times\int_S\int\mathbf{G}_{Q\pm}(\boldsymbol{r}-\boldsymbol{r}';t-t')\cdot\left[\mathcal{Z}_\pm(\mathcal{Y}^*_\pm\boldsymbol{J}^m_S-i\eta_0\boldsymbol{J}^e_S)\right](\boldsymbol{r}',t')\,dt'\,dS',
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
&\left[\mathcal{K}_\pm(\mathcal{Y}^*_\pm\boldsymbol{J}^m_S-i\eta_0\boldsymbol{J}^e_S)\right](\boldsymbol{r},t)=4\left[\mathcal{K}_\pm\mathcal{Z}^{-1}_\pm(\boldsymbol{u}_n\times\boldsymbol{Q}_{i\pm})\right](\boldsymbol{r},t)\\
&\quad\pm2i\boldsymbol{u}_n\times\int_S\int\nabla\mathcal{E}_\pm(\boldsymbol{r}-\boldsymbol{r}';t-t')\nabla'\cdot\left[(\mathcal{Y}^*_\pm\boldsymbol{J}^m_S-i\eta_0\boldsymbol{J}^e_S)\right](\boldsymbol{r}',t')\,dt'\,dS' \qquad (7.11)\\
&\quad\mp2i\boldsymbol{u}_n\times\int_S\int\left[(\mathbf{I}\mathcal{K}^2_\pm+i\nabla\times\mathbf{I}\mathcal{K}_\pm)\mathcal{E}_\pm\right](\boldsymbol{r}-\boldsymbol{r}';t-t')\cdot\left[(\mathcal{Y}^*_\pm\boldsymbol{J}^m_S-i\eta_0\boldsymbol{J}^e_S)\right](\boldsymbol{r}',t')\,dt'\,dS'.
\end{aligned}$$

The equations above can be used to calculate the unknown surface fields $\boldsymbol{J}^e_S(\boldsymbol{r},t)$ and $\boldsymbol{J}^m_S(\boldsymbol{r},t)$. Unfortunately, they suffer from the same problem as the equation (7.7) — the integral kernel $\nabla\nabla\mathcal{E}$ (or, to be more exact, its real part $\nabla\nabla\mathcal{E}_{\rm co}$), which appears in the second term on the right-hand side after integrating by parts, is too singular. However, in case when the materials in the domains V_- and V_+ have the same value of $N(0+)$ ($N_+(0+) = N_-(0+)$), the equations above can be combined to obtain a system of integral equations for $\boldsymbol{J}^e_S(\boldsymbol{r},t)$ and $\boldsymbol{J}^m_S(\boldsymbol{r},t)$ which does not contain highly singular kernels. (Observe that a similar condition was needed in Refs. 2, 4.) To achieve this, apply the operator $\mathcal{Y}^{-1}_{{\rm co}\pm}$ to the imaginary parts of the equations (7.11) and add the results to get

$$\begin{aligned}
&\left[\mathrm{Im}\left\{\mathcal{K}_+\mathcal{Y}^*_+\mathcal{Y}^{-1}_{\rm co+}+\mathcal{K}_-\mathcal{Y}^*_-\mathcal{Y}^{-1}_{\rm co-}\right\}\boldsymbol{J}^m_S-\mathrm{Re}\left\{\mathcal{K}_+\mathcal{Y}^{-1}_{\rm co+}+\mathcal{K}_-\mathcal{Y}^{-1}_{\rm co-}\right\}\eta_0\boldsymbol{J}^e_S\right](\boldsymbol{r},t)\\
&\quad=4\mathrm{Im}\left\{\left[\mathcal{K}_+\mathcal{Z}^{-1}_+\mathcal{Y}^{-1}_{\rm co+}(\boldsymbol{u}_n\times\boldsymbol{Q}_{i+})+\mathcal{K}_-\mathcal{Z}^{-1}_-\mathcal{Y}^{-1}_{\rm co-}(\boldsymbol{u}_n\times\boldsymbol{Q}_{i-})\right](\boldsymbol{r},t)\right\}\\
&\quad+2\boldsymbol{u}_n\times\int_S\int\nabla\nabla[\mathcal{E}_{\rm co+}-\mathcal{E}_{\rm co-}](\boldsymbol{r}-\boldsymbol{r}';t-t')\cdot\boldsymbol{J}^m_S(\boldsymbol{r}',t')\,dt'\,dS' \qquad (7.12)\\
&\quad+2\boldsymbol{u}_n\times\int_S\int\nabla\nabla\left[\mathcal{Y}^{-1}_{\rm co+}\mathcal{Y}_{\rm cr+}\mathcal{E}_{\rm cr+}-\mathcal{Y}^{-1}_{\rm co-}\mathcal{Y}_{\rm cr-}\mathcal{E}_{\rm cr-}\right](\boldsymbol{r}-\boldsymbol{r}';t-t')\cdot\boldsymbol{J}^m_S(\boldsymbol{r}',t')\,dt'\,dS'\\
&\quad+2\boldsymbol{u}_n\times\int_S\int\nabla\nabla\left[\mathcal{Y}^{-1}_{\rm co+}\mathcal{E}_{\rm cr+}-\mathcal{Y}^{-1}_{\rm co-}\mathcal{E}_{\rm cr-}\right](\boldsymbol{r}-\boldsymbol{r}';t-t')\cdot\eta_0\boldsymbol{J}^e_S(\boldsymbol{r}',t')\,dt'\,dS'\\
&\quad-2\boldsymbol{u}_n\times\int_S\int\mathrm{Re}\Big\{\left[(\mathbf{I}\mathcal{K}^2_++i\nabla\times\mathbf{I}\mathcal{K}_+)\mathcal{Y}^{-1}_{\rm co+}\mathcal{E}_+\right](\boldsymbol{r}-\boldsymbol{r}';t-t')\cdot\left[(\mathcal{Y}^*_+\boldsymbol{J}^m_S-i\eta_0\boldsymbol{J}^e_S)\right](\boldsymbol{r}',t')\\
&\qquad-\left[(\mathbf{I}\mathcal{K}^2_-+i\nabla\times\mathbf{I}\mathcal{K}_-)\mathcal{Y}^{-1}_{\rm co-}\mathcal{E}_-\right](\boldsymbol{r}-\boldsymbol{r}';t-t')\cdot\left[(\mathcal{Y}^*_-\boldsymbol{J}^m_S-i\eta_0\boldsymbol{J}^e_S)\right](\boldsymbol{r}',t')\Big\}\,dt'\,dS',
\end{aligned}$$

where the integration by parts has been performed. Similarly, applying the operator $((\mathcal{Y}_\pm^{-1})_{\rm co})^{-1}(\mathcal{Y}_\pm^{-1})^*$ to the real parts of the equations (7.11) and adding the results give

$$
\begin{aligned}
&\big[\mathrm{Re}\left\{\mathcal{K}_+((\mathcal{Y}_+^{-1})_{\rm co})^{-1}+\mathcal{K}_-((\mathcal{Y}_-^{-1})_{\rm co})^{-1}\right\}\boldsymbol{J}_S^m\\
&\qquad+\mathrm{Im}\left\{\mathcal{K}_+(\mathcal{Y}_+^{-1})^*((\mathcal{Y}_+^{-1})_{\rm co})^{-1}+\mathcal{K}_-(\mathcal{Y}_-^{-1})^*((\mathcal{Y}_-^{-1})_{\rm co})^{-1}\right\}\eta_0\boldsymbol{J}_S^e\big](\boldsymbol{r},t)\\
&\quad=4\mathrm{Re}\big\{\big[\mathcal{K}_+\mathcal{Z}_+^{-1}(\mathcal{Y}_+^{-1})^*((\mathcal{Y}_+^{-1})_{\rm co})^{-1}(\boldsymbol{u}_n\times\boldsymbol{Q}_{i+})\\
&\qquad\qquad+\mathcal{K}_-\mathcal{Z}_-^{-1}(\mathcal{Y}_-^{-1})^*((\mathcal{Y}_-^{-1})_{\rm co})^{-1}(\boldsymbol{u}_n\times\boldsymbol{Q}_{i-})\big](\boldsymbol{r},t)\big\}\\
&\quad+2\boldsymbol{u}_n\times\int_S\int\nabla\nabla[\mathcal{E}_{\rm co+}-\mathcal{E}_{\rm co-}](\boldsymbol{r}-\boldsymbol{r}';t-t')\cdot\eta_0\boldsymbol{J}_S^e(\boldsymbol{r}',t')\,dt'\,dS'\\
&\quad+2\boldsymbol{u}_n\times\int_S\int\nabla\nabla\big[((\mathcal{Y}_+^{-1})_{\rm co})^{-1}(\mathcal{Y}_+^{-1})_{\rm cr}\mathcal{E}_{\rm cr+}\\
&\qquad\qquad-((\mathcal{Y}_-^{-1})_{\rm co})^{-1}(\mathcal{Y}_-^{-1})_{\rm cr}\mathcal{E}_{\rm cr-}\big](\boldsymbol{r}-\boldsymbol{r}';t-t')\cdot\eta_0\boldsymbol{J}_S^e(\boldsymbol{r}',t')\,dt'\,dS'\\
&\quad-2\boldsymbol{u}_n\times\int_S\int\nabla\nabla\big[((\mathcal{Y}_+^{-1})_{\rm co})^{-1}\mathcal{E}_{\rm cr+}-((\mathcal{Y}_-^{-1})_{\rm co})^{-1}\mathcal{E}_{\rm cr-}\big](\boldsymbol{r}-\boldsymbol{r}';t-t')\cdot\boldsymbol{J}_S^m(\boldsymbol{r}',t')\,dt'\,dS'\\
&\quad+2\boldsymbol{u}_n\times\int_S\int\mathrm{Im}\Big\{\big[(\mathbf{I}\mathcal{K}_+^2+i\nabla\times\mathbf{I}\mathcal{K}_+)((\mathcal{Y}_+^{-1})_{\rm co})^{-1}\mathcal{E}_+\big](\boldsymbol{r}-\boldsymbol{r}';t-t')\cdot\\
&\qquad\qquad\qquad\big[(\boldsymbol{J}_S^m-i(\mathcal{Y}_+^{-1})^*\eta_0\boldsymbol{J}_S^e)\big](\boldsymbol{r}',t')\\
&\qquad\qquad-\big[(\mathbf{I}\mathcal{K}_-^2+i\nabla\times\mathbf{I}\mathcal{K}_-)((\mathcal{Y}_-^{-1})_{\rm co})^{-1}\mathcal{E}_-\big](\boldsymbol{r}-\boldsymbol{r}';t-t')\cdot\\
&\qquad\qquad\qquad\big[(\boldsymbol{J}_S^m-i(\mathcal{Y}_-^{-1})^*\eta_0\boldsymbol{J}_S^e)\big](\boldsymbol{r}',t')\Big\}\,dt'\,dS'.
\end{aligned}
\tag{7.13}
$$

From the results in the previous subsection it follows that the integral kernels $\nabla\nabla\mathcal{E}_{\rm cr\pm}$ have at most $1/r$ singularity at the origin. Furthermore, due to the assumption that $N_+(0+)=N_-(0+)$, the singularity of the kernel $\nabla\nabla(\mathcal{E}_{\rm co+}-\mathcal{E}_{\rm co-})$ can be estimated by $1/r$ as well (*cf.* Ref. 4). Thus, equations (7.12)–(7.13) build a system of integral equation for the fields $\boldsymbol{J}_S^e(\boldsymbol{r},t)$ and $\boldsymbol{J}_S^m(\boldsymbol{r},t)$ which can be used in numerical calculations.

8 Conclusion

In this paper, the Green dyadics for temporally dispersive bi-isotropic media are analyzed using time-domain techniques.

The use of the complex time-dependent electromagnetic field simplifies the analysis significantly. Advantages are more evident here then in the case of isotropic materials [4]. This depends on the fact that the electric and the magnetic fields in bi-isotropic media are coupled in a more intricate way.

The Green dyadics for bi-isotropic materials are introduced and given in an explicit form using Schwartz' pseudo-functions. The derivation of the equation for the Green dyadics differs slightly from the one used in Ref. 4. No electromagnetic potentials are needed in the present work.

The example of Section 5 shows that the second forerunner approximation to the dipole fields in an unbounded, temporally dispersive, bi-isotropic medium gives reasonably good results. It is also seen that the agreement between the numerical result and the approximation becomes better with increasing distance from the source. The main advantage of the proposed technique is that no time- or memory-consuming computations are involved. If it takes hours to compute the numerical values of the fields, it takes only seconds to obtain the approximation. Obviously, the introduced method can be used to obtain fields due to other time-dependent sources (antennas, *etc*).

The surface integral equations derived in Section 7 have reasonably regular integral kernels, and it is conjectured that they can be solved numerically with the help of the standard techniques (*e.g.*, the method of moments). Note that after the surface fields are obtained, all quantities on the right-hand sides of (6.3)–(6.4) are known and these integral relations can be used to calculate the fields at any point.

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