



LUND UNIVERSITY

Estimates of scattered electromagnetic fields

Wellander, Niklas; Kristensson, Gerhard

2013

[Link to publication](#)

Citation for published version (APA):

Wellander, N., & Kristensson, G. (2013). *Estimates of scattered electromagnetic fields*. (Technical Report LUTEDX/(TEAT-7226)/1-29/(2013); Vol. TEAT-7226). The Department of Electrical and Information Technology.

Total number of authors:

2

General rights

Unless other specific re-use rights are stated the following general rights apply:

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Read more about Creative commons licenses: <https://creativecommons.org/licenses/>

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

CODEN:LUTEDX/(TEAT-7226)/1-29/(2013)

Revision No. 1: August 2014

Estimates of scattered electromagnetic fields

Niklas Wellander and Gerhard Kristensson

Electromagnetic Theory
Department of Electrical and Information Technology
Lund University
Sweden



Niklas Wellander
niklas.wellander@foi.se

Swedish Defence Research Agency, FOI
P.O. Box 1165
SE-581 11 Linköping
Sweden

and

Department of Electrical and Information Technology
Electromagnetic Theory
Lund University
P.O. Box 118
SE-221 00 Lund
Sweden

Gerhard Kristensson
Gerhard.Kristensson@eit.lth.se

Department of Electrical and Information Technology
Electromagnetic Theory
Lund University
P.O. Box 118
SE-221 00 Lund
Sweden

Abstract

We present some general estimates of the scattered electromagnetic fields for a general bounded scattering domain Ω in the anisotropic materials setting. In particular, it is shown that the $\|\cdot\|_{L^2(\Omega_s; \mathbb{C}^3)}$ -norm and sup-norm of the scattered field in an arbitrary finite exterior domain Ω_s are bounded by the $H(\text{curl}, \Omega)$ norm of the incident field. Moreover, several estimates of the traces of the scattered field on the boundary and a circumscribing sphere are presented.

1 Introduction

Estimates of the scattered field in terms of the excited (incident) field play a crucial role in controlling how much energy an obstacle scatters. In some applications the aim is to maximize this energy in some specified directions or sectors. In other situations, the objective is to create a scatterer such that the scattered energy is as small as possible — preferably zero. In particular, invisibility devices, so called cloaks, which have attracted quite some attention lately, see *e.g.*, [6, 7], is an application where the goal is to enclose the scatterer such that the scattered field is as small as possible. To be successful in this endeavor, the control of the scattered field is needed.

In Section 2, the scattering problem is formulated in a variational setting and the existence and uniqueness of the solution to the scattering problem for an anisotropic scatterer is proved. Some of this technique is well known, see *e.g.*, Refs [9, 12]. One of the cornerstones in this approach is the coercivity and the *a-priori* estimates of the solution to the problem. Existence and uniqueness of the scattering problem are proved in Section 3, and the estimates of the scattered field are presented in Section 4. The paper is closed with some conclusions in Section 5 followed by a few appendices with definitions and more technical details.

2 Formulation of the scattering problem

In this section, we present the geometry of the problem, the scattered field, and its solution in a weak setting.

2.1 The interior problem with anisotropic material

Let Ω be a bounded, open simply connected set in \mathbb{R}^3 with $C^{1,1}$ boundary, $\partial\Omega$.¹ The outward pointing unit normal is denoted by $\hat{\nu}$. We denote the exterior of the domain Ω by $\Omega_e = \mathbb{R}^3 \setminus \overline{\Omega}$, which is assumed to be vacuous and simply connected. See Figure 1 for a typical geometry.

¹Several of the results presented in this paper holds also for Lipschitz boundaries.

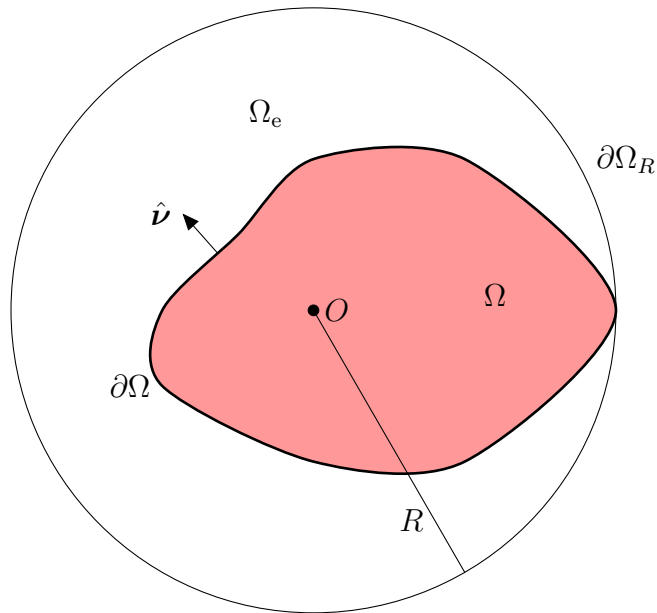


Figure 1: Typical geometry of the scattering problem in this paper. The domain Ω , its boundary $\partial\Omega$, and the least circumscribing sphere $\partial\Omega_R$ are shown.

The Maxwell equations in the anisotropic case are given by² (we adopt the time convention $e^{-i\omega t}$)

$$\begin{cases} \nabla \times \mathbf{E}(\mathbf{x}) = ik_0 \boldsymbol{\mu}(\mathbf{x}) \cdot \mathbf{H}(\mathbf{x}) \\ \nabla \times \mathbf{H}(\mathbf{x}) = -ik_0 \boldsymbol{\epsilon}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}) \end{cases} \quad \mathbf{x} \in \Omega \quad (2.1)$$

The wave number in vacuum is $k_0 = \omega/c_0$, where ω is the angular frequency of the fields, and c_0 is the speed of light in vacuum. The dimensionless entries of $\boldsymbol{\epsilon}(\mathbf{x})$ and $\boldsymbol{\mu}(\mathbf{x})$ are elements in $L^\infty(\Omega; \mathbb{C})$. Throughout this paper vectors in \mathbb{C}^3 are typed in italic boldface, and linear transformations (dyadics) on \mathbb{C}^3 in roman boldface. Unit vectors are denoted by a caret ($\hat{\cdot}$).

We are looking for solutions \mathbf{E} and \mathbf{H} of (2.1) in the space $H(\text{curl}, \Omega)$, see Appendix A for details and definitions of function spaces. A weak formulation of this problem is given in Section 3. The boundary conditions on $\partial\Omega$, that connect the solutions in the domains Ω and Ω_e , are introduced in the formulation of the exterior problem, and the pertinent radiation conditions presented in Section 2.3.

The incident field, \mathbf{E}_i or \mathbf{H}_i , is assumed to have its sources in a bounded region $\Omega_i \subset \Omega_e$, *i.e.*, $\Omega \cap \Omega_i = \emptyset$. Outside this region, the fields satisfy the time-harmonic Maxwell equations in vacuum, and they are assumed to have traces on

²We use scaled electric and magnetic fields in this paper, *i.e.*, the SI-unit fields \mathbf{E}_{SI} and \mathbf{H}_{SI} are related to the fields \mathbf{E} and \mathbf{H} used in this paper by

$$\mathbf{E}_{\text{SI}}(\mathbf{x}) = \frac{\mathbf{E}(\mathbf{x})}{\sqrt{\epsilon_0}}, \quad \mathbf{H}_{\text{SI}}(\mathbf{x}) = \frac{\mathbf{H}(\mathbf{x})}{\sqrt{\mu_0}}$$

where the permittivity and permeability of vacuum are denoted by ϵ_0 and μ_0 , respectively.

$\partial\Omega$ belonging to $H^{-1/2}(\text{div}, \partial\Omega)$, more precisely $(\boldsymbol{\gamma}(\mathbf{E}_i), \boldsymbol{\gamma}(\mathbf{H}_i)) \in H^{-1/2}(\text{div}, \partial\Omega) \times H^{-1/2}(\text{div}, \partial\Omega)$, see Appendix A. Otherwise, the incident fields are arbitrary.

The material properties, for the given fixed frequency $k_0 > 0$, are assumed to be bounded and passive, *i.e.*, $\boldsymbol{\epsilon}, \boldsymbol{\mu} \in L^\infty(\Omega; \mathbb{C}^{3 \times 3})$ and the material properties are assumed to satisfy the coercivity condition

$$\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}^\dagger \cdot \begin{pmatrix} -ik_0(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^\dagger) & \mathbf{0} \\ \mathbf{0} & -ik_0(\boldsymbol{\mu} - \boldsymbol{\mu}^\dagger) \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix} \geq c_m (|\mathbf{a}|^2 + |\mathbf{b}|^2) \quad (2.2)$$

almost everywhere for all fields $\mathbf{a}, \mathbf{b} \in \mathbb{C}^3$ and a given constant $c_m > 0$, which depends on the frequency. The dagger \dagger denotes the conjugate transpose. This condition is equivalent to the assumption that the material is lossy almost everywhere in Ω .

2.2 The exterior problem

The presence of the material in the domain Ω distorts the incident fields \mathbf{E}_i and \mathbf{H}_i . This distortion is denoted by the scattered fields, \mathbf{E}_s and \mathbf{H}_s , which belong to $H_{\text{loc}}(\text{curl}, \overline{\Omega}_e)$.

In Ω_e , the sum of the incident and the scattered fields is defined as the total field, *i.e.*,

$$\begin{cases} \mathbf{E}_t(\mathbf{x}) = \mathbf{E}_i(\mathbf{x}) + \mathbf{E}_s(\mathbf{x}) \\ \mathbf{H}_t(\mathbf{x}) = \mathbf{H}_i(\mathbf{x}) + \mathbf{H}_s(\mathbf{x}) \end{cases} \quad \mathbf{x} \in \Omega_e$$

The boundary conditions on $\partial\Omega$ are

$$\begin{cases} \boldsymbol{\gamma}_+(\mathbf{E}_i + \mathbf{E}_s) = \boldsymbol{\gamma}_-(\mathbf{E}) \\ \boldsymbol{\gamma}_+(\mathbf{H}_i + \mathbf{H}_s) = \boldsymbol{\gamma}_-(\mathbf{H}) \end{cases} \quad (2.3)$$

where the trace operator from the outside (inside) of Ω is denoted by $\boldsymbol{\gamma}_+$ ($\boldsymbol{\gamma}_-$). The trace operator is given as $\boldsymbol{\gamma}_-(\mathbf{E}) = \hat{\boldsymbol{\nu}} \times \mathbf{E}|_{\partial\Omega}$ if \mathbf{E} is in $C(\overline{\Omega}; \mathbb{C}^3)$, and analogous for the trace from the outside, $\boldsymbol{\gamma}_+$. The traces belong to $H^{-1/2}(\text{div}, \partial\Omega)$, see Theorem A.1.

The exterior Calderón operator or the admittance operator C^e is defined as the mapping of the tangential component of the scattered electric field to the tangential component of the scattered magnetic field on the boundary of Ω . We use the solution of a specific exterior problem to make the definition precise. This solution is found in the following section.

2.3 An auxiliary exterior problem (R)

Consider the following exterior problem where the tangential component of the scattered electric field on the boundary is given by a fixed vector $\mathbf{m} \in H^{-1/2}(\text{div}, \partial\Omega)$,³

$$\begin{aligned}
& 1) (\mathbf{E}_s, \mathbf{H}_s) \in H_{\text{loc}}(\text{curl}, \overline{\Omega}_e) \times H_{\text{loc}}(\text{curl}, \overline{\Omega}_e) \\
& 2) \begin{cases} \nabla \times \mathbf{E}_s(\mathbf{x}) = ik_0 \mathbf{H}_s(\mathbf{x}) \\ \nabla \times \mathbf{H}_s(\mathbf{x}) = -ik_0 \mathbf{E}_s(\mathbf{x}) \end{cases} \quad \mathbf{x} \in \Omega_e \\
& 3) \begin{cases} \hat{\mathbf{x}} \times \mathbf{E}_s(\mathbf{x}) - \mathbf{H}_s(\mathbf{x}) = o(1/x) \\ \text{or} \\ \hat{\mathbf{x}} \times \mathbf{H}_s(\mathbf{x}) + \mathbf{E}_s(\mathbf{x}) = o(1/x) \end{cases} \quad \text{as } x \rightarrow \infty \quad (\text{Problem(R)}) \quad (2.4) \\
& 4) \gamma_+(\mathbf{E}_s) = \mathbf{m} \in H^{-1/2}(\text{div}, \partial\Omega)
\end{aligned}$$

This problem has a unique solution [1, 3].

Definition 2.1. *The exterior Calderón operator C^e is defined as*

$$C^e : \mathbf{m} \mapsto \gamma_+(\mathbf{H}_s), \quad H^{-1/2}(\text{div}, \partial\Omega) \mapsto H^{-1/2}(\text{div}, \partial\Omega),$$

where $\mathbf{m} = \gamma_+(\mathbf{E}_s)$ and the fields \mathbf{E}_s and \mathbf{H}_s satisfy Problem (R) in (2.4).

We notice that the exterior Calderón operator C^e is uniquely defined for all $\mathbf{m} \in H^{-1/2}(\text{div}, \partial\Omega)$, since Problem (R) has a unique solution in $H_{\text{loc}}(\text{curl}, \overline{\Omega}_e) \times H_{\text{loc}}(\text{curl}, \overline{\Omega}_e)$ for any $\mathbf{m} \in H^{-1/2}(\text{div}, \partial\Omega)$ and $\gamma_+(\mathbf{H}_s) \in H^{-1/2}(\text{div}, \partial\Omega)$. If we want to emphasize dependence on the bounding surface of the scatterer, we adopt the notation $C_{\partial\Omega}^e$. Details on the space $H^{-1/2}(\text{div}, \partial\Omega)$ and its dual space $H^{-1/2}(\text{curl}, \partial\Omega)$ are given in Monk [9].

Theorem 2.1. *The exterior Calderón operator defined in Definition 2.1 has the following properties [3, 15]:*

1. *The exterior Calderón operator satisfies the positivity condition*

$$\text{Re} \int_{\partial\Omega} C^e(\mathbf{m}) \cdot (\hat{\nu} \times \overline{\mathbf{m}}) \, dS \geq 0 \quad \text{for all } \mathbf{m} \in H^{-1/2}(\text{div}, \partial\Omega). \quad (2.5)$$

where dS denotes the surface measure of $\partial\Omega$, and the bar denotes the complex conjugate.

2. *The exterior Calderón operator satisfies*

$$(C^e)^2 = -\mathbf{I} \text{ on } H^{-1/2}(\text{div}, \partial\Omega),$$

3. *The exterior Calderón operator is an isomorphism in $H^{-1/2}(\text{div}, \partial\Omega)$, and consequently there exist constants $0 < c_C \leq C_C$, such that*

$$c_C \|\mathbf{m}\|_{H^{-1/2}(\text{div}, \partial\Omega)} \leq \|C^e(\mathbf{m})\|_{H^{-1/2}(\text{div}, \partial\Omega)} \leq C_C \|\mathbf{m}\|_{H^{-1/2}(\text{div}, \partial\Omega)}$$

³The source \mathbf{m} can be interpreted as a magnetic current density.

4. The exterior Calderón operator is independent of the material properties inside the domain Ω .

From item 2 we conclude that the norm of the exterior Calderón operator satisfies $\|C^e\|_{H^{-1/2}(\text{div}, \partial\Omega)} \geq 1$, and also that the constants in item 3 can be chosen as $c_C = 1/\|C^e\|_{H^{-1/2}(\text{div}, \partial\Omega)}$ and $C_C = \|C^e\|_{H^{-1/2}(\text{div}, \partial\Omega)}$.

2.4 Sesquilinear form and weak formulation

The sesquilinear form for an anisotropic obstacle is [3, 15] defined next.

Definition 2.2.

$$A(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \left\{ \frac{i}{k_0} (\overline{\nabla \times \mathbf{v}}) \cdot \boldsymbol{\mu}^{-1} \cdot (\nabla \times \mathbf{u}) - ik_0 \overline{\mathbf{v}} \cdot \boldsymbol{\epsilon} \cdot \mathbf{u} \right\} d\mathbf{v} \\ - \int_{\partial\Omega} \overline{\boldsymbol{\pi}_-(\mathbf{v})} \cdot C^e(\boldsymbol{\gamma}_-(\mathbf{u})) dS, \quad \mathbf{u}, \mathbf{v} \in H(\text{curl}, \Omega) \quad (2.6)$$

where the trace operators $\boldsymbol{\pi}_-$ and $\boldsymbol{\gamma}_-$ on $H(\text{curl}, \Omega)$ are defined in Theorem A.1, and where dS denotes the surface measure on the bounding surface $\partial\Omega$, and $d\mathbf{v}$ denotes the volume measure in \mathbb{R}^3 .

Substituting $\mathbf{m} = \boldsymbol{\gamma}_-(\mathbf{u})$ in the inequality (2.5) gives

$$- \text{Re} \int_{\partial\Omega} \overline{\boldsymbol{\pi}_-(\mathbf{u})} \cdot C^e(\boldsymbol{\gamma}_-(\mathbf{u})) dS \geq 0$$

The sesquilinear form is continuous, bounded and coercive on $H(\text{curl}, \Omega)$. The continuity and boundedness follows immediately from the properties of the material in Ω , and from (for the definition of the space $H^{-1/2}(\text{curl}, \partial\Omega)$, see Appendix A)

$$\left| \int_{\partial\Omega} \overline{\boldsymbol{\pi}_-(\mathbf{v})} \cdot C^e(\boldsymbol{\gamma}_-(\mathbf{u})) dS \right| \leq C_C \|\boldsymbol{\pi}_-(\mathbf{v})\|_{H^{-1/2}(\text{curl}, \partial\Omega)} \|\boldsymbol{\gamma}_-(\mathbf{u})\|_{H^{-1/2}(\text{div}, \partial\Omega)} \\ \leq C_C C_\pi C_\gamma \|\mathbf{v}\|_{H(\text{curl}, \Omega)} \|\mathbf{u}\|_{H(\text{curl}, \Omega)}$$

where we have used the continuity properties of the trace operators $\boldsymbol{\pi}_-$ and $\boldsymbol{\gamma}_-$ on $H(\text{curl}, \Omega)$ in Theorem A.1. The coercivity is proved in [15], but will be restated in the next section in a slightly more distinct formulation. We note that the sesquilinear form couples the interior problem with the solution to the exterior problem through the exterior Calderón operator C^e .

The scattering problem can now be written in a weak form, *i.e.*, the interior problem (2.1) supplied with the boundary conditions (2.3), which couples the interior problem with the exterior problem (2.4), supplied with its radiation conditions.

$$A(\mathbf{E}, \mathbf{v}) = f(\mathbf{v}), \quad \forall \mathbf{v} \in H(\text{curl}, \Omega) \quad (2.7)$$

where

$$f(\mathbf{v}) = \int_{\partial\Omega} (\gamma_+(\mathbf{H}_i) - C^e(\gamma_+(\mathbf{E}_i))) \cdot \overline{\boldsymbol{\pi}_-(\mathbf{v})} \, dS, \quad \forall \mathbf{v} \in H(\text{curl}, \Omega)$$

Note that the given incident fields $\mathbf{E}_i, \mathbf{H}_i$ act as a bounded linear functional on $H(\text{curl}, \Omega)$.

3 Existence and uniqueness of solution

In this section, we characterize the sesquilinear form, which yields existence and uniqueness of solution due to Lax-Milgram's theorem (for proofs see [15]).

3.1 Coercivity

We have the following coercivity properties for the sesquilinear form (2.6):

Proposition 3.1. *For fixed frequency k_0 , the sesquilinear form A in (2.6) is coercive on $H(\text{curl}, \Omega)$ in the following sense*

$$\text{Re } A(\mathbf{u}, \mathbf{u}) \geq C_1 \|\mathbf{u}\|_{L^2(\Omega; \mathbb{C}^3)}^2 + C_2 \|\nabla \times \mathbf{u}\|_{L^2(\Omega; \mathbb{C}^3)}^2 \geq C_A \|\mathbf{u}\|_{H(\text{curl}, \Omega)}^2 \quad (3.1)$$

where the constants

$$C_1 = \inf_{\Omega} \min \left(\text{Eig} \left(-ik_0 \left(\boldsymbol{\epsilon}(\hat{\mathbf{x}}) - \boldsymbol{\epsilon}^\dagger(\hat{\mathbf{x}}) \right) \right) \right) \geq c_m$$

and

$$\begin{aligned} C_2 &= \sup_{\Omega} \max \left(\text{Eig} \left(-ik_0 \left(\boldsymbol{\mu}(\hat{\mathbf{x}}) - \boldsymbol{\mu}^\dagger(\hat{\mathbf{x}}) \right) \right) \right) \\ &= \inf_{\Omega} \min \left(\text{Eig} \left(ik_0^{-1} \left(\boldsymbol{\mu}^{-1}(\hat{\mathbf{x}}) - \boldsymbol{\mu}^{-1\dagger}(\hat{\mathbf{x}}) \right) \right) \right) \end{aligned}$$

and $C_A = \min(C_1, C_2)$.

The following Corollary follows at once, since $|A(\mathbf{u}, \mathbf{u})|$ dominates $\text{Re } A(\mathbf{u}, \mathbf{u})$.

Corollary 3.1.

$$|A(\mathbf{u}, \mathbf{u})| \geq C_1 \|\mathbf{u}\|_{L^2(\Omega; \mathbb{C}^3)}^2 + C_2 \|\nabla \times \mathbf{u}\|_{L^2(\Omega; \mathbb{C}^3)}^2 \geq C_A \|\mathbf{u}\|_{H(\text{curl}, \Omega)}^2$$

3.2 Existence of solution

The weak formulation (2.7) has, for given material properties in the domain Ω , a unique solution due to Lax-Milgram's theorem, see Theorem A.2 in Appendix A and [5].

Theorem 3.1. Equation (2.7) has a unique solution \mathbf{E} in $H(\text{curl}, \Omega)$ satisfying

$$\|\mathbf{E}\|_{H(\text{curl}, \Omega)} \leq C \left(\|\boldsymbol{\gamma}_+(\mathbf{H}_i)\|_{H^{-1/2}(\text{div}, \partial\Omega)} + \|\boldsymbol{\gamma}_+(\mathbf{E}_i)\|_{H^{-1/2}(\text{div}, \partial\Omega)} \right)$$

where the constant C depends on the material parameters in Ω , the geometry (the shape of Ω), and the frequency. In fact, $C = C_C C_\pi / C_A$, where C_A is given in Proposition 3.1.

Proof. The right-hand side of (2.7) is bounded by

$$\begin{aligned} \|f\|_{H'(\text{curl}, \Omega)} &= \sup_{\|\mathbf{v}\|_{H(\text{curl}, \Omega)}=1} \left| \int_{\partial\Omega} (\boldsymbol{\gamma}_+(\mathbf{H}_i) - C^e(\boldsymbol{\gamma}_+(\mathbf{E}_i))) \cdot \overline{\boldsymbol{\pi}_-(\mathbf{v})} \, dS \right| \\ &\leq \max(1, C_C) C_\pi \left(\|\boldsymbol{\gamma}_+(\mathbf{H}_i)\|_{H^{-1/2}(\text{div}, \partial\Omega)} + \|\boldsymbol{\gamma}_+(\mathbf{E}_i)\|_{H^{-1/2}(\text{div}, \partial\Omega)} \right) \end{aligned}$$

which immediately implies the result of the theorem by the use of Proposition 3.1 and Lax-Milgram's theorem, see Theorem A.2. Notice that $\max(1, C_C) = C_C$ by Theorem 2.1. \square

The right-hand side of (2.7), and consequently the bound in Theorem 3.1, is bounded by the energy of the incident field in the scattering domain. We state this in a Corollary.

Corollary 3.2. Equation (2.7) has a unique solution \mathbf{E} in $H(\text{curl}, \Omega)$ satisfying

$$\|\mathbf{E}\|_{H(\text{curl}, \Omega)} \leq C \left(\|\mathbf{E}_i\|_{H(\text{curl}, \Omega)} + \|\mathbf{H}_i\|_{H(\text{curl}, \Omega)} \right)$$

where $C = C_C C_\pi C_\gamma / C_A$.

Proof. The statement follows at once, since the trace norm is bounded by the norm, see Theorem A.1. \square

If the incident field is the plane wave, *i.e.*,

$$\begin{cases} \mathbf{E}_i(\mathbf{x}) = \mathbf{E}_0 e^{ik_0 \hat{\mathbf{k}}_i \cdot \mathbf{x}} \\ \mathbf{H}_i(\mathbf{x}) = \hat{\mathbf{k}}_i \times \mathbf{E}_0 e^{ik_0 \hat{\mathbf{k}}_i \cdot \mathbf{x}} \end{cases} \quad \mathbf{x} \in \mathbb{R}^3$$

where $\mathbf{E}_0 \cdot \hat{\mathbf{k}}_i = 0$, then we have the following corollary:

Corollary 3.3. If the incident field is a plane wave, then equation (2.7) has a unique solution \mathbf{E} in $H(\text{curl}, \Omega)$ satisfying

$$\|\mathbf{E}\|_{H(\text{curl}, \Omega)} \leq C |\mathbf{E}_0| |\Omega|^{1/2}$$

where $C = 2C_C (1 + k_0^2)^{1/2} C_\pi C_\gamma / C_A$. The volume of Ω is denoted by $|\Omega|$.

Proof. The proof follows from Corollary 3.2, and the following bound on the terms in the right-hand side of the result in Corollary 3.2, *i.e.*,

$$\|\mathbf{E}_i\|_{H(\text{curl}, \Omega)}^2 = \|\mathbf{H}_i\|_{H(\text{curl}, \Omega)}^2 \leq (1 + k_0^2) |\mathbf{E}_0|^2 |\Omega|$$

\square

From the existence of the interior electric field, we construct the corresponding magnetic field $\mathbf{H}(\mathbf{x})$ as⁴

$$\begin{cases} \mathbf{H}(\mathbf{x}) = \frac{1}{ik_0} \boldsymbol{\mu}^{-1}(\mathbf{x}) \cdot (\nabla \times \mathbf{E}(\mathbf{x})) \\ \nabla \times \mathbf{H}(\mathbf{x}) = -ik_0 \boldsymbol{\epsilon}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}) \end{cases} \quad \mathbf{x} \in \Omega$$

With the internal fields $\mathbf{E}(\mathbf{x})$ and $\mathbf{H}(\mathbf{x})$, we apply the trace operator from the inside, γ_- , and construct the tangential scattered electric field from the outside, $\gamma_+(\mathbf{E}_s) = \gamma_-(\mathbf{E} - \mathbf{E}_i)$. An application of the Calderón operator on $\gamma_+(\mathbf{E}_s)$ then gives the proper exterior solution.

3.3 Alternative existence proof

Theorem 3.2. *Equation (2.7) has a unique solution \mathbf{E} in $H(\text{curl}, \Omega)$ satisfying*

$$\|\mathbf{E}\|_{H(\text{curl}, \Omega)} \leq C \|\mathbf{E}_i\|_{H(\text{curl}, \Omega)}$$

where the constants $C = 1 + C_3/C_A$, and where

$$C_3 = \max \left(k_0 \|\boldsymbol{\epsilon} - \mathbf{I}_3\|_{L^\infty(\Omega; \mathbb{C}^{3 \times 3})}, k_0^{-1} \|\boldsymbol{\mu}^{-1} - \mathbf{I}_3\|_{L^\infty(\Omega; \mathbb{C}^{3 \times 3})} \right)$$

and C_A is given in Proposition 3.1.

Notice that the constant C in this estimate does not contain the constants C_C , C_π and C_γ as in Corollary 3.2.

Proof. Apply the disturbance of the interior field due to the scatterer, *i.e.*, the difference between the internal solution and the incident field, $\mathbf{u}' = \mathbf{u} - \mathbf{E}_i$ to the sesquilinear form (2.6). We get

$$\begin{aligned} A(\mathbf{u}', \mathbf{v}) &= \int_{\partial\Omega} \overline{\boldsymbol{\pi}_-(\mathbf{v})} \cdot C^e(\boldsymbol{\gamma}_-(\mathbf{E}_i)) \, dS \\ &\quad - \int_{\Omega} \left\{ \frac{i}{k_0} \overline{\nabla \times \mathbf{v}} \cdot \boldsymbol{\mu}^{-1} \cdot \nabla \times \mathbf{E}_i - ik_0 \overline{\mathbf{v}} \cdot \boldsymbol{\epsilon} \cdot \mathbf{E}_i \right\} \, dv, \quad \forall \mathbf{v} \in H(\text{curl}, \Omega) \end{aligned}$$

The weak formulation of the difference between the internal solution and the incident field, $\mathbf{E}' = \mathbf{E} - \mathbf{E}_i$ then is

$$A(\mathbf{E}', \mathbf{v}) = h(\mathbf{v}), \quad \forall \mathbf{v} \in H(\text{curl}, \Omega)$$

⁴This construction is consistent, since $-ik_0 \boldsymbol{\epsilon}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x})$ is the weak curl of $\mathbf{H}(\mathbf{x}) = \frac{1}{ik_0} \boldsymbol{\mu}^{-1}(\mathbf{x}) \cdot \nabla \times \mathbf{E}(\mathbf{x})$. In fact, we have

$$(\mathbf{H}, \nabla \times \boldsymbol{\phi})_{L^2(\Omega; \mathbb{C}^3)} + ik_0 (\boldsymbol{\epsilon}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}), \boldsymbol{\phi})_{L^2(\Omega; \mathbb{C}^3)} = 0 \quad \forall \boldsymbol{\phi} \in D(\Omega; \mathbb{C}^3)$$

since $A(\mathbf{E}, \boldsymbol{\phi}) = 0, \forall \boldsymbol{\phi} \in D(\Omega; \mathbb{C}^3)$.

where the right-hand side is a linear, bounded functional on $H(\text{curl}, \Omega)$. In fact,

$$\begin{aligned} h(\mathbf{v}) &= \int_{\partial\Omega} \overline{\boldsymbol{\pi}_-(\mathbf{v})} \cdot C^e(\boldsymbol{\gamma}_-(\mathbf{E}_i)) \, dS \\ &\quad - \int_{\Omega} \left\{ \frac{i}{k_0} \overline{\nabla \times \mathbf{v}} \cdot \boldsymbol{\mu}^{-1} \cdot \nabla \times \mathbf{E}_i - ik_0 \overline{\mathbf{v}} \cdot \boldsymbol{\epsilon} \cdot \mathbf{E}_i \right\} \, dv \\ &\quad + \int_{\partial\Omega} (\boldsymbol{\gamma}_+(\mathbf{H}_i) - C^e(\boldsymbol{\gamma}_+(\mathbf{E}_i))) \cdot \overline{\boldsymbol{\pi}_-(\mathbf{v})} \, dS \end{aligned}$$

or since $\boldsymbol{\gamma}_-(\mathbf{E}_i) = \boldsymbol{\gamma}_+(\mathbf{E}_i)$

$$\begin{aligned} h(\mathbf{v}) &= ik_0 \int_{\Omega} \overline{\mathbf{v}} \cdot (\boldsymbol{\epsilon} - \mathbf{I}_3) \cdot \mathbf{E}_i \, dv - \frac{i}{k_0} \int_{\Omega} \overline{(\nabla \times \mathbf{v})} \cdot (\boldsymbol{\mu}^{-1} - \mathbf{I}_3) \cdot (\nabla \times \mathbf{E}_i) \, dv \\ &\quad + ik_0 \int_{\Omega} \overline{\mathbf{v}} \cdot \mathbf{E}_i \, dv - \frac{i}{k_0} \int_{\Omega} \overline{(\nabla \times \mathbf{v})} \cdot (\nabla \times \mathbf{E}_i) \, dv + \int_{\partial\Omega} \boldsymbol{\gamma}_+(\mathbf{H}_i) \cdot \overline{\boldsymbol{\pi}_-(\mathbf{v})} \, dS \end{aligned}$$

which we rewrite as

$$h(\mathbf{v}) = ik_0 \int_{\Omega} \overline{\mathbf{v}} \cdot (\boldsymbol{\epsilon} - \mathbf{I}_3) \cdot \mathbf{E}_i \, dv - \frac{i}{k_0} \int_{\Omega} \overline{(\nabla \times \mathbf{v})} \cdot (\boldsymbol{\mu}^{-1} - \mathbf{I}_3) \cdot (\nabla \times \mathbf{E}_i) \, dv$$

We have here used the properties of the incident electric field, *viz.*, $\nabla \times (\nabla \times \mathbf{E}_i) = k_0^2 \mathbf{E}_i$, and

$$\begin{aligned} & ik_0 \int_{\Omega} \overline{\mathbf{v}} \cdot \mathbf{E}_i \, dv - \frac{i}{k_0} \int_{\Omega} \overline{(\nabla \times \mathbf{v})} \cdot (\nabla \times \mathbf{E}_i) \, dv + \int_{\partial\Omega} \boldsymbol{\gamma}_+(\mathbf{H}_i) \cdot \overline{\boldsymbol{\pi}_-(\mathbf{v})} \, dS \\ &= -\frac{i}{k_0} \int_{\Omega} \nabla \cdot \{\overline{\mathbf{v}} \times (\nabla \times \mathbf{E}_i)\} \, dv + \int_{\partial\Omega} \boldsymbol{\gamma}_+(\mathbf{H}_i) \cdot \overline{\boldsymbol{\pi}_-(\mathbf{v})} \, dS \\ &= \int_{\partial\Omega} \hat{\boldsymbol{\nu}} \cdot \{\overline{\mathbf{v}} \times \mathbf{H}_i\} \, dS + \int_{\partial\Omega} \boldsymbol{\gamma}_+(\mathbf{H}_i) \cdot \overline{\boldsymbol{\pi}_-(\mathbf{v})} \, dS = 0 \end{aligned}$$

This leads to the estimate

$$\begin{aligned} |h(\mathbf{v})| &\leq k_0 \|\mathbf{v}\|_{L^2(\Omega; \mathbb{C}^3)} \|\boldsymbol{\epsilon} - \mathbf{I}_3\|_{L^\infty(\Omega; \mathbb{C}^{3 \times 3})} \|\mathbf{E}_i\|_{L^2(\Omega; \mathbb{C}^3)} \\ &\quad + k_0^{-1} \|\nabla \times \mathbf{v}\|_{L^2(\Omega; \mathbb{C}^3)} \|\boldsymbol{\mu}^{-1} - \mathbf{I}_3\|_{L^\infty(\Omega; \mathbb{C}^{3 \times 3})} \|\nabla \times \mathbf{E}_i\|_{L^2(\Omega; \mathbb{C}^3)} \end{aligned}$$

and due to the assumptions made on the material parameters, we get

$$|h(\mathbf{v})| \leq C_3 \|\mathbf{v}\|_{H(\text{curl}, \Omega)} \|\mathbf{E}_i\|_{H(\text{curl}, \Omega)}$$

where the constant

$$C_3 = \max(k_0 \|\boldsymbol{\epsilon} - \mathbf{I}_3\|_{L^\infty(\Omega; \mathbb{C}^{3 \times 3})}, k_0^{-1} \|\boldsymbol{\mu}^{-1} - \mathbf{I}_3\|_{L^\infty(\Omega; \mathbb{C}^{3 \times 3})})$$

Consequently, the norm of h in the dual space of $H(\text{curl}, \Omega)$ is

$$\|h\|_{H'(\text{curl}, \Omega)} = \sup_{\|\mathbf{v}\|_{H(\text{curl}, \Omega)}=1} |h(\mathbf{v})| \leq C_3 \|\mathbf{E}_i\|_{H(\text{curl}, \Omega)}$$

Therefore, by Lax-Milgram's theorem, we can estimate the disturbance as

$$\|\mathbf{E}'\|_{H(\text{curl}, \Omega)} \leq C \|\mathbf{E}_i\|_{H(\text{curl}, \Omega)} \quad (3.2)$$

where $C = C_3/C_A$, where C_A is given in Proposition 3.1, and the proposition is proved by the use of $\mathbf{E} = \mathbf{E}' + \mathbf{E}_i$. \square

We have also proved

Corollary 3.4. *The solution $\mathbf{E}' = \mathbf{E} - \mathbf{E}_i$ satisfies*

$$\|\mathbf{E}'\|_{H(\text{curl}, \Omega)} \leq C \|\mathbf{E}_i\|_{H(\text{curl}, \Omega)}$$

where the constants $C = C_3/C_A$, where C_3 is given in Theorem 3.2, and C_A is given in Proposition 3.1.

Corollary 3.5. *If the incident field is a plane wave, then equation (2.7) has a unique solution \mathbf{E} in $H(\text{curl}, \Omega)$ satisfying*

$$\|\mathbf{E}\|_{H(\text{curl}, \Omega)} \leq C |\mathbf{E}_0| |\Omega|^{1/2}$$

where $C = (1 + C_3/C_A)(1 + k_0^2)^{1/2}$. The volume of Ω is denoted by $|\Omega|$.

Proof. The proof follows from Theorem 3.2, and

$$\|\mathbf{E}_i\|_{H(\text{curl}, \Omega)}^2 \leq (1 + k_0^2) |\mathbf{E}_0|^2 |\Omega|$$

\square

4 Representation of the scattered fields

Outside any circumscribing sphere of the domain Ω , *i.e.*, any sphere with radius $x \geq R$, the scattered field can be represented in an infinite series of spherical vector waves $\mathbf{u}_{\tau n}(k_0 \mathbf{x})$. The spherical vector waves are defined in Appendix B, see also [15].

We start with some general expressions and representations of the scattered field in Ω_e . The scattered field $\mathbf{E}_s(\mathbf{x})$, *i.e.*, the general solution to Problem (R) in (2.4), has an integral representation [4, 13]

$$\begin{aligned} \mathbf{E}_s(\mathbf{x}) = & \frac{i}{k_0} \nabla \times \left\{ \nabla \times \int_{\partial\Omega} \mathbf{I}_3 g(|\mathbf{x} - \mathbf{x}'|) \cdot C^e(\boldsymbol{\gamma}_+(\mathbf{E}_s))(\mathbf{x}') \, dS' \right\} \\ & + \nabla \times \int_{\partial\Omega} \mathbf{I}_3 g(|\mathbf{x} - \mathbf{x}'|) \cdot \boldsymbol{\gamma}_+(\mathbf{E}_s)(\mathbf{x}') \, dS', \quad \mathbf{x} \in \mathbb{R}^3 \setminus \bar{\Omega} \end{aligned} \quad (4.1)$$

where we have used the notation

$$g(y) = \frac{e^{ik_0 y}}{4\pi y}$$

Moreover, outside the least circumscribing sphere (radius R) of Ω , the solution can be represented in terms of outgoing spherical vector waves [13].

$$\begin{cases} \mathbf{E}_s(\mathbf{x}) = \sum_{\tau n} f_{\tau n} \mathbf{u}_{\tau n}(k_0 \mathbf{x}) \\ \mathbf{H}_s(\mathbf{x}) = -\frac{i}{k_0} \sum_{\tau n} f_{\tau n} \nabla \times \mathbf{u}_{\tau n}(k_0 \mathbf{x}) = -i \sum_{\tau n} f_{\tau n} \mathbf{u}_{\bar{\tau} n}(k_0 \mathbf{x}) \end{cases} \quad x > R \quad (4.2)$$

where n is a multi-index, and where the index $\bar{\tau}$ is the dual index of τ , defined by $\bar{1} = 2$ and $\bar{2} = 1$. The explicit expressions of the electric and magnetic fields in (4.2) for $x > R$ are, see Appendix B

$$\begin{cases} \mathbf{E}_s(\mathbf{x}) = \sum_n \left(f_{1n} \frac{\xi_l(k_0 x)}{k_0 x} \mathbf{A}_{1n}(\hat{\mathbf{x}}) \right. \\ \quad \left. + f_{2n} \left(\frac{\xi'_l(k_0 x)}{k_0 x} \mathbf{A}_{2n}(\hat{\mathbf{x}}) + \sqrt{l(l+1)} \frac{\xi_l(k_0 x)}{k_0^2 x^2} \mathbf{A}_{3n}(\hat{\mathbf{x}}) \right) \right) \\ \mathbf{H}_s(\mathbf{x}) = -i \sum_n \left(f_{1n} \left(\frac{\xi'_l(k_0 x)}{k_0 x} \mathbf{A}_{2n}(\hat{\mathbf{x}}) + \sqrt{l(l+1)} \frac{\xi_l(k_0 x)}{k_0^2 x^2} \mathbf{A}_{3n}(\hat{\mathbf{x}}) \right) \right. \\ \quad \left. + f_{2n} \frac{\xi_l(k_0 x)}{k_0 x} \mathbf{A}_{1n}(\hat{\mathbf{x}}) \right) \end{cases} \quad (4.3)$$

and

$$\begin{cases} \gamma_+(\mathbf{E}_s)(\mathbf{x}) = \hat{\mathbf{x}} \times \mathbf{E}_s(\mathbf{x}) = \sum_n \left(f_{1n} \frac{\xi_l(k_0 x)}{k_0 x} \mathbf{A}_{2n}(\hat{\mathbf{x}}) - f_{2n} \frac{\xi'_l(k_0 x)}{k_0 x} \mathbf{A}_{1n}(\hat{\mathbf{x}}) \right) \\ C^e(\gamma_+(\mathbf{E}_s))(\mathbf{x}) = \gamma_+(\mathbf{H}_s)(\mathbf{x}) = \hat{\mathbf{x}} \times \mathbf{H}_s(\mathbf{x}) \\ \quad = i \sum_n \left(f_{1n} \frac{\xi'_l(k_0 x)}{k_0 x} \mathbf{A}_{1n}(\hat{\mathbf{x}}) - f_{2n} \frac{\xi_l(k_0 x)}{k_0 x} \mathbf{A}_{2n}(\hat{\mathbf{x}}) \right) \end{cases} \quad (4.4)$$

and

$$\begin{cases} \pi_+(\mathbf{E}_s)(\mathbf{x}) = \sum_n \left(f_{1n} \frac{\xi_l(k_0 x)}{k_0 x} \mathbf{A}_{1n}(\hat{\mathbf{x}}) + f_{2n} \frac{\xi'_l(k_0 x)}{k_0 x} \mathbf{A}_{2n}(\hat{\mathbf{x}}) \right) \\ \pi_+(\mathbf{H}_s)(\mathbf{x}) = -i \sum_n \left(f_{1n} \frac{\xi'_l(k_0 x)}{k_0 x} \mathbf{A}_{2n}(\hat{\mathbf{x}}) + f_{2n} \frac{\xi_l(k_0 x)}{k_0 x} \mathbf{A}_{1n}(\hat{\mathbf{x}}) \right) \end{cases}$$

Here, $\mathbf{A}_{\tau n}$ are vector spherical harmonics, see Appendix B and [15]. The coefficients $f_{\tau n}$ are determined by [13], *i. e.*,

$$f_{\tau n} = -k_0^2 \int_{\partial\Omega} \mathbf{v}_{\tau n}(k_0 \mathbf{x}) \cdot C^e(\gamma_+(\mathbf{E}_s))(\mathbf{x}) \, dS + ik_0^2 \int_{\partial\Omega} \mathbf{v}_{\bar{\tau} n}(k_0 \mathbf{x}) \cdot \gamma_+(\mathbf{E}_s)(\mathbf{x}) \, dS \quad (4.5)$$

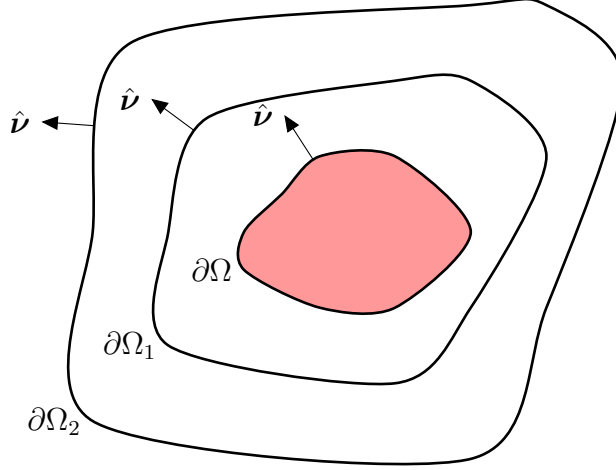


Figure 2: Typical geometry of the geometry in Lemm 4.1.

where $\mathbf{v}_{\tau n}(k_0 \mathbf{x})$ denotes the regular spherical vector waves, see [15]. In fact, the representations in (4.2) and (4.5) are direct consequences of the expansion [2]

$$\mathbf{I}_3 g(R) = ik_0 \sum_{\tau=1,2,3}^n \mathbf{u}_{\tau n}(k_0 \mathbf{x}_>) \mathbf{v}_{\tau n}(k_0 \mathbf{x}_<) = ik_0 \sum_{\tau=1,2,3}^n \mathbf{v}_{\tau n}(k_0 \mathbf{x}_<) \mathbf{u}_{\tau n}(k_0 \mathbf{x}_>)$$

where $R = |\mathbf{x} - \mathbf{x}'|$, and where the argument $\mathbf{x}_>$ ($\mathbf{x}_<$) denotes the vector with the larger (smaller) magnitude of \mathbf{x} and \mathbf{x}' . The coefficients $f_{\tau n}$ do not depend on which surface $\partial\Omega$ they are evaluated on in the following respect:

Lemma 4.1. *Let $\Omega \subseteq \Omega_1 \subseteq \Omega_2$ be two open sets in \mathbb{R}^3 , and assume that $\partial\Omega_1$ and $\partial\Omega_2$ are $C^{1,1}$ surfaces, see Figure 2 for a typical geometry. Then*

$$\begin{aligned} f_{\tau n} &= -k_0^2 \int_{\partial\Omega_1} \mathbf{v}_{\tau n}(k_0 \mathbf{x}) \cdot C^e(\boldsymbol{\gamma}_+(\mathbf{E}_s))(\mathbf{x}) \, dS + ik_0^2 \int_{\partial\Omega_1} \mathbf{v}_{\bar{\tau} n}(k_0 \mathbf{x}) \cdot \boldsymbol{\gamma}_+(\mathbf{E}_s)(\mathbf{x}) \, dS \\ &= -k_0^2 \int_{\partial\Omega_2} \mathbf{v}_{\tau n}(k_0 \mathbf{x}) \cdot C^e(\boldsymbol{\gamma}_+(\mathbf{E}_s))(\mathbf{x}) \, dS + ik_0^2 \int_{\partial\Omega_2} \mathbf{v}_{\bar{\tau} n}(k_0 \mathbf{x}) \cdot \boldsymbol{\gamma}_+(\mathbf{E}_s)(\mathbf{x}) \, dS \end{aligned}$$

Note that the Calderón operators and the trace operators on the surfaces $\partial\Omega_1$ and $\partial\Omega_2$ are different, but, to avoid cumbersome notation, we adopt the same symbols on the two surfaces.

Proof. We start by noting that for $\mathbf{x} \in \Omega_e$, we have

$$\begin{cases} ik_0 \mathbf{H}_s(\mathbf{x}) = \nabla \times \mathbf{E}_s(\mathbf{x}) \\ k_0 \mathbf{v}_{\bar{\tau} n}(k_0 \mathbf{x}) = \nabla \times \mathbf{v}_{\tau n}(k_0 \mathbf{x}) \end{cases} \quad \begin{cases} \nabla \times (\nabla \times \mathbf{E}_s(\mathbf{x})) = k_0^2 \mathbf{E}_s(\mathbf{x}) \\ \nabla \times (\nabla \times \mathbf{v}_{\tau n}(k_0 \mathbf{x})) = k_0^2 \mathbf{v}_{\tau n}(k_0 \mathbf{x}) \end{cases}$$

By the divergence theorem (Gauss' theorem), the difference between the integrals in the lemma are

$$\begin{aligned}
& -k_0^2 \int_{\partial\Omega_2 - \partial\Omega_1} \mathbf{v}_{\tau n}(k_0\mathbf{x}) \cdot C^e(\boldsymbol{\gamma}_+(\mathbf{E}_s))(\mathbf{x}) \, dS + ik_0^2 \int_{\partial\Omega_2 - \partial\Omega_1} \mathbf{v}_{\bar{\tau}n}(k_0\mathbf{x}) \cdot \boldsymbol{\gamma}_+(\mathbf{E}_s)(\mathbf{x}) \, dS \\
& = -ik_0 \int_{\partial\Omega_2 - \partial\Omega_1} \hat{\boldsymbol{\nu}}(\mathbf{x}) \cdot (\mathbf{v}_{\tau n}(k_0\mathbf{x}) \times (\nabla \times \mathbf{E}_s(\mathbf{x})) + (\nabla \times \mathbf{v}_{\tau n}(k_0\mathbf{x})) \times \mathbf{E}_s(\mathbf{x})) \, dS \\
& = -ik_0 \int_{\Omega_2 \setminus \bar{\Omega}_1} \nabla \cdot (\mathbf{v}_{\tau n}(k_0\mathbf{x}) \times (\nabla \times \mathbf{E}_s(\mathbf{x})) + (\nabla \times \mathbf{v}_{\tau n}(k_0\mathbf{x})) \times \mathbf{E}_s(\mathbf{x})) \, dv
\end{aligned}$$

However, the integrand in the last integral is zero, since for $\mathbf{x} \in \Omega_2 \setminus \bar{\Omega}_1 \subseteq \Omega_e$, we have

$$\begin{aligned}
& \nabla \cdot (\mathbf{v}_{\tau n}(k_0\mathbf{x}) \times (\nabla \times \mathbf{E}_s(\mathbf{x})) + (\nabla \times \mathbf{v}_{\tau n}(k_0\mathbf{x})) \times \mathbf{E}_s(\mathbf{x})) \\
& = (\nabla \times \mathbf{v}_{\tau n}(k_0\mathbf{x})) \times (\nabla \times \mathbf{E}_s(\mathbf{x})) - k_0^2 \mathbf{v}_{\tau n}(k_0\mathbf{x}) \cdot \mathbf{E}_s(\mathbf{x}) \\
& \quad + k_0^2 \mathbf{v}_{\tau n}(k_0\mathbf{x}) \cdot \mathbf{E}_s(\mathbf{x}) - (\nabla \times \mathbf{v}_{\tau n}(k_0\mathbf{x})) \times (\nabla \times \mathbf{E}_s(\mathbf{x})) = 0
\end{aligned}$$

and the lemma is proved. \square

From this lemma, we see that the coefficients $f_{\tau n}$ only depend on the bounding surface $\partial\Omega$, the material inside Ω , and the incident field (due to the dependence on $C^e(\boldsymbol{\gamma}_+(\mathbf{E}_s))$ and $\boldsymbol{\gamma}_+(\mathbf{E}_s)$).

4.1 Surface field estimates

Proposition 4.1. *The scattered field on the surface $\partial\Omega$ satisfies*

$$\|\boldsymbol{\gamma}_+(\mathbf{E}_s)\|_{H^{-1/2}(\text{div}, \partial\Omega)} \leq C \|\mathbf{E}_i\|_{H(\text{curl}, \Omega)}$$

and

$$\|C^e(\boldsymbol{\gamma}_+(\mathbf{E}_s))\|_{H^{-1/2}(\text{div}, \partial\Omega)} \leq C_C C \|\mathbf{E}_i\|_{H(\text{curl}, \Omega)}$$

where the constants $C = C_\gamma C_3 / C_A$, and where

$$C_3 = \max(k_0 \|\boldsymbol{\epsilon} - \mathbf{I}_3\|_{L^\infty(\Omega; \mathbb{C}^{3 \times 3})}, k_0^{-1} \|\boldsymbol{\mu}^{-1} - \mathbf{I}_3\|_{L^\infty(\Omega; \mathbb{C}^{3 \times 3})})$$

C_A is given in Proposition 3.1, and C_C is the operator norm of the exterior Calderón operator, which depend only of the shape of Ω , and the frequency k_0 .

Proof. We apply the result of Corollary 3.4, *i.e.*,

$$\|\mathbf{E}'\|_{H(\text{curl}, \Omega)} \leq C \|\mathbf{E}_i\|_{H(\text{curl}, \Omega)}$$

where $C = C_3 / C_A$, and where the disturbance of the interior field due to the scatterer is $\mathbf{E}' = \mathbf{E} - \mathbf{E}_i$. The trace space norm dependence on the volume norm, $H(\text{curl}, \Omega)$, and the boundary conditions imply

$$\|\boldsymbol{\gamma}_+(\mathbf{E}_s)\|_{H^{-1/2}(\text{div}, \partial\Omega)} = \|\boldsymbol{\gamma}_-(\mathbf{E}')\|_{H^{-1/2}(\text{div}, \partial\Omega)} \leq C_\gamma \|\mathbf{E}'\|_{H(\text{curl}, \Omega)} \leq C_\gamma C \|\mathbf{E}_i\|_{H(\text{curl}, \Omega)}$$

The bound on the exterior Calderón operator gives

$$\begin{aligned} \|C^e(\gamma_+(\mathbf{E}_s))\|_{H^{-1/2}(\text{div},\partial\Omega)} &= \|C^e(\gamma_-(\mathbf{E}'))\|_{H^{-1/2}(\text{div},\partial\Omega)} \leq C_C C_\gamma \|\mathbf{E}'\|_{H(\text{curl},\Omega)} \\ &\leq C_C C_\gamma C \|\mathbf{E}_i\|_{H(\text{curl},\Omega)} \end{aligned}$$

and the proposition is proved. \square

If the incident field is a plane wave, the following corollary holds:

Corollary 4.1. *For an incident plane wave, the scattered field on the surface $\partial\Omega$ satisfies*

$$\|\gamma_+(\mathbf{E}_s)\|_{H^{-1/2}(\text{div},\partial\Omega)} \leq C (1 + k_0^2)^{1/2} |\mathbf{E}_0| |\Omega|^{1/2}$$

and

$$\|C^e(\gamma_+(\mathbf{E}_s))\|_{H^{-1/2}(\text{div},\partial\Omega)} \leq C_C C (1 + k_0^2)^{1/2} |\mathbf{E}_0| |\Omega|^{1/2}$$

where the constants C is given in Proposition 4.1. The volume of Ω is denoted by $|\Omega|$.

Proof. The result follows immediately from the proof of Corollary 3.3. \square

4.2 Scattered field estimates

Proposition 4.2. *We have for any bounded domain $\Omega_s \subset \Omega_e$*

$$\|\mathbf{E}_s\|_{L^2(\Omega_s;\mathbb{C}^3)} \leq C_4 \|\mathbf{E}_i\|_{H(\text{curl},\Omega)}$$

and

$$\sup_{\mathbf{x} \in \Omega_s} |\mathbf{E}_s(\mathbf{x})| \leq C_5 \|\mathbf{E}_i\|_{H(\text{curl},\Omega)}$$

where

$$C_4 = C \left\| \left(C_C \|\mathbf{F}_1(\mathbf{x}, \cdot)\|_{H^{-1/2}(\text{curl},\partial\Omega)} + \|\mathbf{F}_2(\mathbf{x}, \cdot)\|_{H^{-1/2}(\text{curl},\partial\Omega)} \right) \right\|_{L^2(\Omega_s;\mathbb{C}^3)}$$

$$C_5 = C \sup_{\mathbf{x} \in \Omega_s} \left(C_C \|\mathbf{F}_1(\mathbf{x}, \cdot)\|_{H^{-1/2}(\text{curl},\partial\Omega)} + \|\mathbf{F}_2(\mathbf{x}, \cdot)\|_{H^{-1/2}(\text{curl},\partial\Omega)} \right)$$

C is given in Proposition 4.1, and

$$\begin{cases} \mathbf{F}_1(\mathbf{x}, \mathbf{x}') = \frac{i}{k_0} \nabla \times \{\nabla \times \mathbf{I}_3 g(|\mathbf{x} - \mathbf{x}'|)\} \\ \mathbf{F}_2(\mathbf{x}, \mathbf{x}') = \nabla \times \mathbf{I}_3 g(|\mathbf{x} - \mathbf{x}'|) \end{cases}$$

Proof. The integral representation of the scattered field in (4.1) reads

$$\begin{aligned} \mathbf{E}_s(\mathbf{x}) &= \int_{\partial\Omega} \mathbf{F}_1(\mathbf{x}, \mathbf{x}') \cdot C^e(\gamma_+(\mathbf{E}_s))(\mathbf{x}') \, dS' \\ &\quad + \int_{\partial\Omega} \mathbf{F}_2(\mathbf{x}, \mathbf{x}') \cdot \gamma_+(\mathbf{E}_s)(\mathbf{x}') \, dS', \quad \mathbf{x} \in \Omega_e \end{aligned}$$

where

$$\begin{cases} \mathbf{F}_1(\mathbf{x}, \mathbf{x}') = \frac{i}{k_0} \nabla \times \{\nabla \times \mathbf{I}_3 g(|\mathbf{x} - \mathbf{x}'|)\} \\ \mathbf{F}_2(\mathbf{x}, \mathbf{x}') = \nabla \times \mathbf{I}_3 g(|\mathbf{x} - \mathbf{x}'|) \end{cases}$$

Since $\gamma_+(\mathbf{E}_s)$ and $C^e(\gamma_+(\mathbf{E}_s))$ are bounded in $H^{-1/2}(\text{div}, \partial\Omega)$, we only have to verify that $\mathbf{F}_i(\mathbf{x}, \mathbf{x}')$, $i = 1, 2$, are bounded in the dual space $H^{-1/2}(\text{curl}, \partial\Omega)$, see *e.g.*, [3, p. 38], for all $\mathbf{x} \in \Omega_s$. In fact, the kernels are infinitely differentiable whenever $\mathbf{x} \notin \partial\Omega$, which implies

$$\begin{aligned} |\mathbf{E}_s(\mathbf{x})| &\leq \|\mathbf{F}_1(\mathbf{x}, \cdot)\|_{H^{-1/2}(\text{curl}, \partial\Omega)} \|C^e(\gamma_+(\mathbf{E}_s))\|_{H^{-1/2}(\text{div}, \partial\Omega)} \\ &\quad + \|\mathbf{F}_2(\mathbf{x}, \cdot)\|_{H^{-1/2}(\text{curl}, \partial\Omega)} \|\gamma_+(\mathbf{E}_s)\|_{H^{-1/2}(\text{div}, \partial\Omega)} \\ &\leq C'(\mathbf{x}) \|\gamma_+(\mathbf{E}_s)\|_{H^{-1/2}(\text{div}, \partial\Omega)} \end{aligned}$$

where

$$C'(\mathbf{x}) = C_C \|\mathbf{F}_1(\mathbf{x}, \cdot)\|_{H^{-1/2}(\text{curl}, \partial\Omega)} + \|\mathbf{F}_2(\mathbf{x}, \cdot)\|_{H^{-1/2}(\text{curl}, \partial\Omega)}$$

Taking sup over all $\mathbf{x} \in \Omega_s$ yields

$$\sup_{\mathbf{x} \in \Omega_s} |\mathbf{E}_s(\mathbf{x})| \leq \sup_{\mathbf{x} \in \Omega_s} C'(\mathbf{x}) \|\gamma_+(\mathbf{E}_s)\|_{H^{-1/2}(\text{div}, \partial\Omega)} \leq C'' C \|\mathbf{E}_i\|_{H(\text{curl}, \Omega)}$$

where

$$C'' = \sup_{\mathbf{x} \in \Omega_s} C'(\mathbf{x})$$

and where we have employed the result in Proposition 4.1. This ends the second estimate in the proposition.

The first estimate also follows immediately. We have

$$\|\mathbf{E}_s\|_{L^2(\Omega_s; \mathbb{C}^3)}^2 = \int_{\Omega_s} |\mathbf{E}_s(\mathbf{x})|^2 \, dv \leq C^2 \|C'\|_{L^2(\Omega_s; \mathbb{C}^3)}^2 \|\mathbf{E}_i\|_{H(\text{curl}, \Omega)}^2$$

where we have employed the result in Proposition 4.1, and the proposition follows. \square

Corollary 4.2. *Plane incident fields yield*

$$\|\mathbf{E}_s\|_{L^2(\Omega_s; \mathbb{C}^3)} \leq C_4 (1 + k_0^2)^{1/2} |\mathbf{E}_0| |\Omega|^{1/2} |\Omega_s|^{1/2}$$

and

$$\sup_{\mathbf{x} \in \Omega_s} |\mathbf{E}_s(\mathbf{x})| \leq C_5 (1 + k_0^2)^{1/2} |\mathbf{E}_0| |\Omega|^{1/2}$$

where constants C_4, C_5 are given in Proposition 4.2.

Proof. The result follows immediately from Proposition 4.2 and the proof of Corollary 3.3. \square

4.3 Far field amplitude

The far field amplitude, $\mathbf{F}(\hat{\mathbf{x}})$, defined by [4]

$$\mathbf{F}(\hat{\mathbf{x}}) = \lim_{x \rightarrow \infty} x e^{-ik_0 x} \mathbf{E}_s(\mathbf{x})$$

is generally given in terms of the traces on the enclosing surface by

$$\mathbf{F}(\hat{\mathbf{x}}) = i \frac{k}{4\pi} \hat{\mathbf{x}} \times \int_{\partial\Omega} \{ \boldsymbol{\gamma}_+(\mathbf{E}_s) - \hat{\mathbf{x}} \times \boldsymbol{\gamma}_+(\mathbf{H}_s) \} e^{-ik\hat{\mathbf{x}} \cdot \mathbf{x}'} dS'$$

From the expansion of the scattered electric field outside the least circumscribing sphere, (4.2), we also have

$$\mathbf{F}(\hat{\mathbf{x}}) = \frac{1}{k_0} \sum_{\tau n} f_{\tau n} e^{-i(l+2-\tau)\pi/2} \mathbf{A}_{\tau n}(\hat{\mathbf{x}}) \quad (4.6)$$

where the coefficients f_{1n} and f_{2n} are determined by the orthonormal properties of the vector spherical harmonics, *i.e.*,

$$f_{\tau n} = k_0 e^{i(l+2-\tau)\pi/2} \int_{\Gamma} \mathbf{A}_{\tau n}(\hat{\mathbf{x}}) \cdot \mathbf{F}(\hat{\mathbf{x}}) d\Gamma$$

where Γ is the unit sphere in \mathbb{R}^3 .

From (4.3) and (4.4), we get by orthogonality of the vector spherical harmonics, and the definition of the trace norm $\|\cdot\|_{H^{-1/2}(\text{div}, \partial B_x)}$, see (B.2) in Appendix B and Ref. 8.

Lemma 4.2. *The norm of the far field amplitude is*

$$\|\mathbf{F}\|_{\Gamma}^2 = \frac{1}{k_0^2} \sum_n \{ |f_{1n}|^2 + |f_{2n}|^2 \}$$

where the norm on the unit sphere Γ is defined in (B.1) in Appendix B. Outside the least circumscribing sphere, $x \geq R$, the norms of the tangential fields are ($B_x = B(\mathbf{0}, x)$ denotes the ball of radius x , centered at the origin, and $\kappa = k_0 x$)

$$\begin{cases} \|\boldsymbol{\gamma}(\mathbf{E}_s)\|_{L^2(\partial B_x; \mathbb{C}^3)}^2 = \frac{1}{k_0^2} \sum_n \{ |\xi_l(\kappa)|^2 |f_{1n}|^2 + |\xi'_l(\kappa)|^2 |f_{2n}|^2 \} \\ \|\boldsymbol{\gamma}(\mathbf{E}_s)\|_{H^{-1/2}(\text{div}, \partial B_x)}^2 = \frac{1}{k_0^2} \sum_n \left\{ \sqrt{1+l(l+1)} |\xi_l(\kappa)|^2 |f_{1n}|^2 + \frac{|\xi'_l(\kappa)|^2 |f_{2n}|^2}{\sqrt{1+l(l+1)}} \right\} \end{cases} \quad (4.7)$$

and

$$\begin{cases} \|\boldsymbol{\gamma}(\mathbf{H}_s)\|_{L^2(\partial B_x; \mathbb{C}^3)}^2 = \frac{1}{k_0^2} \sum_n \{ |\xi'_l(\kappa)|^2 |f_{1n}|^2 + |\xi_l(\kappa)|^2 |f_{2n}|^2 \} \\ \|\boldsymbol{\gamma}(\mathbf{H}_s)\|_{H^{-1/2}(\text{div}, \partial B_x)}^2 = \frac{1}{k_0^2} \sum_n \left\{ \frac{|\xi'_l(\kappa)|^2 |f_{1n}|^2}{\sqrt{1+l(l+1)}} + \sqrt{1+l(l+1)} |\xi_l(\kappa)|^2 |f_{2n}|^2 \right\} \end{cases}$$

and the full fields have the norms

$$\begin{cases} \|\mathbf{E}_s\|_{L^2(\partial B_x; \mathbb{C}^3)}^2 = \frac{1}{k_0^2} \sum_n \left\{ |\xi_l(\kappa)|^2 |f_{1n}|^2 + \left(|\xi'_l(\kappa)|^2 + l(l+1) \frac{|\xi_l(\kappa)|^2}{\kappa^2} \right) |f_{2n}|^2 \right\} \\ \|\mathbf{H}_s\|_{L^2(\partial B_x; \mathbb{C}^3)}^2 = \frac{1}{k_0^2} \sum_n \left\{ \left(|\xi'_l(\kappa)|^2 + l(l+1) \frac{|\xi_l(\kappa)|^2}{\kappa^2} \right) |f_{1n}|^2 + |\xi_l(\kappa)|^2 |f_{2n}|^2 \right\} \end{cases}$$

The result of this lemma can also be rewritten in a more symmetric form that is more useful in the analysis. We have

$$\begin{aligned} \|\gamma(\mathbf{H}_s)\|_{H^{-1/2}(\text{div}, \partial B_x)}^2 &= \frac{1}{k_0^2} \sum_n \left\{ \sqrt{1 + l(l+1)} |t_l(\kappa)|^2 |\xi_l(\kappa)|^2 |f_{1n}|^2 \right. \\ &\quad \left. + \frac{|\xi'_l(\kappa)|^2 |f_{2n}|^2}{|t_l(\kappa)|^2 \sqrt{1 + l(l+1)}} \right\} \end{aligned} \quad (4.8)$$

where

$$t_l(\kappa) = \frac{1}{\sqrt{1 + l(l+1)}} \frac{\xi'_l(\kappa)}{\xi_l(\kappa)}$$

In Ref. 8 the following lemma is proved, and for convenience the proof is repeated:

Lemma 4.3. *For each fixed κ , the sequence $t_l(\kappa)$, $l = 1, 2, 3, \dots$ satisfies*

$$c(\kappa) \leq |t_l(\kappa)| \leq C(\kappa)$$

where the constants $0 < c(\kappa) \leq C(\kappa)$ are independent l , but they depend on κ . Moreover, for fixed $\kappa_0 > 0$, $\kappa c(\kappa)$ and $\kappa C(\kappa)$ are uniformly bounded from above and below for all $\kappa > \kappa_0$.

Proof. Since neither $\xi_l(z)$ nor $\xi'_l(z)$ have any real zeroes $z = x$, $t_l(\kappa)$ stay away from zero and infinity for all finite l , and it suffices to check if the limit as $l \rightarrow \infty$ is not zero or infinite to prove the lemma. The Riccati-Bessel function $\xi_l(z)$ is an entire function of z , and it has a finite series representation [11]

$$\xi_l(z) = e^{iz - i(l+1)\pi/2} \sum_{k=0}^l \frac{(l+k)!}{k!(l-k)!} \frac{1}{(-2iz)^k}$$

and it satisfies the recursion relation

$$\xi'_l(z) = \xi_{l-1}(z) - \frac{l}{z} \xi_l(z), \quad \xi_{l+1}(z) = \frac{2l+1}{z} \xi_l(z) - \xi_{l-1}(z)$$

which implies

$$\frac{\xi'_l(\kappa)}{\xi_l(\kappa)} = -\frac{l}{\kappa} + \frac{\xi_{l-1}(\kappa)}{\xi_l(\kappa)}$$

We now focus on how the quotient $f_l(\kappa) = \xi_{l+1}(\kappa)/\xi_l(\kappa)$ grows as $l \rightarrow \infty$. From the recursion formula we have

$$f_l(\kappa) = \frac{2l+1}{\kappa} - \frac{1}{f_{l-1}(\kappa)}, \quad f_0(\kappa) = -i + \frac{1}{\kappa}$$

Repeated use of this recursion formula leads to (continued fraction)

$$f_l(\kappa) = \frac{2l+1}{\kappa} (1 + O(1/l))$$

which implies

$$\left| \frac{\xi'_l(\kappa)}{\xi_l(\kappa)} \right| = \frac{l}{\kappa} (1 + O(1/l^2))$$

and

$$|t_l(\kappa)| = \frac{l}{\kappa \sqrt{1+l(l+1)}} (1 + O(1/l^2)) \rightarrow \frac{1}{\kappa} \text{ as } l \rightarrow \infty$$

and the lemma is proved. \square

Lemma 4.4. *The norm of the exterior Calderón operator is $\|C^e\|_{H^{-1/2}(\text{div}, \partial\Omega)} = \sup_l \max\{|t_l(\kappa)|, 1/|t_l(\kappa)|\}$. The norm depends on $\kappa = k_0x$.*

Proof. We have

$$\begin{aligned} \|C^e\|_{H^{-1/2}(\text{div}, \partial B_x)} &= \sup_{\|\boldsymbol{\gamma}(\mathbf{E}_s)\|_{H^{-1/2}(\text{div}, \partial B_x)} \neq 0} \frac{\|C^e(\boldsymbol{\gamma}(\mathbf{E}_s))\|_{H^{-1/2}(\text{div}, \partial B_x)}}{\|\boldsymbol{\gamma}(\mathbf{E}_s)\|_{H^{-1/2}(\text{div}, \partial B_x)}} \\ &= \sup_{\|\boldsymbol{\gamma}(\mathbf{E}_s)\|_{H^{-1/2}(\text{div}, \partial B_x)} \neq 0} \frac{\|\boldsymbol{\gamma}(\mathbf{H}_s)\|_{H^{-1/2}(\text{div}, \partial B_x)}}{\|\boldsymbol{\gamma}(\mathbf{E}_s)\|_{H^{-1/2}(\text{div}, \partial B_x)}} \end{aligned}$$

The result of the lemma follows by comparing coefficients in (4.7) and (4.8) applying Lemma 4.3, which certifies the boundedness of the expression. \square

In Figure 3, we depict the norm $\|C^e\|_{H^{-1/2}(\text{div}, \partial B_x)}$ as a function of the radius, or the frequency of the applied field. The shape of the plot is due to the coupling between the different modes in the expansions in the norms in (4.7) and (4.8). Each local minima at the cusps correspond to a shift from one mode to the next one. Larger radius and higher frequencies correspond to higher order modes at which the mapping between the tangential electric and magnetic fields is the strongest. The minimum at $k_0x \approx 0.6490$ is the root of

$$\frac{1}{\kappa} = |t_\infty| = \frac{1}{|t_1|} = \frac{\sqrt{3\kappa}\sqrt{1+\kappa^2}}{\sqrt{1-\kappa^2+\kappa^4}} \Rightarrow 3\kappa^6 + 2\kappa^4 + \kappa^2 - 1 = 0$$

Lemma 4.5. *Outside the least circumscribing sphere, $x \geq R$, we have for any spherical surface $R \leq x_1 \leq x_2$*

$$\begin{aligned} 2\|\mathbf{F}\|_\Gamma^2 &\leq \|\boldsymbol{\gamma}_+(\mathbf{E}_s)\|_{L_t^2(\partial B(\mathbf{0}, x_2); \mathbb{C}^3)}^2 + \|\boldsymbol{\gamma}_+(\mathbf{H}_s)\|_{L_t^2(\partial B(\mathbf{0}, x_2); \mathbb{C}^3)}^2 \\ &\leq \|\boldsymbol{\gamma}_+(\mathbf{E}_s)\|_{L_t^2(\partial B(\mathbf{0}, x_1); \mathbb{C}^3)}^2 + \|\boldsymbol{\gamma}_+(\mathbf{H}_s)\|_{L_t^2(\partial B(\mathbf{0}, x_1); \mathbb{C}^3)}^2 \end{aligned}$$

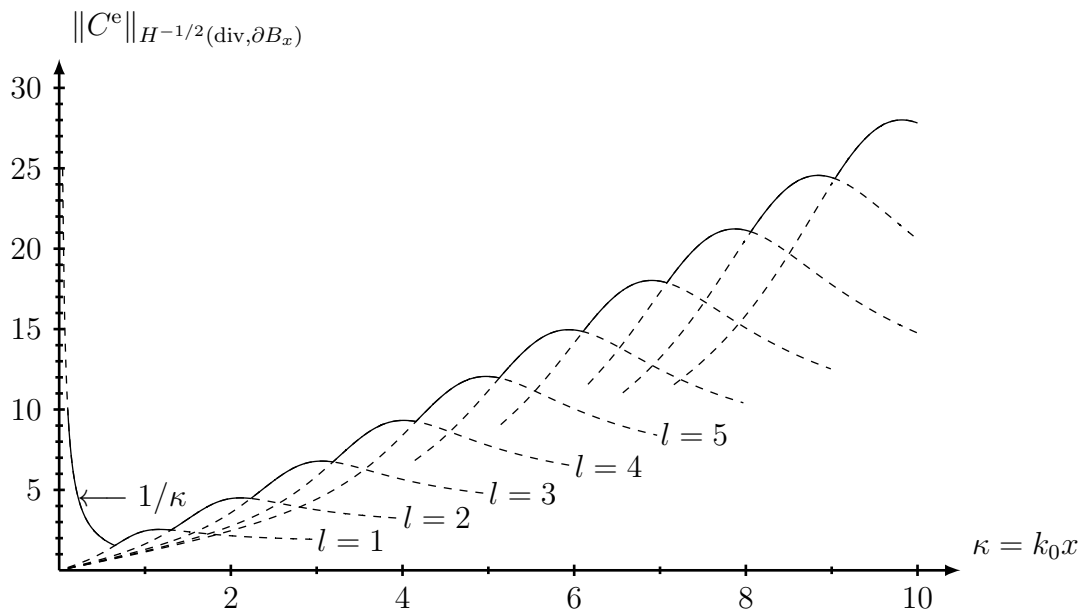


Figure 3: The norm of the exterior Calderón operator $\|C^e\|_{H^{-1/2}(\text{div}, \partial B_x)}$ for a sphere of radius x is depicted. The minimum of $\|C^e\|_{H^{-1/2}(\text{div}, \partial B_x)}(k_0 x) \approx 1.541$ is obtained at $k_0 x \approx 0.6490$. Below the minimum point $|t_l|$ gives the maximum, and above this value $1/|t_l|$ gives the maximum. The dashed lines depict the function $1/|t_l(k_0 x)|$ for $l = 1, 2, 3, \dots, 10$.

and

$$\begin{cases} \|\mathbf{E}_s\|_{L^2(\partial B(\mathbf{0}, x_2); \mathbb{C}^3)} \leq \|\mathbf{E}_s\|_{L^2(\partial B(\mathbf{0}, x_1); \mathbb{C}^3)} \\ \|\mathbf{H}_s\|_{L^2(\partial B(\mathbf{0}, x_2); \mathbb{C}^3)} \leq \|\mathbf{H}_s\|_{L^2(\partial B(\mathbf{0}, x_1); \mathbb{C}^3)} \end{cases}$$

The limits

$$\begin{aligned} \lim_{x \rightarrow \infty} \|\gamma_+(\mathbf{E}_s)\|_{L_t^2(\partial \Omega_x; \mathbb{C}^3)} &= \lim_{x \rightarrow \infty} \|\gamma_+(\mathbf{H}_s)\|_{L_t^2(\partial \Omega_x; \mathbb{C}^3)} \\ &= \lim_{x \rightarrow \infty} \|\mathbf{E}_s\|_{L^2(\partial \Omega_x; \mathbb{C}^3)} = \lim_{x \rightarrow \infty} \|\mathbf{H}_s\|_{L^2(\partial \Omega_x; \mathbb{C}^3)} = \|\mathbf{F}\|_{\Gamma} \end{aligned}$$

hold.

Proof. The lemma follows immediately from Lemma 4.2 and Lemma C.1 in Appendix C. \square

Similarly, for the $H^{-1/2}(\text{div}, \partial \Omega)$ norm, we have

Lemma 4.6. *Outside the least circumscribing sphere, $x \geq R$, we have for any spherical surface $R \leq x_1 \leq x_2$*

$$\begin{aligned} \|\mathbf{F}\|_{H^{-1/2}(\text{div}, \Gamma)}^2 + \|\mathbf{F}\|_{H^{-1/2}(\text{curl}, \Gamma)}^2 &\leq \|\gamma_+(\mathbf{E}_s)\|_{H^{-1/2}(\text{div}, \partial B_{x_2})}^2 + \|\gamma_+(\mathbf{H}_s)\|_{H^{-1/2}(\text{div}, \partial B_{x_2})}^2 \\ &\leq \|\gamma_+(\mathbf{E}_s)\|_{H^{-1/2}(\text{div}, \partial B_{x_1})}^2 + \|\gamma_+(\mathbf{H}_s)\|_{H^{-1/2}(\text{div}, \partial B_{x_1})}^2 \end{aligned}$$

The limit

$$\begin{aligned} \|\mathbf{F}\|_{H^{-1/2}(\operatorname{div},\Gamma)}^2 + \|\mathbf{F}\|_{H^{-1/2}(\operatorname{curl},\Gamma)}^2 \\ = \lim_{\kappa \rightarrow \infty} \|\gamma(\mathbf{E}_s)\|_{H^{-1/2}(\operatorname{div},\partial B_x)}^2 + \|\gamma(\mathbf{H}_s)\|_{H^{-1/2}(\operatorname{div},\partial B_x)}^2 \end{aligned}$$

holds.

Proof. From Lemmas 4.2, we have

$$\begin{aligned} & \|\gamma(\mathbf{E}_s)\|_{H^{-1/2}(\operatorname{div},\partial B_x)}^2 + \|\gamma(\mathbf{H}_s)\|_{H^{-1/2}(\operatorname{div},\partial B_x)}^2 \\ &= \frac{1}{k_0^2} \sum_n \left\{ \frac{|\xi'_l(\kappa)|^2 |f_{2n}|^2}{\sqrt{1+l(l+1)}} + \sqrt{1+l(l+1)} |\xi_l(\kappa)|^2 |f_{1n}|^2 \right\} \\ &+ \frac{1}{k_0^2} \sum_n \left\{ \frac{|\xi'_l(\kappa)|^2 |f_{1n}|^2}{\sqrt{1+l(l+1)}} + \sqrt{1+l(l+1)} |\xi_l(\kappa)|^2 |f_{2n}|^2 \right\} \\ &= \frac{1}{k_0^2} \sum_n \frac{|\xi'_l(\kappa)|^2 + (1+l(l+1)) |\xi_l(\kappa)|^2}{\sqrt{1+l(l+1)}} (|f_{1n}|^2 + |f_{2n}|^2) \\ &\geq \frac{1}{k_0^2} \sum_n \frac{2+l(l+1)}{\sqrt{1+l(l+1)}} (|f_{1n}|^2 + |f_{2n}|^2) = \\ &\|\mathbf{F}\|_{H^{-1/2}(\operatorname{div},\Gamma)}^2 + \|\mathbf{F}\|_{H^{-1/2}(\operatorname{curl},\Gamma)}^2 \end{aligned}$$

Since, by Lemma C.1, the functions $|\xi'_l(\kappa)|^2 + |\xi_l(\kappa)|^2$ and $|\xi_l(\kappa)|^2$ are decreasing functions of κ , the lemma follows. Moreover, in the limit $\kappa \rightarrow \infty$, we obtain

$$\begin{aligned} \lim_{\kappa \rightarrow \infty} \|\gamma(\mathbf{E}_s)\|_{H^{-1/2}(\operatorname{div},\partial B_x)}^2 + \|\gamma(\mathbf{H}_s)\|_{H^{-1/2}(\operatorname{div},\partial B_x)}^2 \\ = \|\mathbf{F}\|_{H^{-1/2}(\operatorname{div},\Gamma)}^2 + \|\mathbf{F}\|_{H^{-1/2}(\operatorname{curl},\Gamma)}^2 \end{aligned}$$

by Lemma C.1, once again. □

5 Conclusions

In this paper, we present some general estimates of the total and scattered fields in a general anisotropic setting. The exterior Calderón operator is employed to solve the exterior scattering problem. The estimates of the fields are expressed in the natural norms $H(\operatorname{curl},\Omega)$ and $H^{-1/2}(\operatorname{div},\partial\Omega)$, for volume and surface estimates, respectively. Estimates on an enclosing spherical surface are treated in detail. Moreover, pointwise estimates of the scattered field in the supremum norm and the L^2 norm on an exterior volume are given.

Appendix A Function spaces

In this appendix, we list the various function spaces used in this paper. Let Ω be a bounded, open, simply connected set in \mathbb{R}^3 with Lipschitz and connected boundary $\partial\Omega$.

The space $C(\Omega)$ is the space of continuous functions in Ω . We also use $C_0(\overline{\Omega})$ which consists of all uniformly continuous functions, which are zero at the boundary. The space $C^\infty(\Omega)$ is the space of infinitely continuously differentiable functions in Ω , and $C_0^\infty(\Omega)$ are the functions in this space with compact support in Ω , which we also denote $D(\Omega)$.

Several function spaces with square integrable functions are used in this paper. The basic space is $(u : \Omega \subset \mathbb{R}^3 \mapsto \mathbb{C})$

$$L^2(\Omega) \stackrel{\text{def}}{=} \left\{ u(\mathbf{x}) : u \text{ Lebesgue integrable in } \Omega, \int_{\Omega} |u(\mathbf{x})|^2 \, dv < \infty \right\}$$

with scalar product and norm

$$(u, v)_{L^2(\Omega)} = \int_{\Omega} u(\mathbf{x}) \overline{v(\mathbf{x})} \, dv, \quad \|u\|_{L^2(\Omega)} = \left\{ \int_{\Omega} |u(\mathbf{x})|^2 \, dv \right\}^{1/2} = (u, u)_{L^2(\Omega)}^{1/2}$$

where bar denotes the complex conjugate. Similarly for vector-valued spaces we have the scalar product

$$(\mathbf{u}, \mathbf{v})_{L^2(\Omega; \mathbb{C}^3)} = \int_{\Omega} \mathbf{u}(\mathbf{x}) \cdot \overline{\mathbf{v}(\mathbf{x})} \, dv$$

and the norm

$$\|\mathbf{u}\|_{L^2(\Omega; \mathbb{C}^3)} = \left\{ \int_{\Omega} |\mathbf{u}(\mathbf{x})|^2 \, dv \right\}^{1/2} = (\mathbf{u}, \mathbf{u})_{L^2(\Omega; \mathbb{C}^3)}^{1/2}$$

where \cdot and $|\cdot|$ denotes the Euclidean scalar product and norm in \mathbb{C}^3 , respectively. We also define the function spaces

$$\begin{cases} H(\text{div}, \Omega) \stackrel{\text{def}}{=} \{ \mathbf{u} \in L^2(\Omega; \mathbb{C}^3) : \nabla \cdot \mathbf{u} \in L^2(\Omega) \} \\ H(\text{curl}, \Omega) \stackrel{\text{def}}{=} \{ \mathbf{u} \in L^2(\Omega; \mathbb{C}^3) : \nabla \times \mathbf{u} \in L^2(\Omega; \mathbb{C}^3) \} \end{cases}$$

which are Hilbert spaces with norms

$$\begin{cases} \|\mathbf{u}\|_{H(\text{div}, \Omega)} = \left(\|\mathbf{u}\|_{L^2(\Omega; \mathbb{C}^3)}^2 + \|\nabla \cdot \mathbf{u}\|_{L^2(\Omega)}^2 \right)^{1/2} \\ \|\mathbf{u}\|_{H(\text{curl}, \Omega)} = \left(\|\mathbf{u}\|_{L^2(\Omega; \mathbb{C}^3)}^2 + \|\nabla \times \mathbf{u}\|_{L^2(\Omega; \mathbb{C}^3)}^2 \right)^{1/2} \end{cases}$$

The curl and the divergence are defined in the weak sense as

$$\begin{cases} (\nabla \times \mathbf{u}, \boldsymbol{\phi})_{L^2(\Omega; \mathbb{C}^3)} = (\mathbf{u}, \nabla \times \boldsymbol{\phi})_{L^2(\Omega; \mathbb{C}^3)}, & \forall \boldsymbol{\phi} \in D(\Omega; \mathbb{C}^3) \\ (\nabla \cdot \mathbf{u}, \phi)_{L^2(\Omega)} = -(\mathbf{u}, \nabla \phi)_{L^2(\Omega; \mathbb{C}^3)}, & \forall \phi \in D(\Omega) \end{cases}$$

In the exterior region, we define spaces of locally integrable functions as

$$\begin{cases} H_{\text{loc}}(\text{div}, \overline{\Omega}_e) \stackrel{\text{def}}{=} \{ \mathbf{u} \in D'(\Omega_e; \mathbb{C}^3) : \xi \mathbf{u} \in H(\text{div}, \Omega_e), \forall \xi \in D(\mathbb{R}^3) \} \\ H_{\text{loc}}(\text{curl}, \overline{\Omega}_e) \stackrel{\text{def}}{=} \{ \mathbf{u} \in D'(\Omega_e; \mathbb{C}^3) : \xi \nabla \times \mathbf{u} \in H(\text{curl}, \Omega_e), \forall \xi \in D(\mathbb{R}^3) \} \end{cases}$$

where $\Omega_e = \mathbb{R}^3 \setminus \overline{\Omega}$ and $D'(\Omega_e)$ is the space of distributions.

A.1 Trace and lifting operators

On the boundary, we have the L^2 spaces

$$L^2(\partial\Omega) = \left\{ u : \int_{\partial\Omega} |u(\mathbf{x})|^2 \, dS < \infty \right\}$$

where dS denotes the surface measure of $\partial\Omega$. For the vector-valued functions, we have

$$\begin{cases} L^2(\partial\Omega; \mathbb{C}^3) = \left\{ \mathbf{u} : \int_{\partial\Omega} |\mathbf{u}(\mathbf{x})|^2 \, dS < \infty \right\} \\ L^2_t(\partial\Omega; \mathbb{C}^3) = \left\{ \mathbf{u} : \mathbf{u} \cdot \hat{\boldsymbol{\nu}} = 0 \text{ and } \int_{\partial\Omega} |\mathbf{u}(\mathbf{x})|^2 \, dS < \infty \right\} \end{cases}$$

The scalar products and norms are

$$(u, v)_{L^2(\partial\Omega)} = \int_{\partial\Omega} u(\mathbf{x}) \overline{v(\mathbf{x})} \, dS, \quad \|u\|_{L^2(\partial\Omega)} = \left\{ \int_{\partial\Omega} |u(\mathbf{x})|^2 \, dS \right\}^{1/2}$$

and

$$(\mathbf{u}, \mathbf{v})_{L^2(\partial\Omega; \mathbb{C}^3)} = \int_{\partial\Omega} \mathbf{u}(\mathbf{x}) \cdot \overline{\mathbf{v}(\mathbf{x})} \, dS, \quad \|\mathbf{u}\|_{L^2(\partial\Omega; \mathbb{C}^3)} = \left\{ \int_{\partial\Omega} |\mathbf{u}(\mathbf{x})|^2 \, dS \right\}^{1/2}$$

The appropriate trace spaces which we use in this paper are $H^{-1/2}(\text{div}, \partial\Omega)$ and $H^{-1/2}(\text{curl}, \partial\Omega)$ defined by

$$\begin{cases} H^{-1/2}(\text{div}, \partial\Omega) \stackrel{\text{def}}{=} \left\{ \mathbf{u} \in H^{-\frac{1}{2}}(\partial\Omega; \mathbb{C}^3), \hat{\boldsymbol{\nu}} \cdot \mathbf{u} = 0, \text{div}_{\partial\Omega} \mathbf{u} \in H^{-\frac{1}{2}}(\partial\Omega) \right\} \\ H^{-1/2}(\text{curl}, \partial\Omega) \stackrel{\text{def}}{=} \left\{ \mathbf{u} \in H^{-\frac{1}{2}}(\partial\Omega; \mathbb{C}^3), \hat{\boldsymbol{\nu}} \cdot \mathbf{u} = 0, \text{curl}_{\partial\Omega} \mathbf{u} \in H^{-\frac{1}{2}}(\partial\Omega) \right\} \end{cases}$$

where the surface divergence, $\text{div}_{\partial\Omega}$, and the surface rotation, $\text{curl}_{\partial\Omega}$, are defined by duality and restriction

$$\begin{cases} (\text{div}_{\partial\Omega} \mathbf{u}, \phi)_{L^2(\partial\Omega)} = -(\mathbf{u}, \text{grad}_{\partial\Omega} \phi)_{L^2(\partial\Omega; \mathbb{C}^3)}, & \forall \phi \in D(\partial\Omega) \\ \text{curl}_{\partial\Omega} \mathbf{u} = \hat{\boldsymbol{\nu}} \cdot (\nabla \times \mathbf{u})|_{\partial\Omega} \end{cases}$$

and the surface gradient, $\text{grad}_{\partial\Omega}$, is defined by the orthogonal projection of ∇ on the surface $\partial\Omega$, *i.e.*, $\text{grad}_{\partial\Omega} \phi = \boldsymbol{\pi}(\nabla \phi)$, where $\boldsymbol{\pi}$ is defined in Theorem A.1 below. With the assumptions made on the boundary $\partial\Omega$, $H^{-1/2}(\text{curl}, \partial\Omega)$ is the dual of $H^{-1/2}(\text{div}, \partial\Omega)$, *i.e.*, $(H^{-1/2}(\text{div}, \partial\Omega))' = H^{-1/2}(\text{curl}, \partial\Omega)$.

The following theorem is proved in [10]:

Theorem A.1. 1. The trace mapping $\boldsymbol{\pi} : H(\text{curl}, \Omega) \mapsto H^{-1/2}(\text{curl}, \partial\Omega)$, that assigns to any $\mathbf{u} \in H(\text{curl}, \Omega)$ its tangential component $\hat{\boldsymbol{\nu}} \times (\mathbf{u} \times \hat{\boldsymbol{\nu}})$, is continuous and surjective from $H(\text{curl}, \Omega)$ onto $H^{-1/2}(\text{curl}, \partial\Omega)$. That is

$$\|\boldsymbol{\pi}(\mathbf{u})\|_{H^{-1/2}(\text{curl}, \partial\Omega)} \leq C_\pi \|\mathbf{u}\|_{H(\text{curl}, \Omega)}, \quad \forall \mathbf{u} \in H(\text{curl}, \Omega)$$

2. The trace mapping $\boldsymbol{\gamma} : H(\text{curl}, \Omega) \mapsto H^{-1/2}(\text{div}, \partial\Omega)$, that takes $\mathbf{u} \in H(\text{curl}, \Omega)$ to its (rotated) tangential component $\hat{\boldsymbol{\nu}} \times \mathbf{u}$, is continuous and surjective from $H(\text{curl}, \Omega)$ onto $H^{-1/2}(\text{div}, \partial\Omega)$. That is

$$\|\boldsymbol{\gamma}(\mathbf{u})\|_{H^{-1/2}(\text{div}, \partial\Omega)} \leq C_\gamma \|\mathbf{u}\|_{H(\text{curl}, \Omega)}, \quad \forall \mathbf{u} \in H(\text{curl}, \Omega)$$

3. In both cases, a continuous lifting with zero divergence for these trace operators in $H(\text{curl}, \Omega)$ exists. More precisely, there exists an operator $\mathcal{R} : H^{-1/2}(\text{div}, \partial\Omega) \mapsto H(\text{curl}, \Omega)$ such that for every $\mathbf{m} \in H^{-1/2}(\text{div}, \partial\Omega)$ there exists a $\mathbf{u} \in H(\text{curl}, \Omega)$ satisfying $\boldsymbol{\gamma}(\mathbf{u}) = \mathbf{m}$, and

$$\|\mathcal{R}(\mathbf{m})\|_{H(\text{curl}, \Omega)} \leq C \|\mathbf{m}\|_{H^{-1/2}(\text{div}, \partial\Omega)}, \quad \forall \mathbf{m} \in H^{-1/2}(\text{div}, \partial\Omega)$$

Moreover, for any $\mathbf{u}, \mathbf{v} \in H(\text{curl}, \Omega)$, the following Stokes' formula holds:

$$(\nabla \times \mathbf{u}, \mathbf{v})_{L^2(\Omega; \mathbb{C}^3)} - (\mathbf{u}, \nabla \times \mathbf{v})_{L^2(\Omega; \mathbb{C}^3)} = (\boldsymbol{\gamma}(\mathbf{u}), \boldsymbol{\pi}(\mathbf{v}))_{L^2(\partial\Omega; \mathbb{C}^3)}$$

A.2 Lax-Milgram's theorem

We conclude this appendix by stating the Lax-Milgram theorem [5].

Theorem A.2 (Lax-Milgram). Assume that H is a Hilbert space, with norm $\|\cdot\|_H$. Moreover, assume

$$B : H \times H \mapsto \mathbb{C}$$

is a sesquilinear functional on H , for which there exists constants $a, b > 0$, such that

$$|B[u, v]| \leq a \|u\|_H \|v\|_H, \quad \forall u, v \in H$$

and

$$b \|u\|_H^2 \leq |B[u, u]|, \quad \forall u \in H$$

Finally, let $f : H \mapsto \mathbb{C}$ be a bounded linear functional on H .

Then there exists a unique $u \in H$ such that

$$B[u, v] = f(v), \quad \forall v \in H.$$

satisfying

$$\|u\|_H \leq \frac{1}{b} \|f\|_{H'}$$

where H' denotes the dual space of H with norm $\|\cdot\|_{H'}$.

Appendix B Vector spherical harmonics and spherical vector waves

The vector spherical harmonics are defined as [2]

$$\begin{cases} \mathbf{A}_{1n}(\hat{\mathbf{x}}) = \frac{1}{\sqrt{l(l+1)}} \nabla \times (\mathbf{x} Y_n(\hat{\mathbf{x}})) = \frac{1}{\sqrt{l(l+1)}} \nabla Y_n(\hat{\mathbf{x}}) \times \mathbf{x} \\ \mathbf{A}_{2n}(\hat{\mathbf{x}}) = \frac{1}{\sqrt{l(l+1)}} \mathbf{x} \nabla Y_n(\hat{\mathbf{x}}) \\ \mathbf{A}_{3n}(\hat{\mathbf{x}}) = \hat{\mathbf{x}} Y_n(\hat{\mathbf{x}}) \end{cases}$$

where the spherical harmonics are denoted by $Y_n(\hat{\mathbf{x}})$. The index n is a multi-index for the integer indices $l = 1, 2, 3, \dots$, $m = 0, 1, \dots, l$, and $\sigma = e, o$ (even and odd in the azimuthal angle). From these definitions we see that the first two vector spherical harmonics, $\mathbf{A}_{1n}(\hat{\mathbf{x}})$ and $\mathbf{A}_{2n}(\hat{\mathbf{x}})$, are tangential to the unit sphere Γ in \mathbb{R}^3 and they are related as

$$\begin{cases} \hat{\mathbf{x}} \times \mathbf{A}_{1n}(\hat{\mathbf{x}}) = \mathbf{A}_{2n}(\hat{\mathbf{x}}) \\ \hat{\mathbf{x}} \times \mathbf{A}_{2n}(\hat{\mathbf{x}}) = -\mathbf{A}_{1n}(\hat{\mathbf{x}}) \end{cases}$$

The vector spherical harmonics form an orthonormal set over the unit sphere Γ in \mathbb{R}^3 , *i. e.*,

$$\int_{\Gamma} \mathbf{A}_{\tau n}(\hat{\mathbf{x}}) \cdot \mathbf{A}_{\tau' n'}(\hat{\mathbf{x}}) \, d\Gamma = \delta_{nn'} \delta_{\tau\tau'}$$

where $d\Gamma$ is the surface measure on the unit sphere.

The L^2 -norm and the scalar product on the unit sphere Γ in \mathbb{R}^3 , $\|\cdot\|_{\Gamma}$ and $\langle \cdot, \cdot \rangle_{\Gamma}$ are

$$\|\mathbf{u}\|_{\Gamma}^2 = \int_{\Gamma} |\mathbf{u}(\hat{\mathbf{x}})|^2 \, d\Gamma \quad \langle \mathbf{u}, \mathbf{v} \rangle_{\Gamma} = \int_{\Gamma} \mathbf{u}(\hat{\mathbf{x}}) \cdot \overline{\mathbf{v}(\hat{\mathbf{x}})} \, d\Gamma \quad (\text{B.1})$$

Any $\mathbf{u}(\hat{\mathbf{x}}) \in L^2(\Gamma)$ has the expansion

$$\mathbf{u}(\hat{\mathbf{x}}) = \sum_{\substack{n \\ \tau=1,2,3}} b_{\tau n} \mathbf{A}_{\tau n}(\hat{\mathbf{x}})$$

with norm

$$\|\mathbf{u}\|_{\Gamma}^2 = \sum_{\substack{n \\ \tau=1,2,3}} |b_{\tau n}|^2$$

For a tangential field, $\mathbf{u}(\hat{\mathbf{x}}) \cdot \hat{\mathbf{x}} = 0$, the summation is only over $\tau = 1, 2$.

Moreover, any $\mathbf{u}(\hat{\mathbf{x}}) \in H^{-1/2}(\text{div}, \partial B_x)$ has the expansion [8, 9]

$$\mathbf{u}(\hat{\mathbf{x}}) = \sum_{\substack{n \\ \tau=1,2}} b_{\tau n} \mathbf{A}_{\tau n}(\hat{\mathbf{x}})$$

with norm

$$\begin{aligned} \|\mathbf{u}\|_{H^{-1/2}(\text{div}, \partial B_x)}^2 &= x^2 \sum_{\substack{n \\ \tau=1,2}} (1+l(l+1))^{\tau-3/2} |b_{\tau n}|^2 \\ &= x^2 \sum_n \left\{ \frac{1}{\sqrt{1+l(l+1)}} |b_{1n}|^2 + \sqrt{1+l(l+1)} |b_{2n}|^2 \right\} \end{aligned} \quad (\text{B.2})$$

The radiating solutions to the Maxwell equations in vacuum are defined as (outgoing spherical vector waves)

$$\begin{cases} \mathbf{u}_{1n}(k_0\mathbf{x}) = \frac{\xi_l(k_0x)}{k_0x} \mathbf{A}_{1n}(\hat{\mathbf{x}}) \\ \mathbf{u}_{2n}(k_0\mathbf{x}) = \frac{1}{k_0} \nabla \times \left(\frac{\xi_l(k_0x)}{k_0x} \mathbf{A}_{1n}(\hat{\mathbf{x}}) \right) \end{cases}$$

Here, we use the Riccati-Bessel functions $\xi_l(x) = x h_l^{(1)}(x)$ where $h_l^{(1)}(k_0x)$ is the spherical Hankel function of the first kind [11]. These vector waves satisfy

$$\nabla \times (\nabla \times \mathbf{u}_{\tau n}(k_0\mathbf{x})) - k_0^2 \mathbf{u}_{\tau n}(k_0\mathbf{x}) = \mathbf{0}, \quad \tau = 1, 2$$

and they also satisfy the radiation condition in (2.4). Another representation of the definition of the vector waves is

$$\begin{cases} \mathbf{u}_{1n}(k_0\mathbf{x}) = \frac{\xi_l(k_0x)}{k_0x} \mathbf{A}_{1n}(\hat{\mathbf{x}}) \\ \mathbf{u}_{2n}(k_0\mathbf{x}) = \frac{\xi'_l(k_0x)}{k_0x} \mathbf{A}_{2n}(\hat{\mathbf{x}}) + \sqrt{l(l+1)} \frac{\xi_l(k_0x)}{(k_0x)^2} \mathbf{A}_{3n}(\hat{\mathbf{x}}) \end{cases}$$

A simple consequence of these definitions is

$$\begin{cases} \mathbf{u}_{1n}(k_0\mathbf{x}) = \frac{1}{k_0} \nabla \times \mathbf{u}_{2n}(k_0\mathbf{x}) \\ \mathbf{u}_{2n}(k_0\mathbf{x}) = \frac{1}{k_0} \nabla \times \mathbf{u}_{1n}(k_0\mathbf{x}). \end{cases}$$

In a similar way, the regular spherical vector waves $\mathbf{v}_{\tau n}(k_0\mathbf{x})$ are defined [2].

$$\begin{cases} \mathbf{v}_{1n}(k_0\mathbf{x}) = j_l(k_0x) \mathbf{A}_{1n}(\hat{\mathbf{x}}) \\ \mathbf{v}_{2n}(k_0\mathbf{x}) = \frac{1}{k_0} \nabla \times (j_l(k_0x) \mathbf{A}_{1n}(\hat{\mathbf{x}})) \end{cases}$$

where $j_l(k_0x)$ is the spherical Bessel function of the first kind [11].

Appendix C Modulus of the spherical Hankel functions

Some useful monotonicity properties of the modulus of the spherical Hankel functions are derived in this appendix.

Lemma C.1. Define $\xi_n(x) = xh_n^{(1,2)}(x)$ for $x > 0$. Then for each fixed positive integer n , the function⁵

$$|\xi_n(x)|^2$$

and the combinations

$$\begin{cases} |\xi'_n(x)|^2 + |\xi_n(x)|^2 \\ |\xi'_n(x)|^2 + n(n+1) \frac{|\xi_n(x)|^2}{x^2} \end{cases}$$

are decreasing functions on the positive real axis. The limits as $x \rightarrow \infty$ are

$$\lim_{x \rightarrow \infty} |\xi_n(x)|^2 = \lim_{x \rightarrow \infty} |\xi'_n(x)|^2 = \lim_{x \rightarrow \infty} \left\{ |\xi'_n(x)|^2 + n(n+1) \frac{|\xi_n(x)|^2}{x^2} \right\} = 1$$

Proof. This non-trivial result is most conveniently proved with Nicholson's integral [14, (1) on p. 444]

$$|h_n^{(1,2)}(x)|^2 = j_n^2(x) + n_n^2(x) = \frac{4}{\pi x} \int_0^\infty K_0(2x \sinh t) \cosh[(2n+1)t] dt, \quad x > 0$$

where $K_0(x)$ is the modified Bessel function [11]. Since $K_0(x)$ is real-valued and decreasing on the positive real axis,⁶ $|h_n^{(1,2)}(x)|^2$ is a decreasing function on the positive real axis,⁷ *i.e.*,

$$\frac{d}{dx} |h_n^{(1,2)}(x)|^2 < 0, \quad x > 0$$

To prove that $|xh_n^{(1,2)}(x)|^2$ are decreasing functions, again use Nicholson's integral and integrate by parts. We get

$$\begin{aligned} \frac{d}{dx} (x^2 j_n^2(x) + x^2 n_n^2(x)) &= \frac{d}{dx} \frac{4x}{\pi} \int_0^\infty K_0(2x \sinh t) \cosh[(2n+1)t] dt \\ &= \frac{4}{\pi} \int_0^\infty \{K_0(2x \sinh t) + 2x \sinh t K_0'(2x \sinh t)\} \cosh[(2n+1)t] dt \\ &= \frac{4}{\pi} \left\{ \tanh t \cosh[(2n+1)t] K_0(2x \sinh t) \Big|_{t=0}^{t=\infty} \right. \\ &\quad \left. + \frac{4}{\pi} \int_0^\infty K_0(2x \sinh t) \left\{ \cosh[(2n+1)t] - \frac{d}{dt} (\tanh t \cosh[(2n+1)t]) \right\} dt \right\} \end{aligned}$$

⁵ $|xh_n^{(j)}(x)|$, $j = 1, 2$, are identical functions on the real axis.

⁶This is obvious from the integral representation

$$K_0(x) = \int_0^\infty e^{-x \cosh t} dt$$

⁷In fact, $|h_n^{(1,2)}(x)|^2$ and $x|h_n^{(1,2)}(x)|^2$ are decreasing for all real n .

The first term on the right-hand side vanishes, and we obtain

$$\begin{aligned} & \frac{d}{dx} (x^2 j_n^2(x) + x^2 n_n^2(x)) \\ &= \frac{4}{\pi} \int_0^\infty K_0(2x \sinh t) \cosh[(2n+1)t] \tanh t \{ \tanh t - (2n+1) \tanh[(2n+1)t] \} dt \end{aligned}$$

The integrand is negative, since

$$(2n+1) \tanh[(2n+1)t] - \tanh t$$

is positive for $t \geq 0$ and n a positive integer, and we conclude that

$$\frac{d}{dx} |xh_n^{(1,2)}(x)|^2 = \frac{d}{dx} (x^2 j_n^2(x) + x^2 n_n^2(x)) < 0, \quad x > 0$$

and the first part of the lemma is proved.

Using the differential equations, we obtain

$$\begin{aligned} \frac{d}{dx} |(xh_n^{(1)}(x))'|^2 &= (xh_n^{(1)}(x))''(xh_n^{(2)}(x))' + (xh_n^{(1)}(x))'(xh_n^{(2)}(x))'' \\ &= \left(\frac{n(n+1)}{x^2} - 1 \right) \{ xh_n^{(1)}(x)(xh_n^{(2)}(x))' + (xh_n^{(1)}(x))'xh_n^{(2)}(x) \} \\ &= \left(\frac{n(n+1)}{x^2} - 1 \right) \frac{d}{dx} |xh_n^{(1)}(x)|^2 \end{aligned}$$

From this identity we see that $|(xh_n^{(1)}(x))'|^2$ is not monotonic,⁸ but, on the other hand for $n = 1, 2, \dots$

$$\frac{d}{dx} \left\{ |(xh_n^{(1)}(x))'|^2 + |xh_n^{(1)}(x)|^2 \right\} = \frac{n(n+1)}{x^2} \frac{d}{dx} |xh_n^{(1)}(x)|^2 < 0, \quad x > 0$$

and

$$\begin{aligned} & \frac{d}{dx} \left\{ |(xh_n^{(1)}(x))'|^2 + n(n+1) |h_n^{(1)}(x)|^2 \right\} \\ &= \left(\frac{n(n+1)}{x^2} - 1 \right) \frac{d}{dx} |xh_n^{(1)}(x)|^2 + n(n+1) \frac{d}{dx} |xh_n^{(1)}(x)|^2 - n(n+1)2x |h_n^{(1)}(x)|^2 \\ &= \left(\frac{n(n+1)}{x^2} - 1 + n(n+1) \right) \frac{d}{dx} |xh_n^{(1)}(x)|^2 - n(n+1)2x |h_n^{(1)}(x)|^2 < 0, \quad x > 0 \end{aligned}$$

and the remaining two parts of the monotonicity are proved. The limit values are easily proved by the properties of the spherical Hankel functions for large real arguments. \square

⁸In fact, it has a minimum at $x = \sqrt{n(n+1)}$.

References

- [1] M. Artola. Homogenization and electromagnetic wave propagation in composite media with high conductivity inclusions. In G. Dal Maso and G. Dell’Antonio, editors, *Proceedings of the Second Workshop on Composite Media & Homogenization Theory*, Singapore, 1995. World Scientific Publisher.
- [2] A. Boström, G. Kristensson, and S. Ström. Transformation properties of plane, spherical and cylindrical scalar and vector wave functions. In V. V. Varadan, A. Lakhtakia, and V. K. Varadan, editors, *Field Representations and Introduction to Scattering*, Acoustic, Electromagnetic and Elastic Wave Scattering, chapter 4, pages 165–210. Elsevier Science Publishers, Amsterdam, 1991.
- [3] M. Cessenat. *Mathematical Methods in Electromagnetism*. Series on Advances in Mathematics for Applied Sciences — Vol. 41. World Scientific Publisher, Singapore, 1996.
- [4] D. Colton and R. Kress. *Integral Equation Methods in Scattering Theory*. John Wiley & Sons, New York, 1983.
- [5] L. C. Evans. *Partial Differential Equations*. American Mathematical Society, Providence, Rhode Island, 1998.
- [6] A. Greenleaf, Y. Kurylev, M. Lassas, and G. Uhlmann. Invisibility and inverse problems. *Bulletin of the American Mathematical Society*, **46**(1), 55–97, 2009.
- [7] A. Greenleaf, Y. Kurylev, M. Lassas, and G. Uhlmann. Cloaking devices, electromagnetic wormholes, and transformation optics. *SIAM Review*, **51**(1), 3–33, 2009.
- [8] A. Kirsch and P. Monk. A finite element/spectral method for approximating the time-harmonic Maxwell system in \mathbb{R}^3 . *SIAM J. Appl. Math.*, **55**(5), 1324–1344, 1995.
- [9] P. Monk. *Finite Element Methods for Maxwell’s Equations*. Oxford University Press, Oxford, 2003.
- [10] J. C. Nédélec. *Acoustic and electromagnetic equations: integral representations for harmonic problems*. Springer-Verlag, Berlin, 2001.
- [11] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark. *NIST Handbook of mathematical functions*. Cambridge University Press, New York, 2010.
- [12] G. Roach, I. Stratis, and A. Yannacopoulos. *Mathematical Analysis of Deterministic and Stochastic Problems in Complex Media Electromagnetics*. Princeton University Press, 2012.

- [13] S. Ström. Introduction to integral representations and integral equations for time-harmonic acoustic, electromagnetic and elastodynamic wave fields. In V. V. Varadan, A. Lakhtakia, and V. K. Varadan, editors, *Field Representations and Introduction to Scattering*, volume 1 of *Handbook on Acoustic, Electromagnetic and Elastic Wave Scattering*, chapter 2, pages 37–141. Elsevier Science Publishers, Amsterdam, 1991.
- [14] G. N. Watson. *A Treatise on the Theory of Bessel Functions*. Cambridge University Press, Cambridge, U.K., second edition, 1966.
- [15] N. Wellander and G. Kristensson. Homogenization of the Maxwell equations at fixed frequency. *SIAM J. Appl. Math.*, **64**(1), 170–195, 2003. doi: 10.1137/S0036139902403366.