Evaluation of some integrals relevant to multiple scattering by randomly distributed obstacles

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Evaluation of some integrals relevant to multiple scattering by randomly distributed obstacles

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Abstract

This paper analyzes and solves an integral and its indefinite Fourier transform of importance in multiple scattering problems of randomly distributed scatterers. The integrand contains a radiating spherical wave, and the two-dimensional domain of integration excludes a circular region of varying size. A solution of the integral in terms of radiating spherical waves is demonstrated. The method employs the Erdélyi operators, which leads to a recursion relation. This recursion relation is solved in terms of a finite sum of radiating spherical waves. The solution of the indefinite Fourier transform of the integral contains the indefinite Fourier transforms of the Legendre polynomials, which are solved by a closed formula.

1 Introduction

In recent years, the electromagnetic scattering problem by randomly distributed objects has been successfully formulated and solved. Some important contributions in the field are found in e.g., [3-8, 10, 11, 13, 16-19, 21-25]. These references refer to various aspects of the topic, and more references can be found in these papers. The topic is also treated in several textbooks, see e.g., [12, 14, 20], which can be consulted for a comprehensive treatment of the various multiple scattering theories.

Of critical importance for the solution of a specific scattering problem with hole-corrections (HC) is an integral of the form [9, 18, 20]

\[ I_l(z) = \frac{k^2}{2\pi} \int_{\mathbb{R}^2} H(r-a)h^{(1)}_l(kr)P_l(\cos \theta) \, dx \, dy, \quad z \in \mathbb{R} \]  

(1.1)

where \( H(x) \) denotes the Heaviside function, \( h^{(1)}_l(kr) \) the spherical Hankel function, and \( P_l(x) \) the Legendre polynomial of order \( l \), respectively. We have also adopted the spherical coordinates, \( r = \sqrt{x^2 + y^2 + z^2} \) and \( \theta (\cos \theta = z/r) \), and the wave number \( k \). The domain of integration is the plane \( z = \) constant, excluding the sphere of radius \( a > 0 \) at the center, see Figure 1. For a given value of \( |z| \leq a \), the radius of the excluded circle is \( \sqrt{a^2 - z^2} \). For \( |z| \geq a \) the integration is the entire \( x-y \) plane. This integral, for a given \( a > 0 \), is a non-trivial function of \( z \in \mathbb{R} \). To ensure convergence of the integral at infinity, we assume the wave number \( k \) has an arbitrarily small imaginary part. The explicit solution of this integral, as a function of \( z \) and the index \( l = 0, 1, 2, \ldots \), is the aim of this paper, and the goal is to express the solutions in a form that is attractive from a numerical computation point of view.

The solution of the integral \( I_l(z) \) is developed in Sections 2 and 3. The indefinite Fourier transform of \( I_l(z) \) is also essential for a successful solution of the multiple scattering problem with hole-corrections, and this analysis is found in Sections 4 and 5. The paper is concluded with a short summary in Section 6.
2 The integral $I_l(z)$

Rewrite the integral $I_l(z)$ in (1.1) in cylindrical coordinates and perform the integration in the azimuthal angle. We get from (1.1)

$$I_l(z) = k^2 \int_{h(z)}^{\infty} h_l^{(1)} \left( k \sqrt{\rho^2 + z^2} \right) P_l \left( \frac{z}{\sqrt{\rho^2 + z^2}} \right) \rho \, d\rho, \quad z \in \mathbb{R} \quad (2.1)$$

where

$$h(z) = \begin{cases} \sqrt{a^2 - z^2}, & -a \leq z \leq a \\ 0, & |z| > a \end{cases}$$

From the parity of the Legendre polynomials, $P_l(-x) = (-1)^l P_l(x)$, we see that also $I_l(-z) = (-1)^l I_l(z)$. Thus, it suffices to evaluate the integral for $z > 0$. In particular, $I_l(0) = 0$, if $l$ is an odd integer. From (2.1) we also easily compute the integral for $l = 0$, viz.,

$$I_0(z) = \begin{cases} e^{-ikz}, & z \leq -a \\ ikah_0^{(1)}(ka) = e^{ika}, & -a \leq z \leq a \\ e^{ikz}, & z \geq a \end{cases}$$

2.1 Solution outside the interval $[-a, a]$

In the interval $z > a$, the integral is evaluated with the use of the transformation of the outgoing scalar spherical wave in terms of planar waves [2, p. 180], i.e., for a
general value of $z \neq 0$

$$
\left( k \sqrt{\rho^2 + z^2} \right) P_l \left( \pm z / \sqrt{\rho^2 + z^2} \right) = \frac{i^{-l}}{2\pi} \int_{\mathbb{R}^2} P_l \left( \pm k_z / k \right) e^{ik \cdot \rho} \frac{k}{k_z} \frac{dk_x dk_y}{k^2}, \quad z \geq 0
$$

where $\rho = x \hat{x} + y \hat{y}$, $k_t = k_x \hat{x} + k_y \hat{y}$, $k_t = |k_t|$, and $k_z$ is defined by

$$
k_z = (k^2 - k_t^2)^{1/2} = \begin{cases} 
\sqrt{k^2 - k_t^2} & \text{for } k_t < k \\
\sqrt{k_t^2 - k^2} & \text{for } k_t > k
\end{cases}
$$

For $z > a$, we get from (1.1)

$$
I_l(z) = \frac{k^2}{2\pi} \int_{\mathbb{R}^2} \left( \frac{1}{2\pi} \int_{\mathbb{R}^2} P_l \left( k_z / k \right) e^{ik \cdot \rho} \frac{k}{k_z} \frac{dk_x dk_y}{k^2} \right) dx dy
$$

by orthogonality or completeness of the planar waves.\(^3\) This implies that the integral for $z > a$ is

$$
I_l(z) = i^{-l} e^{ikz}, \quad z > a
$$

and consequently, by parity, or analogous calculations

$$
I_l(z) = i^l e^{-ikz}, \quad z < -a
$$

We observe that the integral outside the interval $[-a, a]$ is not singular as $a \to 0$. In fact, the module is constant 1.

### 3 Solution of the integral $I_l(\eta), -a \leq z \leq a$

We have already obtained a solution of the integral in the interval $|z| > a$, and we now concentrate on finding a solution of the integral in the interval $-a \leq z \leq a$.

The Erdélyi operators $\mathcal{Y}_n^m$ in Ref. 12 are instrumental in finding a closed formula for the integral $I_l(z)$. From [12, Th. 3.13], we have the following very useful result:

$$
D \left( h_l^{(1)}(kr) P_l(\cos \theta) \right) = \frac{l + 1}{2l + 1} h_{l+1}^{(1)}(kr) P_{l+1}(\cos \theta) - \frac{l - 1}{2l + 1} h_{l-1}^{(1)}(kr) P_{l-1}(\cos \theta)
$$

where $D = -k^{-1}(\partial / \partial z)$. The $D$ operator and the Erdélyi operators are related by $\mathcal{Y}^0 = \sqrt{\frac{2}{4\pi}} D^0 = \sqrt{\frac{2}{4\pi}} D$.

\(^3\)To ensure convergence of the integral at infinity, assume the wave number $k$ has an arbitrary small, positive imaginary part.
Apply the differential operator \( D \) to the integral \( I_l(z) \) in (2.1), and use the relation above. We obtain, since \( h'(z)h(z) = -z \), the following recursion relation:

\[
DI_l(z) = -kzh_l^{(1)}(ka)P_l(z/a) + \frac{l+1}{2l+1}I_{l+1}(z) - \frac{l}{2l+1}I_l(z), \quad -a \leq z \leq a
\]

with initial condition \( I_0(z) = ikh_0^{(1)}(ka) \).

In the dimensionless variables \( \eta = kz \) and \( \xi = ka > 0 \), this leads to the recursion relation, \( l = 0, 1, 2, \ldots \) (note the mild change in notation)

\[
I_{l+1}(\eta) = \frac{2l+1}{l+1}\xi h_l^{(1)}(\xi)\frac{\eta}{\xi}P_l(\eta/\xi) - \frac{2l+1}{l+1}\frac{d}{d\eta}I_l(\eta) + \frac{l}{l+1}I_{l-1}(\eta), \quad -\xi \leq \eta \leq \xi
\]

The recursion relation is conveniently put in a more generic form by introducing the variable \( x = \eta/\xi \in [-1, 1] \). The dependent variable is now \( x \), and \( \xi \) is a parameter. Retaining the same notation for the integral, but with a change of the independent variable, we get

\[
I_{l+1}(x) = \frac{2l+1}{l+1}\xi h_l^{(1)}(\xi)xP_l(x) - \frac{2l+1}{\xi(l+1)}I_l'(x) + \frac{l}{l+1}I_{l-1}(x), \quad -1 \leq x \leq 1
\]

The following proposition states the surprisingly simple and elegant solution of this recursion relation.

**Proposition 3.1.** The recursion relation

\[
I_{l+1}(x) = \frac{2l+1}{l+1}\xi h_l^{(1)}(\xi)xP_l(x) - \frac{2l+1}{\xi(l+1)}I_l'(x) + \frac{l}{l+1}I_{l-1}(x), \quad l = 0, 1, 2, \ldots \quad (3.1)
\]

with initial condition

\[
I_0(z) = i\xi h_0^{(1)}(\xi)
\]

has the solution

\[
I_l(x) = -\xi h_l^{(1)}(\xi)P_l(x)
\]

\[
+ \sum_{k=0}^{[l/2]} (-1)^k \left( \xi h_{l+1-2k}^{(1)}(\xi) + \xi h_{l-1-2k}^{(1)}(\xi) \right) P_{l-2k}(x), \quad l = 0, 1, 2, \ldots \quad (3.2)
\]

\(^2\)Outside the interval \( z \in [-a, a] \) the recursion relation reads

\[
I_{l+1}(z) = \frac{2l+1}{l+1}DI_l(z) + \frac{l}{l+1}I_{l-1}(z), \quad I_0(z) = e^{ikz} \quad z \geq a
\]

which is easily solved by induction over the integer \( l \). The result is

\[
I_l(z) = i^{-l}e^{ikz}, \quad z \geq a
\]

in agreement with the result above.
**Proof.** We prove the proposition by induction over the integer \( l \). The recursion relation (3.2) is true for \( l = 0 \), due to the properties of the spherical Hankel functions [15, 10.16.1]. We have from (3.2)

\[
I_0(x) = \xi h^{(1)}_0(\xi) = \xi \left( \frac{\pi}{2\xi} \right)^{1/2} H^{(1)}_{-1/2}(\xi) = i\xi \left( \frac{\pi}{2\xi} \right)^{1/2} H^{(1)}_{1/2}(\xi) = i\xi h^{(1)}_0(\xi)
\]

Now assume the solution (3.2) holds for all integers less than or equal to \( l \), and we want to prove that it holds for \( l + 1 \). We have from (3.1) and the induction assumption

\[
I_{l+1}(x) = \frac{2l+1}{l+1} \xi h^{(1)}_l(\xi) x P_l(x) - \frac{2l+1}{\xi(l+1)} I'_l(x) + \frac{l}{l+1} I_{l-1}(x)
\]

\[
= \xi h^{(1)}_l(\xi) P_{l+1}(x) + \frac{2l+1}{\xi(l+1)} \xi h^{(1)}_{l+1}(\xi) P'_l(x)
\]

\[
- \frac{2l+1}{\xi(l+1)} \sum_{k=0}^{[l/2]} (-1)^k \left( \xi h^{(1)}_{l+1-2k}(\xi) + \xi h^{(1)}_{l-1-2k}(\xi) \right) P_{l-2k}(x)
\]

\[
+ \frac{l}{l+1} \sum_{k=0}^{[(l-1)/2]} (-1)^k \left( \xi h^{(1)}_{l-2k}(\xi) + \xi h^{(1)}_{l-2-2k}(\xi) \right) P_{l-2k}(x)
\]

where we used the following recursion relation for the Legendre polynomials:

\[
(2l+1)x P_l(x) = (l+1) P_{l+1}(x) + l P_{l-1}(x)
\]

We conclude that \( I_{l+1}(x) \) is a polynomial in \( x \) of the order \( l + 1 \), and therefore can be expanded in a series of Legendre polynomials. The form is

\[
I_{l+1}(x) = \sum_{n=0}^{[(l+1)/2]} a_n P_{l+1-2n}(x)
\]

where \( a_n \) depends on \( l \) and \( \xi \). The coefficients \( a_n \) are determined by orthogonality of the Legendre polynomials.

\[
a_n = \frac{2l+3-4n}{2} \int_{-1}^{1} I_{l+1}(x) P_{l+1-2n}(x) \, dx
\]

The first coefficient is special.

\[
a_0 = \xi h^{(1)}_l(\xi) = -\xi h^{(1)}_{l+2}(\xi) + \left( \xi h^{(1)}_{l+2}(\xi) + \xi h^{(1)}_l(\xi) \right)
\]

Proceed in the same way with the remaining coefficients, \( n = 1, 2, \ldots, [(l+1)/2] \).

\[
a_n = \frac{2l+1}{l+1} \frac{2l+3-4n}{2} h^{(1)}_{l+1}(\xi) I_{l+1-2n}
\]

\[
- \frac{2l+1}{l+1} \frac{2l+3-4n}{2} \sum_{k=0}^{[l/2]} (-1)^k \left( h^{(1)}_{l+1-2k}(\xi) + h^{(1)}_{l-1-2k}(\xi) \right) I_{l-2k,l+1-2n}
\]

\[
+ \frac{l}{l+1} \sum_{k=0}^{[(l-1)/2]} (-1)^k \left( \xi h^{(1)}_{l-2k}(\xi) + \xi h^{(1)}_{l-2-2k}(\xi) \right) \delta_{k,n-1}
\]
where we used the notion
\[
I_{k,n} = \int_{-1}^{1} P'_k(x)P_n(x) \, dx = \begin{cases} 0, & 0 \leq k \leq n \\ 1 - (-1)^{k+n}, & 0 \leq n < k \end{cases}
\]

Use this result, and the following recursion relation for the spherical Hankel functions:
\[
(2l + 1)h_{l}^{(1)}(\xi) = \xi h_{l+1}^{(1)}(\xi) + \xi h_{l-1}^{(1)}(\xi)
\]  
(3.3)

We get
\[
a_n = \frac{2l + 1}{l + 1} (2l + 3 - 4n) \left( h_{l+1}^{(1)}(\xi) - \sum_{k=0}^{n-1} (-1)^k \left( h_{l+1-2k}^{(1)}(\xi) + h_{l-1-2k}^{(1)}(\xi) \right) \right)
\]
\[
+ \frac{l}{l+1} (-1)^{n-1} \left( \xi h_{l+2-2n}^{(1)}(\xi) + \xi h_{l-2n}^{(1)}(\xi) \right)
\]
\[
= \frac{2l + 1}{l + 1} (-1)^n (2l + 3 - 4n)h_{l+1}^{(1)}(\xi) - \frac{l}{l+1} (-1)^n \left( \xi h_{l+2-2n}^{(1)}(\xi) + \xi h_{l-2n}^{(1)}(\xi) \right)
\]
\[
= (-1)^n \left( \xi h_{l+2-2n}^{(1)}(\xi) + \xi h_{l-2n}^{(1)}(\xi) \right)
\]

Collecting the results gives
\[
I_{l+1}(x) = \sum_{n=0}^{[(l+1)/2]} a_n P_{l+1-2n}(x)
\]
\[
= -\xi h_{l+2}^{(1)}(\xi)P_{l+1}(x) + \sum_{n=0}^{[(l+1)/2]} (-1)^n \left( \xi h_{l+2-2n}^{(1)}(\xi) + \xi h_{l-2n}^{(1)}(\xi) \right) P_{l+1-2n}(x)
\]

which is the statement (3.2) for \( l + 1 \), and the proposition is proved. \( \square \)

Alternative expressions of the integral \( I(z) \) in the interval \( z \in [-a, a] \) can be found. The following corollary shows some.

**Corollary 3.1.** The integral \( I(z) \) in Proposition 3.1 has the following alternative expressions:

\[
I_l(x) = -\xi h_{l+1}^{(1)}(\xi)P_l(x) + \sum_{k=0}^{[l/2]} (-1)^k (2l - 4k + 1)h_{l-2k}^{(1)}(\xi)P_{l-2k}(x), \quad l = 0, 1, 2, \ldots
\]  
(3.4)

and

\[
I_l(x) = i^{l-1}\xi h_{l}^{(1)}(\xi)\left( P_{l}^{(1)}(x) - P_{l}^{(2)}(x) \right) + \sum_{k=0}^{[l/2]-1} (-1)^k \xi h_{l-2k-1}^{(1)}(\xi) \left( P_{l-2k}(x) - P_{l-2k-2}(x) \right), \quad l = 0, 1, 2, \ldots
\]  
(3.5)
and

\[ I_l(x) = i^{l-1} \xi h_0^{(1)}(\xi) P_{l-2\lfloor l/2 \rfloor}(x) \]
\[ - \sum_{k=0}^{[l/2]-1} (-1)^k \xi h_{l-2k-1}^{(1)}(\xi) \frac{2l - 4k - 1}{(l - 2k)(l - 2k)} P_{l-2k-1}(x), \quad l = 0, 1, 2, \ldots \tag{3.6} \]

where the two last sums are zero for \( l = 0, 1 \).

**Proof.** The solution in (3.4) is equivalent to (3.2), which is easily seen since the spherical Hankel functions \( h_l^{(1)}(\xi) \) satisfy the recursion relation (3.3). The representation in (3.5) is simply a rearrangement of the sum in (3.2). We obtain from (3.2)

\( I_l(x) = \sum_{k=0}^{[l/2]-1} (-1)^k \xi h_{l-1-2k}^{(1)}(\xi) (P_{l-2k}(x) - P_{l-2-2k}(x)) \)
\[ + (-1)^{[l/2]} \xi h_{l-2\lfloor l/2 \rfloor}^{(1)}(\xi) P_{l-2\lfloor l/2 \rfloor}(x) \]
\[ = \sum_{k=0}^{[l/2]-1} (-1)^k \xi h_{l-2k}^{(1)}(\xi) (P_{l-2k}(x) - P_{l-2-2k}(x)) \]
\[ + i^{l-1} \xi h_0^{(1)}(\xi) P_{l-2\lfloor l/2 \rfloor}(x) \]

where we used [15, 10.16.1]

\[ h_l^{(1)}(\xi) = i h_0^{(1)}(\xi) \]

Finally, the relation (3.6) from (3.5) with the use of the recursion relation

\[ l(l+1) (P_{l+1}(x) - P_{l-1}(x)) = -(2l + 1)(1 - x^2) P'_l(x) \]

\[ \square \]

In the original variables \( z \) and \( a \), we have

\[ I_l(z) = i^{l-1} k a h_0^{(1)}(ka) P_{l-2\lfloor l/2 \rfloor}(z/a) \]
\[ + \sum_{n=0}^{[l/2]-1} (-1)^n k ah_{l-2n-1}^{(1)}(ka) (P_{l-2n}(z/a) - P_{l-2n-2}(z/a)), \quad l = 0, 1, 2, \ldots \]

or

\[ I_l(z) = -kah_{l+1}^{(1)}(ka) P_{l}(z/a) \]
\[ + \sum_{k=0}^{[l/2]} (-1)^k (2l - 4k + 1) h_{l-2k}^{(1)}(ka) P_{l-2k}(z/a), \quad l = 0, 1, 2, \ldots \]

and we see that the integral \( I_l(z) \) can be written as a finite sum of spherical waves (except the first term). The most singular term in powers of \( ka \) is of the order \((ka)^{l-1}\) (order \( O(1) \) if \( l = 0 \)), which is most easily seen from the representation in (3.5).
4 Fourier transform of $I_l(z)$

The indefinite Fourier transform of the function $I_l(z)$ has also importance in the analysis of [9]. More specifically, our goal in this section is to compute

$$\hat{I}_l^+(z) = k \int_{z_0}^z I_l(t) e^{\pm ikt} \, dt, \quad z \geq z_0, \quad l = 0, 1, 2, \ldots$$

(4.1)

where $z_0$ is a fixed number such that $z_0 < -a$.

The function $I_l(t)$ has explicit forms in the three intervals $[z_0, -a]$, $(-a, a)$, and $[a, \infty)$. The explicit forms are:

$$I_l(t) = i^l e^{-ikt}, \quad t \leq -a$$

and in the interval $t \in (-a, a)$ as a finite sum of spherical waves

$$I_l(t) = i^{l-1} k a h^{(1)}_{l-1/2}(ka) P_{l-1/2}(t/a)$$

$$+ \sum_{n=0}^{[l/2]-1} (-1)^n k a h^{(1)}_{l-2n-1}(ka) (P_{l-2n}(t/a) - P_{l-2n-2}(t/a))$$

In the interval $t \geq a$

$$I_l(t) = i^{-l} e^{ikt}$$

To compute the indefinite Fourier transform we need to calculate the function

$$h^+_l(z) = k \int_{-a}^z P_l(t/a) e^{\pm ikt} \, dt = ka \int_{-1}^{z/a} P_l(t) e^{\pm ikat} \, dt, \quad |z| \leq a$$

(4.2)

For $z = a$ the integral is a spherical Bessel function, viz.,

$$h^+_l(a) = k \int_{-a}^a P_l(t/a) e^{\pm ikt} \, dt = ka \int_{-1}^1 P_l(t) e^{\pm ikat} \, dt = 2ka (\pm i)^l j_l(ka)$$

We divide the interval $[z_0, z]$ in three parts. In the interval $z_0 \leq z < -a$, we have

$$\hat{I}_l^+(z) = i^l k \int_{z_0}^z e^{i(l+1-1)kt} \, dt = i^l \left\{ \frac{k(z - z_0)}{2i} (e^{-2ika} - e^{-2ikz}) \right\}$$

and in the interval $-a < z < a$, we have

$$\hat{I}_l^+(z) = i^l \left\{ \frac{k(-a - z_0)}{2i} (e^{-2ika} - e^{2ika}) + i^{l-1} k a h^{(1)}_{l-2l/2}(z) h^+_l(z) \right\}$$

$$+ \sum_{n=0}^{[l/2]-1} (-1)^n k a h^{(1)}_{l-2n-1}(ka) (h^+_l(z) - h^+_l(z))$$
and in the interval \( a < z \), we have

\[
\hat{i}_l^\pm(z) = i^l \left\{ \frac{k(-a-z_0)}{2\pi i} + \frac{i^{l-2}2(k-2l/2)(j_{l-2}(ka))}{(ka)^2} \right. \\
+ 2(k-2l/2) \sum_{n=0}^{[l/2]} h_{l-2n-1}(ka) (j_{n-2}(ka) + j_{n-2}(ka)) \\
+ \left. i^{l-1} \left( \frac{e^{2kz}}{k(z-a)} - e^{2ika} \right) \right\}
\]

5 Indefinite integral of Legendre polynomials

It remains to find an effective method to compute the functions \( h_l^\pm(z) \) in (4.2). To this end, define

\[
h_l(\eta, \zeta) = \int_{-1}^\eta P_l(t)e^{i\zeta t} dt, \quad |\eta| \leq 1
\]

(5.1)

We see that \( h_l(1, \zeta) = 2i^lj_l(\zeta) \). In terms of the functions \( h_l(\eta, \zeta) \), the functions \( h_l^\pm(z) \) are

\[
h_l^\pm(z) = kah_l(z/a \pm ka)
\]

Our ambition in this section is to find an efficient method to compute the integrals in (5.1). We express the function \( h_l(\eta, \zeta) \) as a recursion relation.

5.1 Solution by recursion

The following recursion relation of Legendre polynomials is useful:

\[
P_l(t) = \frac{1}{2l+1} \left( P_{l+1}'(t) - P_{l-1}'(t) \right)
\]

Integration by parts then implies (\( P_l(-1) = (-1)^l \))

\[
h_l(\eta, \zeta) = \int_{-1}^\eta P_l(t)e^{i\zeta t} dt = \frac{1}{2l+1} \int_{-1}^\eta \left( P_{l+1}'(t) - P_{l-1}'(t) \right) e^{i\zeta t} dt
\]

\[
= \frac{1}{2l+1} \left( P_{l+1}(\eta) - P_{l-1}(\eta) \right) e^{i\zeta \eta} - \frac{i\zeta}{2l+1} \left( h_{l+1}(\eta, \zeta) - h_{l-1}(\eta, \zeta) \right)
\]

or solving for \( h_{l+1}(\eta, \zeta) \)

\[
h_{l+1}(\eta, \zeta) = \frac{1}{i\zeta} \left( P_{l+1}(\eta) - P_{l-1}(\eta) \right) e^{i\zeta \eta} - \frac{2l+1}{i\zeta} h_l(\eta, \zeta) + h_{l-1}(\eta, \zeta), \quad l = 1, 2, 3, \ldots
\]

The functions \( h_l(\eta, \zeta) \) can therefore be found by iteration with starting values

\[
h_0(\eta, \zeta) = \frac{1}{i\zeta} \left( e^{i\zeta \eta} - e^{-i\zeta} \right) = \frac{1}{i\zeta} P_0(\eta)e^{i\zeta \eta} + h_0(\zeta) = \eta h_0^{(1)}(\zeta) + h_0^{(2)}(\zeta)
\]
and

\[
\begin{align*}
  h_1(\eta, \zeta) &= \frac{1}{i\zeta} (\eta e^{i\zeta} + e^{-i\zeta}) + \frac{1}{\zeta^2} (e^{i\zeta} - e^{-i\zeta}) \\
  &= \frac{1}{i\zeta} \left( P_1(\eta) - \frac{1}{i\zeta} P_0(\eta) \right) e^{i\zeta} + ih_1^{(1)}(\zeta) = \eta^2 h_1^{(1)}(\zeta) + ih_1^{(2)}(\zeta)
\end{align*}
\]

To find the general solution to this recursion scheme, we start by solving the homogeneous difference equation.

**Lemma 5.1.** The solution to the homogeneous difference equation

\[
a_{l+1} + \frac{2l + 1}{i\zeta} a_l - a_{l-1} = 0, \quad l = 1, 2, 3, \ldots
\]

given the initial values \(a_0\) and \(a_1\) is

\[
a_l = -\frac{\zeta^2}{2l} \left( a_0 h_0^{(2)'}(\zeta) - ia_1 h_0^{(2)}(\zeta) \right) i^l h_1^{(1)}(\zeta) + \frac{\zeta^2}{2l} \left( a_0 h_0^{(1)'}(\zeta) - ia_1 h_0^{(1)}(\zeta) \right) i^l h_1^{(2)}(\zeta), \quad l = 2, 3, 4, \ldots
\]

**Proof.** Two linearly independent solutions to the homogeneous difference equation in the lemma are \(i^l h_1^{(1)}(\zeta)\) and \(i^l h_1^{(2)}(\zeta)\), which is easily proved by the recursion relation \(f_{l+1}(z) - (2l + 1)f_l(z)/z + f_{l-1}(z) = 0\), where \(f_l(z)\) is any spherical Bessel or Hankel function. The general solution therefore is

\[
a_l = c_1 i^l h_1^{(1)}(\zeta) + c_2 i^l h_1^{(2)}(\zeta), \quad l = 2, 3, 4, \ldots
\]

where \(c_1\) and \(c_2\) are constants determined by the starting values \(a_0\) and \(a_1\). Explicitly, we get

\[
\begin{align*}
  \begin{cases}
    c_1 h_0^{(1)}(\zeta) + c_2 h_0^{(2)}(\zeta) = a_0 \\
    c_1 ih_1^{(1)}(\zeta) + c_2 ih_1^{(2)}(\zeta) = a_1
  \end{cases}
\end{align*}
\]

with solution

\[
\begin{align*}
  c_1 &= -\frac{\zeta^2}{2l} \left( a_0 h_0^{(2)'}(\zeta) - ia_1 h_0^{(2)}(\zeta) \right) \\
  c_2 &= \frac{\zeta^2}{2l} \left( a_0 h_0^{(1)'}(\zeta) - ia_1 h_0^{(1)}(\zeta) \right)
\end{align*}
\]

where we used the Wronskian of the spherical Hankel functions.

\[
h_n^{(2)}(z) h_n^{(1)'}(z) - h_n^{(2)'}(z) h_n^{(1)}(z) = \frac{2i}{z^2}
\]

and 

\[
h_n^{(1,2)'}(z) = -h_n^{(1,2)}(z).\] This completes the proof of the lemma. \(\square\)

We are now ready for the solution to the inhomogeneous difference equation in \(h_l(\eta, \zeta)\) above. We formulate this as a lemma.
Lemma 5.2. Define an iteration scheme by

$$h_{l+1}(\eta, \zeta) = \frac{1}{i\zeta} (P_{l+1}(\eta) - P_{l-1}(\eta)) e^{i\zeta} - \frac{2l + 1}{i\zeta} h_l(\eta, \zeta) + h_{l-1}(\eta, \zeta), \quad l = 1, 2, 3, \ldots$$

with starting values

$$h_0(\eta, \zeta) = \eta h_0^{(1)}(\zeta) + h_0^{(2)}(\zeta)$$

and

$$h_1(\eta, \zeta) = i \left( \eta^2 h_1^{(1)}(\zeta) + h_1^{(2)}(\zeta) \right)$$

The solution is

$$h_l(\eta, \zeta) = f_l(\eta, \zeta) e^{i\zeta} + i^l h_l^{(2)}(\zeta), \quad l = 0, 1, 2, 3, \ldots$$

where

$$f_l(\eta, \zeta) = i^l h_l^{(1)}(\zeta) \left\{ \sum_{k=1}^{l} \frac{1}{\zeta h_{k-1}^{(1)}(\zeta) h_k^{(1)}(\zeta)} \left( - \sum_{n=0}^{k} i^{-n+1} (2n + 1) \frac{h_n^{(1)}(\zeta) P_n(\eta)}{\zeta} - i \frac{P_0(\eta)}{\zeta h_0^{(1)}(\zeta)} \right) \right\}, \quad l = 0, 1, 2, \ldots$$

Proof. We first subtract the part of the solution that contains the spherical Hankel function of the second kind $h_l^{(2)}(\zeta)$ and the exponential function $e^{i\zeta}$. To this end, let $h_l(\eta, \zeta) = f_l(\eta, \zeta) e^{i\zeta} + i^l h_l^{(2)}(\zeta)$. The recursion relation for $f_l(\eta, \zeta)$ is easily found by the use of the recursion relation $h_{l+1}^{(2)}(z) = (2l + 1) h_l^{(2)}(z)/z - h_{l-1}^{(2)}(z)$. We get the new difference equation

$$f_{l+1}(\eta, \zeta) = \frac{1}{i\zeta} (P_{l+1}(\eta) - P_{l-1}(\eta)) - \frac{2l + 1}{i\zeta} f_l(\eta, \zeta) + f_{l-1}(\eta, \zeta), \quad l = 1, 2, 3, \ldots$$

with starting values

$$f_0(\eta, \zeta) = \frac{1}{i\zeta} P_0(\eta)$$

and

$$f_1(\eta, \zeta) = \frac{1}{i\zeta} \left( P_1(\eta) - \frac{1}{i\zeta} P_0(\eta) \right)$$

To simplify the notation, we put the difference equation in a standard form [1].

$$a_{n+2} + p_1(n)a_{n+1} + p_0(n)a_n = q(n), \quad n = 1, 2, \ldots$$

where

$$\begin{cases} a_n = f_{n-1}(\eta, \zeta) \\ p_1(n) = \frac{2n + 1}{i\zeta} \\ p_0(n) = -1 \\ q(n) = \frac{1}{i\zeta} (P_{n+1}(\eta) - P_{n-1}(\eta)) \end{cases}$$
with initial values

\[
\begin{align*}
  a_1 &= \frac{1}{i\zeta} P_0(\eta) \\
  a_2 &= \frac{1}{i\zeta} \left( P_1(\eta) - \frac{1}{i\zeta} P_0(\eta) \right)
\end{align*}
\]

A solution to the homogeneous difference equation is (see Lemma 5.1)

\[ y_l = i^{l-1} h_{l-1}^{(1)}(\zeta) \]

The final solution then is \([1], \ (n = 3, 4, \ldots)\)

\[ a_n = \left( \sum_{k=1}^{n-1} \prod_{j=1}^{k-1} \frac{p_0(j) y_j}{y_{j+2}} \left( \sum_{l=1}^{k-1} \frac{g(l)}{y_{l+2}} \left[ \prod_{m=1}^{l} \frac{p_0(m)y_m}{y_{m+2}} \right] \right)^{-1} + \frac{a_2}{y_2 - \frac{a_1}{y_1}} \right) y_n \]

Insert the explicit values, and we obtain

\[
\begin{align*}
  f_l(\eta, \zeta) &= \left\{ \sum_{k=1}^{l} \frac{1}{\zeta h_{k-1}^{(1)}(\zeta) h_{k}^{(1)}(\zeta)} \left( -\sum_{n=1}^{k-1} h_{n}^{(1)}(\zeta) \frac{P_{n+1}(\eta) - P_{n-1}(\eta)}{i^n} + i h_{1}^{(1)}(\zeta) P_0(\eta) \\
  &\quad - h_0^{(1)}(\zeta) \left( P_1(\eta) + i \frac{1}{\zeta} P_0(\eta) \right) \right) \right\} \left( \sum_{k=1}^{l} \frac{1}{\zeta h_{k}^{(1)}(\zeta)} \left( P_{k-1}(\eta) - P_{k-3}(\eta) \right) - i^{k+1} h_{k-1}^{(1)}(\zeta) \left( P_k(\eta) - P_{k-2}(\eta) \right) \right) \right. \\
  &\quad + \left. \left( h_0^{(1)}(\zeta) + h_2^{(1)}(\zeta) \right) P_1(\eta) + i \left( h_1^{(1)}(\zeta) + h_3^{(1)}(\zeta) \right) P_2(\eta) \right) \\
  &\quad + \left( h_2^{(1)}(\zeta) + h_4^{(1)}(\zeta) \right) P_3(\eta) - i \left( h_3^{(1)}(\zeta) + h_5^{(1)}(\zeta) \right) P_4(\eta) + \ldots \\
  &\quad - i^{k+2} \left( h_{k-2}^{(1)}(\zeta) + h_{k}^{(1)}(\zeta) \right) P_{k-1}(\eta) - i^{k+1} \left( h_{k-1}^{(1)}(\zeta) + h_{k+1}^{(1)}(\zeta) \right) P_k(\eta) \right) \\
  &\quad + \left. i^{k+2} h_{k}^{(1)}(\zeta) P_{k-1}(\eta) + \left( h_{k-1}^{(1)}(\zeta) + h_{k+1}^{(1)}(\zeta) \right) P_k(\eta) \right) \\
  &\quad = -3 \left( \frac{-3}{\zeta} h_{1}^{(1)}(\zeta) P_1(\eta) + 5i \frac{h_2^{(1)}(\zeta)}{\zeta} P_2(\eta) + 7 \frac{h_3^{(1)}(\zeta)}{\zeta} P_3(\eta) - 9i \frac{h_4^{(1)}(\zeta)}{\zeta} P_4(\eta) + \ldots \\
  &\quad - i^{k+1} \left( 2k + 1 \right) \frac{h_{k-1}^{(1)}(\zeta)}{\zeta} P_k(\eta) + i^{k+2} \frac{h_{k}^{(1)}(\zeta)}{\zeta} P_{k-1}(\eta) + i^{k+1} \frac{h_{k+1}^{(1)}(\zeta)}{\zeta} P_k(\eta) \right)
\end{align*}
\]

The recursion relation \( h_{l+1}^{(1)}(z) + h_{l-1}^{(1)}(z) = (2l+1) h_{l}^{(1)}(z)/z \) implies

\[
\begin{align*}
  S &= -3 \left( \frac{-3}{\zeta} h_{1}^{(1)}(\zeta) P_1(\eta) + 5i \frac{h_2^{(1)}(\zeta)}{\zeta} P_2(\eta) + 7 \frac{h_3^{(1)}(\zeta)}{\zeta} P_3(\eta) - 9i \frac{h_4^{(1)}(\zeta)}{\zeta} P_4(\eta) + \ldots \\
  &\quad - i^{k+1} \left( 2k + 1 \right) \frac{h_{k-1}^{(1)}(\zeta)}{\zeta} P_k(\eta) + i^{k+2} \frac{h_{k}^{(1)}(\zeta)}{\zeta} P_{k-1}(\eta) + i^{k+1} \frac{h_{k+1}^{(1)}(\zeta)}{\zeta} P_k(\eta) \right)
\end{align*}
\]
which gives
\[
\begin{align*}
  f_l(\eta, \zeta) &= \left\{ \sum_{k=1}^{l} \frac{1}{\zeta h_{k-1}^{(1)}(\zeta)} \left( -\sum_{n=1}^{k} i^{-n+1} (2n+1) \frac{h_n^{(1)}(\zeta)}{\zeta} P_n(\eta) ight. \\
  &\quad + i^{-k+2} h_k^{(1)}(\zeta) P_{k-1}(\eta) + i^{-k+1} h_{k+1}^{(1)}(\zeta) P_k(\eta) - i \frac{h_0^{(1)}(\zeta)}{\zeta} P_0(\eta) \bigg) \\
  &\quad \left. - i \frac{P_0(\eta)}{\zeta h_0^{(1)}(\zeta)} \right\} i_l h_l^{(1)}(\zeta)
\end{align*}
\]

or
\[
\begin{align*}
  f_l(\eta, \zeta) &= \left\{ \sum_{k=1}^{l} \frac{1}{\zeta h_{k-1}^{(1)}(\zeta)} h_k^{(1)}(\zeta) \left( -\sum_{n=0}^{k} i^{-n+1} (2n+1) \frac{h_n^{(1)}(\zeta)}{\zeta} P_n(\eta) ight. \\
  &\quad + i^{-k+2} h_k^{(1)}(\zeta) P_{k-1}(\eta) + i^{-k+1} h_{k+1}^{(1)}(\zeta) P_k(\eta) \bigg) - i \frac{P_0(\eta)}{\zeta h_0^{(1)}(\zeta)} \right\} i_l h_l^{(1)}(\zeta)
\end{align*}
\]

This completes the lemma. \(\square\)

In conclusion, the functions \(h_l^\pm(z)\) defined in (4.2) can be expressed in the function \(h(\eta, \zeta)\) in (5.1). Specifically, we have
\[h_l^\pm(z) = k a h_l(z/a, \pm ka)\]

6 Summary and explicit terms

This paper contains an evaluation of a non-trivial integral that occurs in the formulation of scattering by randomly distributed obstacles.

To summarize, the integral \(I_l(z)\) in (1.1) has been solved and the solution outside the interval \([-a, a]\) is a simple exponential function in \(kz\), while inside the interval \([-a, a]\), the solution can be found in a finite series of spherical waves. The finite sum of spherical waves depends on the two parameters \(kz\) and \(ka\), or, more precisely, the parameter \(ka\) and a polynomial of the order \(l\) in the parameter \(z/a\). Several equivalent solutions are presented in the paper, one of them is \((l = 0, 1, 2, \ldots)\)

\[
I_l(z) = \begin{cases} 
  i^l e^{-ikz}, & z \leq -a \\
  i^{l-1} k a h_0^{(1)}(ka) P_{l-2\lfloor l/2 \rfloor}(z/a) \\
  + \sum_{n=0}^{\lfloor l/2 \rfloor - 1} (-1)^n k a h_{2n-1}^{(1)}(ka) (P_{l-2n}(z/a) - P_{l-2n-2}(z/a)), & z \in [-a, a] \\
  i^{-l} e^{ikz}, & z \geq a
\end{cases}
\]

The first integrals, \(l = 0, 1, 2\), are of interest for low-frequency expansions. For \(l = 0\) the integral is
\[
I_0(z) = \begin{cases} 
  e^{-ikz}, & z \leq -a \\
  e^{ika}, & z \in [-a, a] \\
  e^{ikz}, & z \geq a
\end{cases}
\]
and for $l = 1$ the result is

$$I_1(z) = \begin{cases} \text{ie}^{-ikz}, & z \leq -a \\ \text{ie}^{ika} \frac{z}{a}, & z \in [-a, a] \\ \text{ie}^{ikz}, & z \geq a \end{cases}$$

For $l = 2$ the result is

$$I_2(z) = \begin{cases} -\text{e}^{-ikz}, & z \leq -a \\ \frac{\text{e}^{ika}(ka)^2(3i + ka) - 3(i + ka)(kz)^2}{2(ka)^3}, & z \in [-a, a] \\ -\text{e}^{ikz}, & z \geq a \end{cases}$$

and we notice that the integral contains a polynomial in $z/a$ of order $l$.

Moreover, the indefinite Fourier transform of $I_l(z)$ has also been investigated. More precisely, the integral, see (4.1)

$$\hat{I}_l^\pm(z) = k \int_{z_0}^{z} I_l(t)e^{\pm ikt} \, dt, \quad z \geq z_0, \quad l = 0, 1, 2, \ldots$$

is shown to have a solution expressed in spherical waves.

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**References**


