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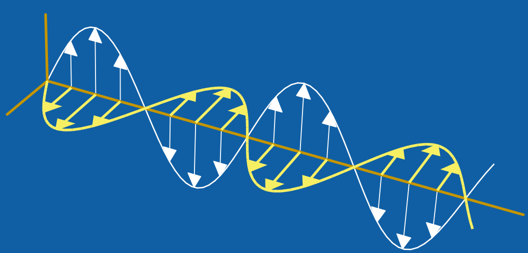

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Abstract

Using the Cayley-Hamilton theorem and unique solubility of scalar Volterra convolution equations of the second kind, the inverse problem of determining the four time-dependent susceptibility dyadics of a linear, homogeneous, bi-anisotropic slab from generic scattering data at oblique incidence is shown to be well posed. An explicit formula for the crucial step is given.

1 Introduction

All materials are characterized by temporal dispersion, which is generally anomalous in the absorption bands [11, 21]. This means that electromagnetic pulses pulses propagate under distortion in non-vacuous regions of space. The most well known transient fields in dispersive media are known as the Sommerfeld and Brillouin precursors or forerunners [1, 4, 13, 21].

For linear, causal, and time-invariant materials, temporal dispersion is modeled by time convolution in the constitutive relations [12]. Since there is generally a coupling between the electric field and the magnetic field in the constitutive relations, the most general bi-anisotropic medium is described by as many as 36 temporal susceptibility kernels [19].

Time-domain direct and inverse scattering problems for bi-anisotropic slabs located in free space have been in focus during the 1990's [3, 5–9, 16–18, 22–24]. Generally the exciting fields have been known normally or obliquely incident fields of both polarizations[1], possibly several ones impinging on the slab under different angles of incidence or different azimuth angles of incidence. The direct problem is then to determine the scattered fields, *i.e.*, the reflected and transmitted fields, given the susceptibility kernels of the medium in the slab. The inverse problem is to determine the unknown susceptibility kernels of the medium, given measured scattered fields obtained at one or several experiments. Usually either the imbedding approach or the Green function technique has been used, both for the direct and the inverse problem. Both methods depend on vacuum wave splitting. Good numerical results of numerical calculations were reported in the references, indicating that the inverse problem basically is well posed, *i.e.*, that the results depend continuously on data. This was, however, not proved.

In [25], using dispersive splitting and referring to unique solubility of Volterra convolution equations of the second kind, the inverse problem for the bi-isotropic (four kernels) slab at normal incidence was shown to be well posed. Moreover, in [6], one of the natural steps in solving the inverse problem for the anisotropic (18 susceptibility kernels) slab at oblique incidence was proved to be well posed. In the present paper, it is shown that the generic[2] inverse problem for the general bi-anisotropic slab under oblique incidence indeed is well posed. The Cayley-Hamilton theorem and wellposedness of scalar Volterra convolution equations of the second

[1] Transverse electric (TE) fields and transverse magnetic (TM) fields are intended.

[2] This excludes the illposedness of obtaining scattering kernels by deconvolution of scattered fields.

kind are referred to, and it is conjectured, that a constructive algorithm of solving the inverse problem can be based on the proof. Pertinent background material is presented systematically and developed for intended purposes in the introductory section 1. The solution of the inverse problem, presented in section 2, turns out to be simple consequence of these facts.

1.1 Notation

A dyadic notation [20] combined with standard block-matrix notation is used. Scalars are typed in italic letters, vectors in italic boldface style, and dyadics in roman boldface style. Thus, three-dimensional vectors are written as

$$\boldsymbol{V} = \hat{\boldsymbol{x}}V_x + \hat{\boldsymbol{y}}V_y + \hat{\boldsymbol{z}}V_z,$$

whereas three-dimensional dyadics are written in the form

$$\begin{aligned}\mathbf{D} =& \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}D_{xx} + \hat{\boldsymbol{x}}\hat{\boldsymbol{y}}D_{xy} + \hat{\boldsymbol{x}}\hat{\boldsymbol{z}}D_{xz}\\ &+\hat{\boldsymbol{y}}\hat{\boldsymbol{x}}D_{yx} + \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}D_{yy} + \hat{\boldsymbol{y}}\hat{\boldsymbol{z}}D_{yz}\\ &+\hat{\boldsymbol{z}}\hat{\boldsymbol{x}}D_{zx} + \hat{\boldsymbol{z}}\hat{\boldsymbol{y}}D_{zy} + \hat{\boldsymbol{z}}\hat{\boldsymbol{z}}D_{zz},\end{aligned}$$

where $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ are the Cartesian basis vectors. Special notation is used for the radius vector, $\boldsymbol{r} = \hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y + \hat{\boldsymbol{z}}z$, and standard notation is used for the gradient, $\nabla = \hat{\boldsymbol{x}}\partial_x + \hat{\boldsymbol{y}}\partial_y + \hat{\boldsymbol{z}}\partial_z$. The identity dyadic $\mathbf{I} = \hat{\boldsymbol{x}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$ has the properties that $\mathbf{I}\cdot\boldsymbol{V} = \boldsymbol{V}\cdot\mathbf{I} = \boldsymbol{V}$ for each vector $\boldsymbol{V}$ and $\mathbf{I}\cdot\mathbf{D} = \mathbf{D}\cdot\mathbf{I} = \mathbf{D}$ for each dyadic $\mathbf{D}$.

For the purpose of analyzing pulse propagation in a preferred direction, $\hat{\boldsymbol{z}}$, in space, it is appropriate to decompose the unit dyadic as $\mathbf{I} = \mathbf{I}_{\perp\perp} + \hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$, where $\mathbf{I}_{\perp\perp} = \hat{\boldsymbol{x}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{y}}$ is the identity dyadic in the x-y plane.

Three-dimensional vectors are decomposed uniquely as

$$\boldsymbol{V} = \boldsymbol{V}_\perp + \hat{\boldsymbol{z}}V_z,$$

where $\boldsymbol{V}_\perp = \mathbf{I}_{\perp\perp}\cdot\boldsymbol{V} = \boldsymbol{V}\cdot\mathbf{I}_{\perp\perp} = \hat{\boldsymbol{x}}V_x + \hat{\boldsymbol{y}}V_y$ is the projection of $\boldsymbol{V}$ on the x-y plane. The zero vector $\mathbf{0}$ is decomposed as $\mathbf{0} = \mathbf{0}_\perp + \hat{\boldsymbol{z}}0$. Three-dimensional dyadics are partitioned uniquely as

$$\mathbf{D} = \mathbf{D}_{\perp\perp} + \boldsymbol{D}_{\perp z}\hat{\boldsymbol{z}} + \hat{\boldsymbol{z}}\boldsymbol{D}_{z\perp} + \hat{\boldsymbol{z}}D_{zz}\hat{\boldsymbol{z}}, \tag{1.1}$$

where

$$\begin{cases}\mathbf{D}_{\perp\perp} = \mathbf{I}_{\perp\perp}\cdot\mathbf{D}\cdot\mathbf{I}_{\perp\perp},\\ \boldsymbol{D}_{z\perp} = \hat{\boldsymbol{z}}\cdot\mathbf{D}\cdot\mathbf{I}_{\perp\perp},\end{cases}\qquad \begin{cases}\boldsymbol{D}_{\perp z} = \mathbf{I}_{\perp\perp}\cdot\mathbf{D}\cdot\hat{\boldsymbol{z}},\\ D_{zz} = \hat{\boldsymbol{z}}\cdot\mathbf{D}\cdot\hat{\boldsymbol{z}}.\end{cases}$$

The dyadic $\mathbf{D}_{\perp\perp}$ is a two-dimensional dyadic in the x-y plane, and the vectors $\boldsymbol{D}_{z\perp}$ and $\boldsymbol{D}_{\perp z}$ are two-dimensional vectors in this plane. The zero dyadic $\mathbf{0}$ is decomposed as $\mathbf{0} = \mathbf{0}_{\perp\perp} + \mathbf{0}_{\perp z}\hat{\boldsymbol{z}} + \hat{\boldsymbol{z}}\mathbf{0}_{z\perp} + \hat{\boldsymbol{z}}0\hat{\boldsymbol{z}}$.

The dyadic notation is often combined with a matrix notation. The pertinent block-matrix representation of a dyadic $\mathbf{D}$ is

$$\begin{pmatrix}\mathbf{D}_{\perp\perp} & \boldsymbol{D}_{\perp z}\\ \boldsymbol{D}_{z\perp} & D_{zz}\end{pmatrix}.$$

When using this notation, $\boldsymbol{D}_{z\perp}$ is to be thought of as a two-dimensional row vector and $\boldsymbol{D}_{\perp z}$ as a two-dimensional column vector. Four our purposes, it is also appropriate to form the four-block-matrices

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}, \begin{pmatrix} \mathbf{A}_{\perp\perp} & \mathbf{B}_{\perp\perp} \\ \mathbf{C}_{\perp\perp} & \mathbf{D}_{\perp\perp} \end{pmatrix}, \begin{pmatrix} \boldsymbol{A}_{\perp z} & \boldsymbol{B}_{\perp z} \\ \boldsymbol{C}_{\perp z} & \boldsymbol{D}_{\perp z} \end{pmatrix}, \begin{pmatrix} \boldsymbol{A}_{z\perp} & \boldsymbol{B}_{z\perp} \\ \boldsymbol{C}_{z\perp} & \boldsymbol{D}_{z\perp} \end{pmatrix}, \text{ and } \begin{pmatrix} A_{zz} & B_{zz} \\ C_{zz} & D_{zz} \end{pmatrix},$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are given three-dimensional dyadics and the other block-matrix entries arise when $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are partitioned in accordance with (1.1). These four-block-matrices are of types 6×6, 4×4, 4×2, 2×4, and 2×2, respectively.

The electric and magnetic fields are denoted by $\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{H} = \boldsymbol{H}(\boldsymbol{r}, t)$, respectively, and the corresponding flux densities are written $\boldsymbol{D} = \boldsymbol{D}(\boldsymbol{r}, t)$ and $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{r}, t)$, where t denotes time. In macroscopic media, the Maxwell equations

$$\begin{cases} \nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}, \\ \nabla \times \boldsymbol{H} = \partial_t \boldsymbol{D}, \end{cases}$$

model the dynamics of the fields. These equations can be written economically as

$$\begin{pmatrix} \mathbf{0} & \nabla \times \mathbf{I} \\ -\nabla \times \mathbf{I} & \mathbf{0} \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{E} \\ \eta_0 \boldsymbol{H} \end{pmatrix} = c_0^{-1} \partial_t \begin{pmatrix} c_0 \eta_0 \boldsymbol{D} \\ c_0 \boldsymbol{B} \end{pmatrix},$$

where $\eta_0 = \sqrt{\mu_0/\epsilon_0}$ and $c_0 = 1/\sqrt{\epsilon_0 \mu_0}$ are the intrinsic impedance of vacuum and the speed of light in vacuum, respectively, and ϵ_0 and μ_0 are the permittivity and permeability of vacuum, respectively. Temporal differentiation is denoted by ∂_t and temporal integration (anti-differentiation) by $\partial_t^{-1} = \int_{-\infty}^{t} dt$. At pulse propagation, all fields are initially quiescent, and these operators commute.

The temporal Heaviside unit step and temporal delta function are denoted by $H = H(t)$ and $\delta = \delta(t)$, respectively.

1.2 Bi-anisotropic media

The general homogeneous[3] bi-anisotropic medium is a linear, complex material comprising 36 different scalar constitutive time-dependent parameters (functions) [19]. The constitutive relations are

$$\begin{cases} \boldsymbol{D} = \epsilon_0 \boldsymbol{\epsilon} \cdot \boldsymbol{E} + \dfrac{1}{c_0} \boldsymbol{\xi} \cdot \boldsymbol{H}, \\ \boldsymbol{B} = \dfrac{1}{c_0} \boldsymbol{\zeta} \cdot \boldsymbol{E} + \mu_0 \boldsymbol{\mu} \cdot \boldsymbol{H}, \end{cases} \tag{1.2}$$

or, in a compact form,

$$\begin{pmatrix} c_0 \eta_0 \boldsymbol{D} \\ c_0 \boldsymbol{B} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\epsilon} & \boldsymbol{\xi} \\ \boldsymbol{\zeta} & \boldsymbol{\mu} \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{E} \\ \eta_0 \boldsymbol{H} \end{pmatrix}.$$

[3]In this section medium parameters may, more generally, depend on position, *i.e.*, $\boldsymbol{\chi}^{ij} = \boldsymbol{\chi}^{ij}(\boldsymbol{r}, t)$, $i, j \in \{e, m\}$.

The dyadic operators $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ in (1.2) are the permittivity and the permeability operators of the medium, respectively, which for anisotropic materials are general, that is, comprising nine parameters each. For isotropic media, $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ are proportional to the identity dyadic $\mathbf{I}$. In bi-isotropic media, which are the simplest complex materials that involve the cross-coupling terms $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$, all the constitutive dyadics are proportional to the identity dyadic. One should bear in mind, that the partitioning into isotropic, bi-isotropic, anisotropic, and bi-anisotropic materials, by definition, is mutually exclusive. The dyadics $\boldsymbol{\epsilon}$, $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$, and $\boldsymbol{\mu}$ are dimensionless. Introducing the constitutive relations in the Maxwell equations gives

$$\begin{pmatrix} \mathbf{0} & \nabla\times\mathbf{I} \\ -\nabla\times\mathbf{I} & \mathbf{0} \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{E} \\ \eta_0 \boldsymbol{H} \end{pmatrix} = c_0^{-1}\partial_t \begin{pmatrix} \boldsymbol{\epsilon} & \boldsymbol{\xi} \\ \boldsymbol{\zeta} & \boldsymbol{\mu} \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{E} \\ \eta_0 \boldsymbol{H} \end{pmatrix}. \tag{1.3}$$

The relative permittivity and permeability operators of the medium are[4]

$$\begin{cases} \boldsymbol{\epsilon} = \mathbf{I} + (\boldsymbol{\chi}^{ee} H)\circ = (\mathbf{I}\delta + \boldsymbol{\chi}^{ee} H)\circ, \\ \boldsymbol{\mu} = \mathbf{I} + (\boldsymbol{\chi}^{mm} H)\circ = (\mathbf{I}\delta + \boldsymbol{\chi}^{mm} H)\circ, \end{cases}$$

whereas the relative cross-coupling operators are

$$\begin{cases} \boldsymbol{\xi} = (\boldsymbol{\chi}^{em} H)\circ, \\ \boldsymbol{\zeta} = (\boldsymbol{\chi}^{me} H)\circ, \end{cases}$$

where the circle ($\circ$) denotes temporal convolution and the circle endowed with a dot ($\odot$) denotes temporal convolution combined with the dyadic dot product ($\cdot$):

$$\begin{cases} [\boldsymbol{\epsilon}\cdot\boldsymbol{E}](\boldsymbol{r},t) = ((\mathbf{I}\delta + \boldsymbol{\chi}^{ee} H)\odot \boldsymbol{E})(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r},t) + \displaystyle\int_{-\infty}^{t} \boldsymbol{\chi}^{ee}(t-t')\cdot\boldsymbol{E}(\boldsymbol{r},t')\,dt', \\ [\boldsymbol{\xi}\cdot\eta_0\boldsymbol{H}](\boldsymbol{r},t) = ((\boldsymbol{\chi}^{em} H)\odot \eta_0\boldsymbol{H})(\boldsymbol{r},t) = \displaystyle\int_{-\infty}^{t} \boldsymbol{\chi}^{em}(t-t')\cdot\eta_0\boldsymbol{H}(\boldsymbol{r},t')\,dt'. \end{cases}$$

The four dyadic integral kernels $\boldsymbol{\chi}^{ij} = \boldsymbol{\chi}^{ij}(t)$, $i,j\in\{e,m\}$, are the susceptibility functions of the medium. They are assumed to be bounded and smooth (infinitely differentiable). Well-known examples, applicable to non-magnetic, isotropic materials, are the Lorentz model (the resonance model)

$$\boldsymbol{\chi}^{ee}(t) = \mathbf{I}\frac{\omega_p^2}{\sqrt{\omega_0^2 - \left(\frac{\nu}{2}\right)^2}} \exp\left(-\frac{\nu t}{2}\right) \sin\left(\sqrt{\omega_0^2 - \left(\frac{\nu}{2}\right)^2}\, t\right)$$

and the Drude model (set $\omega_0 = 0$ in the Lorentz model)

$$\boldsymbol{\chi}^{ee}(t) = \mathbf{I}\frac{\omega_p^2}{\nu}\left(1 - \exp\left(-\nu t\right)\right),$$

[4]Causality implies that the integral kernels are identically zero for $t<0$. This is modeled by the Heaviside unit step: $(\boldsymbol{\chi}^{ij} H)(t) = \boldsymbol{\chi}^{ee}(t)H(t)$, $i,j\in\{e,m\}$.

where ω_p is the plasma frequency, ω_0 is the harmonic frequency, and ν is the collision frequency [15]. The former model applies to bound electrons in insulators and the latter to free electrons in conductors. Other well-known models are the Debye model for polar liquids and Ohm's law; however, these models violate the condition $\boldsymbol{\chi}^{ee}(0) = \mathbf{0}$, which has been claimed to be "unphysical" in a major textbook concerning electromagnetics [11]. Therefore, it is adequate to impose the more general condition

$$\boldsymbol{\chi}^{ij}(0) = \mathbf{0} \qquad (i, j \in \{e, m\}) \tag{1.4}$$

on the susceptibility kernels of the medium. Another motivation for (1.4) is that the inverse problem to be studied in this paper is not generally uniquely solvable if (1.4) is violated. In the short-wave-length limit, the constitutive relations reduce to the ones in vacuum, $\boldsymbol{D} = \epsilon_0 \boldsymbol{E}$ and $\boldsymbol{B} = \mu_0 \boldsymbol{H}$, provided that the susceptibility kernels are absolutely integrable (the Riemann-Lebesgue lemma).

There is a profound difference between the operators $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ on one hand and the operators $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ on the other[5]. For instance, if $\boldsymbol{G} = \boldsymbol{G}(t)$ is a well-behaved (*e.g.*, causal, bounded, and smooth) temporal function, then the equation $\boldsymbol{\epsilon} \cdot \boldsymbol{F} = \boldsymbol{G}$ is a linear Volterra convolution equation of the second kind with well-behaved solution $\boldsymbol{F} = \boldsymbol{\epsilon}^{-1} \cdot \boldsymbol{G}$. Solving such equations are well posed problems [14]. In fact, generalizing the method of successive approximations presented in [26] gives

$$\boldsymbol{F}(t) = \boldsymbol{G}(t) + \sum_{n=1}^{\infty} (-1)^n \left(\boldsymbol{\chi}_n^{ee} \odot \boldsymbol{G}\right)(t),$$

where the introduced kernels are given by

$$\boldsymbol{\chi}_n^{ee} = \begin{cases} \boldsymbol{\chi}^{ee}(t) H(t) & (n = 1), \\ (\boldsymbol{\chi}_{n-1}^{ee} \odot \boldsymbol{\chi}_1^{ee})(t) & (n > 1), \end{cases}$$

This series converges owing to the fact that the susceptibility kernel be bounded. The equation $\boldsymbol{\xi} \cdot \boldsymbol{F} = \boldsymbol{G}$, however, is a linear Volterra convolution equation of the first kind, and does not necessarily have this property. Nonetheless, Volterra convolution equations of the first kind can sometimes be transformed to Volterra convolution equations of the second kind by (first or repeated) differentiation [14]. For instance, if $\frac{d}{dt}\boldsymbol{\chi}^{em}(0)$ is non-singular, then two-fold differentiation of both members of the first-kind equation $\boldsymbol{\xi} \cdot \boldsymbol{F} = \boldsymbol{G}$ gives the second-kind equation

$$\boldsymbol{F}(t) + \int_{-\infty}^{t} \left(\left(\frac{d}{dt} \boldsymbol{\chi}^{em}(0) \right)^{-1} \cdot \frac{d^2}{dt^2} \boldsymbol{\chi}^{em}(t - t') \right) \cdot \boldsymbol{F}(t') \, dt' = \left(\frac{d}{dt} \boldsymbol{\chi}^{em}(0) \right)^{-1} \cdot \frac{d^2}{dt^2} \boldsymbol{G}(t).$$

Thus, the solution of $\boldsymbol{\xi} \cdot \boldsymbol{F} = \boldsymbol{G}$ can be written formally as $\boldsymbol{F} = (\partial_t^2 \boldsymbol{\xi})^{-1} \cdot \partial_t^2 \boldsymbol{G}$, provided $\frac{d}{dt}\boldsymbol{\chi}^{em}(0)$ is non-singular.

[5]Observe that the results presented here for the 3D case hold for the 2D and 1D cases.

1.3 Fundamental equation at oblique incidence

At oblique incidence on a homogeneous[6] bi-anisotropic slab $|z| < d/2$ embedded in vacuum, the up-going and down-going incident electric and magnetic fields traveling in the unit directions $\hat{\boldsymbol{n}}^{\pm}$ can be written in the forms

$$\begin{cases} \boldsymbol{E}_i^{\pm}(\boldsymbol{r},t) = \boldsymbol{E}_{i0}^{\pm}\left(t - \hat{\boldsymbol{n}}^{\pm}\cdot(\boldsymbol{r}\pm\hat{\boldsymbol{z}}d/2)/c_0\right) \\ \boldsymbol{H}_i^{\pm}(\boldsymbol{r},t) = \boldsymbol{H}_{i0}^{\pm}\left(t - \hat{\boldsymbol{n}}^{\pm}\cdot(\boldsymbol{r}\pm\hat{\boldsymbol{z}}d/2)/c_0\right) \end{cases} \qquad (\pm z < -d/2),$$

where $\boldsymbol{E}_{i0}^{+}(t-\boldsymbol{n}_\perp\cdot\boldsymbol{r}_\perp/c_0)$ $(\boldsymbol{H}_{i0}^{+}(t-\boldsymbol{n}_\perp\cdot\boldsymbol{r}_\perp/c_0))$ is the incident electric (magnetic) field at $z = -d/2$ and $\boldsymbol{E}_{i0}^{-}(t-\boldsymbol{n}_\perp\cdot\boldsymbol{r}_\perp/c_0)$ $(\boldsymbol{H}_{i0}^{-}(t-\boldsymbol{n}_\perp\cdot\boldsymbol{r}_\perp/c_0))$ is the incident electric (magnetic) field at $z = d/2$. The directions of propagation are assumed to have the same transverse component, *i.e.*, , they can be decomposed as $(n_z > 0)$

$$\hat{\boldsymbol{n}}^{\pm} = \boldsymbol{n}_\perp \pm \hat{\boldsymbol{z}}n_z = \boldsymbol{n}_\perp \pm \hat{\boldsymbol{z}}\sqrt{1-\boldsymbol{n}_\perp\cdot\boldsymbol{n}_\perp},$$

where the common transverse direction of propagation $\boldsymbol{n}_\perp$ is fixed but arbitrary and the positive square root is intended. Under these circumstances, the up-going and down-going scattered electric and magnetic fields must satisfy

$$\begin{cases} \boldsymbol{E}_s^{\pm}(\boldsymbol{r},t) = \boldsymbol{E}_{s0}^{\pm}\left(t - \hat{\boldsymbol{n}}^{\pm}\cdot(\boldsymbol{r}\mp\hat{\boldsymbol{z}}d/2)/c_0\right) \\ \boldsymbol{H}_s^{\pm}(\boldsymbol{r},t) = \boldsymbol{H}_{s0}^{\pm}\left(t - \hat{\boldsymbol{n}}^{\pm}\cdot(\boldsymbol{r}\mp\hat{\boldsymbol{z}}d/2)/c_0\right) \end{cases} \qquad (\pm z > d/2),$$

where $\boldsymbol{E}_{s0}^{+}(t-\boldsymbol{n}_\perp\cdot\boldsymbol{r}_\perp/c_0)$ $(\boldsymbol{H}_{s0}^{+}(t-\boldsymbol{n}_\perp\cdot\boldsymbol{r}_\perp/c_0))$ is the scattered electric (magnetic) field at $z = d/2$ and $\boldsymbol{E}_{s0}^{-}(t-\boldsymbol{n}_\perp\cdot\boldsymbol{r}_\perp/c_0)$ $(\boldsymbol{H}_{s0}^{-}(t-\boldsymbol{n}_\perp\cdot\boldsymbol{r}_\perp/c_0))$ is the scattered electric (magnetic) field at $z = -d/2$. Therefore, the appropriate Ansatz for the total fields throughout space is

$$\begin{cases} \boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}(z, t-\boldsymbol{n}_\perp\cdot\boldsymbol{r}_\perp/c_0) \\ \boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{H}(z, t-\boldsymbol{n}_\perp\cdot\boldsymbol{r}_\perp/c_0) \end{cases} \qquad (-\infty < z < \infty).$$

Substituting ∇ for $-\boldsymbol{n}_\perp c_0^{-1}\partial_t + \hat{\boldsymbol{z}}\partial_z$ into the Maxwell equations (1.3) gives

$$\partial_z \begin{pmatrix} \mathbf{0} & \hat{\boldsymbol{z}}\times\mathbf{I} \\ -\hat{\boldsymbol{z}}\times\mathbf{I} & \mathbf{0} \end{pmatrix}\cdot\begin{pmatrix} \boldsymbol{E} \\ \eta_0\boldsymbol{H} \end{pmatrix} = c_0^{-1}\partial_t \begin{pmatrix} \boldsymbol{\epsilon} & \boldsymbol{\xi}+\boldsymbol{n}_\perp\times\mathbf{I} \\ \boldsymbol{\zeta}-\boldsymbol{n}_\perp\times\mathbf{I} & \boldsymbol{\mu} \end{pmatrix}\cdot\begin{pmatrix} \boldsymbol{E} \\ \eta_0\boldsymbol{H} \end{pmatrix}, \quad (1.5)$$

which is a integro partial differential equation (PDE) in z and t.

As in the time-harmonic case, the longitudinal components are uniquely determined by the transverse fields; specifically,

$$\begin{pmatrix} \epsilon_{zz} & \xi_{zz} \\ \zeta_{zz} & \mu_{zz} \end{pmatrix}\begin{pmatrix} E_z \\ \eta_0 H_z \end{pmatrix} = -\begin{pmatrix} \boldsymbol{\epsilon}_{z\perp} & \boldsymbol{\xi}_{z\perp}+\hat{\boldsymbol{z}}\times\boldsymbol{n}_\perp \\ \boldsymbol{\zeta}_{z\perp}-\hat{\boldsymbol{z}}\times\boldsymbol{n}_\perp & \boldsymbol{\mu}_{z\perp} \end{pmatrix}\cdot\begin{pmatrix} \boldsymbol{E}_\perp \\ \eta_0\boldsymbol{H}_\perp \end{pmatrix}, \quad (1.6)$$

where

$$\begin{pmatrix} \epsilon_{zz} & \xi_{zz} \\ \zeta_{zz} & \mu_{zz} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \chi_{zz}^{ee}(t) & \chi_{zz}^{em}(t) \\ \chi_{zz}^{me}(t) & \chi_{zz}^{mm}(t) \end{pmatrix}\circ. \quad (1.7)$$

[6]In this section the medium may be plane-stratified, *i.e.*, $\boldsymbol{\chi}^{ij} = \boldsymbol{\chi}^{ij}(z,t)$, $i,j \in \{e,m\}$.

In analogy with the results presented in section 1.2, the inverse of the operator (1.7) exists, and (1.6) is a system of linear Volterra convolution equations of the second kind in the longitudinal components in terms of the transverse fields. Applying

$$\begin{pmatrix} \mathbf{0} & \hat{\boldsymbol{z}}\times\mathbf{I} \\ -\hat{\boldsymbol{z}}\times\mathbf{I} & \mathbf{0} \end{pmatrix}$$

on both members of (1.5) gives

$$\begin{aligned}\partial_z\begin{pmatrix}\boldsymbol{E}_\perp\\ \eta_0\boldsymbol{H}_\perp\end{pmatrix} &= c_0^{-1}\partial_t\begin{pmatrix}\mathbf{0}_{\perp\perp} & \hat{\boldsymbol{z}}\times\mathbf{I}_{\perp\perp}\\ -\hat{\boldsymbol{z}}\times\mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp}\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{\epsilon}_{\perp\perp} & \boldsymbol{\xi}_{\perp\perp}\\ \boldsymbol{\zeta}_{\perp\perp} & \boldsymbol{\mu}_{\perp\perp}\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{E}_\perp\\ \eta_0\boldsymbol{H}_\perp\end{pmatrix}\\ &+ c_0^{-1}\partial_t\begin{pmatrix}\mathbf{0}_{\perp\perp} & \hat{\boldsymbol{z}}\times\mathbf{I}_{\perp\perp}\\ -\hat{\boldsymbol{z}}\times\mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp}\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{\epsilon}_{\perp z} & \boldsymbol{\xi}_{\perp z}+\boldsymbol{n}_\perp\times\hat{\boldsymbol{z}}\\ \boldsymbol{\zeta}_{\perp z}-\boldsymbol{n}_\perp\times\hat{\boldsymbol{z}} & \boldsymbol{\mu}_{\perp z}\end{pmatrix}\begin{pmatrix}E_z\\ \eta_0 H_z\end{pmatrix}.\end{aligned}$$

Eliminating the longitudinal components using (1.6) gives the fundamental equation at oblique incidence, which relates the transverse electric and magnetic fields in the complex slab to one another:

$$\partial_z\begin{pmatrix}\boldsymbol{E}_\perp\\ \eta_0\boldsymbol{H}_\perp\end{pmatrix} = c_0^{-1}\partial_t\begin{pmatrix}\mathbf{0}_{\perp\perp} & \hat{\boldsymbol{z}}\times\mathbf{I}_{\perp\perp}\\ -\hat{\boldsymbol{z}}\times\mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp}\end{pmatrix}\cdot\begin{pmatrix}\widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}}\\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}}\end{pmatrix}\cdot\begin{pmatrix}\boldsymbol{E}_\perp\\ \eta_0\boldsymbol{H}_\perp\end{pmatrix}, \tag{1.8}$$

where[7]

$$\begin{aligned}\begin{pmatrix}\widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}}\\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}}\end{pmatrix} &= \begin{pmatrix}\boldsymbol{\epsilon}_{\perp\perp} & \boldsymbol{\xi}_{\perp\perp}\\ \boldsymbol{\zeta}_{\perp\perp} & \boldsymbol{\mu}_{\perp\perp}\end{pmatrix}\\ &- \begin{pmatrix}\boldsymbol{\epsilon}_{\perp z} & \boldsymbol{\xi}_{\perp z}+\boldsymbol{n}_\perp\times\hat{\boldsymbol{z}}\\ \boldsymbol{\zeta}_{\perp z}-\boldsymbol{n}_\perp\times\hat{\boldsymbol{z}} & \boldsymbol{\mu}_{\perp z}\end{pmatrix}\begin{pmatrix}\epsilon_{zz} & \xi_{zz}\\ \zeta_{zz} & \mu_{zz}\end{pmatrix}^{-1}\begin{pmatrix}\boldsymbol{\epsilon}_{z\perp} & \boldsymbol{\xi}_{z\perp}+\hat{\boldsymbol{z}}\times\boldsymbol{n}_\perp\\ \boldsymbol{\zeta}_{z\perp}-\hat{\boldsymbol{z}}\times\boldsymbol{n}_\perp & \boldsymbol{\mu}_{z\perp}\end{pmatrix}.\end{aligned} \tag{1.9}$$

The medium operator defined by (1.9) depends on the transverse direction of propagation, $\boldsymbol{n}_\perp$. Emphasizing this, (1.9) can be written as

$$\begin{aligned}\begin{pmatrix}\widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}}\\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}}\end{pmatrix}(\boldsymbol{n}_\perp) &= \begin{pmatrix}\widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}}\\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}}\end{pmatrix}(\mathbf{0}_\perp)\\ &- \begin{pmatrix}\boldsymbol{\epsilon}_{\perp z} & \boldsymbol{\xi}_{\perp z}\\ \boldsymbol{\zeta}_{\perp z} & \boldsymbol{\mu}_{\perp z}\end{pmatrix}\begin{pmatrix}\epsilon_{zz} & \xi_{zz}\\ \zeta_{zz} & \mu_{zz}\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{0}_{z\perp} & \hat{\boldsymbol{z}}\times\boldsymbol{n}_\perp\\ -\hat{\boldsymbol{z}}\times\boldsymbol{n}_\perp & \mathbf{0}_{z\perp}\end{pmatrix}\\ &- \begin{pmatrix}\mathbf{0}_{\perp z} & \boldsymbol{n}_\perp\times\hat{\boldsymbol{z}}\\ -\boldsymbol{n}_\perp\times\hat{\boldsymbol{z}} & \mathbf{0}_{\perp z}\end{pmatrix}\begin{pmatrix}\epsilon_{zz} & \xi_{zz}\\ \zeta_{zz} & \mu_{zz}\end{pmatrix}^{-1}\begin{pmatrix}\boldsymbol{\epsilon}_{z\perp} & \boldsymbol{\xi}_{z\perp}\\ \boldsymbol{\zeta}_{z\perp} & \boldsymbol{\mu}_{z\perp}\end{pmatrix}\\ &- \begin{pmatrix}\mathbf{0}_{\perp z} & \boldsymbol{n}_\perp\times\hat{\boldsymbol{z}}\\ -\boldsymbol{n}_\perp\times\hat{\boldsymbol{z}} & \mathbf{0}_{\perp z}\end{pmatrix}\begin{pmatrix}\epsilon_{zz} & \xi_{zz}\\ \zeta_{zz} & \mu_{zz}\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{0}_{z\perp} & \hat{\boldsymbol{z}}\times\boldsymbol{n}_\perp\\ -\hat{\boldsymbol{z}}\times\boldsymbol{n}_\perp & \mathbf{0}_{z\perp}\end{pmatrix},\end{aligned} \tag{1.10}$$

where the medium operator at normal incidence is

$$\begin{pmatrix}\widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}}\\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}}\end{pmatrix}(\mathbf{0}_\perp) = \begin{pmatrix}\boldsymbol{\epsilon}_{\perp\perp} & \boldsymbol{\xi}_{\perp\perp}\\ \boldsymbol{\zeta}_{\perp\perp} & \boldsymbol{\mu}_{\perp\perp}\end{pmatrix} - \begin{pmatrix}\boldsymbol{\epsilon}_{\perp z} & \boldsymbol{\xi}_{\perp z}\\ \boldsymbol{\zeta}_{\perp z} & \boldsymbol{\mu}_{\perp z}\end{pmatrix}\begin{pmatrix}\epsilon_{zz} & \xi_{zz}\\ \zeta_{zz} & \mu_{zz}\end{pmatrix}^{-1}\begin{pmatrix}\boldsymbol{\epsilon}_{z\perp} & \boldsymbol{\xi}_{z\perp}\\ \boldsymbol{\zeta}_{z\perp} & \boldsymbol{\mu}_{z\perp}\end{pmatrix}.$$

[7] Notice that this matrix operator is of type 4×4.

The fundamental equation at pulse propagation in a preferred direction in an unbounded medium is also obtained by setting $\boldsymbol{n}_\perp = \mathbf{0}_\perp$.

For given incident electric fields, $\boldsymbol{E}_{i0}^{\pm}$, at the boundaries $z = \mp d/2$, the direct problem (1.8) has a unique solution. This is a consequence of the vacuum wave splitting, which transforms the scattering problem defined by (1.8) into (1.27) subject to the boundary relations (1.26), for which there is a theory available [22]. The solution of (1.8) can be written in the form[8]

$$\begin{pmatrix} \boldsymbol{E}_\perp \\ \eta_0 \boldsymbol{H}_\perp \end{pmatrix}(z) = \begin{pmatrix} \boldsymbol{\Pi}^{ee} & \boldsymbol{\Pi}^{em} \\ \boldsymbol{\Pi}^{me} & \boldsymbol{\Pi}^{mm} \end{pmatrix}(z, z') \cdot \begin{pmatrix} \boldsymbol{E}_\perp \\ \eta_0 \boldsymbol{H}_\perp \end{pmatrix}(z'),$$

where a temporal convolution operator referred to as the wave propagator for the transverse electric and magnetic fields in the medium has been introduced. This operator, which takes the transverse fields at point z' to point z, satisfies

$$\partial_z \begin{pmatrix} \boldsymbol{\Pi}^{ee} & \boldsymbol{\Pi}^{em} \\ \boldsymbol{\Pi}^{me} & \boldsymbol{\Pi}^{mm} \end{pmatrix}(z, z') = c_0^{-1} \partial_t \begin{pmatrix} \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\zeta}} & \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\mu}} \\ -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\epsilon}} & -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\xi}} \end{pmatrix}(z) \cdot \begin{pmatrix} \boldsymbol{\Pi}^{ee} & \boldsymbol{\Pi}^{em} \\ \boldsymbol{\Pi}^{me} & \boldsymbol{\Pi}^{mm} \end{pmatrix}(z, z'),$$

where the possibility that the medium be stratified has been stressed and

$$\begin{pmatrix} \boldsymbol{\Pi}^{ee} & \boldsymbol{\Pi}^{em} \\ \boldsymbol{\Pi}^{me} & \boldsymbol{\Pi}^{mm} \end{pmatrix}(z', z') = \begin{pmatrix} \mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp} \end{pmatrix}.$$

It should be pointed out that also the direct problem defined by these matrix equations is uniquely solvable [22].

1.4 Reflections on the inverse problem

The inverse problem is to obtain the dyadics $\boldsymbol{\epsilon}$, $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$, and $\boldsymbol{\mu}$ from scattering data. Suppose, that, for fixed but arbitrary non-trivial vector $\boldsymbol{n}_\perp$, the material dyadics

$$\begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix}(\pm \boldsymbol{n}_\perp), \qquad \begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix}(\pm \hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp),$$

defined by (1.10) can be determined from a series of inverse experiments. These matrices correspond to a fixed angle of incidence ($\theta = \arcsin(|\boldsymbol{n}_\perp|)$) and four different azimuth angles of incidence ($\phi = \pi/2 \mp \pi/2$ and $\phi = \pm\pi/2$, respectively). Then

$$\begin{aligned} \begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix}^{+}(\boldsymbol{n}_\perp) &:= \begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix}(\boldsymbol{n}_\perp) + \begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix}(-\boldsymbol{n}_\perp) - 2 \begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix}(\mathbf{0}_\perp) \\ \begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix}^{-}(\boldsymbol{n}_\perp) &:= \begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix}(\boldsymbol{n}_\perp) - \begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix}(-\boldsymbol{n}_\perp) \end{aligned}$$

[8]The dependence on the parameter $s = t - \boldsymbol{n}_\perp \cdot \boldsymbol{r}_\perp / c_0$ is relaxed since it is not essential here.

and analogously

$$\begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix}^{+} (\hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp), \qquad \begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix}^{-} (\hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp)$$

can be formed and the inverse problem can be solved easily step by step. For anisotropic materials this was demonstrated in [6], wherein this part of the inverse problem was referred to as Retrieval of Internal Parameters (RIP).

To show this in the general homogeneous[9] bi-anisotropic case, one forms

$$\begin{aligned}
&\begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix}^{+} (\boldsymbol{n}_\perp) - \begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix}^{+} (\hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp) \\
&= -2 \begin{pmatrix} \mathbf{0}_{\perp z} & \boldsymbol{n}_\perp \times \hat{\boldsymbol{z}} \\ -\boldsymbol{n}_\perp \times \hat{\boldsymbol{z}} & \mathbf{0}_{\perp z} \end{pmatrix} \begin{pmatrix} \epsilon_{zz} & \xi_{zz} \\ \zeta_{zz} & \mu_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0}_{z\perp} & \hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp \\ -\hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp & \mathbf{0}_{z\perp} \end{pmatrix} \\
&\quad + 2 \begin{pmatrix} \mathbf{0}_{\perp z} & \boldsymbol{n}_\perp \\ -\boldsymbol{n}_\perp & \mathbf{0}_{\perp z} \end{pmatrix} \begin{pmatrix} \epsilon_{zz} & \xi_{zz} \\ \zeta_{zz} & \mu_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0}_{z\perp} & -\boldsymbol{n}_\perp \\ \boldsymbol{n}_\perp & \mathbf{0}_{z\perp} \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix}^{-} (\boldsymbol{n}_\perp) &= -2 \begin{pmatrix} \boldsymbol{\epsilon}_{\perp z} & \boldsymbol{\xi}_{\perp z} \\ \boldsymbol{\zeta}_{\perp z} & \boldsymbol{\mu}_{\perp z} \end{pmatrix} \begin{pmatrix} \epsilon_{zz} & \xi_{zz} \\ \zeta_{zz} & \mu_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0}_{z\perp} & \hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp \\ -\hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp & \mathbf{0}_{z\perp} \end{pmatrix} \\
&\quad - 2 \begin{pmatrix} \mathbf{0}_{\perp z} & \boldsymbol{n}_\perp \times \hat{\boldsymbol{z}} \\ -\boldsymbol{n}_\perp \times \hat{\boldsymbol{z}} & \mathbf{0}_{\perp z} \end{pmatrix} \begin{pmatrix} \epsilon_{zz} & \xi_{zz} \\ \zeta_{zz} & \mu_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{\epsilon}_{z\perp} & \boldsymbol{\xi}_{z\perp} \\ \boldsymbol{\zeta}_{z\perp} & \boldsymbol{\mu}_{z\perp} \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix}^{-} (\hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp) &= -2 \begin{pmatrix} \boldsymbol{\epsilon}_{\perp z} & \boldsymbol{\xi}_{\perp z} \\ \boldsymbol{\zeta}_{\perp z} & \boldsymbol{\mu}_{\perp z} \end{pmatrix} \begin{pmatrix} \epsilon_{zz} & \xi_{zz} \\ \zeta_{zz} & \mu_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0}_{z\perp} & -\boldsymbol{n}_\perp \\ \boldsymbol{n}_\perp & \mathbf{0}_{z\perp} \end{pmatrix} \\
&\quad - 2 \begin{pmatrix} \mathbf{0}_{\perp z} & \boldsymbol{n}_\perp \\ -\boldsymbol{n}_\perp & \mathbf{0}_{\perp z} \end{pmatrix} \begin{pmatrix} \epsilon_{zz} & \xi_{zz} \\ \zeta_{zz} & \mu_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{\epsilon}_{z\perp} & \boldsymbol{\xi}_{z\perp} \\ \boldsymbol{\zeta}_{z\perp} & \boldsymbol{\mu}_{z\perp} \end{pmatrix},
\end{aligned}$$

where the left members are known. Using orthogonality in the first equation gives

$$\begin{aligned}
&\begin{pmatrix} \epsilon_{zz} & \xi_{zz} \\ \zeta_{zz} & \mu_{zz} \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} \mathbf{0}_{z\perp} & -\frac{\boldsymbol{n}_\perp}{\boldsymbol{n}_\perp \cdot \boldsymbol{n}_\perp} \\ \frac{\boldsymbol{n}_\perp}{\boldsymbol{n}_\perp \cdot \boldsymbol{n}_\perp} & \mathbf{0}_{z\perp} \end{pmatrix} \\
&\cdot \left(\begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix}^{+} (\boldsymbol{n}_\perp) - \begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix}^{+} (\hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp) \right) \cdot \begin{pmatrix} \mathbf{0}_{\perp z} & \frac{\boldsymbol{n}_\perp}{\boldsymbol{n}_\perp \cdot \boldsymbol{n}_\perp} \\ -\frac{\boldsymbol{n}_\perp}{\boldsymbol{n}_\perp \cdot \boldsymbol{n}_\perp} & \mathbf{0}_{\perp z} \end{pmatrix}
\end{aligned}$$

and combining the two latter equations using orthogonality and the identity (1.22)

[9] The medium may be plane-stratified in the z-direction.

yields

$$
\begin{aligned}
&\begin{pmatrix} \boldsymbol{\epsilon}_{\perp z} & \boldsymbol{\xi}_{\perp z} \\ \boldsymbol{\zeta}_{\perp z} & \boldsymbol{\mu}_{\perp z} \end{pmatrix} \begin{pmatrix} \epsilon_{zz} & \xi_{zz} \\ \zeta_{zz} & \mu_{zz} \end{pmatrix}^{-1} \\
&= -\frac{1}{2} \begin{pmatrix} \frac{\boldsymbol{n}_\perp \boldsymbol{n}_\perp}{\boldsymbol{n}_\perp \cdot \boldsymbol{n}_\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \frac{\boldsymbol{n}_\perp \boldsymbol{n}_\perp}{\boldsymbol{n}_\perp \cdot \boldsymbol{n}_\perp} \end{pmatrix} \cdot \begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix}^{-} (\boldsymbol{n}_\perp) \cdot \begin{pmatrix} \mathbf{0}_{\perp z} & \frac{\boldsymbol{n}_\perp \times \hat{\boldsymbol{z}}}{\boldsymbol{n}_\perp \cdot \boldsymbol{n}_\perp} \\ -\frac{\boldsymbol{n}_\perp \times \hat{\boldsymbol{z}}}{\boldsymbol{n}_\perp \cdot \boldsymbol{n}_\perp} & \mathbf{0}_{\perp z} \end{pmatrix} \\
&\quad -\frac{1}{2} \begin{pmatrix} \frac{(\hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp)(\hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp)}{\boldsymbol{n}_\perp \cdot \boldsymbol{n}_\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \frac{(\hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp)(\hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp)}{\boldsymbol{n}_\perp \cdot \boldsymbol{n}_\perp} \end{pmatrix} \cdot \begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix}^{-} (\hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp) \cdot \begin{pmatrix} \mathbf{0}_{\perp z} & \frac{\boldsymbol{n}_\perp}{\boldsymbol{n}_\perp \cdot \boldsymbol{n}_\perp} \\ -\frac{\boldsymbol{n}_\perp}{\boldsymbol{n}_\perp \cdot \boldsymbol{n}_\perp} & \mathbf{0}_{\perp z} \end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
&\begin{pmatrix} \epsilon_{zz} & \xi_{zz} \\ \zeta_{zz} & \mu_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{\epsilon}_{z\perp} & \boldsymbol{\xi}_{z\perp} \\ \boldsymbol{\zeta}_{z\perp} & \boldsymbol{\mu}_{z\perp} \end{pmatrix} \\
&= -\frac{1}{2} \begin{pmatrix} \mathbf{0}_{z\perp} & \frac{\hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp}{\boldsymbol{n}_\perp \cdot \boldsymbol{n}_\perp} \\ -\frac{\hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp}{\boldsymbol{n}_\perp \cdot \boldsymbol{n}_\perp} & \mathbf{0}_{z\perp} \end{pmatrix} \cdot \begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix}^{-} (\boldsymbol{n}_\perp) \cdot \begin{pmatrix} \frac{\boldsymbol{n}_\perp \boldsymbol{n}_\perp}{\boldsymbol{n}_\perp \cdot \boldsymbol{n}_\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \frac{\boldsymbol{n}_\perp \boldsymbol{n}_\perp}{\boldsymbol{n}_\perp \cdot \boldsymbol{n}_\perp} \end{pmatrix} \\
&\quad -\frac{1}{2} \begin{pmatrix} \mathbf{0}_{z\perp} & -\frac{\boldsymbol{n}_\perp}{\boldsymbol{n}_\perp \cdot \boldsymbol{n}_\perp} \\ \frac{\boldsymbol{n}_\perp}{\boldsymbol{n}_\perp \cdot \boldsymbol{n}_\perp} & \mathbf{0}_{z\perp} \end{pmatrix} \cdot \begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix}^{-} (\hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp) \cdot \begin{pmatrix} \frac{(\hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp)(\hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp)}{\boldsymbol{n}_\perp \cdot \boldsymbol{n}_\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \frac{(\hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp)(\hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp)}{\boldsymbol{n}_\perp \cdot \boldsymbol{n}_\perp} \end{pmatrix}.
\end{aligned}
$$

These Volterra convolution equations of the second kind determine the operators

$$
\begin{pmatrix} \epsilon_{zz} & \xi_{zz} \\ \zeta_{zz} & \mu_{zz} \end{pmatrix}, \qquad \begin{pmatrix} \boldsymbol{\epsilon}_{\perp z} & \boldsymbol{\xi}_{\perp z} \\ \boldsymbol{\zeta}_{\perp z} & \boldsymbol{\mu}_{\perp z} \end{pmatrix}, \qquad \begin{pmatrix} \boldsymbol{\epsilon}_{z\perp} & \boldsymbol{\xi}_{z\perp} \\ \boldsymbol{\zeta}_{z\perp} & \boldsymbol{\mu}_{z\perp} \end{pmatrix}.
$$

Finally, by definition,

$$
\begin{pmatrix} \boldsymbol{\epsilon}_{\perp\perp} & \boldsymbol{\xi}_{\perp\perp} \\ \boldsymbol{\zeta}_{\perp\perp} & \boldsymbol{\mu}_{\perp\perp} \end{pmatrix} = \begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix} (\mathbf{0}_\perp) + \begin{pmatrix} \boldsymbol{\epsilon}_{\perp z} & \boldsymbol{\xi}_{\perp z} \\ \boldsymbol{\zeta}_{\perp z} & \boldsymbol{\mu}_{\perp z} \end{pmatrix} \begin{pmatrix} \epsilon_{zz} & \xi_{zz} \\ \zeta_{zz} & \mu_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{\epsilon}_{z\perp} & \boldsymbol{\xi}_{z\perp} \\ \boldsymbol{\zeta}_{z\perp} & \boldsymbol{\mu}_{z\perp} \end{pmatrix},
$$

where the first term in the right member can be computed using (1.10).

As a consequence of the analysis, one can concentrate on obtaining the operator

$$
\begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix} (\boldsymbol{n}_\perp) \tag{1.11}
$$

for one single transverse directional vector $\boldsymbol{n}_\perp$ (*i.e.*, one fixed angle of incidence and one fixed azimuth angle of incidence) from scattering data. This quantity comprises 16 parameters. The general idea of obtaining these parameters is almost obvious: for each one of the four types of excitations — up-going and down-going TE and TM pulses — four scattering parameters are measured, namely the scattered up-going and down-going TE and TM pulses.

1.5 Wave propagator for the homogeneous slab

For a homogeneous, bi-anisotropic layer, the wave propagator is translation invariant, *i.e.*, the pair of arguments (z, z') can be replaced by the single argument $(z-z')$. Thus, the propagator satisfies the integro PDE

$$\partial_z \begin{pmatrix} \mathbf{\Pi}^{ee} & \mathbf{\Pi}^{em} \\ \mathbf{\Pi}^{me} & \mathbf{\Pi}^{mm} \end{pmatrix}(z) = c_0^{-1}\partial_t \begin{pmatrix} \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\zeta}} & \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\mu}} \\ -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\epsilon}} & -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\xi}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{\Pi}^{ee} & \mathbf{\Pi}^{em} \\ \mathbf{\Pi}^{me} & \mathbf{\Pi}^{mm} \end{pmatrix}(z), \tag{1.12}$$

where

$$\begin{pmatrix} \mathbf{\Pi}^{ee} & \mathbf{\Pi}^{em} \\ \mathbf{\Pi}^{me} & \mathbf{\Pi}^{mm} \end{pmatrix}(0) = \begin{pmatrix} \mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp} \end{pmatrix}.$$

Formally, the wave propagator can be written as

$$\begin{pmatrix} \mathbf{\Pi}^{ee} & \mathbf{\Pi}^{em} \\ \mathbf{\Pi}^{me} & \mathbf{\Pi}^{mm} \end{pmatrix}(z) = \exp\left(z c_0^{-1}\partial_t \begin{pmatrix} \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\zeta}} & \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\mu}} \\ -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\epsilon}} & -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\xi}} \end{pmatrix} \cdot \right) \begin{pmatrix} \mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp} \end{pmatrix}. \tag{1.13}$$

A more explicit but not very useful expression is given by the Mclaurin series

$$\begin{pmatrix} \mathbf{\Pi}^{ee} & \mathbf{\Pi}^{em} \\ \mathbf{\Pi}^{me} & \mathbf{\Pi}^{mm} \end{pmatrix}(z) = \sum_{m=0}^{\infty} \frac{\left(z c_0^{-1}\partial_t\right)^m}{m!} \left(\begin{pmatrix} \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\zeta}} & \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\mu}} \\ -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\epsilon}} & -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\xi}} \end{pmatrix} \cdot \right)^m \begin{pmatrix} \mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp} \end{pmatrix},$$

which is obtained by solving (1.12) using the method of successive approximations starting with

$$\begin{pmatrix} \mathbf{\Pi}^{ee} & \mathbf{\Pi}^{em} \\ \mathbf{\Pi}^{me} & \mathbf{\Pi}^{mm} \end{pmatrix}(z) = \begin{pmatrix} \mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp} \end{pmatrix}.$$

The propagator for the homogeneous layer can be decomposed using basically pure algebraic concepts such as the Cayley-Hamilton theorem and the Lagrange-Sylvester interpolation polynomials, see appendix A. After having decomposed the wave propagator, solving a number of scalar integral equations remains; in particular, a number (at most four) of scalar wave propagators have to be calculated.

To apply the theory, one needs the eigenoperators λ of the matrix operator

$$\begin{pmatrix} \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\zeta}} & \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\mu}} \\ -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\epsilon}} & -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\xi}} \end{pmatrix}$$

obtained by solving the scalar Volterra convolution equation of the second kind

$$0 = \det \begin{pmatrix} \lambda \mathbf{I}_{\perp\perp} - \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\zeta}} & -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\mu}} \\ \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\epsilon}} & \lambda \mathbf{I}_{\perp\perp} + \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\xi}} \end{pmatrix}$$

or[10]

$$\begin{aligned} 0 = &\det((\lambda \mathbf{I}_{\perp\perp} - \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\zeta}}) \cdot (\lambda \mathbf{I}_{\perp\perp} + \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\xi}}) \\ &+ (\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\mu}}) \cdot (\lambda \mathbf{I}_{\perp\perp} + \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\xi}})^{-1} \cdot (\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\epsilon}}) \cdot (\lambda \mathbf{I}_{\perp\perp} + \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\xi}})). \end{aligned} \tag{1.14}$$

[10] The determinant of a (square) matrix with square diagonal blocks is

$$\det \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} = \det\left(\mathbf{A}_{11} \cdot \mathbf{A}_{22} - \mathbf{A}_{12} \cdot \mathbf{A}_{22}^{-1} \cdot \mathbf{A}_{21} \cdot \mathbf{A}_{22}\right),$$

provided that $\mathbf{A}_{22}$ is non-singular.

Four eigenoperators of the form

$$\begin{cases} \lambda_+^- = n_z + (N_+^- H)\circ, \\ \lambda_-^- = n_z + (N_-^- H)\circ, \\ \lambda_+^+ = -n_z + (N_+^+ H)\circ, \\ \lambda_-^+ = -n_z + (N_-^+ H)\circ, \end{cases} \tag{1.15}$$

where $N_i^k = N_i^k(t)$, $i, k \in \{+, -\}$, are bounded and smooth scalar kernels, are thus obtained. In vacuum, these kernels are all equal to zero. The superscript +(-) refers to up-going (down-going) waves[11]. It can happen that $\lambda_+^- = \lambda_-^-$ or $\lambda_+^- = \lambda_-^-$ as in vacuum, isotropic materials, and Tellegen materials [19], see section 1.5.3. This is however the only possible degeneration for a general complex medium when $n_z > 0$.

Consider the case when $\lambda_1^- \neq \lambda_2^-$ and $\lambda_+^+ \neq \lambda_-^+$. This means that there is a least integer degree of differentiation $m^+ \geq 0$ such that $(N_+^\pm)^{(m^+)}(0) \neq (N_-^\pm)^{(m^+)}(0)$ and a least integer degree of differentiation $m^- \geq 0$ such that $(N_+^-)^{(m^-)}(0) \neq (N_-^-)^{(m^-)}(0)$. Actually, $m^\pm$ are positive since

$$N_i^k(0) = 0 \qquad (i, k \in \{+, -\}) \tag{1.16}$$

due to (1.4) and (1.14). In view of the results in appendix A, one has to define what is meant by inverses of operators of the form $\lambda_i^k - \lambda_j^l$, $i, j, k, l, \in \{+, -\}$. This offers no difficulty except in the two cases when the superscripts coincide, since $\lambda_+^\pm - \lambda_-^\pm$ are Volterra convolution operators of the first kind and all other operators are Volterra convolution operators of the second kind. One can, however, apply the technique of transforming a first-order equation to a second-order equation by differentiation presented in section 1.2 and factor as

$$\lambda_+^\pm - \lambda_-^\pm := \partial_t^{(-m^\pm - 1)} \left(\partial_t^{(m^\pm + 1)} (\lambda_+^\pm - \lambda_-^\pm) \right),$$

where

$$\partial_t^{(m^\pm + 1)} (\lambda_+^\pm - \lambda_-^\pm) = N_+^{(m^\pm)}(0) - N_-^{(m^\pm)}(0) + (N_+^{(m^\pm + 1)} - N_-^{(m^\pm + 1)})\circ$$

is a second-kind operator. The inverses of these operators are now well defined:

$$(\lambda_+^\pm - \lambda_-^\pm)^{-1} := \partial_t^{(m^\pm + 1)} \left(\partial_t^{(m^\pm + 1)} (\lambda_+^\pm - \lambda_-^\pm) \right)^{-1}.$$

Similar considerations have to be made in the degenerate cases. The theory in appendix A can now be fully adopted.

1.5.1 Non-degenerate case

If f is entire and $\lambda_-^- \neq \lambda_-^-$ and $\lambda_+^+ \neq \lambda_-^+$, then

$$f\left(\frac{z}{c_0} \partial_t \begin{pmatrix} \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\zeta}} & \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\mu}} \\ -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\epsilon}} & -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\xi}} \end{pmatrix} \cdot \right) \begin{pmatrix} \mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp} \end{pmatrix} = \sum_{i,k \in \{+,-\}} \begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}_i^k f_i^k(z),$$

[11]The subscripts are not of the essence here; however, at fixed frequency, they represent right-hand elliptically polarized waves and left-hand elliptically polarized waves.

where the spectral projections are given by[12]

$$\begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}_i^k = \prod_{j \neq i, l \neq k} \left(\begin{pmatrix} \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\zeta}} & \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\mu}} \\ -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\epsilon}} & -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\xi}} \end{pmatrix} - \begin{pmatrix} \mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp} \end{pmatrix} \lambda_j^l \right) (\lambda_i^k - \lambda_j^l)^{-1}$$

and the scalar convolution operators are given by

$$f_i^k(z) = f(zc_0^{-1}\partial_t \lambda_i^k).$$

In particular, the wave propagator is

$$\begin{pmatrix} \mathbf{\Pi}^{ee} & \mathbf{\Pi}^{em} \\ \mathbf{\Pi}^{me} & \mathbf{\Pi}^{mm} \end{pmatrix}(z) = \sum_{i,k \in \{+,-\}} \begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}_i^k \Pi_i^k(z),$$

where the elementary propagators are

$$\Pi_i^k(z) = \exp\left(zc_0^{-1}\partial_t \lambda_i^k\right).$$

Moreover, by taking $f(z) = 1$ and $f(z) = z^m$, $m = 1, 2, 3$, the useful identities

$$\begin{cases} \begin{pmatrix} \mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp} \end{pmatrix} = \sum_{i,k \in \{+,-\}} \begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}_i^k, \\ \left(\begin{pmatrix} \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\zeta}} & \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\mu}} \\ -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\epsilon}} & -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\xi}} \end{pmatrix} \cdot \right)^m \begin{pmatrix} \mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp} \end{pmatrix} = \sum_{i,k \in \{+,-\}} \begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}_i^k \left(\lambda_i^k\right)^m \end{cases}$$

are obtained. The spectral projections have the orthogonality property[13]

$$\begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}_i^k \cdot \begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}_j^l = \begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}_i^k \delta_{ij}\delta_{kl}, \tag{1.17}$$

where δ_{kl} denotes the Kronecker delta.

The elementary propagators

$$\Pi_i^k(z) = \exp\left(zc_0^{-1}\partial_t \lambda_i^k\right) = \sum_{m=0}^{\infty} \frac{\left(zc_0^{-1}\partial_t \lambda_i^k\right)^m}{m!} \tag{1.18}$$

are easy to calculate [13]. The operators λ_i^k are given by (1.15). In view of

$$\exp\left(\mp z n_z c_0^{-1}\partial_t\right)\phi(t) = \sum_{n=0}^{\infty} \frac{\left(\mp z n_z c_0^{-1}\right)^n}{n!} \phi^{(n)}(t) = \phi(t \mp n_z z/c_0) \qquad (\phi \in C_0^\infty(R)),$$

one obtains

$$\exp\left(\mp z n_z c_0^{-1}\partial_t\right) = \delta(t \mp n_z z/c_0)\circ = \delta_{\pm t_z}\circ,$$

[12]Observe that operators of the form $zc_0^{-1}\partial_t$ cancel.

[13]The case when $k \neq l$ is a direct consequence of the Cayley-Hamilton theorem. Using this result and the first identity in (1.5.1) gives the desired result.

where

$$t_z = n_z z / c_0,$$

and, consequently, using the characteristic property of the exponential,

$$\begin{cases} \Pi_+^-(z) = Q_+^-(z)\delta_{-t_z} \circ \exp\left(zc_0^{-1}(H\partial_t N_+^-)\circ\right)), \\ \Pi_-^-(z) = Q_-^-(z)\delta_{-t_z} \circ \exp\left(zc_0^{-1}(H\partial_t N_-^-)\circ\right)), \\ \Pi_+^+(z) = Q_-^-(z)\delta_{t_z} \circ \exp\left(zc_0^{-1}(H\partial_t N_+^+)\circ\right)), \\ \Pi_-^+(z) = Q_-^+(z)\delta_{t_z} \circ \exp\left(zc_0^{-1}(H\partial_t N_-^+)\circ\right)), \end{cases} \tag{1.19}$$

where

$$Q_i^k(z) = \exp\left(zc_0^{-1}N_i^k(0)\right)$$

are damping factors. The operator

$$\exp\left(zc_0^{-1}(H\partial_t N_i^k)\circ\right)) = \sum_{m=0}^{\infty} \frac{\left(zc_0^{-1}(H\partial_t N_i^k)\circ\right)^m}{m!}$$

can be written as

$$\exp\left(zc_0^{-1}(H\partial_t N_i^k)(\cdot)\circ\right)) = 1 + P_i^k(z,\cdot)\circ .$$

The kernel $P_i^k = P_i^k(z,t)$ is determined in terms of the kernel $H\partial_t N_i^k = (H\partial_t N_i^k)(t)$ by the Volterra temporal convolution equation of the second kind [13]

$$zc_0^{-1}M_i^k(t) + zc_0^{-1}(M_i^k(\cdot) \circ P_i^k(z,\cdot))(t) = tP_i^k(z,t), \tag{1.20}$$

where $M_i^k(t) = t(H\partial_t N_i^k)(t) = tH(t)\partial_t N_i^k(t)$. However, one deduces from (1.20), that the kernel $H\partial_t N_i^k = (H\partial_t N_i^k)(t)$ is also determined in terms of the kernel $P_i^k = P_i^k(z,t)$ by another the Volterra temporal convolution equation of the second kind. This observation is importance for the inverse problem, since the kernel $N_i^k(t)$ can be determined by integration provided the value $N_i^k(0)$ is known *a priori*. This is, however, in view of (1.16), the case. Otherwise, there is an ambiguity arising from the complex logarithm:

$$zc_0^{-1}N_i^k(0) = \ln\left(Q_i^k(z)\right).$$

Thus, the operator equation

$$\Pi_i^k(z) = \exp\left(zc_0^{-1}\partial_t \lambda_i^k\right)$$

has a unique solution for a given left member $\Pi_i^k(z)$; the solution is written

$$zc_0^{-1}\partial_t \lambda_i^k = \ln\left(\Pi_i^k(z)\right).$$

Actually, the solution is given by the series expansion

$$\ln\left(\Pi_i^k(z)\right) = \sum_{m=1}^{\infty} \frac{\left(1 - \Pi_i^k(z)\right)^m}{m}, \tag{1.21}$$

which easily can be checked by substitution into (1.18). Equation (1.20) is easy to solve numerically both ways for a fixed but arbitrary penetration depth z [13].

1.5.2 degenerate cases

The degenerate cases, which arise 1) when the eigenoperators $\lambda^+_\pm$ coincide, 2) when the eigenoperators $\lambda^-_\pm$ coincide, or 3) when $\lambda^+_\pm$ coincide and $\lambda^-_\pm$ coincide, can be found by applying the theory in appendix A. However, a limit procedure, based on spectral projections used in the non-degenerate case, applies just as well [10]. The results and notation given below reflect the effects of such a procedure. It is appropriate to make the following definition:

$$\begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^{\pm} := \begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^{\pm}_{+} + \begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^{\pm}_{-}.$$

1. $\lambda^+_\pm = \lambda^+$ and $\lambda^-_+ \neq \lambda^-_-$:

$$\begin{aligned}
&f\left(\frac{z}{c_0}\partial_t \begin{pmatrix} \hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\zeta}} & \hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\mu}} \\ -\hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\epsilon}} & -\hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\xi}} \end{pmatrix}\cdot\right)\begin{pmatrix} \mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp} \end{pmatrix} = \\
&f(zc_0^{-1}\partial_t\lambda^-_+)\lim_{\lambda^+_\pm\to\lambda^+}\begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^{-}_{+} + f(zc_0^{-1}\partial_t\lambda^-_-)\lim_{\lambda^+_\pm\to\lambda^+}\begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^{-}_{-} \\
&+ f(zc_0^{-1}\partial_t\lambda^+)\lim_{\lambda^+_\pm\to\lambda^+}\begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^{+} + zc_0^{-1}\partial_t f'(zc_0^{-1}\partial_t\lambda^+) \\
&\cdot\left(\begin{pmatrix} \hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\zeta}} & \hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\mu}} \\ -\hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\epsilon}} & -\hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\xi}} \end{pmatrix} - \begin{pmatrix} \mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp} \end{pmatrix}\lambda^+\right)\cdot\lim_{\lambda^+_\pm\to\lambda^+}\begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^{+}.
\end{aligned}$$

2. $\lambda^-_\pm = \lambda^-$ and $\lambda^+_+ \neq \lambda^+_-$:

$$\begin{aligned}
&f\left(\frac{z}{c_0}\partial_t \begin{pmatrix} \hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\zeta}} & \hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\mu}} \\ -\hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\epsilon}} & -\hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\xi}} \end{pmatrix}\cdot\right)\begin{pmatrix} \mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp} \end{pmatrix} = \\
&f(zc_0^{-1}\partial_t\lambda^+_+)\lim_{\lambda^-_\pm\to\lambda^-}\begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^{+}_{+} + f(zc_0^{-1}\partial_t\lambda^+_-)\lim_{\lambda^-_\pm\to\lambda^-}\begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^{+}_{-} \\
&+ f(zc_0^{-1}\partial_t\lambda^-)\lim_{\lambda^-_\pm\to\lambda^-}\begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^{-} + zc_0^{-1}\partial_t f'(zc_0^{-1}\partial_t\lambda^-) \\
&\cdot\left(\begin{pmatrix} \hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\zeta}} & \hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\mu}} \\ -\hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\epsilon}} & -\hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\xi}} \end{pmatrix} - \begin{pmatrix} \mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp} \end{pmatrix}\lambda^-\right)\cdot\lim_{\lambda^-_\pm\to\lambda^-}\begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^{-}.
\end{aligned}$$

3. $\lambda_{\pm}^{+} = \lambda^{+}$ and $\lambda_{\pm}^{-} = \lambda^{-}$:

$$
\begin{aligned}
&f\left(\frac{z}{c_0}\partial_t\begin{pmatrix}\hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\zeta}} & \hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\mu}}\\ -\hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\epsilon}} & -\hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\xi}}\end{pmatrix}\cdot\right)\begin{pmatrix}\mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp}\\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp}\end{pmatrix}\\
&= f(zc_0^{-1}\partial_t\lambda^{+})\lim_{\lambda_{\pm}^{\pm}\to\lambda^{\pm}}\begin{pmatrix}\mathbf{W}^{ee} & \mathbf{W}^{em}\\ \mathbf{W}^{me} & \mathbf{W}^{mm}\end{pmatrix}^{+} + zc_0^{-1}\partial_t f'(zc_0^{-1}\partial_t\lambda^{+})\\
&\cdot\left(\left(\begin{pmatrix}\hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\zeta}} & \hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\mu}}\\ -\hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\epsilon}} & -\hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\xi}}\end{pmatrix} - \begin{pmatrix}\mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp}\\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp}\end{pmatrix}\lambda^{+}\right)\cdot\lim_{\lambda_{\pm}^{\pm}\to\lambda^{\pm}}\begin{pmatrix}\mathbf{W}^{ee} & \mathbf{W}^{em}\\ \mathbf{W}^{me} & \mathbf{W}^{mm}\end{pmatrix}^{+}\right.\\
&+ f(zc_0^{-1}\partial_t\lambda^{-})\lim_{\lambda_{\pm}^{\pm}\to\lambda^{\pm}}\begin{pmatrix}\mathbf{W}^{ee} & \mathbf{W}^{em}\\ \mathbf{W}^{me} & \mathbf{W}^{mm}\end{pmatrix}^{-} + zc_0^{-1}\partial_t f'(zc_0^{-1}\partial_t\lambda^{-})\\
&\cdot\left(\left(\begin{pmatrix}\hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\zeta}} & \hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\mu}}\\ -\hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\epsilon}} & -\hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\xi}}\end{pmatrix} - \begin{pmatrix}\mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp}\\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp}\end{pmatrix}\lambda^{-}\right)\cdot\lim_{\lambda_{\pm}^{\pm}\to\lambda^{\pm}}\begin{pmatrix}\mathbf{W}^{ee} & \mathbf{W}^{em}\\ \mathbf{W}^{me} & \mathbf{W}^{mm}\end{pmatrix}^{-}\right. .
\end{aligned}
$$

In all cases, the explicit values of the appearing limits of spectral projections can be found in appendix A. They can also be computed by setting $f = 1$, $f = z$, and $f = z^2$ and solving the thus obtained system of linear equations. The orthogonality properties of the spectral projections is preserved.

1.5.3 Wave propagator for a homogeneous, isotropic material

The homogeneous, isotropic material attracts interest from two reasons: it represents the common situation, and it is degenerate implying that the spectral projections concept cannot be applied in a straightforward manner.

In a homogeneous, isotropic material, where

$$
\begin{pmatrix}\widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}}\\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}}\end{pmatrix} = \begin{pmatrix}\mathbf{I}_{\perp\perp}\epsilon + \hat{\boldsymbol{z}}\times\boldsymbol{n}_{\perp}\boldsymbol{n}_{\perp}\times\hat{\boldsymbol{z}}\mu^{-1} & \mathbf{0}_{\perp\perp}\\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp}\mu + \hat{\boldsymbol{z}}\times\boldsymbol{n}_{\perp}\boldsymbol{n}_{\perp}\times\hat{\boldsymbol{z}}\epsilon^{-1}\end{pmatrix},
$$

the eigenoperator equation becomes

$$
\begin{aligned}
0 &= \det(\lambda^2\mathbf{I}_{\perp\perp} + (\hat{\boldsymbol{z}}\times\mathbf{I}_{\perp\perp}\mu - \boldsymbol{n}_{\perp}\boldsymbol{n}_{\perp}\times\hat{\boldsymbol{z}}\epsilon^{-1})\cdot(\hat{\boldsymbol{z}}\times\mathbf{I}_{\perp\perp}\epsilon - \boldsymbol{n}_{\perp}\boldsymbol{n}_{\perp}\times\hat{\boldsymbol{z}}\mu^{-1}))\\
&= \det((\lambda^2 - \mu\epsilon)\mathbf{I}_{\perp\perp} - \hat{\boldsymbol{z}}\times\boldsymbol{n}_{\perp}\boldsymbol{n}_{\perp}\times\hat{\boldsymbol{z}} + \boldsymbol{n}_{\perp}\boldsymbol{n}_{\perp})\\
&= \det((\lambda^2 - \mu\epsilon + \boldsymbol{n}_{\perp}\cdot\boldsymbol{n}_{\perp})\mathbf{I}_{\perp\perp}) = (\lambda^2 - \mu\epsilon + \boldsymbol{n}_{\perp}\cdot\boldsymbol{n}_{\perp})^2
\end{aligned}
$$

owing to the identity

$$
\mathbf{I}_{\perp\perp}(\boldsymbol{n}_{\perp}\cdot\boldsymbol{n}_{\perp}) = \boldsymbol{n}_{\perp}\boldsymbol{n}_{\perp} - \hat{\boldsymbol{z}}\times\boldsymbol{n}_{\perp}\boldsymbol{n}_{\perp}\times\hat{\boldsymbol{z}}, \tag{1.22}
$$

which reflects the fact that the vectors $\boldsymbol{n}_{\perp}$ and $\hat{\boldsymbol{z}}\times\boldsymbol{n}_{\perp}$ constitute a linear basis for vectors in the x-y plane. The eigenoperators are, where

$$
\lambda_{+}^{-} = \lambda_{+}^{-} = -\lambda_{+}^{+} = -\lambda_{-}^{+} := \lambda = \sqrt{\mu\epsilon - \boldsymbol{n}_{\perp}\cdot\boldsymbol{n}_{\perp}} = n_z + (HN)\circ .
$$

The kernel $N(t)$ satisfies the Volterra convolution equation of the second kind

$$2n_z H(t)N(t) + ((HN) \circ (HN))(t) = H(t)(\chi^{ee}(t) + \chi^{ee}(t)) + ((H\chi^{ee}) \circ (H\chi^{mm}))(t),$$

where $\boldsymbol{\chi}^{ee} = \mathbf{I}\chi^{ee}$ and $\boldsymbol{\chi}^{mm} = \mathbf{I}\chi^{mm}$.

Obviously, the isotropic medium represents a degenerate case, and straightforward computations show that

$$\begin{pmatrix} \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\zeta}} & \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\mu}} \\ -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\epsilon}} & -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\xi}} \end{pmatrix} \cdot \begin{pmatrix} \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\zeta}} & \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\mu}} \\ -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\epsilon}} & -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\xi}} \end{pmatrix} = \begin{pmatrix} \mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp} \end{pmatrix} \lambda^2$$

Consequently, for any entire function f, using the result for this degenerate case given in 1.5.2 (choose first $f = z^2$) or in appendix A, one has

$$\begin{aligned} f\left(\frac{z}{c_0}\partial_t \begin{pmatrix} \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\zeta}} & \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\mu}} \\ -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\epsilon}} & -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\xi}} \end{pmatrix} \cdot \right) &= \begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^{-} f(zc_0^{-1}\partial_t \lambda) \\ &+ \begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^{+} f(-zc_0^{-1}\partial_t \lambda), \end{aligned} \tag{1.23}$$

where the introduced spectral projections[14]

$$\begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^{\pm}$$

can be obtained by straightforward computation. However, it is somewhat easier to apply the result (1.23) to $f = 1$ and to $f = z$ and obtain the system of equations

$$\begin{cases} \begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^{+} + \begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^{-} = \begin{pmatrix} \mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp} \end{pmatrix}, \\ -\lambda \begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^{+} + \lambda \begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^{-} = \begin{pmatrix} \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\zeta}} & \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\mu}} \\ -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\epsilon}} & -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\xi}} \end{pmatrix}, \end{cases}$$

which can be solved giving

$$\begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^{\mp} = \frac{1}{2}\left(\begin{pmatrix} \mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp} \end{pmatrix} \pm \begin{pmatrix} \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\zeta}} & \hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\mu}} \\ -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\epsilon}} & -\hat{\boldsymbol{z}} \times \widetilde{\boldsymbol{\xi}} \end{pmatrix} \lambda^{-1} \right),$$

i.e.,

$$\begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^{\mp} = \frac{1}{2}\begin{pmatrix} \mathbf{I}_{\perp\perp} & \pm\hat{\boldsymbol{z}} \times \mathbf{Z}_{\perp\perp} \\ \mp\hat{\boldsymbol{z}} \times \mathbf{Y}_{\perp\perp} & \mathbf{I}_{\perp\perp} \end{pmatrix},$$

where the relative impedance and admittance operators, respectively, are given by

$$\begin{cases} \mathbf{Z}_{\perp\perp} = \left(\mathbf{I}_{\perp\perp}\mu + \hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp \boldsymbol{n}_\perp \times \hat{\boldsymbol{z}}\epsilon^{-1}\right)\lambda^{-1}, \\ \mathbf{Y}_{\perp\perp} = \left(\mathbf{I}_{\perp\perp}\epsilon + \hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp \boldsymbol{n}_\perp \times \hat{\boldsymbol{z}}\mu^{-1}\right)\lambda^{-1}. \end{cases}$$

[14] A simplified notation is adopted here.

The orthogonality property, which is a reminiscent of (1.17)

$$\begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^k \cdot \begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^l = \delta_{k,l} \qquad (k, l \in \{+, -\})$$

justifies calling these operators spectral projections. In particular, the wave propagator is

$$\begin{aligned} \begin{pmatrix} \mathbf{\Pi}^{ee} & \mathbf{\Pi}^{em} \\ \mathbf{\Pi}^{me} & \mathbf{\Pi}^{mm} \end{pmatrix}(z) = &\begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^- \exp\left(zc_0^{-1}\partial_t\lambda\right) \\ &+ \begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^+ \exp\left(-zc_0^{-1}\partial_t\lambda\right). \end{aligned}$$

Using these properties of the spectral projections, the fundamental equation (1.8) at oblique incidence for the isotropic slab becomes

$$\partial_z \begin{pmatrix} \boldsymbol{E}_\perp \\ \eta_0 \boldsymbol{H}_\perp \end{pmatrix} = c_0^{-1}\partial_t \left\{ \lambda \left(\begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^- - \begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^+ \right) \cdot \begin{pmatrix} \boldsymbol{E}_\perp \\ \eta_0 \boldsymbol{H}_\perp \end{pmatrix} \right\}.$$

Moreover,

$$\partial_z \begin{pmatrix} \boldsymbol{E}_\perp^\pm \\ \eta_0 \boldsymbol{H}_\perp^\pm \end{pmatrix} = \mp c_0^{-1}\partial_t \left\{ \lambda \cdot \begin{pmatrix} \boldsymbol{E}_\perp^\pm \\ \eta_0 \boldsymbol{H}_\perp^\pm \end{pmatrix} \right\},$$

where

$$\begin{pmatrix} \boldsymbol{E}_\perp^\pm \\ \eta_0 \boldsymbol{H}_\perp^\pm \end{pmatrix} = \begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^\pm \cdot \begin{pmatrix} \boldsymbol{E}_\perp \\ \eta_0 \boldsymbol{H}_\perp \end{pmatrix}$$

and the + (-) sign represents the up-going (down-going) fields. Also,

$$\begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^\mp \cdot \begin{pmatrix} \boldsymbol{E}_\perp^\pm \\ \eta_0 \boldsymbol{H}_\perp^\pm \end{pmatrix} = \begin{pmatrix} \mathbf{0}_\perp \\ \mathbf{0}_\perp \end{pmatrix}.$$

In particular,

$$\begin{cases} \boldsymbol{E}_\perp^\pm = \dfrac{1}{2}\left(\boldsymbol{E}_\perp \mp \hat{\boldsymbol{z}} \times \mathbf{Z}_{\perp\perp} \cdot \eta_0 \boldsymbol{H}_\perp\right), \\ \mathbf{0}_\perp = \mp \hat{\boldsymbol{z}} \times \mathbf{Y}_{\perp\perp} \cdot \boldsymbol{E}_\perp^\pm + \eta_0 \boldsymbol{H}_\perp^\pm, \end{cases}$$

and, hence

$$\begin{cases} \boldsymbol{E}_\perp = \boldsymbol{E}_\perp^+ + \boldsymbol{E}_\perp^-, \\ \eta_0 \boldsymbol{H}_\perp = \eta_0 \boldsymbol{H}_\perp^+ + \eta_0 \boldsymbol{H}_\perp^- = \hat{\boldsymbol{z}} \times \mathbf{Y}_{\perp\perp} \cdot \boldsymbol{E}_\perp^+ - \hat{\boldsymbol{z}} \times \mathbf{Y}_{\perp\perp} \cdot \boldsymbol{E}_\perp^-. \end{cases}$$

1.5.4 Wave propagator in vacuum

In the vacuum regions,

$$\begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix} = \begin{pmatrix} \mathbf{I}_{\perp\perp} + \hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp \boldsymbol{n}_\perp \times \hat{\boldsymbol{z}} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp} + \hat{\boldsymbol{z}} \times \boldsymbol{n}_\perp \boldsymbol{n}_\perp \times \hat{\boldsymbol{z}} \end{pmatrix}.$$

The wave propagator is

$$\begin{pmatrix} \mathbf{\Pi}^{ee} & \mathbf{\Pi}^{em} \\ \mathbf{\Pi}^{me} & \mathbf{\Pi}^{mm} \end{pmatrix}(z) = \begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^{-} \delta\left(\frac{t+n_z z}{c_0}\right)\circ + \begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^{+} \delta\left(\frac{t-n_z z}{c_0}\right)\circ,$$

where

$$\begin{pmatrix} \mathbf{W}^{ee} & \mathbf{W}^{em} \\ \mathbf{W}^{me} & \mathbf{W}^{mm} \end{pmatrix}^{\mp} = \frac{1}{2}\begin{pmatrix} \mathbf{I}_{\perp\perp} & \pm\frac{\hat{\boldsymbol{z}}\times(\mathbf{I}_{\perp\perp}+\hat{\boldsymbol{z}}\times\boldsymbol{n}_\perp\boldsymbol{n}_\perp\times\hat{\boldsymbol{z}})}{n_z} \\ \mp\frac{\hat{\boldsymbol{z}}\times(\mathbf{I}_{\perp\perp}+\hat{\boldsymbol{z}}\times\boldsymbol{n}_\perp\boldsymbol{n}_\perp\times\hat{\boldsymbol{z}})}{n_z} & \mathbf{I}_{\perp\perp} \end{pmatrix}.$$

The electric and magnetic fields are

$$\begin{cases} \boldsymbol{E}_\perp = \boldsymbol{E}_\perp^+ + \boldsymbol{E}_\perp^-, \\ \eta_0 \boldsymbol{H}_\perp = \hat{\boldsymbol{z}}\times\mathbf{O}^{-1}\cdot\boldsymbol{E}_\perp^+ - \hat{\boldsymbol{z}}\times\mathbf{O}^{-1}\cdot\boldsymbol{E}_\perp^-, \end{cases}$$

where the impedance (and, in fact, admittance) operator in vacuum is (apply (1.22))

$$\mathbf{O}^{-1} = \frac{1}{n_z}\left(\mathbf{I}_{\perp\perp} + \hat{\boldsymbol{z}}\times\boldsymbol{n}_\perp\boldsymbol{n}_\perp\times\hat{\boldsymbol{z}}\right) = \frac{1}{n_z}\left(\mathbf{I}_{\perp\perp}n_z^2 + \boldsymbol{n}_\perp\boldsymbol{n}_\perp\right) \tag{1.24}$$

and the up- and down-going electric fields

$$\boldsymbol{E}_\perp^\pm = \frac{1}{2}\left(\boldsymbol{E}_\perp \mp \hat{\boldsymbol{z}}\times\mathbf{O}^{-1}\cdot\eta_0\boldsymbol{H}_\perp\right)$$

satisfy the PDEs

$$\partial_z \boldsymbol{E}_\perp^\pm = \mp n_z c_0^{-1}\partial_t \boldsymbol{E}_\perp^\pm$$

with plane-wave solutions $\boldsymbol{E}_\perp^\pm(\boldsymbol{r},t) = \boldsymbol{E}_\perp^\pm\left(t - \hat{\boldsymbol{n}}^\pm\cdot\boldsymbol{r}/c_0\right)$ as expected.

The above expressions can be manipulated using

$$\mathbf{O}\times\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}\times\mathbf{O}^{-1}$$

where (apply (1.22))

$$\mathbf{O} = \frac{1}{n_z}\left(\mathbf{I}_{\perp\perp} - \boldsymbol{n}_\perp\boldsymbol{n}_\perp\right) = n_z\left(\mathbf{I}_{\perp\perp} - \frac{1}{n_z^2}\hat{\boldsymbol{z}}\times\boldsymbol{n}_\perp\boldsymbol{n}_\perp\times\hat{\boldsymbol{z}}\right) \tag{1.25}$$

Other useful representations of the operators $\mathbf{O}^{-1}$ and $\mathbf{O}$ at oblique incidence are

$$\begin{cases} \mathbf{O} = \dfrac{\boldsymbol{n}_\perp\boldsymbol{n}_\perp n_z^2 - \hat{\boldsymbol{z}}\times\boldsymbol{n}_\perp\boldsymbol{n}_\perp\times\hat{\boldsymbol{z}}}{n_z\boldsymbol{n}_\perp\cdot\boldsymbol{n}_\perp} \\ \mathbf{O}^{-1} = \dfrac{\boldsymbol{n}_\perp\boldsymbol{n}_\perp - \hat{\boldsymbol{z}}\times\boldsymbol{n}_\perp\boldsymbol{n}_\perp\times\hat{\boldsymbol{z}}n_z^2}{n_z\boldsymbol{n}_\perp\cdot\boldsymbol{n}_\perp} \end{cases}$$

At normal incidence, $\mathbf{O} = \mathbf{O}^{-1} = \mathbf{I}_{\perp\perp}$.

1.6 Vacuum wave splitting

One way to organize efficiently the input to and the output from the homogeneous[15] bi-anisotropic scatterer is to introduce a wave splitting. A wave splitting is a one-to-one correspondence between the dependent vector field variables, *i.e.*, the transverse electric field and the transverse magnetic field, and two new so called split vector field variables, commonly denoted by $\boldsymbol{F}^+$ and $\boldsymbol{F}^-$, that represent the up-going waves and the down-going waves, respectively. Usually, $\boldsymbol{F}^+$ and $\boldsymbol{F}^-$ are taken to be the up-going and down-going transverse electric fields in the absence of the scatterer.

Using matrix notation, the vacuum wave splitting is

$$\begin{pmatrix} \boldsymbol{F}^+ \\ \boldsymbol{F}^- \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathbf{I}_{\perp\perp} & -\mathbf{O}\times\hat{\boldsymbol{z}} \\ \mathbf{I}_{\perp\perp} & \mathbf{O}\times\hat{\boldsymbol{z}} \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{E}_\perp \\ \eta_0 \boldsymbol{H}_\perp \end{pmatrix}$$

with inverse

$$\begin{pmatrix} \boldsymbol{E}_\perp \\ \eta_0 \boldsymbol{H}_\perp \end{pmatrix} = \begin{pmatrix} \mathbf{I}_{\perp\perp} & \mathbf{I}_{\perp\perp} \\ \hat{\boldsymbol{z}}\times\mathbf{O}^{-1} & -\hat{\boldsymbol{z}}\times\mathbf{O}^{-1} \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{F}^+ \\ \boldsymbol{F}^- \end{pmatrix}$$

where the two-dimensional dyadics $\mathbf{O}$ and $\mathbf{O}^{-1}$ are given by (1.25) and (1.24). Notice that only the transverse field variables appear in these transformations and that

$$\begin{pmatrix} \boldsymbol{F}^+ \\ \boldsymbol{F}^- \end{pmatrix}(-d/2) = \begin{pmatrix} \boldsymbol{E}^+_{i0,\perp} \\ \boldsymbol{E}^-_{s0,\perp} \end{pmatrix}, \qquad \begin{pmatrix} \boldsymbol{F}^+ \\ \boldsymbol{F}^- \end{pmatrix}(d/2) = \begin{pmatrix} \boldsymbol{E}^+_{s0,\perp} \\ \boldsymbol{E}^-_{i0,\perp} \end{pmatrix}. \tag{1.26}$$

The fundamental equation at oblique incidence (1.8) is transformed into

$$\partial_z \begin{pmatrix} \boldsymbol{F}^+ \\ \boldsymbol{F}^- \end{pmatrix} = n_z c_0^{-1} \partial_t \begin{pmatrix} -\mathbf{I}_{\perp\perp} + (H\mathbf{N}^{++})\circ & (H\mathbf{N}^{+-})\circ \\ (H\mathbf{N}^{-+})\circ & \mathbf{I}_{\perp\perp} + (H\mathbf{N}^{--})\circ \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{F}^+ \\ \boldsymbol{F}^- \end{pmatrix} \tag{1.27}$$

where the temporal convolution operator

$$\begin{aligned} &n_z \begin{pmatrix} -\mathbf{I}_{\perp\perp} + (H\mathbf{N}^{++})\circ & (H\mathbf{N}^{+-})\circ \\ (H\mathbf{N}^{-+})\circ & \mathbf{I}_{\perp\perp} + (H\mathbf{N}^{--})\circ \end{pmatrix} \\ &\quad = \frac{1}{2}\begin{pmatrix} -\mathbf{O} & \hat{\boldsymbol{z}}\times\mathbf{I}_{\perp\perp} \\ \mathbf{O} & \hat{\boldsymbol{z}}\times\mathbf{I}_{\perp\perp} \end{pmatrix} \cdot \begin{pmatrix} \widetilde{\boldsymbol{\epsilon}} & \widetilde{\boldsymbol{\xi}} \\ \widetilde{\boldsymbol{\zeta}} & \widetilde{\boldsymbol{\mu}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{I}_{\perp\perp} & \mathbf{I}_{\perp\perp} \\ \hat{\boldsymbol{z}}\times\mathbf{O}^{-1} & -\hat{\boldsymbol{z}}\times\mathbf{O}^{-1} \end{pmatrix} \end{aligned}$$

and the kernels $\mathbf{N}^{kl}(t)$ are smooth.

The solution to the propagation problem for a sub-slab of the medium is

$$\begin{pmatrix} \boldsymbol{F}^+ \\ \boldsymbol{F}^- \end{pmatrix}(z) = \begin{pmatrix} \boldsymbol{\Pi}^{++} & \boldsymbol{\Pi}^{+-} \\ \boldsymbol{\Pi}^{-+} & \boldsymbol{\Pi}^{--} \end{pmatrix}(z,z') \cdot \begin{pmatrix} \boldsymbol{F}^+ \\ \boldsymbol{F}^- \end{pmatrix}(z')$$

where the wave propagator for the vacuum-split fields is related to the wave propagator for the transverse electric and magnetic fields by (use same argument (z,z'))

$$\begin{pmatrix} \boldsymbol{\Pi}^{ee} & \boldsymbol{\Pi}^{em} \\ \boldsymbol{\Pi}^{me} & \boldsymbol{\Pi}^{mm} \end{pmatrix} = \begin{pmatrix} \mathbf{I}_{\perp\perp} & \mathbf{I}_{\perp\perp} \\ \hat{\boldsymbol{z}}\times\mathbf{O}^{-1} & -\hat{\boldsymbol{z}}\times\mathbf{O}^{-1} \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{\Pi}^{++} & \boldsymbol{\Pi}^{+-} \\ \boldsymbol{\Pi}^{-+} & \boldsymbol{\Pi}^{--} \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} \mathbf{I}_{\perp\perp} & -\mathbf{O}\times\hat{\boldsymbol{z}} \\ \mathbf{I}_{\perp\perp} & \mathbf{O}\times\hat{\boldsymbol{z}} \end{pmatrix}.$$

[15] The medium may be plane-stratified in the z-direction.

The wave propagator satisfies

$$\partial_z \begin{pmatrix} \mathbf{\Pi}^{++} & \mathbf{\Pi}^{+-} \\ \mathbf{\Pi}^{-+} & \mathbf{\Pi}^{--} \end{pmatrix} (z, z')$$
$$= n_z c_0^{-1} \partial_t \begin{pmatrix} -\mathbf{I}_{\perp\perp} + \mathbf{N}^{++}\circ & \mathbf{N}^{+-}\circ \\ \mathbf{N}^{-+}\circ & \mathbf{I}_{\perp\perp} + \mathbf{N}^{--}\circ \end{pmatrix} (z) \cdot \begin{pmatrix} \mathbf{\Pi}^{++} & \mathbf{\Pi}^{+-} \\ \mathbf{\Pi}^{-+} & \mathbf{\Pi}^{--} \end{pmatrix} (z, z')$$

subjected to

$$\begin{pmatrix} \mathbf{\Pi}^{++} & \mathbf{\Pi}^{+-} \\ \mathbf{\Pi}^{-+} & \mathbf{\Pi}^{--} \end{pmatrix} (z', z') = \begin{pmatrix} \mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp} \end{pmatrix}.$$

For the homogeneous slab, the pair of argument (z, z') can be replaced by the argument $(z - z')$ and the wave propagator can be written formally as

$$\begin{pmatrix} \mathbf{\Pi}^{++} & \mathbf{\Pi}^{+-} \\ \mathbf{\Pi}^{-+} & \mathbf{\Pi}^{--} \end{pmatrix} (z)\cdot = \exp\left(\frac{z n_z}{c_0} \partial_t \begin{pmatrix} -\mathbf{I}_{\perp\perp} + (H\mathbf{N}^{++})\circ & (H\mathbf{N}^{+-})\circ \\ (H\mathbf{N}^{-+})\circ & \mathbf{I}_{\perp\perp} + (H\mathbf{N}^{--})\circ \end{pmatrix} \cdot \right),$$

which can be calculated using the Cayley-Hamilton theorem.

1.7 Scattering operators

The scattering relation for the electric field at oblique incidence on a bi-anisotropic slab, $|z| < d/2$, can be written as

$$\begin{pmatrix} \boldsymbol{E}^+_{s0,\perp} \\ \boldsymbol{E}^-_{s0,\perp} \end{pmatrix} = \begin{pmatrix} \delta_{t_d} \circ (\mathbf{Q}^+ + \mathbf{T}^+\circ) & \mathbf{R}^-\circ \\ \mathbf{R}^+\circ & \delta_{t_d} \circ (\mathbf{Q}^- + \mathbf{T}^-\circ) \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{E}^+_{i0,\perp} \\ \boldsymbol{E}^-_{i0,\perp} \end{pmatrix}$$

where the causal scattering kernels $\mathbf{R}^{\pm}(t)$ and $\mathbf{T}^{\pm}(t)$ lack direct terms. The wave-front propagators, $\mathbf{Q}^{\pm}$, for up- and down-going fields, respectively, are, in view of (1.4) or (1.16), equal to identity, *i.e.*,

$$\mathbf{Q}^{\pm} = \mathbf{I}_{\perp\perp}.$$

In the inverse scattering problem, by definition, these scattering operators are regarded as known; specifically, they have been obtained by deconvolution of scattered fields at excitation with up-going and down-going incident TM and TE pulses.

In the inverse scattering problem, the wave propagator for the split fields for the total slab can be obtained from scattering data as

$$\begin{pmatrix} \mathbf{\Pi}^{++} & \mathbf{\Pi}^{+-} \\ \mathbf{\Pi}^{-+} & \mathbf{\Pi}^{--} \end{pmatrix} (d) = \begin{pmatrix} \delta_{t_d} \circ (\mathbf{Q}^+ + \mathbf{T}^+\circ) & \mathbf{R}^-\circ \\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{R}^+\circ & \delta_{t_d} \circ (\mathbf{Q}^- + \mathbf{T}^-\circ) \end{pmatrix}^{-1} \tag{1.28}$$

or

$$\begin{cases} \mathbf{\Pi}^{++}(d) = \delta_{t_d} \circ (\mathbf{Q}^+ + \mathbf{T}^+\circ) - \delta_{-t_d} \circ \mathbf{R}^- \odot \left(\mathbf{Q}^- + \mathbf{T}^-\circ\right)^{-1} \cdot \mathbf{R}^+\circ, \\ \mathbf{\Pi}^{+-}(d) = \delta_{-t_d} \circ \mathbf{R}^- \odot \left(\mathbf{Q}^- + \mathbf{T}^-\circ\right)^{-1}, \\ \mathbf{\Pi}^{-+}(d) = -\delta_{-t_d} \circ \left(\mathbf{Q}^- + \mathbf{T}^-\circ\right)^{-1} \cdot \mathbf{R}^+\circ, \\ \mathbf{\Pi}^{--}(d) = \delta_{-t_d} \circ \left(\mathbf{Q}^- + \mathbf{T}^-\circ\right)^{-1}. \end{cases}$$

The eigenoperators $\Pi(d) = \Pi_i^k(d)$, $i,k \in \{+,-\}$, of (1.28), are, in view of (1.19), given by (2.2), where $Q_\pm^-(d)$ are the eigenvalues of $\mathbf{Q}^-$ and $Q_\pm^+(d)$ are the eigenvalues of $\mathbf{Q}^+$ and the kernels $P_\pm^-(d,t)$ and $P_\pm^+(d,t)$ are causal. The kernels P_i^k, $i,k \in \{+,-\}$, are obtained by solving the scalar, non-linear Volterra integral equation of the second kind

$$\begin{aligned}0 = \det((\Pi(d)\mathbf{I}_{\perp\perp} - \mathbf{\Pi}^{++}(d))\cdot(\Pi(d)\mathbf{I}_{\perp\perp} - \mathbf{\Pi}^{--}(d))\\ - \mathbf{\Pi}^{+-}(d)\cdot(\Pi(d)\mathbf{I}_{\perp\perp} - \mathbf{\Pi}^{--}(d))^{-1}\cdot\mathbf{\Pi}^{-+}(d)\cdot(\Pi(d)\mathbf{I}_{\perp\perp} - \mathbf{\Pi}^{--}(d))).\end{aligned}$$

From scattering data one can obtain

$$\begin{pmatrix}\mathbf{\Pi}^{ee} & \mathbf{\Pi}^{em}\\ \mathbf{\Pi}^{me} & \mathbf{\Pi}^{mm}\end{pmatrix}(d) = \begin{pmatrix}\mathbf{I}_{\perp\perp} & \mathbf{I}_{\perp\perp}\\ \hat{\boldsymbol{z}}\times\mathbf{O}^{-1} & -\hat{\boldsymbol{z}}\times\mathbf{O}^{-1}\end{pmatrix}\cdot\begin{pmatrix}\mathbf{\Pi}^{++} & \mathbf{\Pi}^{+-}\\ \mathbf{\Pi}^{-+} & \mathbf{\Pi}^{--}\end{pmatrix}(d)\cdot\frac{1}{2}\begin{pmatrix}\mathbf{I}_{\perp\perp} & -\mathbf{O}\times\hat{\boldsymbol{z}}\\ \mathbf{I}_{\perp\perp} & \mathbf{O}\times\hat{\boldsymbol{z}}\end{pmatrix}.$$

2 Solution of inverse scattering problem based the Cayley-Hamilton theorem

Attention is now focused on the inverse problem for the homogeneous bi-anisotropic slab at oblique incidence. In the non-degenerate case, the operator

$$\begin{pmatrix}\mathbf{\Pi}^{ee} & \mathbf{\Pi}^{em}\\ \mathbf{\Pi}^{me} & \mathbf{\Pi}^{mm}\end{pmatrix}(d) = \sum_{i,k\in\{+,-\}}\begin{pmatrix}\mathbf{W}^{ee} & \mathbf{W}^{em}\\ \mathbf{W}^{me} & \mathbf{W}^{mm}\end{pmatrix}_i^k \Pi_i^k(d) \tag{2.1}$$

is known and the operator

$$\begin{pmatrix}\hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\zeta}} & \hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\mu}}\\ -\hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\epsilon}} & -\hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\xi}}\end{pmatrix} = \sum_{i,k\in\{+,-\}}\begin{pmatrix}\mathbf{W}^{ee} & \mathbf{W}^{em}\\ \mathbf{W}^{me} & \mathbf{W}^{mm}\end{pmatrix}_i^k \lambda_i^k$$

is sought. Clearly, these operators have the same spectral projections

$$\begin{pmatrix}\mathbf{W}^{ee} & \mathbf{W}^{em}\\ \mathbf{W}^{me} & \mathbf{W}^{mm}\end{pmatrix}_i^k$$

but different eigenoperators. The eigenoperators

$$\begin{cases}\Pi_+^-(d) = \delta_{-t_d}Q_+^-(d)(1+P_+^-(d,\cdot)\circ),\\ \Pi_-^-(d) = \delta_{-t_d}Q_-^-(d)(1+P_-^-(d,\cdot)\circ),\\ \Pi_+^+(d) = \delta_{t_d}Q_+^+(d)(1+P_+^+(d,\cdot)\circ),\\ \Pi_-^+(d) = \delta_{t_d}Q_-^+(d)(1+P_-^+(d,\cdot)\circ),\end{cases} \tag{2.2}$$

of (2.1) are computed first as explained in Section 1.7, and the spectral projections are then obtained as

$$\begin{pmatrix}\mathbf{W}^{ee} & \mathbf{W}^{em}\\ \mathbf{W}^{me} & \mathbf{W}^{mm}\end{pmatrix}_i^k = \prod_{j\neq i,l\neq k}\left(\begin{pmatrix}\mathbf{\Pi}^{ee} & \mathbf{\Pi}^{em}\\ \mathbf{\Pi}^{me} & \mathbf{\Pi}^{ee}\end{pmatrix}(d) - \begin{pmatrix}\mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp}\\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp}\end{pmatrix}\Pi_j^l(d)\right)(\Pi_i^k(d)-\Pi_j^l(d))^{-1}.$$

It merely remains to compute the eigenoperators

$$\begin{cases} dc_0^{-1}\partial_t\lambda_+^- = \ln\left(\Pi_+^-(d)\right) = -dn_d c_0^{-1}\partial_t + \ln Q_+^-(d) + dc_0^{-1}(H\partial_t N_+^-)\circ, \\ dc_0^{-1}\partial_t\lambda_-^- = \ln\left(\Pi_-^-(d)\right) = -dn_d c_0^{-1}\partial_t + \ln Q_-^-(d) + dc_0^{-1}(H\partial_t N_-^-)\circ, \\ dc_0^{-1}\partial_t\lambda_+^+ = \ln\left(\Pi_+^+(d)\right) = dn_d c_0^{-1}\partial_t + \ln Q_-^-(d) + dc_0^{-1}(H\partial_t N_+^+)\circ, \\ dc_0^{-1}\partial_t\lambda_-^+ = \ln\left(\Pi_-^+(d)\right) = dn_d c_0^{-1}\partial_t + \ln Q_-^+(d) + dc_0^{-1}(H\partial_t N_-^+)\circ \end{cases}$$

by solving the Volterra integral equation of the second kind (1.20) for the kernels $M_i^k(t) = tH(t)\partial_t N_i^k(t)$. Recall that the wave front propagators Q_i^k, $i,k \in \{+,-\}$, are equal to one by definition, see (1.4) and (1.16).

The isotropic case is solved analogously and the other degenerate cases are solved similarly. Appropriate formulae can be found in section 1.5.3 in the former case and in section 1.5.2 in the latter cases.

3 Summary

Under the presumably not very severe restriction (1.4) on the susceptibility kernels, the inverse problem for the homogeneous bi-anisotropic slab at oblique incidence has been shown to be well posed. This has been surmised in earlier investigations but never actually proved. The result follows from the wellposedness of solutions of Volterra convolution equations of the second kind. Moreover, the proof is, to a large extent, based on the elementary but extremely powerful Cayley-Hamilton theorem. The proof consists of two steps, described in section 1.4 and in section 2. The unique solution is basically given by the intriguing formula

$$dc_0^{-1}\partial_t \begin{pmatrix} \hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\zeta}} & \hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\mu}} \\ -\hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\epsilon}} & -\hat{\boldsymbol{z}}\times\widetilde{\boldsymbol{\xi}} \end{pmatrix} = \ln\begin{pmatrix} \boldsymbol{\Pi}^{ee} & \boldsymbol{\Pi}^{em} \\ \boldsymbol{\Pi}^{me} & \boldsymbol{\Pi}^{mm} \end{pmatrix}(d), \tag{3.1}$$

where the logarithm is defined by the series expansion (cf. (1.21))

$$\ln\begin{pmatrix} \boldsymbol{\Pi}^{ee} & \boldsymbol{\Pi}^{em} \\ \boldsymbol{\Pi}^{me} & \boldsymbol{\Pi}^{mm} \end{pmatrix}(z) = \sum_{j=1}^{\infty} \frac{\left(\left(\begin{pmatrix} \mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp} \end{pmatrix} - \begin{pmatrix} \boldsymbol{\Pi}^{ee} & \boldsymbol{\Pi}^{em} \\ \boldsymbol{\Pi}^{me} & \boldsymbol{\Pi}^{mm} \end{pmatrix}(z)\right)\cdot\right)^j \begin{pmatrix} \mathbf{I}_{\perp\perp} & \mathbf{0}_{\perp\perp} \\ \mathbf{0}_{\perp\perp} & \mathbf{I}_{\perp\perp} \end{pmatrix}}{j}$$

and can be computed using the Cayley-Hamilton theorem and by solving at most four scalar Volterra integral equations of the second kind (namely equation (1.20)). Formula (3.1) should be compared to formula (1.13), which applies to the direct scattering problem. The steps used in the proof are conjectured to be effective ways of resolving the inverse problem also in practical situations. In practise, owing to non-accurate measurements and trivial or non-trivial deconvolution techniques, the non-degenerate case is most likely to arise in any case, also when the sample is known to be an isotropic material. The degenerate cases are therefore highly hypothetical and have been discussed only for the sake of completeness.

Appendix A The Cayley-Hamilton theorem

Theorem A.1 (Cayley-Hamilton). *A quadratic matrix* $\mathbf{A}$ *satisfies its own characteristic equation:*

$$\textit{If } p_{\mathbf{A}}(\lambda) = \det(\lambda \mathbf{I} - \mathbf{A}), \ \textit{then } p_{\mathbf{A}}(\mathbf{A}) = \mathbf{0}.$$

From this theorem, one can prove the following important theorem.

Theorem A.2. *Let* $\lambda_1, \ldots, \lambda_p$ *be the different eigenvalues of the* $n \times n$ *matrix* $\mathbf{A}$*, and let* $n_1, \ldots, n_p$ *be their multiplicity. If* $f(z)$ *is an entire analytic function, then*

$$f(\mathbf{A}) = q(\mathbf{A}),$$

where the uniquely defined polynomial q *of degree* $\leq n-1$ *is defined by the conditions*

$$\frac{d^j q}{dz^j}(\lambda_k) = \frac{d^j f}{dz^j}(\lambda_k), \qquad j = 0, \ldots, n_k - 1, k = 1, \ldots, p.$$

The theorem holds for the case $f(\lambda)$ is a complex function such that $f^{(j)}(\lambda_k)$ exists for $j = 0, \cdots, n_k - 1$ and $k = 1, \cdots, p$, see Gantmacher [10]. The polynomial $q(\lambda)$ is referred to as the Lagrange-Sylvester interpolation polynomial for $f(\lambda)$ on $\mathrm{sp}(\mathbf{A})$, see [10].

Recall that the spectrum of $\mathbf{A}$ is the point set $\mathrm{sp}(\mathbf{A}) = \{\lambda_1, \cdots, \lambda_p\}$ and that the characteristic polynomial of $\mathbf{A}$ is

$$p_{\mathbf{A}}(\lambda) = (\lambda - \lambda_1)^{n_1} \cdots (\lambda - \lambda_p)^{n_p} \qquad (n_1 + \cdots + n_p = n).$$

We seek a general formula for $q(\lambda)$. Fractional decomposition gives

$$\frac{q(\lambda)}{p_{\mathbf{A}}(\lambda)} = \sum_{k=1}^{p} \sum_{j=0}^{n_k - 1} \frac{a_{kj}}{(\lambda - \lambda_k)^{n_k - j}} \qquad (\lambda \neq \lambda_1, \cdots, \lambda_p).$$

We define the polynomial

$$p_{\mathbf{A}}^k(\lambda) := \frac{p_{\mathbf{A}}(\lambda)}{(\lambda - \lambda_k)^{n_k}} = (\lambda - \lambda_1)^{n_1} \cdots (\lambda - \lambda_{k-1})^{n_{k-1}} (\lambda - \lambda_{k+1})^{n_{k+1}} \cdots (\lambda - \lambda_p)^{n_p}$$

and find by multiplication with $(\lambda - \lambda_k)^{n_k}$ that

$$\frac{q(\lambda)}{p_{\mathbf{A}}^k(\lambda)} = \sum_{j=0}^{n_k - 1} a_{kj} (\lambda - \lambda_k)^j + (\lambda - \lambda_k)^{n_k} r_k(\lambda) \qquad (k = 1, \cdots, p),$$

where $r_k(\lambda)$ is rational, defined for $\lambda \neq \lambda_1, \cdots, \lambda_{k-1}, \lambda_{k+1}, \cdots \lambda_p$. Hence

$$a_{kj} = \frac{1}{j!} \left[\frac{q(\lambda)}{p_{\mathbf{A}}^k(\lambda)} \right]_{\lambda = \lambda_k}^{(j)} = \frac{1}{j!} \left[\frac{f(\lambda)}{p_{\mathbf{A}}^k(\lambda)} \right]_{\lambda = \lambda_k}^{(j)} \qquad (j = 0, \cdots, n_k - 1, \quad k = 1, \cdots, p).$$

With these values of the coefficients a_{kj} one arrives at

$$q(\lambda) = \sum_{k=1}^{p} p_{\mathbf{A}}^k(\lambda) \sum_{j=0}^{n_k-1} a_{kj}(\lambda-\lambda_k)^j = \sum_{k=1}^{p} p_{\mathbf{A}}^k(\lambda) \sum_{j=0}^{n_k-1} \frac{1}{j!} \left[\frac{f(\lambda)}{p_{\mathbf{A}}^k(\lambda)}\right]^{(j)}_{\lambda=\lambda_k} (\lambda-\lambda_k)^j. \quad \text{(A.1)}$$

With the aid of the Leibniz formula, one gets

$$a_{kj} = \frac{1}{j!} \sum_{i=0}^{j} \binom{j}{i} f^{(j-i)}(\lambda_k) \left(\frac{1}{p_{\mathbf{A}}^k(\lambda_k)}\right)^{(i)} \qquad (j = 0, \cdots, n_k - 1, \quad k = 1, \cdots, p),$$

where, for fixed k, the derivatives

$$\left(\frac{1}{p_{\mathbf{A}}^k(\lambda_k)}\right)^{(i)} = \frac{d^i}{d\lambda^i} \left(\frac{1}{p_{\mathbf{A}}^k(\lambda)}\right)(\lambda_k) \qquad (i = 0, \cdots, n_k - 1)$$

can be determined recursively:

$$\frac{1}{p_{\mathbf{A}}^k(\lambda_k)} = \prod_{l=1, l\neq k}^{p} \frac{1}{(\lambda_k - \lambda_l)^{n_l}},$$

$$\left(\frac{1}{p_{\mathbf{A}}^k(\lambda_k)}\right)' = -\left(\frac{1}{p_{\mathbf{A}}^k(\lambda_k)}\right) \left(\sum_{l=1, l\neq k}^{p} \frac{n_l}{\lambda_k - \lambda_l}\right),$$

and, consequently,

$$\left(\frac{1}{p_{\mathbf{A}}^k(\lambda_k)}\right)^{(i)} = \sum_{m=0}^{i-1} \binom{i-1}{m} \left(\frac{1}{p_{\mathbf{A}}^k(\lambda_k)}\right)^{(i-1-m)} (-1)^{m+1} m! \left(\sum_{l=1, l\neq k}^{p} \frac{n_l}{(\lambda_k - \lambda_l)^{m+1}}\right).$$

If f is the exponential function, then the Lagrange-Sylvester interpolation polynomial is given by

$$q(\lambda) = \sum_{k=1}^{p} \exp(\lambda_k) p_{\mathbf{A}}^k(\lambda) \sum_{j=0}^{n_k-1} (\lambda - \lambda_k)^j \frac{1}{j!} \sum_{i=0}^{j} \binom{j}{i} \frac{d^i}{dz^i} \left(\frac{1}{p_{\mathbf{A}}^k(z)}\right)(\lambda_k).$$

If all the eigenvalues are are simple, that is, if $\mathrm{sp}(\mathbf{A}) = \{\lambda_1, \cdots, \lambda_n\}$, then the Lagrange interpolation polynomial for $f(\lambda)$ on $\mathrm{sp}(\mathbf{A})$ is

$$q(\lambda) = \sum_{k=1}^{n} \frac{p_{\mathbf{A}}^k(\lambda)}{p_{\mathbf{A}}^k(\lambda_k)} f(\lambda_k). \quad \text{(A.2)}$$

If $\mathrm{sp}(\mathbf{A}) = \{\lambda_1\}$, then the Lagrange-Sylvester interpolation polynomial for $f(\lambda)$ on $\mathrm{sp}(\mathbf{A})$ equals the n first terms in the Taylor-expansion of $f(\lambda)$ about λ_1:

$$q(\lambda) = \sum_{j=0}^{n-1} \frac{1}{j!} f^{(j)}(\lambda_1)(\lambda - \lambda_1)^j. \quad \text{(A.3)}$$

If $\mathrm{sp}(\mathbf{A}) = \{\lambda_1, \lambda_2\}$ and consequently $p_{\mathbf{A}}(\lambda) = (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2}$, then the Lagrange-Sylvester interpolation polynomial for $f(\lambda)$ on $\mathrm{sp}(\mathbf{A})$ is

$$q(\lambda) = \sum_{k=1}^{2} (\lambda - \lambda_{\bar{k}})^{n_{\bar{k}}} \sum_{j=0}^{n_k - 1} a_{kj}(\lambda - \lambda_k)^j, \tag{A.4}$$

where

$$a_{kj} = \frac{1}{j!} \sum_{i=0}^{j} \binom{j}{i} (-1)^i \frac{(n_{\bar{k}} - 1 + i)!}{(n_{\bar{k}} - 1)!} \frac{f^{(j-i)}(\lambda_k)}{(\lambda_k - \lambda_{\bar{k}})^{n_{\bar{k}} + i}},$$

and $\bar{k}$ is the index dual to k.[16] If, particularly, $n_2 = 1$, then

$$q(\lambda) = (\lambda - \lambda_1)^{n_1} a_{20} + (\lambda - \lambda_2) \sum_{j=0}^{n_1 - 1} a_{1j}(\lambda - \lambda_1)^j, \tag{A.5}$$

where

$$a_{20} = \frac{f(\lambda_2)}{(\lambda_2 - \lambda_1)^{n_1}}, \qquad a_{1j} = \sum_{i=0}^{j} \frac{(-1)^i}{(j-i)!} \frac{f^{(j-i)}(\lambda_1)}{(\lambda_1 - \lambda_2)^{1+i}}.$$

The cases $n = 2$, $n = 3$, and $n = 4$, which are of special interest for many applications in mathematical physics, are displayed below.

A.1 Case $n = 2$

From the formulae (A.2) and (A.3) follows that

$$q(\lambda) = \begin{cases} \frac{\lambda - \lambda_2}{\lambda_1 - \lambda_2} f(\lambda_1) + \frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1} f(\lambda_2) & (n_1 = 1, n_2 = 1)\,, \\ f(\lambda_1) + (\lambda - \lambda_1) f'(\lambda_1) & (n_1 = 2, n_2 = 0)\,. \end{cases}$$

If f is the exponential function, then one obtains

$$q(\lambda) = \begin{cases} \frac{\lambda - \lambda_2}{\lambda_1 - \lambda_2} \exp(\lambda_1) + \frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1} \exp(\lambda_2) & (n_1 = 1, n_2 = 1)\,, \\ (1 + \lambda - \lambda_1) \exp(\lambda_1) & (n_1 = 2, n_2 = 0)\,. \end{cases}$$

A.2 Case $n = 3$

From the formulae (A.2), (A.3) and (A.5) follows that

[16] $\bar{1} = 2$ and $\bar{2} = 1$.

$$q(\lambda) = \begin{cases} \frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}\frac{\lambda-\lambda_3}{\lambda_1-\lambda_3} f(\lambda_1) \\ +\frac{\lambda-\lambda_3}{\lambda_2-\lambda_3}\frac{\lambda-\lambda_1}{\lambda_2-\lambda_1} f(\lambda_2) \\ +\frac{\lambda-\lambda_1}{\lambda_3-\lambda_1}\frac{\lambda-\lambda_2}{\lambda_3-\lambda_2} f(\lambda_3) & (n_1 = 1, n_2 = 1, n_3 = 1)\,, \\ \frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}\left(1 + \frac{\lambda-\lambda_1}{\lambda_2-\lambda_1}\right) f(\lambda_1) \\ +\frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}(\lambda-\lambda_1) f'(\lambda_1) \\ +\frac{(\lambda-\lambda_1)^2}{(\lambda_2-\lambda_1)^2} f(\lambda_2) & (n_1 = 2, n_2 = 1, n_3 = 0)\,, \\ f(\lambda_1) \\ +(\lambda-\lambda_1) f'(\lambda_1) \\ +\frac{1}{2}(\lambda-\lambda_1)^2 f''(\lambda_1) & (n_1 = 3, n_2 = 0, n_3 = 0)\,. \end{cases}$$

If f is the exponential function, then

$$q(\lambda) = \begin{cases} \frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}\frac{\lambda-\lambda_3}{\lambda_1-\lambda_3} \exp(\lambda_1) \\ +\frac{\lambda-\lambda_3}{\lambda_2-\lambda_3}\frac{\lambda-\lambda_1}{\lambda_2-\lambda_1} \exp(\lambda_2) \\ +\frac{\lambda-\lambda_1}{\lambda_3-\lambda_1}\frac{\lambda-\lambda_2}{\lambda_3-\lambda_2} \exp(\lambda_3) & (n_1 = 1, n_2 = 1, n_3 = 1)\,, \\ \frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}\left(1 + \lambda - \lambda_1 + \frac{\lambda-\lambda_1}{\lambda_2-\lambda_1}\right) \exp(\lambda_1) \\ +\frac{(\lambda-\lambda_1)^2}{(\lambda_2-\lambda_1)^2} \exp(\lambda_2) & (n_1 = 2, n_2 = 1, n_3 = 0)\,, \\ (1 + \lambda - \lambda_1 + \frac{1}{2}(\lambda-\lambda_1)^2) \exp(\lambda_1) & (n_1 = 3, n_2 = 0, n_3 = 0)\,. \end{cases}$$

A.3 Case $n = 4$

Surmise that

$$p_{\mathbf{A}}(\lambda) = (\lambda-\lambda_1)^2(\lambda-\lambda_2)(\lambda-\lambda_3) \qquad (\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1).$$

Then the coefficients in formula (A.1) are given by

$$a_{11} = \sum_{i=0}^{1} f^{(1-i)}(\lambda_1)\left(\frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}\right)^{(i)}(\lambda_1) = f'(\lambda_1)\frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} \\ - f(\lambda_1)\left(\frac{1}{(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)} + \frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)^2}\right)$$

and

$$a_{k0} = \frac{f(\lambda_k)}{p^k_{\mathbf{A}}(\lambda_k)} = \begin{cases} \frac{f(\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} & (k = 1), \\ \frac{f(\lambda_2)}{(\lambda_2-\lambda_1)^2(\lambda_2-\lambda_3)} & (k = 2), \\ \frac{f(\lambda_3)}{(\lambda_3-\lambda_1)^2(\lambda_3-\lambda_2)} & (k = 3), \end{cases}$$

and the Lagrange-Sylvester interpolation polynomial for $f(\lambda)$ on $\mathrm{sp}(\mathbf{A})$ becomes

$$q(\lambda) = \sum_{k=1}^{3} p^k_{\mathbf{A}}(\lambda) \sum_{j=0}^{n_k-1} a_{kj}(\lambda-\lambda_k)^j = (\lambda-\lambda_2)(\lambda-\lambda_3)(\lambda-\lambda_1)a_{11} \\ + (\lambda-\lambda_2)(\lambda-\lambda_3)a_{10} + (\lambda-\lambda_1)^2(\lambda-\lambda_3)a_{20} + (\lambda-\lambda_1)^2(\lambda-\lambda_2)a_{30}.$$

Suppose that

$$p_{\mathbf{A}}(\lambda) = (\lambda - \lambda_1)^2(\lambda - \lambda_2)^2 \qquad (\lambda_1 \neq \lambda_2).$$

Then the coefficients in formula (A.4) are

$$a_{k0} = f(\lambda_k)\frac{1}{(\lambda_k - \lambda_{\bar{k}})^2} \qquad (k = 1, 2)$$

and

$$a_{k1} = \frac{f'(\lambda_k)}{(\lambda_k - \lambda_{\bar{k}})^2} - \frac{2f(\lambda_k)}{(\lambda_k - \lambda_{\bar{k}})^3} \qquad (k = 1, 2),$$

and the Lagrange-Sylvester interpolation polynomial for $f(\lambda)$ on $\mathrm{sp}(\mathbf{A})$ becomes

$$\begin{aligned} q(\lambda) = \sum_{k=1}^{2} (\lambda - \lambda_{\bar{k}})^2 \left(a_{k0} + a_{k1}(\lambda - \lambda_k)\right) &= (\lambda - \lambda_2)^2 \left(a_{10} + a_{11}(\lambda - \lambda_1)\right) \\ &+ (\lambda - \lambda_1)^2 \left(a_{20} + a_{21}(\lambda - \lambda_2)\right). \end{aligned}$$

Assume that

$$p_{\mathbf{A}}(\lambda) = (\lambda - \lambda_1)^3(\lambda - \lambda_2) \qquad (\lambda_1 \neq \lambda_2).$$

Then the coefficients in formula (A.5) are

$$a_{1j} = \frac{1}{j!} \sum_{i=0}^{j} \binom{j}{i} (-1)^i i! \frac{f^{(j-i)}(\lambda_1)}{(\lambda_1 - \lambda_2)^{i+1}} \qquad (j = 0, 1, 2)$$

and

$$a_{20} = \frac{f(\lambda_2)}{(\lambda_2 - \lambda_1)^3},$$

and the Lagrange-Sylvester interpolation polynomial for $f(\lambda)$ on $\mathrm{sp}(\mathbf{A})$ becomes

$$q(\lambda) = (\lambda - \lambda_2) \sum_{j=0}^{2} a_{1j}(\lambda - \lambda_1)^j + (\lambda - \lambda_1)^3 a_{20}.$$

The remaining cases are given by the formulae (A.2) and (A.3). Summary:

$$q(\lambda) = \begin{cases} \frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}\frac{\lambda-\lambda_3}{\lambda_1-\lambda_3}\frac{\lambda-\lambda_4}{\lambda_1-\lambda_4} f(\lambda_1) \\ +\frac{\lambda-\lambda_3}{\lambda_2-\lambda_3}\frac{\lambda-\lambda_4}{\lambda_2-\lambda_4}\frac{\lambda-\lambda_1}{\lambda_2-\lambda_1} f(\lambda_2) \\ +\frac{\lambda-\lambda_4}{\lambda_3-\lambda_4}\frac{\lambda-\lambda_1}{\lambda_3-\lambda_1}\frac{\lambda-\lambda_2}{\lambda_3-\lambda_2} f(\lambda_3) \\ +\frac{\lambda-\lambda_1}{\lambda_4-\lambda_1}\frac{\lambda-\lambda_2}{\lambda_4-\lambda_2}\frac{\lambda-\lambda_3}{\lambda_4-\lambda_3} f(\lambda_4) & (n_1=1, n_2=1, n_3=1, n_4=1)\,, \\ \frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}\frac{\lambda-\lambda_3}{\lambda_1-\lambda_3}\left(1+\frac{\lambda-\lambda_1}{\lambda_2-\lambda_1}+\frac{\lambda-\lambda_1}{\lambda_3-\lambda_1}\right) f(\lambda_1) \\ +\frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}\frac{\lambda-\lambda_3}{\lambda_1-\lambda_3}(\lambda-\lambda_1) f'(\lambda_1) \\ +\frac{(\lambda-\lambda_1)^2}{(\lambda_2-\lambda_1)^2}\frac{\lambda-\lambda_3}{\lambda_2-\lambda_3} f(\lambda_2) \\ +\frac{(\lambda-\lambda_1)^2}{(\lambda_3-\lambda_1)^2}\frac{\lambda-\lambda_2}{\lambda_3-\lambda_2} f(\lambda_3) & (n_1=2, n_2=1, n_3=1, n_4=0)\,, \\ \frac{(\lambda-\lambda_2)^2}{(\lambda_1-\lambda_2)^2}\left(1+2\frac{\lambda-\lambda_1}{\lambda_2-\lambda_1}\right) f(\lambda_1) \\ +\frac{(\lambda-\lambda_2)^2}{(\lambda_1-\lambda_2)^2}(\lambda-\lambda_1) f'(\lambda_1) \\ +\frac{(\lambda-\lambda_1)^2}{(\lambda_2-\lambda_1)^2}\left(1+2\frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}\right) f(\lambda_2) \\ +\frac{(\lambda-\lambda_1)^2}{(\lambda_2-\lambda_1)^2}(\lambda-\lambda_2) f'(\lambda_2) & (n_1=2, n_2=2, n_3=0, n_4=0) \end{cases}$$

and

$$q(\lambda) = \begin{cases} \frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}\left(1+\frac{\lambda-\lambda_1}{\lambda_2-\lambda_1}+\frac{(\lambda-\lambda_1)^2}{(\lambda_2-\lambda_1)^2}\right) f(\lambda_1) \\ +\frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}\left(1+\frac{\lambda-\lambda_1}{\lambda_2-\lambda_1}\right)(\lambda-\lambda_1) f'(\lambda_1) \\ +\frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}\frac{1}{2}(\lambda-\lambda_1)^2 f''(\lambda_1) \\ +\frac{(\lambda-\lambda_1)^3}{(\lambda_2-\lambda_1)^3} f(\lambda_2) & (n_1=3, n_2=1, n_3=0, n_4=0)\,, \\ f(\lambda_1) \\ +(\lambda-\lambda_1) f'(\lambda_1) \\ +\frac{1}{2}(\lambda-\lambda_1)^2 f''(\lambda_1) \\ +\frac{1}{6}(\lambda-\lambda_1)^3 f'''(\lambda_1) & (n_1=4, n_2=0, n_3=0, n_3=0)\,. \end{cases}$$

If f is the exponential function, then

$$q(\lambda) = \begin{cases} \frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}\frac{\lambda-\lambda_3}{\lambda_1-\lambda_3}\frac{\lambda-\lambda_4}{\lambda_1-\lambda_4} \exp(\lambda_1) \\ +\frac{\lambda-\lambda_3}{\lambda_2-\lambda_3}\frac{\lambda-\lambda_4}{\lambda_2-\lambda_4}\frac{\lambda-\lambda_1}{\lambda_2-\lambda_1} \exp(\lambda_2) \\ +\frac{\lambda-\lambda_4}{\lambda_3-\lambda_4}\frac{\lambda-\lambda_1}{\lambda_3-\lambda_1}\frac{\lambda-\lambda_2}{\lambda_3-\lambda_2} \exp(\lambda_3) \\ +\frac{\lambda-\lambda_1}{\lambda_4-\lambda_1}\frac{\lambda-\lambda_2}{\lambda_4-\lambda_2}\frac{\lambda-\lambda_3}{\lambda_4-\lambda_3} \exp(\lambda_4) & (n_1=1, n_2=1, n_3=1, n_4=1)\,, \\ \frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}\frac{\lambda-\lambda_3}{\lambda_1-\lambda_3} \\ \cdot\left(1+\lambda-\lambda_1+\frac{\lambda-\lambda_1}{\lambda_2-\lambda_1}+\frac{\lambda-\lambda_1}{\lambda_3-\lambda_1}\right) \exp(\lambda_1) \\ +\frac{(\lambda-\lambda_1)^2}{(\lambda_2-\lambda_1)^2}\frac{\lambda-\lambda_3}{\lambda_2-\lambda_3} \exp(\lambda_2) \\ +\frac{(\lambda-\lambda_1)^2}{(\lambda_3-\lambda_1)^2}\frac{\lambda-\lambda_2}{\lambda_3-\lambda_2} \exp(\lambda_3) & (n_1=2, n_2=1, n_3=1, n_4=0)\,, \\ \frac{(\lambda-\lambda_2)^2}{(\lambda_1-\lambda_2)^2}\left(1+\lambda-\lambda_1+2\frac{\lambda-\lambda_1}{\lambda_2-\lambda_1}\right) \exp(\lambda_1) \\ +\frac{(\lambda-\lambda_1)^2}{(\lambda_2-\lambda_1)^2}\left(1+\lambda-\lambda_2+2\frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}\right) \exp(\lambda_2) & (n_1=2, n_2=2, n_3=0, n_4=0) \end{cases}$$

and

$$q(\lambda) = \begin{cases} \frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}\left(1+\lambda-\lambda_1+\frac{1}{2}(\lambda-\lambda_1)^2+(1+\lambda-\lambda_1)\frac{\lambda-\lambda_1}{\lambda_2-\lambda_1}+\frac{(\lambda-\lambda_1)^2}{(\lambda_2-\lambda_1)^2}\right)\exp(\lambda_1) \\ +\frac{(\lambda-\lambda_1)^3}{(\lambda_2-\lambda_1)^3}\exp(\lambda_2) \qquad (n_1=3, n_2=1, n_3=0, n_4=0)\,, \\ (1+\lambda-\lambda_1+\frac{1}{2}(\lambda-\lambda_1)^2+\frac{1}{6}(\lambda-\lambda_1)^3) \\ \cdot\exp(\lambda_1) \qquad (n_1=4, n_2=0, n_3=0, n_3=0)\,. \end{cases}$$

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