Low frequency scattering by passive periodic structures for oblique incidence: low pass case

Sjöberg, Daniel

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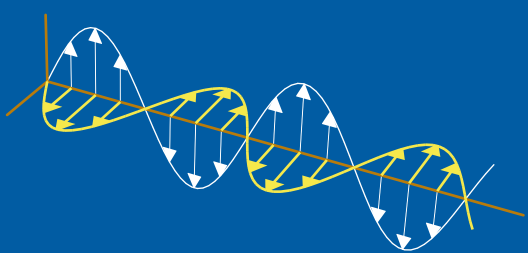
Daniel Sjöberg

Electromagnetic Theory
Department of Electrical and Information Technology
Lund University
Sweden



Daniel Sjöberg
daniel.sjoberg@eit.lth.se

Department of Electrical and Information Technology
Electromagnetic Theory
P.O. Box 118
SE-221 00 Lund
Sweden

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Abstract

We derive the low frequency behavior of the scattering coefficients from a low pass structure which periodic in a plane, and finite in the normal direction. The analysis is for oblique incidence of arbitrary polarization on a structure which can be anisotropic in both electric and magnetic material properties, and may contain metal inclusions. The metal inclusions can be modeled both as perfect electric conductors (PEC), and with a finite conductivity. It is found that the low frequency reflection and transmission coefficients are proportional to the sum and difference of the electric and magnetic polarizability per unit area of the periodic structure. If the metal inclusions are modeled as PEC instead of as a finite conductivity, the first order low frequency reflection is larger whereas the first order transmission is smaller.

1 Introduction

Periodic structures are often used as spatial filters, or frequency selective surfaces. They are typically either band pass or band stop. Band pass structures usually consist of one or several metal sheets with periodic arrays of apertures, whereas the band stop structures are usually periodic arrays of metal inclusions; in the case of one sheet, the two concepts can be considered as complementary structures via Babinet's principle.

In a series of papers, physical limitations on the amount of electromagnetic interaction available for antennas, materials, and general scatterers have been derived based solely on the principles of linearity, causality, and energy conservation [3, 11–14]. There, it is demonstrated that the low frequency behavior of the structure under consideration provides a measure of the total electromagnetic interaction available for all frequencies. It is anticipated that the same kind of relations can be derived for many kinds of periodic structures as well, which makes it interesting to take an explicit look at low frequency scattering for such structures.

In this paper, we limit ourselves to the band stop case, since this is the one most easily analyzed. The reason for this is that in the static limit of band stop structures, the tangential electric and magnetic fields are continuous. In the band pass case, the possibility of an interelement current in the metal sheets provides a possibility for discontinuous tangential magnetic fields, which must be handled separately.

Our analysis is for oblique incidence with arbitrary polarization, and includes fully anisotropic permittivities and permeabilities, as well as metal inclusions.

2 Notation

Let the periodic structure be situated between $0 < z < d$, with periodicity described by two basis vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ in the xy plane as in Figure 1. These are the sides of the unit cell U with area $A = \hat{\boldsymbol{z}} \cdot (\boldsymbol{a}_1 \times \boldsymbol{a}_2)$. An arbitrary lattice vector is then described by

$$\boldsymbol{x}_{\boldsymbol{n}} = n_1\boldsymbol{a}_1 + n_2\boldsymbol{a}_2 \tag{2.1}$$

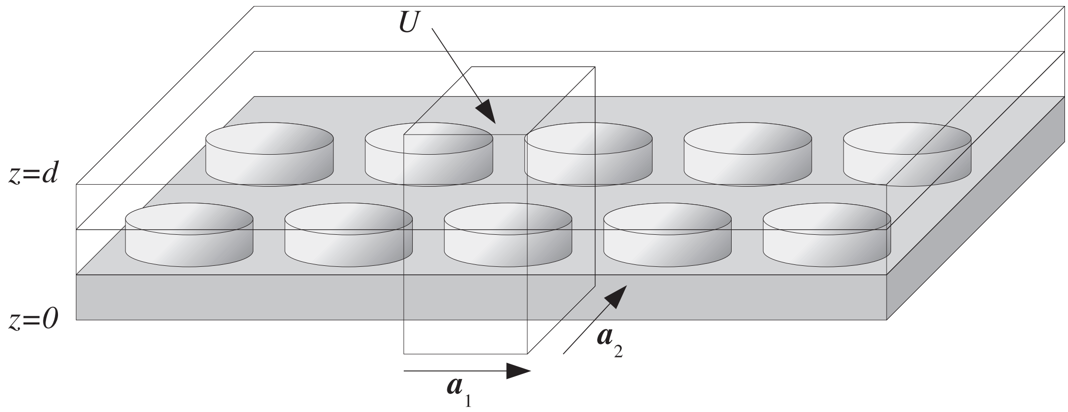

Figure 1: Typical geometry of the periodic structure.

with n_1 and n_2 being integers. The material parameters are U-periodic, *i.e.*, $\boldsymbol{\epsilon}(\boldsymbol{x}+\boldsymbol{x}_{\boldsymbol{n}})=\boldsymbol{\epsilon}(\boldsymbol{x})$ and $\boldsymbol{\mu}(\boldsymbol{x}+\boldsymbol{x}_{\boldsymbol{n}})=\boldsymbol{\mu}(\boldsymbol{x})$ for all $\boldsymbol{n}=(n_1,n_2)$, where $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ are the permittivity and permeability matrices, respectively. In the regions $z<0$ and $z>d$ we have $\boldsymbol{\epsilon}(\boldsymbol{x})=\epsilon_0\mathbf{I}$ and $\boldsymbol{\mu}(\boldsymbol{x})=\mu_0\mathbf{I}$, where ϵ_0 and μ_0 are the permittivity and permeability of vacuum, respectively. Let the incident field be a plane wave (time convention $\mathrm{e}^{-\mathrm{i}\omega t}$)

$$\boldsymbol{E}^{\mathrm{i}}(\boldsymbol{x})=\boldsymbol{E}_0\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}},\quad \boldsymbol{H}^{\mathrm{i}}(\boldsymbol{x})=\boldsymbol{H}_0\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}=\eta_0^{-1}\frac{\boldsymbol{k}}{k}\times\boldsymbol{E}^{\mathrm{i}}(\boldsymbol{x}) \tag{2.2}$$

where c is the speed of light in vacuum, the constant complex vector $\boldsymbol{E}_0$ is the polarization, $\eta_0=\sqrt{\mu_0/\epsilon_0}=377\,\Omega$ is the intrinsic wave impedance in vacuum, and $\boldsymbol{k}$ is the wave vector of the incident wave, with amplitude $|\boldsymbol{k}|=k=\omega/c$. The wave vector can be separated in one normal and one transverse part,

$$\boldsymbol{k}=\boldsymbol{k}_\perp+k_z\hat{\boldsymbol{z}},\quad \frac{|\boldsymbol{k}_\perp|}{k}=\sin\theta,\quad \frac{k_z}{k}=\cos\theta \tag{2.3}$$

where θ is the angle of incidence and $\boldsymbol{k}_\perp$ is a vector in the xy plane.

Due to the use of a plane wave as excitation and the periodicity of the structure, the fields (including incident and scattered fields) satisfy the following translation property:

$$\boldsymbol{E}(\boldsymbol{x}+\boldsymbol{x}_{\boldsymbol{n}})=\boldsymbol{E}(\boldsymbol{x})\mathrm{e}^{\mathrm{i}\boldsymbol{k}_\perp\cdot\boldsymbol{x}_{\boldsymbol{n}}} \tag{2.4}$$

where $\boldsymbol{x}_{\boldsymbol{n}}$ is an arbitrary lattice vector. This property implies that the field

$$\widetilde{\boldsymbol{E}}(\boldsymbol{x})=\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_\perp\cdot\boldsymbol{x}}\boldsymbol{E}(\boldsymbol{x}) \tag{2.5}$$

is U-periodic in $\boldsymbol{x}$. The periodic field $\widetilde{\boldsymbol{E}}(\boldsymbol{x})$ is called the Bloch amplitude of the field $\boldsymbol{E}(\boldsymbol{x})$ [9, 10].

3 Low frequency behavior

Maxwell's equations for time-harmonic fields are (where the possibly anisotropic matrices $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ are the permittivity and permeability of the material, respectively)

$$\nabla \times \boldsymbol{E} = \mathrm{i}\omega\boldsymbol{\mu}(\boldsymbol{x})\boldsymbol{H} \tag{3.1}$$

$$\nabla \times \boldsymbol{H} = -\mathrm{i}\omega\boldsymbol{\epsilon}(\boldsymbol{x})\boldsymbol{E} \tag{3.2}$$

Multiplying these fields with the transverse phase factor of the incident field, $\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_\perp\cdot\boldsymbol{x}}$, we obtain the equations for the Bloch amplitudes

$$(\nabla + \mathrm{i}\boldsymbol{k}_\perp) \times \widetilde{\boldsymbol{E}} = \mathrm{i}\omega\boldsymbol{\mu}\widetilde{\boldsymbol{H}} \tag{3.3}$$

$$(\nabla + \mathrm{i}\boldsymbol{k}_\perp) \times \widetilde{\boldsymbol{H}} = -\mathrm{i}\omega\boldsymbol{\epsilon}\widetilde{\boldsymbol{E}} \tag{3.4}$$

Integrating over $(x, y) \in U$ and $z_1 < z < z_2$, where $z_1 < 0$ and $z_2 > d$ are chosen so that the structure is enclosed, implies

$$\hat{\boldsymbol{z}} \times \Big(\int_U \widetilde{\boldsymbol{E}}(z_2)\,\mathrm{d}S - \int_U \widetilde{\boldsymbol{E}}(z_1)\,\mathrm{d}S \Big) = \int_{z_1}^{z_2} \int_U \Big(-\mathrm{i}\boldsymbol{k}_\perp \times \widetilde{\boldsymbol{E}} + \mathrm{i}\omega\boldsymbol{\mu}\widetilde{\boldsymbol{H}} \Big)\,\mathrm{d}S\,\mathrm{d}z \tag{3.5}$$

$$\hat{\boldsymbol{z}} \times \Big(\int_U \widetilde{\boldsymbol{H}}(z_2)\,\mathrm{d}S - \int_U \widetilde{\boldsymbol{H}}(z_1)\,\mathrm{d}S \Big) = \int_{z_1}^{z_2} \int_U \Big(-\mathrm{i}\boldsymbol{k}_\perp \times \widetilde{\boldsymbol{H}} - \mathrm{i}\omega\boldsymbol{\epsilon}\widetilde{\boldsymbol{E}} \Big)\,\mathrm{d}S\,\mathrm{d}z \tag{3.6}$$

We use the following notation for the mean value of the fields (where $h = z_2 - z_1$)

$$\bar{\boldsymbol{E}} = \frac{1}{Ah}\int_{z_1}^{z_2}\int_U \widetilde{\boldsymbol{E}}\,\mathrm{d}S\,\mathrm{d}z \qquad \bar{\boldsymbol{E}}_{1,2} = \frac{1}{A}\int_U \widetilde{\boldsymbol{E}}(z_{1,2})\,\mathrm{d}S \tag{3.7}$$

$$\bar{\boldsymbol{H}} = \frac{1}{Ah}\int_{z_1}^{z_2}\int_U \widetilde{\boldsymbol{H}}\,\mathrm{d}S\,\mathrm{d}z \qquad \bar{\boldsymbol{H}}_{1,2} = \frac{1}{A}\int_U \widetilde{\boldsymbol{H}}(z_{1,2})\,\mathrm{d}S \tag{3.8}$$

The following matrices $\boldsymbol{\gamma}_\mathrm{e}$ and $\boldsymbol{\gamma}_\mathrm{m}$ exist and are bounded as $\omega \to 0$ since they represent the response of a linear system on an excitation $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ (see Section 4 for computing the matrices in the static limit and generalization to the case of metallic inclusions):

$$\int_{z_1}^{z_2}\int_U (\boldsymbol{\epsilon}/\epsilon_0 - \mathbf{I})\widetilde{\boldsymbol{E}}\,\mathrm{d}S\,\mathrm{d}z \stackrel{\mathrm{def}}{=} \boldsymbol{\gamma}_\mathrm{e}\boldsymbol{E}_0 \tag{3.9}$$

$$\int_{z_1}^{z_2}\int_U (\boldsymbol{\mu}/\mu_0 - \mathbf{I})\widetilde{\boldsymbol{H}}\,\mathrm{d}S\,\mathrm{d}z \stackrel{\mathrm{def}}{=} \boldsymbol{\gamma}_\mathrm{m}\boldsymbol{H}_0 \tag{3.10}$$

The equations are then

$$\hat{\boldsymbol{z}} \times \Big(\bar{\boldsymbol{E}}_2 - \bar{\boldsymbol{E}}_1\Big) = -\mathrm{i}\boldsymbol{k}_\perp h \times \bar{\boldsymbol{E}} + \mathrm{i}\omega\mu_0 h\bar{\boldsymbol{H}} + \mathrm{i}\omega\mu_0 A^{-1}\boldsymbol{\gamma}_\mathrm{m}\boldsymbol{H}_0 \tag{3.11}$$

$$\hat{\boldsymbol{z}} \times \Big(\bar{\boldsymbol{H}}_2 - \bar{\boldsymbol{H}}_1\Big) = -\mathrm{i}\boldsymbol{k}_\perp h \times \bar{\boldsymbol{H}} - \mathrm{i}\omega\epsilon_0 h\bar{\boldsymbol{E}} - \mathrm{i}\omega\epsilon_0 A^{-1}\boldsymbol{\gamma}_\mathrm{e}\boldsymbol{E}_0 \tag{3.12}$$

where the factor $\boldsymbol{k}_\perp h$ is dimensionless, the factors $\mathrm{i}\omega\mu_0 h$ and $\mathrm{i}\omega\mu_0 A^{-1}\boldsymbol{\gamma}_\mathrm{m}$ have dimension of impedance, and the factors $\mathrm{i}\omega\epsilon_0 h$ and $\mathrm{i}\omega\epsilon_0 A^{-1}\boldsymbol{\gamma}_\mathrm{e}$ have dimension of admittance.

Since the right hand side of each of these equations is proportional to ω, this shows us that in the static limit we have

$$\lim_{\omega\to 0} \hat{\boldsymbol{z}} \times (\bar{\boldsymbol{E}}_2 - \bar{\boldsymbol{E}}_1) = \mathbf{0} \quad \text{and} \quad \lim_{\omega\to 0} \hat{\boldsymbol{z}} \times (\bar{\boldsymbol{H}}_2 - \bar{\boldsymbol{H}}_1) = \mathbf{0} \tag{3.13}$$

i.e., the static tangential fields are continuous across the structure.

3.1 Rewriting in transverse components

Since the left hand sides of our equations are orthogonal to $\hat{\boldsymbol{z}}$ due to the cross product with $\hat{\boldsymbol{z}}$, the z component of the right hand sides must be zero,

$$0 = -\hat{\boldsymbol{z}} \cdot (\mathrm{i}\boldsymbol{k}_\perp h \times \bar{\boldsymbol{E}}_\perp) + \mathrm{i}\omega\mu_0 h \bar{H}_z + \mathrm{i}\omega\mu_0 A^{-1}(\boldsymbol{\gamma}_\mathrm{m}\boldsymbol{H}_0)_z \tag{3.14}$$

$$0 = -\hat{\boldsymbol{z}} \cdot (\mathrm{i}\boldsymbol{k}_\perp h \times \bar{\boldsymbol{H}}_\perp) - \mathrm{i}\omega\epsilon_0 h \bar{E}_z - \mathrm{i}\omega\epsilon_0 A^{-1}(\boldsymbol{\gamma}_\mathrm{e}\boldsymbol{E}_0)_z \tag{3.15}$$

From this we extract the z components as

$$\bar{E}_z = \frac{-\hat{\boldsymbol{z}} \cdot (\mathrm{i}\boldsymbol{k}_\perp \times \bar{\boldsymbol{H}}_\perp)}{\mathrm{i}\omega\epsilon_0} - \frac{(\boldsymbol{\gamma}_\mathrm{e}\boldsymbol{E}_0)_z}{Ah} = \frac{-\mathrm{i}\boldsymbol{k}'_\perp \cdot \bar{\boldsymbol{H}}_\perp}{\mathrm{i}\omega\epsilon_0} - \frac{(\boldsymbol{\gamma}_\mathrm{e}\boldsymbol{E}_0)_z}{Ah} \tag{3.16}$$

$$\bar{H}_z = \frac{\hat{\boldsymbol{z}} \cdot (\mathrm{i}\boldsymbol{k}_\perp \times \bar{\boldsymbol{E}}_\perp)}{\mathrm{i}\omega\mu_0} - \frac{(\boldsymbol{\gamma}_\mathrm{m}\boldsymbol{H}_0)_z}{Ah} = \frac{\mathrm{i}\boldsymbol{k}'_\perp \cdot \bar{\boldsymbol{E}}_\perp}{\mathrm{i}\omega\mu_0} - \frac{(\boldsymbol{\gamma}_\mathrm{m}\boldsymbol{H}_0)_z}{Ah} \tag{3.17}$$

where we used $\hat{\boldsymbol{z}} \cdot (\boldsymbol{k}_\perp \times \bar{\boldsymbol{E}}_\perp) = (\hat{\boldsymbol{z}} \times \boldsymbol{k}_\perp) \cdot \bar{\boldsymbol{E}}_\perp = \boldsymbol{k}'_\perp \cdot \bar{\boldsymbol{E}}_\perp$. Inserting this into the transverse part of the equations implies

$$\begin{aligned} \hat{\boldsymbol{z}} \times \left(\bar{\boldsymbol{E}}_2 - \bar{\boldsymbol{E}}_1\right) &= \underbrace{-\mathrm{i}\boldsymbol{k}_\perp h \times \hat{\boldsymbol{z}}\bar{E}_z}_{=\mathrm{i}\boldsymbol{k}'_\perp h \bar{E}_z} + \mathrm{i}\omega\mu_0 h \bar{\boldsymbol{H}}_\perp + \mathrm{i}\omega\mu_0 A^{-1}(\boldsymbol{\gamma}_\mathrm{m}\boldsymbol{H}_0)_\perp \\ &= \frac{\boldsymbol{k}'_\perp\boldsymbol{k}'_\perp}{\mathrm{i}\omega\epsilon_0} h \cdot \bar{\boldsymbol{H}}_\perp - \mathrm{i}\boldsymbol{k}'_\perp \frac{(\boldsymbol{\gamma}_\mathrm{e}\boldsymbol{E}_0)_z}{A} + \mathrm{i}\omega\mu_0 h \bar{\boldsymbol{H}}_\perp + \mathrm{i}\omega\mu_0 \frac{(\boldsymbol{\gamma}_\mathrm{m}\boldsymbol{H}_0)_\perp}{A} \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} \hat{\boldsymbol{z}} \times \left(\bar{\boldsymbol{H}}_2 - \bar{\boldsymbol{H}}_1\right) &= \underbrace{-\mathrm{i}\boldsymbol{k}_\perp h \times \hat{\boldsymbol{z}}\bar{H}_z}_{=\mathrm{i}\boldsymbol{k}'_\perp h \bar{H}_z} - \mathrm{i}\omega\epsilon_0 h \bar{\boldsymbol{E}}_\perp - \mathrm{i}\omega\epsilon_0 A^{-1}(\boldsymbol{\gamma}_\mathrm{e}\boldsymbol{E}_0)_\perp \\ &= -\frac{\boldsymbol{k}'_\perp\boldsymbol{k}'_\perp}{\mathrm{i}\omega\mu_0} h \cdot \bar{\boldsymbol{E}}_\perp - \mathrm{i}\boldsymbol{k}'_\perp \frac{(\boldsymbol{\gamma}_\mathrm{m}\boldsymbol{H}_0)_z}{A} - \mathrm{i}\omega\epsilon_0 h \bar{\boldsymbol{E}}_\perp - \mathrm{i}\omega\epsilon_0 \frac{(\boldsymbol{\gamma}_\mathrm{e}\boldsymbol{E}_0)_\perp}{A} \end{aligned} \tag{3.19}$$

3.2 Reflection and transmission

In the surrounding air, the transverse components of the electric and magnetic fields in a propagating plane wave are related by an impedance matrix

$$\boldsymbol{E}_\perp = \pm\mathbf{Z}(-\hat{\boldsymbol{z}} \times \boldsymbol{H}_\perp) \quad \Leftrightarrow \quad \boldsymbol{H}_\perp = \pm\hat{\boldsymbol{z}} \times \mathbf{Z}^{-1}\boldsymbol{E}_\perp \tag{3.20}$$

where the upper sign is for waves propagating in the positive z direction, and the lower sign is for waves propagating in the negative z direction. The impedance matrix $\mathbf{Z}$ has eigenvalues $\eta_0/\cos\theta$ (TE case) and $\eta_0\cos\theta$ (TM case). When $\boldsymbol{k}_\perp \neq \mathbf{0}$ it can be given the explicit representation

$$\mathbf{Z} = \frac{\eta_0}{\cos\theta}\frac{\boldsymbol{k}'_\perp\boldsymbol{k}'_\perp}{|\boldsymbol{k}'_\perp|^2} + \eta_0\cos\theta\frac{\boldsymbol{k}_\perp\boldsymbol{k}_\perp}{|\boldsymbol{k}_\perp|^2} \tag{3.21}$$

and for $\boldsymbol{k}_\perp = \mathbf{0}$, corresponding to $\cos\theta = 1$, we have $\mathbf{Z} = \eta_0\mathbf{I}$. If $\mathbf{r}$ denotes the reflection matrix for the tangential electric field, *i.e.*, $\boldsymbol{E}_{\mathrm{r}\perp} = \mathbf{r}\boldsymbol{E}_{\mathrm{i}\perp}$, the amplitude of the reflected magnetic field is then

$$\boldsymbol{H}_{\mathrm{r}\perp} = -\hat{\boldsymbol{z}}\times\mathbf{Z}^{-1}\boldsymbol{E}_{\mathrm{r}\perp} = -\hat{\boldsymbol{z}}\times\mathbf{Z}^{-1}\mathbf{r}\boldsymbol{E}_{\mathrm{i}\perp} = -\hat{\boldsymbol{z}}\times\mathbf{Z}^{-1}\mathbf{r}\mathbf{Z}(-\hat{\boldsymbol{z}}\times\boldsymbol{H}_{\mathrm{i}\perp}) \tag{3.22}$$

If the structure does not mix TE and TM modes in the scattering (*i.e.*, $\mathbf{r}$ is diagonal in the $\boldsymbol{k}_\perp/\boldsymbol{k}'_\perp$ basis), the impedance matrices cancel each other in the last expression. We now write the fields on either side of the periodic structure as a sum of incident and scattered fields

$$\bar{\boldsymbol{E}}_{1\perp} = \boldsymbol{E}_{0\perp}\mathrm{e}^{\mathrm{i}k_z z_1} + \mathbf{r}\boldsymbol{E}_{0\perp}\mathrm{e}^{-\mathrm{i}k_z z_1} \qquad \bar{\boldsymbol{H}}_{1\perp} = \hat{\boldsymbol{z}}\times\mathbf{Z}^{-1}(\mathbf{I}\mathrm{e}^{\mathrm{i}k_z z_1} - \mathbf{r}\mathrm{e}^{-\mathrm{i}k_z z_1})\boldsymbol{E}_{0\perp} \tag{3.23}$$

$$\bar{\boldsymbol{E}}_{2\perp} = \mathbf{t}\boldsymbol{E}_{0\perp}\mathrm{e}^{\mathrm{i}k_z z_2} \qquad \bar{\boldsymbol{H}}_{2\perp} = \hat{\boldsymbol{z}}\times\mathbf{Z}^{-1}\mathbf{t}\boldsymbol{E}_{0\perp}\mathrm{e}^{\mathrm{i}k_z z_2} \tag{3.24}$$

where $\mathbf{r}$ and $\mathbf{t}$ are the reflection and transmission matrices with reference plane at $z = 0$, respectively. So far no approximations due to the low frequency have been made. When considering the low frequency limit, the reflection and transmission matrices can be expanded in a formal power series in ω as

$$\mathbf{r} = \mathbf{r}_0 + \omega\mathbf{r}_1 + \cdots \tag{3.25}$$

$$\mathbf{t} = \mathbf{t}_0 + \omega\mathbf{t}_1 + \cdots \tag{3.26}$$

Since the static tangential fields are continuous across the structure according to (3.13), it is immediately seen that $\mathbf{r}_0 = \mathbf{0}$ and $\mathbf{t}_0 = \mathbf{I}$, as expected for a low pass structure. In this paper, we are only interested in terms up to $\mathbf{r}_1$ and $\mathbf{t}_1$, which means it is sufficient to keep only terms up to first order in $\bar{\boldsymbol{E}}_{1,2}$ and $\bar{\boldsymbol{H}}_{1,2}$ (we suppress the expansion of $\mathbf{r}$ and $\mathbf{t}$ for brevity)

$$\bar{\boldsymbol{E}}_{1\perp} = (\mathbf{I} + \mathbf{r} + \mathrm{i}k_z z_1\mathbf{I})\boldsymbol{E}_{0\perp} \qquad \bar{\boldsymbol{H}}_{1\perp} = \hat{\boldsymbol{z}}\times\mathbf{Z}^{-1}(\mathbf{I} - \mathbf{r} + \mathrm{i}k_z z_1\mathbf{I})\boldsymbol{E}_{0\perp} \tag{3.27}$$

$$\bar{\boldsymbol{E}}_{2\perp} = (\mathbf{t} + \mathrm{i}k_z z_2\mathbf{I})\boldsymbol{E}_{0\perp} \qquad \bar{\boldsymbol{H}}_{2\perp} = \hat{\boldsymbol{z}}\times\mathbf{Z}^{-1}(\mathbf{t} + \mathrm{i}k_z z_2\mathbf{I})\boldsymbol{E}_{0\perp} \tag{3.28}$$

$$\bar{\boldsymbol{E}}_{\perp} = \boldsymbol{E}_{0\perp} \qquad \bar{\boldsymbol{H}}_{\perp} = \boldsymbol{H}_{0\perp} = \hat{\boldsymbol{z}}\times\mathbf{Z}^{-1}\boldsymbol{E}_{0\perp} \tag{3.29}$$

The fields $\bar{\boldsymbol{E}}_\perp$ and $\bar{\boldsymbol{H}}_\perp$ are expanded only to zeroth order since in the equations they are multiplied by factors proportional to ω. In order for $\bar{\boldsymbol{E}}_\perp = \boldsymbol{E}_{0\perp}$ and $\bar{\boldsymbol{H}}_\perp = \hat{\boldsymbol{z}}\times\mathbf{Z}^{-1}\boldsymbol{E}_{0\perp}$ to hold to zeroth order, we need to consider a limit process where $h \to \infty$ and $kh \to 0$ simultaneously. This may seem to invalidate the expansions (3.25) and (3.26) since an extra scale is introduced, but a deeper analysis shows the expansions are still valid. The equations (3.18) and (3.19) are now, to first order,

$$\hat{\boldsymbol{z}} \times (\mathbf{t} - \mathbf{I} - \mathbf{r} + \mathrm{i}k_z(z_2 - z_1)\mathbf{I})\boldsymbol{E}_{0\perp} = \left[\frac{\boldsymbol{k}'_\perp \boldsymbol{k}'_\perp}{\mathrm{i}\omega\epsilon_0} h + \mathrm{i}\omega\mu_0 h\mathbf{I}\right] \hat{\boldsymbol{z}} \times \mathbf{Z}^{-1}\boldsymbol{E}_{0\perp} - \mathrm{i}\boldsymbol{k}'_\perp \frac{(\gamma_\mathrm{e}\boldsymbol{E}_0)_z}{A} + \mathrm{i}\omega\mu_0 \frac{(\gamma_\mathrm{m}\boldsymbol{H}_0)_\perp}{A} \tag{3.30}$$

$$-\mathbf{Z}^{-1}(\mathbf{t} - \mathbf{I} + \mathbf{r} + \mathrm{i}k_z(z_2 - z_1)\mathbf{I})\boldsymbol{E}_{0\perp} = -\left[\frac{\boldsymbol{k}'_\perp \boldsymbol{k}'_\perp}{\mathrm{i}\omega\mu_0} h + \mathrm{i}\omega\epsilon_0 h\mathbf{I}\right] \boldsymbol{E}_{0\perp} - \mathrm{i}\boldsymbol{k}'_\perp \frac{(\gamma_\mathrm{m}\boldsymbol{H}_0)_z}{A} - \mathrm{i}\omega\epsilon_0 \frac{(\gamma_\mathrm{e}\boldsymbol{E}_0)_\perp}{A} \tag{3.31}$$

From this equation it is not clear that there is no scattering if there is no material, *i.e.*, we should subtract the parts corresponding to propagation in vacuum. The incident field satisfies (set all polarizability matrices in (3.18) and (3.19) to zero and use $\boldsymbol{E}_1 = \boldsymbol{E}_0\mathrm{e}^{\mathrm{i}k_z z_1}$ and $\boldsymbol{E}_2 = \boldsymbol{E}_0\mathrm{e}^{\mathrm{i}k_z z_2}$ etc, and expand to first order)

$$(\mathrm{e}^{\mathrm{i}k_z z_2} - \mathrm{e}^{\mathrm{i}k_z z_1})\hat{\boldsymbol{z}} \times \boldsymbol{E}_0 = \mathrm{i}k_z(z_2 - z_1)\hat{\boldsymbol{z}} \times \boldsymbol{E}_{0\perp} = \left[\frac{\boldsymbol{k}'_\perp \boldsymbol{k}'_\perp}{\mathrm{i}\omega\epsilon_0} h + \mathrm{i}\omega\mu_0 h\mathbf{I}\right] \boldsymbol{H}_{0\perp} \tag{3.32}$$

$$(\mathrm{e}^{\mathrm{i}k_z z_2} - \mathrm{e}^{\mathrm{i}k_z z_1})\hat{\boldsymbol{z}} \times \boldsymbol{H}_0 = \mathrm{i}k_z(z_2 - z_1)\hat{\boldsymbol{z}} \times \boldsymbol{H}_{0\perp} = -\left[\frac{\boldsymbol{k}'_\perp \boldsymbol{k}'_\perp}{\mathrm{i}\omega\mu_0} h + \mathrm{i}\omega\epsilon_0 h\mathbf{I}\right] \boldsymbol{E}_{0\perp} \tag{3.33}$$

Subtracting this result from the previous equations we find (after multiplying the first equation by $-\hat{\boldsymbol{z}}\times$ and the second by $-\mathbf{Z}$, and observing that $-\hat{\boldsymbol{z}} \times \boldsymbol{k}'_\perp = -\hat{\boldsymbol{z}} \times (\hat{\boldsymbol{z}} \times \boldsymbol{k}_\perp) = \boldsymbol{k}_\perp$, as well as the relations $k = \omega\sqrt{\epsilon_0\mu_0}$ and $\eta_0 = \sqrt{\mu_0/\epsilon_0}$)

$$(\mathbf{t} - \mathbf{I} - \mathbf{r})\boldsymbol{E}_{0\perp} = -\mathrm{i}\boldsymbol{k}_\perp \frac{(\gamma_\mathrm{e}\boldsymbol{E}_0)_z}{A} - \mathrm{i}k\eta_0\hat{\boldsymbol{z}} \times \frac{(\gamma_\mathrm{m}\boldsymbol{H}_0)_\perp}{A} \tag{3.34}$$

$$(\mathbf{t} - \mathbf{I} + \mathbf{r})\boldsymbol{E}_{0\perp} = \mathrm{i}\mathbf{Z} \cdot \boldsymbol{k}'_\perp \frac{(\gamma_\mathrm{m}\boldsymbol{H}_0)_z}{A} + \mathrm{i}k\eta_0^{-1}\mathbf{Z}\frac{(\gamma_\mathrm{e}\boldsymbol{E}_0)_\perp}{A} \tag{3.35}$$

We state again that this equation is only valid asymptotically to first order as $k \to 0$.

3.3 Solving for the reflection and transmission matrices

By adding and subtracting the equations we find

$$2(\mathbf{t} - \mathbf{I})\boldsymbol{E}_{0\perp} = \mathrm{i}k\eta_0^{-1}\mathbf{Z}\frac{(\gamma_\mathrm{e}\boldsymbol{E}_0)_\perp}{A} - \mathrm{i}k\eta_0\hat{\boldsymbol{z}} \times \frac{(\gamma_\mathrm{m}\boldsymbol{H}_0)_\perp}{A} - \mathrm{i}\boldsymbol{k}_\perp \frac{(\gamma_\mathrm{e}\boldsymbol{E}_0)_z}{A} + \mathrm{i}\mathbf{Z} \cdot \boldsymbol{k}'_\perp \frac{(\gamma_\mathrm{m}\boldsymbol{H}_0)_z}{A} \tag{3.36}$$

$$2\mathbf{r}\boldsymbol{E}_{0\perp} = \mathrm{i}k\eta_0^{-1}\mathbf{Z}\frac{(\gamma_\mathrm{e}\boldsymbol{E}_0)_\perp}{A} + \mathrm{i}k\eta_0\hat{\boldsymbol{z}} \times \frac{(\gamma_\mathrm{m}\boldsymbol{H}_0)_\perp}{A} + \mathrm{i}\boldsymbol{k}_\perp \frac{(\gamma_\mathrm{e}\boldsymbol{E}_0)_z}{A} + \mathrm{i}\mathbf{Z} \cdot \boldsymbol{k}'_\perp \frac{(\gamma_\mathrm{m}\boldsymbol{H}_0)_z}{A} \tag{3.37}$$

To find explicit expressions for the transmission and reflection matrices, we must express all the field components in the right hand sides in $\boldsymbol{E}_{0\perp}$, so that this factor can be eliminated. This can be done from the knowledge that the incident field is a plane wave in the surrounding medium, which implies the following formulas:

$$E_{0z} = -\eta_0 \frac{\boldsymbol{k}_\perp \cdot \mathbf{Z}^{-1}\boldsymbol{E}_{0\perp}}{k} \tag{3.38}$$

$$\boldsymbol{H}_{0\perp} = \hat{\boldsymbol{z}} \times \mathbf{Z}^{-1}\boldsymbol{E}_{0\perp} \tag{3.39}$$

$$H_{0z} = \eta_0^{-1} \frac{\boldsymbol{k}'_\perp \cdot \boldsymbol{E}_{0\perp}}{k} \tag{3.40}$$

Making use of these representations and the decompositions

$$\boldsymbol{\gamma}_\mathrm{e} = \begin{pmatrix} \gamma_{\mathrm{e}\perp\perp} & \gamma_{\mathrm{e}\perp z} \\ \gamma_{\mathrm{e}z\perp} & \gamma_{\mathrm{e}zz} \end{pmatrix}, \quad \text{and} \quad \boldsymbol{\gamma}_\mathrm{m} = \begin{pmatrix} \gamma_{\mathrm{m}\perp\perp} & \gamma_{\mathrm{m}\perp z} \\ \gamma_{\mathrm{m}z\perp} & \gamma_{\mathrm{m}zz} \end{pmatrix} \tag{3.41}$$

we can write the various components of the dipole moments as

$$(\boldsymbol{\gamma}_\mathrm{e}\boldsymbol{E}_0)_\perp = \boldsymbol{\gamma}_{\mathrm{e}\perp\perp}\boldsymbol{E}_{0\perp} - \eta_0\boldsymbol{\gamma}_{\mathrm{e}\perp z}\frac{\boldsymbol{k}_\perp}{k} \cdot \mathbf{Z}^{-1}\boldsymbol{E}_{0\perp} \tag{3.42}$$

$$(\boldsymbol{\gamma}_\mathrm{e}\boldsymbol{E}_0)_z = \boldsymbol{\gamma}_{\mathrm{e}z\perp}\boldsymbol{E}_{0\perp} - \eta_0\gamma_{\mathrm{e}zz}\frac{\boldsymbol{k}_\perp}{k} \cdot \mathbf{Z}^{-1}\boldsymbol{E}_{0\perp} \tag{3.43}$$

$$(\boldsymbol{\gamma}_\mathrm{m}\boldsymbol{H}_0)_\perp = \boldsymbol{\gamma}_{\mathrm{m}\perp\perp}\hat{\boldsymbol{z}} \times \mathbf{Z}^{-1}\boldsymbol{E}_{0\perp} + \eta_0^{-1}\boldsymbol{\gamma}_{\mathrm{m}\perp z}\frac{\boldsymbol{k}'_\perp}{k} \cdot \boldsymbol{E}_{0\perp} \tag{3.44}$$

$$(\boldsymbol{\gamma}_\mathrm{m}\boldsymbol{H}_0)_z = \boldsymbol{\gamma}_{\mathrm{m}z\perp}\hat{\boldsymbol{z}} \times \mathbf{Z}^{-1}\boldsymbol{E}_{0\perp} + \eta_0^{-1}\gamma_{\mathrm{m}zz}\frac{\boldsymbol{k}'_\perp}{k} \cdot \boldsymbol{E}_{0\perp} \tag{3.45}$$

Collecting all the results, we can write the transmission and reflection matrices as

$$\begin{aligned} \mathbf{t} - \mathbf{I} = \frac{\mathrm{i}k}{2}\Bigg\{ &\eta_0^{-1}\mathbf{Z}\left[\frac{\boldsymbol{\gamma}_{\mathrm{e}\perp\perp}}{A} + \frac{\boldsymbol{k}'_\perp\boldsymbol{k}'_\perp}{k^2}\frac{\gamma_{\mathrm{m}zz}}{A}\right] + \left[-\hat{\boldsymbol{z}} \times \frac{\boldsymbol{\gamma}_{\mathrm{m}\perp\perp}}{A}\hat{\boldsymbol{z}} \times + \frac{\boldsymbol{k}_\perp\boldsymbol{k}_\perp}{k^2}\frac{\gamma_{\mathrm{e}zz}}{A}\right]\mathbf{Z}^{-1}\eta_0 \\ &+ \mathbf{Z}\left[\frac{\boldsymbol{k}'_\perp}{k}\frac{\boldsymbol{\gamma}_{\mathrm{m}z\perp}}{A} - \frac{\boldsymbol{\gamma}_{\mathrm{e}\perp z}}{A}\frac{\boldsymbol{k}'_\perp}{k}\right]\hat{\boldsymbol{z}} \times \mathbf{Z}^{-1} + \hat{\boldsymbol{z}} \times \left[\frac{\boldsymbol{k}'_\perp}{k}\frac{\boldsymbol{\gamma}_{\mathrm{e}z\perp}}{A} - \frac{\boldsymbol{\gamma}_{\mathrm{m}\perp z}}{A}\frac{\boldsymbol{k}'_\perp}{k}\right]\Bigg\} \end{aligned} \tag{3.46}$$

$$\begin{aligned} \mathbf{r} = \frac{\mathrm{i}k}{2}\Bigg\{ &\eta_0^{-1}\mathbf{Z}\left[\frac{\boldsymbol{\gamma}_{\mathrm{e}\perp\perp}}{A} + \frac{\boldsymbol{k}'_\perp\boldsymbol{k}'_\perp}{k^2}\frac{\gamma_{\mathrm{m}zz}}{A}\right] - \left[-\hat{\boldsymbol{z}} \times \frac{\boldsymbol{\gamma}_{\mathrm{m}\perp\perp}}{A}\hat{\boldsymbol{z}} \times + \frac{\boldsymbol{k}_\perp\boldsymbol{k}_\perp}{k^2}\frac{\gamma_{\mathrm{e}zz}}{A}\right]\mathbf{Z}^{-1}\eta_0 \\ &+ \mathbf{Z}\left[\frac{\boldsymbol{k}'_\perp}{k}\frac{\boldsymbol{\gamma}_{\mathrm{m}z\perp}}{A} - \frac{\boldsymbol{\gamma}_{\mathrm{e}\perp z}}{A}\frac{\boldsymbol{k}'_\perp}{k}\right]\hat{\boldsymbol{z}} \times \mathbf{Z}^{-1} - \hat{\boldsymbol{z}} \times \left[\frac{\boldsymbol{k}'_\perp}{k}\frac{\boldsymbol{\gamma}_{\mathrm{e}z\perp}}{A} - \frac{\boldsymbol{\gamma}_{\mathrm{m}\perp z}}{A}\frac{\boldsymbol{k}'_\perp}{k}\right]\Bigg\} \end{aligned} \tag{3.47}$$

For normal incidence the result simplifies to

$$\mathbf{t} - \mathbf{I} = \frac{\mathrm{i}k}{2}\left\{\frac{\boldsymbol{\gamma}_{\mathrm{e}\perp\perp}}{A} - \hat{\boldsymbol{z}} \times \frac{\boldsymbol{\gamma}_{\mathrm{m}\perp\perp}}{A}\hat{\boldsymbol{z}}\times\right\} \tag{3.48}$$

$$\mathbf{r} = \frac{\mathrm{i}k}{2}\left\{\frac{\boldsymbol{\gamma}_{\mathrm{e}\perp\perp}}{A} + \hat{\boldsymbol{z}} \times \frac{\boldsymbol{\gamma}_{\mathrm{m}\perp\perp}}{A}\hat{\boldsymbol{z}}\times\right\} \tag{3.49}$$

Note that since the operation $\hat{\boldsymbol{z}}\times$ can be identified with a skew-symmetric matrix which is its own (negative) inverse, the matrix $-\hat{\boldsymbol{z}} \times \boldsymbol{\gamma}_{\mathrm{m}\perp\perp}\hat{\boldsymbol{z}}\times = (\hat{\boldsymbol{z}}\times)^{-1}\boldsymbol{\gamma}_{\mathrm{m}\perp\perp}\hat{\boldsymbol{z}}\times$ is a similarity transform of $\boldsymbol{\gamma}_{\mathrm{m}\perp\perp}$. This demonstrates that the first order correction to the static transmission and reflection coefficients is given by the sum and difference of the electric and magnetic polarizability per unit area of the structure, multiplied by $\mathrm{i}k/2$. Note that the expressions contain both co- and cross-polarization results.

Example: dielectric film. For a dielectric, nonmagnetic film we can compute both the polarizability matrices and the transmission and reflection coefficients explicitly. The film is contained in the region $0 < z < d$, and has isotropic permittivity $\boldsymbol{\epsilon} = \epsilon_\mathrm{r}\epsilon_0\mathbf{I}$ and vacuum permeability $\boldsymbol{\mu} = \mu_0\mathbf{I}$. The magnetic polarizability is then $\boldsymbol{\gamma}_\mathrm{m} = \mathbf{0}$, and the electric is

$$\boldsymbol{\gamma}_\mathrm{e} = \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_1 & 0 \\ 0 & 0 & \gamma_2 \end{pmatrix}, \quad \text{where} \quad \begin{cases} \gamma_1 = (\epsilon_\mathrm{r} - 1)Ad \\ \gamma_2 = (1 - \epsilon_\mathrm{r}^{-1})Ad \end{cases} \tag{3.50}$$

The expressions (3.46) and (3.47) are then

$$\begin{aligned} \mathbf{t} - \mathbf{I} &= \frac{\mathrm{i}k}{2}\left\{\eta_0^{-1}\mathbf{Z}\frac{\boldsymbol{\gamma}_{\mathrm{e}\perp\perp}}{A} + \frac{\boldsymbol{k}_\perp\boldsymbol{k}_\perp}{k^2}\frac{\gamma_{\mathrm{e}zz}}{A}\mathbf{Z}^{-1}\eta_0\right\} \\ &= \frac{\mathrm{i}kd}{2}\left\{\frac{\epsilon_\mathrm{r} - 1}{\cos\theta}\frac{\boldsymbol{k}'_\perp\boldsymbol{k}'_\perp}{|\boldsymbol{k}'_\perp|^2} + \cos\theta\left[\epsilon_\mathrm{r} - 1 + (1 - \epsilon_\mathrm{r}^{-1})\tan^2\theta\right]\frac{\boldsymbol{k}_\perp\boldsymbol{k}_\perp}{|\boldsymbol{k}_\perp|^2}\right\} \end{aligned} \tag{3.51}$$

$$\begin{aligned} \mathbf{r} &= \frac{\mathrm{i}k}{2}\left\{\eta_0^{-1}\mathbf{Z}\frac{\boldsymbol{\gamma}_{\mathrm{e}\perp\perp}}{A} - \frac{\boldsymbol{k}_\perp\boldsymbol{k}_\perp}{k^2}\frac{\gamma_{\mathrm{e}zz}}{A}\mathbf{Z}^{-1}\eta_0\right\} \\ &= \frac{\mathrm{i}kd}{2}\left\{\frac{\epsilon_\mathrm{r} - 1}{\cos\theta}\frac{\boldsymbol{k}'_\perp\boldsymbol{k}'_\perp}{|\boldsymbol{k}'_\perp|^2} + \cos\theta\left[\epsilon_\mathrm{r} - 1 - (1 - \epsilon_\mathrm{r}^{-1})\tan^2\theta\right]\frac{\boldsymbol{k}_\perp\boldsymbol{k}_\perp}{|\boldsymbol{k}_\perp|^2}\right\} \end{aligned} \tag{3.52}$$

which can be confirmed to be the proper low frequency expansion of the transmission and reflection coefficients for a dielectric film [2, p. 65].

4 Polarizability matrix

We now turn to the problem of computing the polarizability matrices $\boldsymbol{\gamma}_\mathrm{e}$ and $\boldsymbol{\gamma}_\mathrm{m}$ in the static limit.

4.1 Finite material parameters with no conductivity

We use Stevenson's method [15] to extract the low frequency equations, as is traditional in homogenization theory [1]. A formal expansion of the fields in a power series in ω, *i.e.*,

$$\widetilde{\boldsymbol{E}} = \widetilde{\boldsymbol{E}}^{(0)} + \omega\widetilde{\boldsymbol{E}}^{(1)} + \cdots \tag{4.1}$$

$$\widetilde{\boldsymbol{H}} = \widetilde{\boldsymbol{H}}^{(0)} + \omega\widetilde{\boldsymbol{H}}^{(1)} + \cdots \tag{4.2}$$

and identifying similar powers of ω in the equations, implies that Maxwell's equations reduce to the static equations for the zeroth order fields (where the material parameters must be understood as the static limit, *i.e.*, $\boldsymbol{\epsilon}(\boldsymbol{x}) = \lim_{\omega\to 0} \boldsymbol{\epsilon}(\boldsymbol{x},\omega)$ and $\boldsymbol{\mu}(\boldsymbol{x}) = \lim_{\omega\to 0} \boldsymbol{\mu}(\boldsymbol{x},\omega)$)

$$\nabla \times \widetilde{\boldsymbol{E}}^{(0)} = \boldsymbol{0} \qquad \nabla \cdot [\boldsymbol{\epsilon}\widetilde{\boldsymbol{E}}^{(0)}] = 0 \tag{4.3}$$

$$\nabla \times \widetilde{\boldsymbol{H}}^{(0)} = \boldsymbol{0} \qquad \nabla \cdot [\boldsymbol{\mu}\widetilde{\boldsymbol{H}}^{(0)}] = 0 \tag{4.4}$$

with periodic boundary conditions in the xy plane. In the z-direction, we require that $\widetilde{\boldsymbol{E}}^{(0)}$ and $\widetilde{\boldsymbol{H}}^{(0)}$ go to constants $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$ as $z \to \pm\infty$. The zero curl condition implies

$$\widetilde{\boldsymbol{E}}^{(0)} = \boldsymbol{E}_0 - \nabla\phi_{\mathrm{e}} \quad \text{and} \quad \widetilde{\boldsymbol{H}}^{(0)} = \boldsymbol{H}_0 - \nabla\phi_{\mathrm{m}} \tag{4.5}$$

where the potentials ϕ_{e} and ϕ_{m} are U-periodic functions in x and y with zero mean over U, and $\nabla\phi_{\mathrm{e}}$ and $\nabla\phi_{\mathrm{m}}$ both decay to zero as $z \to \pm\infty$ and are square integrable. Note that we are not requiring ϕ_{e} and ϕ_{m} to be zero at infinity. That this cannot be the general case is seen from a dielectric film subjected to a field in the z direction. The discontinuous polarization in the z direction induces surface charges on the boundaries of the film, which in turn implies a potential difference between the sides of the film. Thus, the potential cannot in general be zero on both sides.

We can now summarize the low frequency problem as two separate local problems in the unit cell,

$$\nabla \cdot [\boldsymbol{\epsilon}(\boldsymbol{E}_0 - \nabla\phi_{\mathrm{e}})] = 0 \tag{4.6}$$

$$\nabla \cdot [\boldsymbol{\mu}(\boldsymbol{H}_0 - \nabla\phi_{\mathrm{m}})] = 0 \tag{4.7}$$

for prescribed constant fields $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$. These are elliptic equations for the potentials ϕ_{e} and ϕ_{m}, which are solvable with standard numerical methods such as the finite element method, as long as these are implemented with the proper boundary conditions. The potentials depend linearly on $\boldsymbol{E}_0$ and $\boldsymbol{H}_0$, which defines linear operators $\boldsymbol{\gamma}_{\mathrm{e}}$ and $\boldsymbol{\gamma}_{\mathrm{m}}$ according to the integrals

$$\int_{-\infty}^{\infty}\int_U (\boldsymbol{\epsilon}/\epsilon_0 - \mathbf{I})(\boldsymbol{E}_0 - \nabla\phi_{\mathrm{e}})\,\mathrm{d}S\,\mathrm{d}z \stackrel{\mathrm{def}}{=} \boldsymbol{\gamma}_{\mathrm{e}}\boldsymbol{E}_0 \tag{4.8}$$

$$\int_{-\infty}^{\infty}\int_U (\boldsymbol{\mu}/\mu_0 - \mathbf{I})(\boldsymbol{H}_0 - \nabla\phi_{\mathrm{m}})\,\mathrm{d}S\,\mathrm{d}z \stackrel{\mathrm{def}}{=} \boldsymbol{\gamma}_{\mathrm{m}}\boldsymbol{H}_0 \tag{4.9}$$

These matrices are the polarizability matrices in the static limit, used in the preceding section.

The polarizabilities can be defined as the minimum of an energy functional. It is shown in [5] that if $\boldsymbol{\epsilon}(\boldsymbol{x}) \le \boldsymbol{\epsilon}(\boldsymbol{x})'$ for all $\boldsymbol{x}$, the corresponding polarizabilities satisfy $\boldsymbol{\gamma}_{\mathrm{e}} \le \boldsymbol{\gamma}_{\mathrm{e}}'$. Even though the derivation in [5] is for a single isotropic particle, the arguments are valid for an anisotropic periodic setting as well as seen in [8], and the corresponding result applies also to $\boldsymbol{\gamma}_{\mathrm{m}}$. Thus, the polarizabilities are monotone in the material parameters. In addition, we have the simple estimates [8]

$$\int_{-\infty}^{\infty}\int_U (\mathbf{I} - \boldsymbol{\epsilon}^{-1}\epsilon_0)\,\mathrm{d}S\,\mathrm{d}z \le \boldsymbol{\gamma}_{\mathrm{e}} \le \int_{-\infty}^{\infty}\int_U (\boldsymbol{\epsilon}/\epsilon_0 - \mathbf{I})\,\mathrm{d}S\,\mathrm{d}z \tag{4.10}$$

$$\int_{-\infty}^{\infty}\int_U (\mathbf{I} - \boldsymbol{\mu}^{-1}\mu_0)\,\mathrm{d}S\,\mathrm{d}z \le \boldsymbol{\gamma}_{\mathrm{m}} \le \int_{-\infty}^{\infty}\int_U (\boldsymbol{\mu}/\mu_0 - \mathbf{I})\,\mathrm{d}S\,\mathrm{d}z \tag{4.11}$$

corresponding to the harmonic and arithmetic means of the material parameters. In classical homogenization theory, the corresponding bounds are known as the Wiener bounds [16].

4.2 PEC inclusions

With some small modifications, the above reasoning applies also for metal inclusions in the unit cell. Modelling the metal as a perfect electric conductor (PEC), the equations should then be interpreted as being valid in the domain $U \times \mathbb{R} \setminus \Omega$, where Ω denotes the PEC region and the boundary conditions $\hat{\boldsymbol{n}} \times \widetilde{\boldsymbol{E}}^{(0)} = \boldsymbol{0}$ and $\hat{\boldsymbol{n}} \cdot (\boldsymbol{\mu} \widetilde{\boldsymbol{H}}^{(0)}) = 0$ applies on $\partial\Omega$ [4, p. 204]. This corresponds to taking the limits $\boldsymbol{\epsilon} \to \infty$ and $\boldsymbol{\mu} \to \boldsymbol{0}$ in the PEC region.

We identify the $\boldsymbol{\gamma}_{\mathrm{e}}$ and $\boldsymbol{\gamma}_{\mathrm{m}}$ matrices as giving the total electric and magnetic dipole moment, respectively. Their definitions are then replaced with (using that the surface charge density is $\rho_{\mathrm{S}} = \hat{\boldsymbol{n}} \cdot (\boldsymbol{\epsilon} \widetilde{\boldsymbol{E}}^{(0)})$ and the surface current density is $\boldsymbol{J}_{\mathrm{S}} = \hat{\boldsymbol{n}} \times \widetilde{\boldsymbol{H}}^{(0)}$)

$$\boldsymbol{\gamma}_{\mathrm{e}} \boldsymbol{E}_0 \overset{\mathrm{def}}{=} \int\limits_{U \times \mathbb{R} \setminus \Omega} (\boldsymbol{\epsilon}/\epsilon_0 - \mathbf{I}) \widetilde{\boldsymbol{E}}^{(0)} \, \mathrm{d}V + \oint_{\partial\Omega} \boldsymbol{x} \hat{\boldsymbol{n}} \cdot \frac{\boldsymbol{\epsilon} \widetilde{\boldsymbol{E}}^{(0)}}{\epsilon_0} \, \mathrm{d}S \tag{4.12}$$

$$\boldsymbol{\gamma}_{\mathrm{m}} \boldsymbol{H}_0 \overset{\mathrm{def}}{=} \int\limits_{U \times \mathbb{R} \setminus \Omega} (\boldsymbol{\mu}/\mu_0 - \mathbf{I}) \widetilde{\boldsymbol{H}}^{(0)} \, \mathrm{d}V + \frac{1}{2} \oint_{\partial\Omega} \boldsymbol{x} \times (\hat{\boldsymbol{n}} \times \widetilde{\boldsymbol{H}}^{(0)}) \, \mathrm{d}S \tag{4.13}$$

It is shown in [6] that the magnetic polarizability $\boldsymbol{\gamma}_{\mathrm{m}}$ for PEC bodies in vacuum is negative. The electric and magnetic polarizabilities are monotone with the volume in the respect that $\boldsymbol{\gamma}_{\mathrm{e}} \leq \boldsymbol{\gamma}_{\mathrm{e}}'$ and $-\boldsymbol{\gamma}_{\mathrm{m}} \leq -\boldsymbol{\gamma}_{\mathrm{m}}'$ if $V \leq V'$, where V and V' are the corresponding volumes [7]. In [8] it is shown that these results apply also when the PEC body is surrounded by a fixed anisotropic medium. Furthermore, we have the following estimates for PEC bodies in vacuum [7]

$$3V' \leq \boldsymbol{\gamma}_{\mathrm{e}} \leq 3V'' \tag{4.14}$$

$$3V'/2 \leq -\boldsymbol{\gamma}_{\mathrm{m}} \leq 3V''/2 \tag{4.15}$$

where V' is the volume of the largest sphere contained in the body, and V'' is the volume of the smallest sphere containing the body.

In the following subsection, we show that the last term in (4.13) is absent if the metal is modeled with a finite conductivity instead of PEC.

4.3 Conducting inclusions

The conductivity case, where $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}' + \boldsymbol{\sigma}/(-\mathrm{i}\omega)$, is fundamentally different since the electric current has a zeroth order term in the ω expansion due to

$$-\mathrm{i}\omega \boldsymbol{\epsilon} \widetilde{\boldsymbol{E}} = \boldsymbol{\sigma} \widetilde{\boldsymbol{E}} - \mathrm{i}\omega \boldsymbol{\epsilon}' \widetilde{\boldsymbol{E}} \tag{4.16}$$

The formal expansions (4.1) and (4.2) then implies the following equations for the zeroth order fields:

$$\nabla \times \widetilde{\boldsymbol{E}}^{(0)} = \boldsymbol{0} \qquad \nabla \cdot (\boldsymbol{\sigma} \widetilde{\boldsymbol{E}}^{(0)}) = 0 \tag{4.17}$$

$$\nabla \times \widetilde{\boldsymbol{H}}^{(0)} = \boldsymbol{\sigma} \widetilde{\boldsymbol{E}}^{(0)} \qquad \nabla \cdot (\boldsymbol{\mu} \widetilde{\boldsymbol{H}}^{(0)}) = 0 \tag{4.18}$$

Assuming $\boldsymbol{\sigma} \neq \boldsymbol{0}$ only inside the region Ω implies the boundary condition $\hat{\boldsymbol{n}} \cdot (\boldsymbol{\sigma} \widetilde{\boldsymbol{E}}^{(0)}) = 0$ at $\partial\Omega$. In simply connected regions Ω there can be no static current, which implies $\boldsymbol{\sigma} \widetilde{\boldsymbol{E}}^{(0)} = \boldsymbol{0}$. This can be seen in a more formal way by considering the quadratic form (using that $\nabla \times \widetilde{\boldsymbol{E}}^{(0)} = \boldsymbol{0}$ implies the representation $\widetilde{\boldsymbol{E}}^{(0)} = \boldsymbol{E}_0 - \nabla\phi_{\mathrm{e}}$ in a simply connected region)

$$\begin{aligned}\int_\Omega (\boldsymbol{E}_0 - \nabla\phi_{\mathrm{e}}) \cdot [\boldsymbol{\sigma}(\boldsymbol{E}_0 - \nabla\phi_{\mathrm{e}})] \, \mathrm{d}V = \boldsymbol{E}_0 \cdot \int_\Omega \boldsymbol{\sigma}(\boldsymbol{E}_0 - \nabla\phi_{\mathrm{e}}) \, \mathrm{d}V + \int_\Omega \phi_{\mathrm{e}} \nabla \cdot [\boldsymbol{\sigma}(\boldsymbol{E}_0 - \phi_{\mathrm{e}})] \, \mathrm{d}V \\ - \oint_{\partial\Omega} \phi_{\mathrm{e}} \hat{\boldsymbol{n}} \cdot [\boldsymbol{\sigma}(\boldsymbol{E}_0 - \phi_{\mathrm{e}})] \, \mathrm{d}V\end{aligned} \tag{4.19}$$

Each of these integrals are zero: the first because the net static current in a closed region must be zero,[1] the second because of the field equation $\nabla \cdot (\boldsymbol{\sigma} \widetilde{\boldsymbol{E}}^{(0)}) = 0$, and the third and last due to the boundary condition $\hat{\boldsymbol{n}} \cdot (\boldsymbol{\sigma} \widetilde{\boldsymbol{E}}^{(0)}) = 0$. Since the integrand in the left hand side is non-negative, it must be zero almost everywhere, proving that the field $\widetilde{\boldsymbol{E}}^{(0)} = \boldsymbol{E}_0 - \nabla\varphi = \boldsymbol{0}$ in the inclusion geometry Ω. This means the equations for the magnetic field reduce to $\nabla \times \widetilde{\boldsymbol{H}}^{(0)} = \boldsymbol{0}$ and $\nabla \cdot (\boldsymbol{\mu} \widetilde{\boldsymbol{H}}^{(0)}) = 0$, *i.e.*, the metal inclusions do not influence the magnetic field.

To determine $\widetilde{\boldsymbol{E}}^{(0)}$ in regions where $\boldsymbol{\sigma} = \boldsymbol{0}$, we need to consider equations further down the chain,

$$\nabla \times \widetilde{\boldsymbol{E}}^{(1)} = \mathrm{i}\boldsymbol{\mu} \widetilde{\boldsymbol{H}}^{(0)} \tag{4.20}$$

$$\nabla \times \widetilde{\boldsymbol{H}}^{(1)} = -\mathrm{i}\boldsymbol{\epsilon} \widetilde{\boldsymbol{E}}^{(0)} + \boldsymbol{\sigma} \widetilde{\boldsymbol{E}}_1 \tag{4.21}$$

From the last equation it is seen that in regions where $\boldsymbol{\sigma} = \boldsymbol{0}$, *i.e.*, outside Ω, we necessarily have $\nabla \cdot (\boldsymbol{\epsilon} \widetilde{\boldsymbol{E}}^{(0)}) = 0$. Since $\widetilde{\boldsymbol{E}}^{(0)} = \boldsymbol{0}$ inside Ω and the tangential electric field must be continuous, this implies the standard boundary condition $\hat{\boldsymbol{n}} \times \widetilde{\boldsymbol{E}}^{(0)} = \boldsymbol{0}$ on $\partial\Omega$.

To summarize, if the metallic inclusions are modelled with a finite conductivity, the electric polarizability should be calculated just as in the PEC case, but the magnetic polarizability is only due to variations in $\boldsymbol{\mu}$. The physical difference between the two models is that in the PEC case, the low frequency limit is taken

[1] Mathematically, this can be shown by considering the integral $\int_\Omega \nabla \cdot (x\boldsymbol{J}) \, \mathrm{d}V$. Using the divergence theorem, we have $\int_\Omega \nabla \cdot (x\boldsymbol{J}) \, \mathrm{d}V = \oint_{\partial\Omega} \hat{\boldsymbol{n}} \cdot (x\boldsymbol{J}) \, \mathrm{d}S = 0$ due to the boundary condition $\hat{\boldsymbol{n}} \cdot \boldsymbol{J} = 0$. Using that $\nabla \cdot (\varphi \boldsymbol{J}) = \nabla\varphi \cdot \boldsymbol{J} + \varphi \nabla \cdot \boldsymbol{J}$ for any φ, the integral must also be $\int_\Omega J_x \, \mathrm{d}V$. Thus, all components of the current integrate to zero, and we have $\int_\Omega \boldsymbol{J} \, \mathrm{d}V = \boldsymbol{0}$ for closed regions.

such that an infinitesimal skin depth is maintained in the metallic particle, whereas in the conductivity case the limit is taken so that the skin depth is much greater than the particle. From (3.48) we see that the first order transmission coefficient is the sum of electric and magnetic polarizability, and from (3.49) the first order reflection coefficient is the difference. Since the magnetic polarizability is negative for PEC bodies, we conclude that the difference between the two models is that the first order transmission is smaller for the PEC model than for the conductivity model, whereas the first order reflection is larger for the PEC model than for the conductivity model.

5 Conclusions

In this paper, we have derived the asymptotic behavior for the low frequency reflection and transmission coefficients of a low pass periodic structure. The structure can be anisotropic in both electric and magnetic properties, and the angle of incidence as well as the polarization of the incoming wave is arbitrary. It is found that the low frequency behavior is proportional to the static electric and magnetic polarizability per unit area of the periodic structure. The transmission coefficient is associated with the sum of the polarizabilities, and the reflection coefficient with the difference.

The polarizabilities can be considered as minima of energy functionals, which provide simple estimates in terms of easily calculated quantities associated with the harmonic and arithmetic mean of the material parameters. When modeling the metal inclusions with a finite conductivity instead of as PEC, the electric polarizability is unchanged, *i.e.*, a specific dipole moment can be identified for the metal body, whereas the magnetic polarizability only depends on variations in permeability, with no specific contribution from the metal body.

References

[1] A. Bensoussan, J. L. Lions, and G. Papanicolaou. *Asymptotic Analysis for Periodic Structures*, volume 5 of *Studies in Mathematics and its Applications*. North-Holland, Amsterdam, 1978.

[2] M. Born and E. Wolf. *Principles of Optics*. Cambridge University Press, Cambridge, U.K., seventh edition, 1999.

[3] M. Gustafsson, C. Sohl, and G. Kristensson. Physical limitations on antennas of arbitrary shape. *Proc. R. Soc. A*, **463**, 2589–2607, 2007.

[4] J. D. Jackson. *Classical Electrodynamics*. John Wiley & Sons, New York, third edition, 1999.

[5] D. S. Jones. Scattering by inhomogeneous dielectric particles. *Quart. J. Mech. Appl. Math.*, **38**, 135–155, 1985.

[6] J. B. Keller, R. E. Kleinman, and T. B. A. Senior. Dipole moments in Rayleigh scattering. *J. Inst. Maths Applic.*, **9**, 14–22, 1972.

[7] M. Schiffer and G. Szegö. Virtual mass and polarization. *Trans. Amer. Math. Soc.*, **67**(1), 130–205, 1949.

[8] D. Sjöberg. Variational principles for the static electric and magnetic polarizability of anisotropic media with PEC inclusions. Technical Report LUTEDX/(TEAT-7175)/1–21/(2008), Lund University, Department of Electrical and Information Technology, P.O. Box 118, S-221 00 Lund, Sweden, 2008. http://www.eit.lth.se.

[9] D. Sjöberg. Homogenization of dispersive material parameters for Maxwell's equations using a singular value decomposition. *Multiscale Modeling and Simulation*, **4**(3), 760–789, 2005.

[10] D. Sjöberg, C. Engström, G. Kristensson, D. J. N. Wall, and N. Wellander. A Floquet-Bloch decomposition of Maxwell's equations, applied to homogenization. *Multiscale Modeling and Simulation*, **4**(1), 149–171, 2005.

[11] C. Sohl and M. Gustafsson. A priori estimates on the partial realized gain of Ultra-Wideband (UWB) antennas. *Quart. J. Mech. Appl. Math.*, **61**(3), 415–430, 2008.

[12] C. Sohl, M. Gustafsson, and G. Kristensson. Physical limitations on broadband scattering by heterogeneous obstacles. *J. Phys. A: Math. Theor.*, **40**, 11165–11182, 2007.

[13] C. Sohl, M. Gustafsson, and G. Kristensson. Physical limitations on metamaterials: Restrictions on scattering and absorption over a frequency interval. *J. Phys. D: Applied Phys.*, **40**, 7146–7151, 2007.

[14] C. Sohl, C. Larsson, M. Gustafsson, and G. Kristensson. A scattering and absorption identity for metamaterials: experimental results and comparison with theory. *J. Appl. Phys.*, **103**(5), 054906, 2008.

[15] A. F. Stevenson. Solution of electromagnetic scattering problems as power series in the ratio (dimension of scatterer)/wavelength. *J. Appl. Phys.*, **24**(9), 1134–1142, September 1953.

[16] O. Wiener. Die Theorie des Mischkörpers für das Feld des stationären Strömung. *Abh. Math. -Physichen Klasse Königl. Säcsh. Gesel. Wissen*, **32**, 509–604, 1912.