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PHYSICAL LIMITATIONS ON BROADBAND SCATTERING

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Abstract: In this paper, physical limitations on broadband scattering are presented for heterogeneous anisotropic scatterers of arbitrary shape. A measure of broadband scattering in terms of the extinction cross section is derived based on the holomorphic properties of the forward scattering dyadic. An isoperimetric bound for isotropic material parameters is presented and ellipsoidal scatterers are discussed. Finally, the theoretical results are illustrated numerically by a generic scatterer.

INTRODUCTION

As far as the author knows, a broadband measure for scattering of electromagnetic waves was first introduced in Purcell[1] concerning absorption and emission of radiation by interstellar dust. Purcell derived the integrated extinction in the special case of homogeneous spheroidal scatterers via the Kramers-Kronig relations. In this paper, Purcell's results are generalized to a considerable larger class of scatterers. The underlying mathematical description is strongly motivated by the study of causality and dispersion relations in the quantum mechanical scattering theory of particles.

The scattering problem considered in this paper is Fourier-synthesized plane wave scattering by a bounded heterogeneous obstacle of arbitrary shape. The scatterer is modeled by the Maxwell's equations with anisotropic and dispersive constitutive relations in terms of the electric and magnetic susceptibility dyadics, $\boldsymbol{\chi}_\mathrm{e}$ and $\boldsymbol{\chi}_\mathrm{m}$, respectively. The analysis includes the perfectly conducting material model, as well as general dispersion with or without a conductivity term.

THE INTEGRATED EXTINCTION

Consider a scatterer V subject to an incident plane-wave $\boldsymbol{E}_\mathrm{in}(\mathrm{c}_0 t - \hat{\boldsymbol{k}}\cdot\boldsymbol{x})$ impinging in the $\hat{\boldsymbol{k}}$-direction. The scattering properties of V are described by the far field amplitude $\boldsymbol{F}$ via the linear and causal time-translational invariant convolution

$$\boldsymbol{F}(\tau,\hat{\boldsymbol{x}}) = \int_{-\infty}^{\tau} \mathrm{S}_\mathrm{t}(\tau-\tau',\hat{\boldsymbol{k}},\hat{\boldsymbol{x}})\cdot\boldsymbol{E}_\mathrm{in}(\tau')\ \mathrm{d}\tau'.$$

Introduce the forward scattering dyadic S as the Fourier transform of S_t evaluated in the direction $\hat{\boldsymbol{x}} = \hat{\boldsymbol{k}}$. Further, let $\boldsymbol{E}_0$ denote the Fourier amplitude of the incident field and introduce the electric and magnetic polarizations, $\hat{\boldsymbol{p}}_\mathrm{e} = \boldsymbol{E}_0/|\boldsymbol{E}_0|$ and $\hat{\boldsymbol{p}}_\mathrm{m} = \hat{\boldsymbol{k}}\times\hat{\boldsymbol{p}}_\mathrm{e}$, respectively. The function $\varrho(k) = \hat{\boldsymbol{p}}_\mathrm{e}^*\cdot\mathrm{S}(k,\hat{\boldsymbol{k}})\cdot\hat{\boldsymbol{p}}_\mathrm{e}/k^2$ is then holomorphic in the upper half-plane $\mathrm{Im}\, k > 0$ for a large class of dyadics S_t. Cauchy's integral theorem applied to ϱ can be proved to yield

$$\varrho(\mathrm{i}\varepsilon) = \int_0^{\pi} \frac{\varrho(\mathrm{i}\varepsilon - \varepsilon\mathrm{e}^{\mathrm{i}\phi})}{2\pi}\ \mathrm{d}\phi + \int_0^{\pi} \frac{\varrho(\mathrm{i}\varepsilon + R\mathrm{e}^{\mathrm{i}\phi})}{2\pi}\ \mathrm{d}\phi + \int_{\varepsilon<|k|<R} \frac{\varrho(k+\mathrm{i}\varepsilon)}{2\pi\mathrm{i}k}\ \mathrm{d}k, \tag{1}$$

where k in the last integral is real-valued. Relation (1) is implicitly subject to the limits $\varepsilon\to 0$ and $R\to\infty$.

The left hand side of (1) and the integrand in the first integral on the right hand side are given by the electric and magnetic polarizability dyadics, $\boldsymbol{\gamma}_{\mathrm{e}}$ and $\boldsymbol{\gamma}_{\mathrm{m}}$, respectively, *viz.*,

$$\varrho(\varepsilon) = \frac{1}{4\pi}\left(\hat{\boldsymbol{p}}_{\mathrm{e}}^{*}\cdot\boldsymbol{\gamma}_{\mathrm{e}}\cdot\hat{\boldsymbol{p}}_{\mathrm{e}} + \hat{\boldsymbol{p}}_{\mathrm{m}}^{*}\cdot\boldsymbol{\gamma}_{\mathrm{m}}\cdot\hat{\boldsymbol{p}}_{\mathrm{m}}\right) + \mathcal{O}(\varepsilon) \quad \text{as } \varepsilon\to 0. \tag{2}$$

The second integral on the right hand side of (1) approaches zero in the limit $R\to\infty$ due to the extinction paradox $\varrho(k) = -A(\hat{\boldsymbol{k}})/(2\pi\mathrm{i}k) + \mathcal{O}(|k|^{-2})$ as $|k|\to\infty$, where A is a real-valued constant. Further, it is assumed that ϱ is sufficiently regular to extend the contour in the third integral to the real-axis. Under this assumption, the real part of (1) yields

$$\varrho(0) = \frac{1}{2}\varrho(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\operatorname{Im}\varrho(k)}{k}\,\mathrm{d}k = \frac{1}{2}\varrho(0) + \frac{1}{8\pi^2}\int_{-\infty}^{\infty}\frac{\sigma_{\mathrm{ext}}(k)}{k^2}\,\mathrm{d}k \tag{3}$$

where k is real-valued and the optical theorem $\sigma_{\mathrm{ext}}(k) = 4\pi k\operatorname{Im}\varrho(k)$ have been used. Recall that σ_{ext} is non-negative. Relation (2) inserted into (3) using $\lambda = 2\pi/k$ finally yields

$$\int_0^{\infty}\sigma_{\mathrm{ext}}(\lambda)\,\mathrm{d}\lambda = \pi^2\left(\hat{\boldsymbol{p}}_{\mathrm{e}}^{*}\cdot\boldsymbol{\gamma}_{\mathrm{e}}\cdot\hat{\boldsymbol{p}}_{\mathrm{e}} + \hat{\boldsymbol{p}}_{\mathrm{m}}^{*}\cdot\boldsymbol{\gamma}_{\mathrm{m}}\cdot\hat{\boldsymbol{p}}_{\mathrm{m}}\right). \tag{4}$$

Note that the integrated extinction (4) is independent of any temporal dispersion, depending only on the long wavelength limit properties of V.

AN ISOPERIMETRIC BOUND ON BROADBAND SCATTERING

Various isoperimetric bounds on the integrated extinction is discussed in Sohl et al.[2]. An example for isotropic material parameters, $\boldsymbol{\chi}_{\mathrm{e}} = \chi_{\mathrm{e}}\mathbf{I}$ and $\boldsymbol{\chi}_{\mathrm{m}} = \chi_{\mathrm{m}}\mathbf{I}$, is given by

$$\pi^2\int_V \frac{\chi_{\mathrm{e}}(\boldsymbol{x})}{\chi_{\mathrm{e}}(\boldsymbol{x})+1} + \frac{\chi_{\mathrm{m}}(\boldsymbol{x})}{\chi_{\mathrm{m}}(\boldsymbol{x})+1}\,\mathrm{dV}_{\boldsymbol{x}} \le \int_0^{\infty}\sigma_{\mathrm{ext}}(\lambda)\,\mathrm{d}\lambda \le \pi^2\int_V \chi_{\mathrm{e}}(\boldsymbol{x}) + \chi_{\mathrm{m}}(\boldsymbol{x})\,\mathrm{dV}_{\boldsymbol{x}}. \tag{5}$$

Since the right hand side of (5) is bounded from above by $\pi^2|V|\sup_{\boldsymbol{x}\in V}(\chi_{\mathrm{e}}(\boldsymbol{x})+\chi_{\mathrm{m}}(\boldsymbol{x}))$ it follows that the integrated extinction for any heterogeneous scatterer is at most equal to the integrated extinction for the corresponding homogeneous scatterer with maximal susceptibilities $\sup_{\boldsymbol{x}\in V}\chi_{\mathrm{e}}(\boldsymbol{x})$ and $\sup_{\boldsymbol{x}\in V}\chi_{\mathrm{m}}(\boldsymbol{x})$. This observation leads to the fascinating conclusion that there is no fundamental difference from a broadband point of view in scattering by heterogeneous and homogeneous obstacles.

HOMOGENEOUS ELLIPSOIDAL SCATTERERS

Closed-form expressions of $\boldsymbol{\gamma}_{\mathrm{e}}$ and $\boldsymbol{\gamma}_{\mathrm{m}}$ exist for the homogeneous ellipsoidal scatterers, *viz.*, $\boldsymbol{\gamma}_{\mathrm{e}} = |V|\boldsymbol{\chi}_{\mathrm{e}}\cdot(\mathbf{I}+\mathbf{L}\cdot\boldsymbol{\chi}_{\mathrm{e}})^{-1}$ and $\boldsymbol{\gamma}_{\mathrm{m}} = |V|\boldsymbol{\chi}_{\mathrm{m}}\cdot(\mathbf{I}+\mathbf{L}\cdot\boldsymbol{\chi}_{\mathrm{m}})^{-1}$, where $\mathbf{L}$ is the depolarizing dyadic. The eigenvalues of $\mathbf{L}$ are the depolarizing factors L_j. Closed-form expressions of L_j exist in the special case of the prolate and oblate spheroids. For example, the integrated extinction for the homogeneous sphere is equal to $3\pi^2|V|\sum_{i=\mathrm{e,m}}\chi_i/(\chi_i+3)$ independent of the electric and magnetic polarizations. In the long wavelength PEC limit ($\chi_{\mathrm{e}}\to\infty$ and $\chi_{\mathrm{m}}\to -1$) the integrated extinction approaches $3\pi^2|V|/2$ for the sphere.

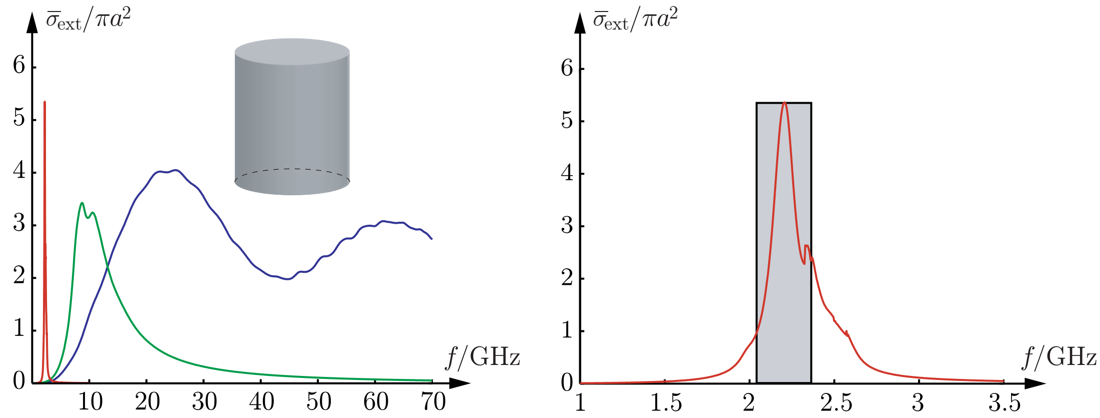

Figure 1: The averaged extinction cross section $\bar{\sigma}_{\text{ext}}$ in units of πa^2 as function of the frequency for a Lorentz dispersive cylinder with volume-equivalent sphere of radius $a = 1\,\text{cm}$. The right figure is a close-up of the $2\,\text{GHz}$ peak and illustrates the physical limitations presented in Sohl et al.[2].

The integrated extinction for the PEC needle of length $2a$ with semi-axis ratio ξ is

$$\int_0^\infty \sigma_{\text{ext}}(\lambda)\ \mathrm{d}\lambda = \frac{4\pi^3 a^3}{3}\frac{f(\theta)}{\ln 2/\xi - 1} + \mathcal{O}(\xi^2) \quad \text{as } \xi \to 0, \tag{6}$$

where θ denotes the angle of incidence and $f(\theta) = \sin^2\theta$ for the TM-polarization and $f(\theta) = 0$ for the TE-polarization. Since the extinction cross section is non-negative, (6) implies that σ_{ext} vanishes almost everywhere except on a set of measure zero consisting of the denumerable resonances for which an integer multiple of $\lambda/2$ approximately coincides with the length $2a$ of the needle.

A NUMERICAL EXAMPLE: THE LORENTZ DISPERSIVE CYLINDER

The extinction cross section averaged over $\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{p}}_{\text{e}}$ for a homogeneous isotropic Lorentz dispersive cylinder is depicted in Figure 1. The ratio of the cylinder length ℓ to its radius b is $\ell/b = 2$. The cylinder is non-magnetic with electric susceptibility given by the Lorentz dispersion model $\chi_{\text{e}}(\omega) = \omega_{\text{p}}^2/(\omega_0^2 - \omega^2 - \mathrm{i}\omega\nu)$. The Lorentz parameters ω_{p}, ω_0 and ν are chosen such that all the three curves in the left figure have the same long wavelength susceptibilities $\chi_{\text{e}}(0) = 1$ and hence the same averaged integrated extinctions, see Sohl et al.[2]. The first two curves correspond to Lorentz resonances $\omega_0/2\pi$ at $2\,\text{GHz}$ and $10\,\text{GHz}$, while the third curve is for a non-dispersive electric susceptibility. The boundary curve of the box in the right figure corresponds to an artificial scatterer with an averaged extinction cross section vanishing everywhere outside the box. The averaged integrated extinction for the boundary of the box and the three curves in the left hand side of Figure 1 coincide.

REFERENCES

[1] E. M. Purcell. "On the absorption and emission of light by interstellar grains". *J. Astrophys.*, vol. 158, pp. 433–440, 1969.

[2] C. Sohl, M. Gustafsson, and G. Kristensson. "Physical limitations on broadband scattering by heterogeneous obstacles". Submitted to Journal of Mathematical Physics.