# What is an Embedding? : A Problem for Category-theoretic Structuralism

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**Abstract**

This paper concerns the proper definition of embeddings in purely category-theoretical terms. It is argued that plain category theory cannot capture what, in the general case, constitutes an embedding of one structure in another. We discuss three available solutions to this problem: variants of monics, concrete categories, and allegories. The first and last of these are found to be unable to solve the problem, and the second to be philosophically unsatisfactory. Instead, we introduce a theory of *forms* and *relators*, which, like allegory theory, attempts to abstract from relation algebras in the way that categories abstract from monoids, but which does not have the shortcomings we have identified in allegories. We show that the theory in question does indeed solve the problem of defining embeddings.

## 1 Introduction

Category theory is often held up as furnishing a mathematical framework for structuralism. Thus Awodey, when answering Hellman's [13] question 'Does Category Theory Provide a Framework for Mathematical Structuralism?' gives the straightforward answer 'yes, obviously' [3]. And this seems quite right; in category theory, we are generally dealing with properties definable only up to isomorphism, as one would expect from a framework for structuralism. Furthermore, many important structural properties are definable entirely in category-theoretic terms.

But a proper linguistic framework for structuralism should ideally fulfill two criteria:

($i$) it should be able to characterize *all* important structural relationships, and

($ii$) it should make it impossible to even talk about of non-structural relationships.

Of these, the second is somewhat less important. Standard category theory, with its identity predicate defined on objects, does allow talk about non-structural relationships, and this may tend to invite pseudo-questions such as whether a natural numbers-object of one category is the same as a natural numbers-object of another. But we can generally see which statements are structural and which are not: for whole categories, a good rule of thumb is that if a property is preserved by category equivalences, then it is structural. For objects *in* a category, we have isomorphisms, which are definable from identities among arrows. For the arrows themselves, 1-dimensonal category theory treats identity as primitive, but higher-dimensional categories allow analyses of structural relationships between arrows as well. In the limit, we have $\infty$-categories, in which it may be possible to do without identity at all. We will, however, not delve into whether this is actually the case in this paper.

On the other hand, the first criterion is of prime importance. If there is an important structural relationship for which we need, say, set theory, to give a description of, then category theory is simply not rich enough to function as an autonomous structuralist framework. I will argue here that we have reason to think that this is indeed the case, and I will sketch a way to handle it. My general purpose is not to attack the program of using category theory as a language of structuralist mathematics, but only to point out a problem that has to be solved if this program is to be carried out.

## 2 Embeddings and Substructures

The structural relationship I will argue is problematic is that of the *embedding* of one structure inside another, or of being a *substructure* of another structure. Informally, we say that structure $A$ is embeddable in structure $B$ iff structure $A$ is isomorphic to a part of structure $B$. Now, isomorphism is beautifully handled by category theory, but, clearly, parthood is not. If there are two morphisms $f, g : A \to B$, there is no way for us to say, category-theoretically, whether it is through $f$ or $g$ that $A$ is *really* inserted in $B$. A morphism is a *way* to fit one object inside another, and the question of which way (if any) is actually instantiated is a non-strucural question.

Looking at it from a structuralist viewpoint, this limitation is thus quite reasonable. If an entity $e$ gets its identity-conditions determined through the relationships to all other entities in the structure, then talking about this entity as apart from that very structure does not make sense. To give an example, consider the natural number 2, defined as the second place in the number series. If any thing is *not* the successor of 1, then this thing is not the number 2. Thus, from this kind of structuralist point of view, it is not strictly correct to say that, e.g. the even numbers form a part of the natural numbers, because the even numbers themselves do not make sense without the odd ones. This is shown by the fact that there can be no 'even natural numbers-object' which is not isomorphic with the natural numbers-object, and thus, according to a popular structuralist interpretation according to which isomorphism implies identity, identical with it. But *proper parthood* is, by definition, irreflexive, so parthood cannot hold between $\mathbb{N}$ and the even numbers. This is rather similar to the standpoint of Frege [11, §23–24], who, against Mill, questions whether it makes sense to talk of abstract objects as standing in stand in parthood relations.

Thinking in set-theoretic terms, though, this conclusion may seem unwarranted. Isn't the *set of even numbers* a subset of $\mathbb{N}$, at least when we have identified $\mathbb{N}$ with some specific set, such as the initial segment of the von Neumann ordinals? Of course it is, but subsethood is not necessairly parthood. $A$'s being a subset of $B$ is dependent on our interpretation of the membership relation, which is given by the whole of set theory, as formulated in, say, ZFC. Thus, interpreting numbers as von Neumann ordinals gives rise not only to the dependence of the number 2 on the number 1, but also its dependence on the ordinal $\epsilon_0$. While subset relations are definable in ZFC, subsets (or subclasses) of a model of ZFC do not thereby constitute independent parts of it, since they cannot be specified without reference to a model of the whole of ZFC.

This points to the lesson that, at least on one interpretation, structuralist mathematics is *holistic* in a very radical sense. The things *having* a structure may very well stand in parthood relations, and the structures these things have determine which things *can be* parts of which through the condition that *a can be* a part of $b$ iff the structure of $a$ is embeddable in the structure of $b$. But the structures themselves should not be said to have parts. As we mentioned, this can be seen just from the fact that if $a$ is a part of $b$, then $a$ is included *in a specific way* in $b$, but there are generally many ways to include one structure in another.

The concept of *substructure* is intimately tied to that of embedding. As we have seen, talking about parts of a structure is, strictly speaking, senseless. But we can still introduce a notion of substructure, which is quite compatible with this viewpoint: we say that a *substructure of $A$* is a structure $B$ for which there is an embedding $m : B \to A$. Perhaps "subobject" would have been a better word, but we will reserve this notion for the standard category-theoretic definition.

Dual to the embedding concept is that of *quotient structure*: informally, a structure in which parts of another structure have been identified. A classic example comes from model theory, where the ultraproduct construction proceeds by taking the quotient of a structure. Because of the dualities of category theory, much of what we say about embeddings will be applicable to quotients as well. We will, however, usually not say how the discussion transfers explicitly.

For our framework, we want both the notion of substructure and that of embedding to designate constructions in the language of category theory. A straightforward intuitive interpretation is the following: take a *structure species*[1] to be a category $\mathcal{C}$, and take the objects of this category to represent *structures of species $\mathcal{C}$*. There are two questions we need to separate here: the first is when one species of structure is embeddable in another, and the second when a

[1] The name comes from Bourbaki [6, ch. IV.4], who use it to denote what is basically a *set* of structures.

structure of one species is embeddable in another. If we take structure species to be categories, the first question is easily answered through the use of functors. For example, the one-object category $\mathbb{N}$, with arrows as natural numbers and composition as addition, has a faithful functor to the one-object category $\mathbb{Z}$, obtained by adding inverses to the arrows in $\mathbb{N}$. Such a functor is very reasonable to take as an embedding.

On the other hand, the second problem is far more difficult: the categorical language treats objects as featureless points, whose properties are given only by their role as domains or codomains for morphisms. The problem is the following: taking the morphisms of $\mathcal{C}$ to represent *structure-preserving mappings* between the objects, we want to identify a subclass $\mathsf{emb}$ of these, and for each object $A$ a class of objects $\mathsf{sub}(A)$, such that

$(i)$ $B \in \mathsf{sub}(A)$ iff there is a (not necessarily unique) $m : B \to A$ such that $m \in \mathsf{emb}$,

$(ii)$ $B \in \mathsf{sub}(A)$ iff $B$ is a substructure of $A$, and

$(iii)$ $m \in \mathsf{emb}$ iff $m$ is an embedding.

The classes, moreover, are to be identified purely through equations among arrows, and their domains and codomains. Furthermore, this identification is to be applicable to all, or at least a large part, of the categories that contain mathematical structures as objects and have structure-preserving maps as morphisms.

The three conditions are just elucidations of what we want, and do not at all tell us which classes $\mathsf{emb}$ and $\mathsf{sub}(A)$ are. What they mean *internally* to the objects in question depends on which category we are considering, and indeed it seems hard to say something that is both general and exact about them. Some attempts are the following:

- An embedding is a map $m : A \to B$ such that the image of $A$ under $m$ is isomorphic to $A$.
- $B$ is a substructure of $A$ iff any object with the structure $A$ has a part with the structure $B$.
- $m$ is an embedding iff, whenever it factors as $g \circ f$, $f$'s being surjective is sufficient for $f$ to be an isomorphism.

All of these, however, also depend on other concepts, which themselves seem no easier to define. In view of these problems, it may be more informative, in practice, to say something like this: *the embeddings and subobjects of a mathematical structure are those that are introduced as embeddings and subobjects in a typical textbook on that type of structure.* Below are a few examples, which will also be useful later on.

- In **Set**, objects are sets and morphisms are functions. Isomorphism is one-to-one correspondence. Embeddings are injective functions. Substructures are subsets.

- In **Ord**, objects are preordered sets and morphisms are monotone functions. Isomorphism is order isomorphism, and embeddings are injective functions $m : A \to B$ such that $m(x) \leqslant m(y) \Leftrightarrow x \leqslant y$. The substructures of $A$ are those subsets of $A$ for which the inclusion function is an embedding.

- In **Top**, objects are topological spaces and morphisms are continuous functions. Isomorphism is homeomorphism. Embeddings are homeomorphisms onto subspaces, with the subspace topology. Substructures are subspaces, again with the subspace topology.

- In **Mod$\boldsymbol{\Sigma}$**, objects are models of a first-order language with signature $\boldsymbol{\Sigma}$ and morphisms are model homomorphisms. Isomorphism is model isomorphism. Embeddings are model embeddings, and substructures are submodels.

It is, of course, easy to give many more examples. What is worth pointing out is that, just as each structure, when introduced in mathematics, is given with its associated structure-preserving mappings, it is also usually given with its notions of isomorphism, embedding, and substructure. Isomorphism, as we have mentioned, is definable from structure-preserving mapping using category theory. Does the same hold for embedding and substructure? I will argue that it does not, and that this means that, if category theory is to serve as an autonomous framework for structuralism, we need to either expand it or to otherwise somehow modify it.

## 3 The Many Shapes of Monomorphism

To a casual reader of the category-theoretic literature, it may seem like category theory *does* contain versions of substructure or embedding, as embodied in the notion of *subobject*. Recall that a subobject of an object $A$ is an equivalence class of monomorphisms $m : X \to A$, with $m_1$ and $m_2$ deemed equivalent iff there is an isomorphism $f : \mathsf{dom}\, m_1 \to \mathsf{dom}\, m_2$ such that $m_2 = m_1 \circ f$. If this does the work of a subset relation, could we not say that an embedding is an isomorphism to a subobject, or perhaps to the domain of some morphism in a subobject? A quick glance at the definition of the equivalence relation in question shows that if any morphism $m$ fulfills this requirement, it is itself a subobject of $A$. Therefore, all that is necessary for $m$ to be an embedding into $A$ in this sense is that it should be a monomorphism with codomain $A$. A substructure, in our sense, would then be the domain of a subobject. Is this a reasonable condition for $m$ to be an embedding?

The answer is, unfortunately, no. Every embedding should be monic: assume that $m : A \to B$ is an embedding, and that $f, g : C \to A$ are different morphisms. Being different, we would expect the result of applying them to $C$ to be different, but $m$, being an embedding, should then insert this result *just as it is* into $B$, and therefore we cannot have that $m \circ f = m \circ g$. But while this is true, many monics are not embeddings. In **Set** they are, as being a monomorphism is sufficient for being an injective function. But the following are some of the exceptions:

- In **Ord**, monomorphisms are injective monotone functions, but not embeddings. Consider the preordered set $A = \{a, b\}$ with the order $\{(a, a),$ $(a, b),$ $(b, b)\}$, and the preordered set $B = \{a', b', c'\}$, with the maximal order in which every element is equivalent. The map $f$ from $A$ to $B$ that takes $a$ to $a'$ and $b$ to $b'$ is a monic, but it is not an embedding.

- In **Top**, monomorphisms are injective continuous functions, and not embeddings (which have to have an injective inverse). Take, for instance, the identity function from a set $A$ with the discrete topology to itself with the indiscrete topology. So long as $|A| > 2$, $f$ will not have a continuous inverse, even if it is a monomorphism.

- In $\mathbf{Mod\Sigma}$, monomorphisms are injective model homomorphisms, but not embeddings. We can have that $f(x)Rf(y)$ without $xRy$ even if $f$ is a monomorphism, but an embedding requires that the same relations hold in the image as those that hold in the domain. As it is sometimes put, a monomorphism *preserves* relations, but does not have to *reflect* them.

To find out what has to gone wrong, it is useful to make a number of informal philosophical distinctions. We say that a mapping $f$ between two structures $A$ and $B$ *reflects identity* if the number of "elements" in the image of $B$ are the same as the number of elements of $A$, i.e. if, figuratively, for each $a, b$ in the image of $f$, $f(a) = f(b)$ implies that $a = b$. We say that it *preserves structure* if, for any $a, b$ in $A$, if $a$ stands in a certain relationship with $b$, then $f(a)$ stands in the same relationship with $f(b)$ in $B$. For this to make sense, $A$ and $B$ must, of course, be structures of the same type, so that it makes sense to talk about 'the same relationship'. Finally, we say that $f$ *reflects* structure if the converse of structure-preservation holds.

We have not introduced the notion of *preserving identity*, since that is supposed to hold for all functions or mappings. Since a morphism is meant to be a structure-preserving mapping, we will furthermore take every morphism to preserve structure. Indeed, it is reasonable to say that what morphisms there are in a category determines what 'structure-preserving' means, and thus also partly determines the meaning of 'structure'. Thus, set structure is structure preserved by functions, algebraic structure is structure preserved by homomorphisms, topological structure is structure preserved by continuous mappings, etc.

Monics can often, reasonably, be said to reflect identity, i.e they do not usually collapse the number of things in a structure.[2] But they do *not*, however,

[2] This does not, actually, hold for quite all categories. In a construct, i.e. a category which is concrete over **Set**, all mappings with injective underlying functions are monics, but even then the converse does not hold unless the category contains a free object [1, p. 143].

have to reflect structure. As in our example with preordered sets, relations may hold in the image of a monic which did not hold in the preimage.

The natural reaction is to search for some stronger notion of monic, which gives us a necessary *and* sufficient criterion for a morphism being an embedding. And there is a large number of such strengthenings available: in order of increasing strength, we have *extremal*, *strong*, *swell*, *strict*, and *regular* monics, and at the top, the *section*, or *split monic*, which is a morphism $f : A \to B$ with has a left inverse, i.e. for which there exists a morphism $g : B \to A$ such that $g \circ f = 1_A$.

In **Set** the question is moot. Here, any monomormism from a non-empty set is also a section, so all the variants more or less collapse into one. In categories similar to **Set**, such as pretoposes, monics coincide with every variant up to the regular one. But requiring every embedding to be a regular monic is too much. In **Haus**, the category of Hausdorff spaces, only embeddings of *closed* subspaces are regular monics, and in **Rng**, the category of rings, some embeddings (such as that of $\mathbb{Z}$ in $\mathbb{Q}$) are non-regular [1, p. 116].

So we had better look for something weaker. Taking a look at the other end of the spectrum, let us consider the *extremal* monics. A monomorphism $m$ is called extremal iff, whenever $m$ factors as $m = f \circ e$ with $e$ being an epimorphism, $e$ is an isomorphism. This is, in many cases, a rather sensible expression of what it means to be an embedding. Consider the category **Ord**, and any monomorphic non-embedding $f : A \to B$ where $B$ has all elements equivalent, but $A$ contains at least two elements $a, b$ such that $a \nleqslant b$. Suppose that $f$ factors as $g \circ e$, with $g$ being a simple insertion into $B$, i.e. an identity function on $f[A]$. Then $e$ is epic, but it cannot be an isormorphism, because that would have meant that $a \leqslant b$ would have held, and thus $f$ is not extremal.

Conversely, let $m : A \to B$ be an order embedding, factorized as $f \circ e$ with $e$ epic. Since $e$ is the first factor of a monic, it has to be monic as well, so set-theoretically it has an inverse. Now assume that $e(a) \leqslant e(b)$ but $a \nleqslant b$. Because $f$ is a morphism, i.e. monotone, we must then have that $f(e(a)) \leqslant f(e(b))$. But this contradicts the facct that $m$ is an embedding, so not only is $e^{-1}$ a well-defined function, but an isomorphism as well.

There are three main problems with taking embeddings to be exactly the extremal monomorphisms. The first concerns their logical properties, the second that they in some cases are too weak, and the third that they are sometimes too strong.

As for the logical problem, it concerns the fact that it is quite possible for the composition of two extremal monics to not itself be extremal. This is different from embeddings, since the composition of two embeddings is always an embedding itself. This may be handled by explicitly requiring transitivity, e.g. by letting an embedding be an extremal monic $m : A \to B$ such that, for any extremal monics $f, g$ such that $\mathsf{cod}\, f = A$ and $\mathsf{dom}\, g = B$, both $m \circ f$ and $g \circ m$ are extremal. One could also use the next strengthening of monics—*strong* monics—instead, as these are closed under composition.

In addition to reinstating closure under composition, taking strong rather than extremal monics to be embeddings may be thought to somewhat lessen the force of the second problem, since it consists in extemal monics sometimes being too weak to be able to capture the notion of embedding. But consider the full subcategory of **Top** which consists of only the set $\{a, b\}$ with the discrete topology, and the set $\{a, b, c\}$ with the indiscrete topology. The insertion of $A$ into $B$ is continuous, and thus a morphism, and it is quickly checked to be a strong monic, but it is still not an embedding in the topological sense.

However, the largest problem is not that of weakness. The known examples of categories in which extremal or strong monics are weaker than embeddings often quite unnatural or, like the one we gave, arbitrary limitations of other categories (cf. [1, p. 135]). Much worse is the fact that in several important categories, embeddings are strictly weaker than extremal monics, and ipso facto also strictly weaker than the isomorphically extremal version.

For an example, consider the category of rings, and the insertion $h$ of $\mathbb{N}$ into $\mathbb{Z}$. This is a classic example of a monomorphic epimorphism that is still not an isomorphism, and it is also an embedding. But take the factorisation of $h$ as $1_{\mathbb{Z}} \circ h$; this gives a diagram of the type used in the definition of extremal monics, but $h$ is not an isomorphism. More generally, such a problem appears in any category which is not *balanced*, i.e. in which an arrow can be monomorphic and epimorphic without being an isomorphism. Conversely, in a balanced category, every monomorphism is extremal, and since monos are closed under composition, so are the extremal monos in such a category.

We could thus try to bite the bullet and say that the notion of embedding strictly only applies in balanced categories, and that in such a category, the monomorphisms are exactly the embeddings. But this is not an attractive limitation, since we would exclude many important categories, such as that of rings, that of Hausdorff spaces, and also the categories **Ord**, **Top** and **Mod$\boldsymbol{\Sigma}$** we described above. Cannot the objects of these categories be embedded in one another?

It is reasonable to object that the notion of embedding has been used informally for a long time, and perhaps we should not expect to find a proper definition exactly coextensive with the class of everything that has been called embeddings. Perhaps (unlike the case of isomorphism) there is no *single* notion of embedding, but a Wittgensteinian family of them? If that is the case, maybe we should settle on one or more *explications*, in the Carnapian sense?

As it turns out such a pessimistic reaction is premature. There *is* a uniform way to treat embeddings by using category theory, although it departs slightly from our requirement of only relying on equations among arrows. To this method wenow turn.

## 4 Substance and Structure

We have discussed the concepts of preservation and reflection of ontology and structure, and noted that a morphism is a structure-preserving, identity-preserving map, a monomorphism is often also identity-reflecting, and an embedding should be all four. What we should not require is that an embedding should cover all of its codomain; that is what isomorphisms are for. The difficult part is to strengthen the monomorphism concept while avoiding ending up in isomorphism.

There is, however, a known way to *expand* category theory in order to go around these problems, which is embodied in the concept of a *concrete* category. Formally, we say that a concrete category is a pair $\mathcal{C}, F$, where $\mathcal{C}$ is a category and $F : \mathcal{C} \to \mathcal{B}$ is a failthful functor to a category we refer to as the *base* category. Historically, the most well-studied concrete categories have been those in which $F$'s codomain is **Set**. Such conrete categories are called *constructs*.[3]

In a concrete category, the base category $\mathcal{B}$ can as the *ontology*, i.e. a description of what the objects in the category and their transformations are. The category $\mathcal{C}$ itself determines a structure *relative* to this ontology. This determination consists in singling out which objects in the ontology have the kind of structure that $\mathcal{C}$ describes, and which of the transformations described in $\mathcal{B}$ preserve this kind of structure.

It is important to note that this division into structure and ontology is not absolute. The category **TopGrp** of topological groups can be considered as concrete over **Grp**, over **Top**, or over **Set**. A preordered set can be taken to be concrete over **Set** or over the trivial category **1**, but not over both (since there is no category that both **1** and **Set** are concrete over). We can also have that two categories, e.g. **Cat** and **Set** can be taken to be concrete over one another, without being equivalent. In such a case, it is a purely conventional matter which of them we consider to be ontology, and which we take to be structure.

Given the notion of a concrete category, there is indeed a working concrete-categorical definition of embedding. In a concrete category $\mathcal{C}, F$, an *initial morphism* $f : A \to B$ is morphism such that, for each $\mathcal{C}$-object $C$ and each $\mathcal{B}$-morphism $g : F(C) \to F(A)$ is in the image of $F$ whenever $F(f) \circ g$ is.

$$\begin{tikzcd}
C \arrow[d, dotted, "F^{-1}(g)"'] \arrow[dr, "h"] & & & F(C) \arrow[d, "g"'] \arrow[dr, "F(f)\circ g=F(h)"] & \\
A \arrow[r, "f"'] & B \arrow[r, phantom, "\overset{\Longrightarrow}{\scriptstyle F}"] & \; & F(A) \arrow[r, "F(f)"'] & F(B)
\end{tikzcd}$$

That $F$ is faithful guarantees that $F^{-1}(g)$ is unique if it exists. A *concrete embedding* (usually just called an *embedding*) is an initial morphism $m$ such that $F(m)$ is monomorphic in $\mathcal{B}$. Since faithful functors reflect monomorphisms, this also entails that $m$ itself must be mono.

A quick informal explanation of why initial morphisms with monic underlying functions are reasonable to take to be embeddings might go as follows, taking $\mathcal{B}$ to be **Set**. Suppose that there is a $\mathcal{B}$-morphism $g$ such that the right-hand side diagram commutes. Then the image of $F(f) \circ g = F(h)$ must lie inside

[3]It is actually common to use the expression 'concrete category' for what we have called constructs, and disregard the possibility to use other categories as bases. However, this seems to reflect nothing more than a common prejudice in favor of set-theoretic ontology, and the general case deserves far more attention than it has received so far. Our usage follows that of [1].

the image of $F(f)$, and since $F(f)$ is injective ($\mathcal{B}$-monic), there has to be an inverse $\mathcal{B}$-morphism $i : B' \to F(A)$, where $B'$ is the subset of $B$ on which $F(f)$ is onto. Composing, we get that $g = i \circ F(h)$. For $f$ to be structure-reflecting, we need $g$ to be a morphism, and this is what the existence of $F^{-1}(g)$ asserts.

The important property of concrete embeddings is that, in constructs, they correspond in more or less all cases precisely to what the embeddings of these structures are usually taken to be, which is a much better result than can be obtained for abstract categories by just using different kinds of monomorphism. As put by [1, p. 114]:

> Orgininally it was believed that monomorphisms would constitute the correct catgorical abstraction of the notion "embeddings of substructures" that exists in various constructs. However, in many instances the concept of monomorphism is too weak; e.g. in **Top** monomorphisms are just injective continuous maps and thus need not be embeddings. Below we introduce two stronger notions (and there are several others in current use [...]) that more frequently correspond with embeddings in categories. However, a satisfactory concept of "embeddings" seems to be possible only in the setting of constructs [...]

The same also holds for many other forms of concrete category, such as **TopGrp**, which we mentioned above. By imposing the functor $F$ to a base category $\mathcal{B}$, we achieve a universal explanation of what an embedding is. Initial morphisms are, fundamentally, a concept that was introduced by Bourbaki in their theory of structure [6]. They were inspired by the topological concept of *initial topology*, and it may of course be just a lucky guess that this should prove to be the right way to generalize the notion of embedding. But perhaps there is some philosophical reason as well?

Bourbaki's theory of structure was indeed very much a type of concrete category theory, with certain sets as objects, and certain classes of functions as morphisms. As such, it is a *two-part* description: what is structured (the sets) is different from what the structure is like, which is determined by which classes of functions are to be counted as morphisms. As an Aristotelian might have said, we have a *substance* (the base category) which is imbued with a *form* (the category $\mathcal{C}$).

Plain category theory, however, makes no such separation between form and substance, between structure and ontology. If the arrows are structure-preserving mappings, there is no way to define a *non*-structure-preserving mapping, for instance. If there is no epimorphism from $A$ to $B$, we cannot know if this is because $B$ is simply "bigger" than $A$, or because $B$'s structure is incompatible with $A$'s.

It is well known that 'monomorphic' does not correspond exactly to 'injective', and that 'epimorphic' is not the same as 'surjective'. But by giving a concrete interpretation of a category in an underlying category where this *does* hold (such as **Set**), we can go around this limitation. This also indicates that, when it comes to concrete embeddings, we should get the right concept whenever $\mathcal{U}$ shares these interpretative properties.

Why would a distinction between structure and ontology necessary in order to describe purely structural relationships, such as embedding or substructure? I do not have a definite answer, other than that without such a distinction, these properties appear hard to define. One thing this *could* indicate is that structure cannot logically stand on its own: we always need to make reference to something which is structured. This may at first remind one of the 'no relations without relata' argument against structuralism (cf. [9, 16]). However, our point is quite different. It is not based on intuitions about the concept of *relation*, but on the suggested difficulty of giving a comprehensive mathematical theory of structure that does not make the distinction. It can be refuted by showing that this is indeed possible, and this is partly a mathematical question, rather than a philosophical one. Indeed, we will attempt to show how it may be solved in the latter half of this paper.

One might also ask if the concrete version still should be counted as a *category-theoretical* characterisation of embeddings. To some part, of course, this is a question of terminology. The notions that are involved (category, functor) are certainly category-theoretical. However, we are no longer talking about objects in **Cat** (or any other general category of categories). Even if $m$ is an embedding in the concrete category $\mathcal{C}, F$, and $\mathcal{D}, G$ is a concrete category which is equivalent to $\mathcal{C}, F$ under the functor $H$, we do not need to have that $H(m)$ is an embedding in $\mathcal{D}, G$. For that, we need a stronger notion of functor (a *concrete* functor) to serve as arrows. So a concrete category is not just a category.

It is also the case that even if we accept the concrete embedding as a category-theoretical solution, it is a thoroughly *unsatisfactory* solution. Picture a mathematician wanting to describe what a subgroup is, and coming up with a definition like '$\mathfrak{H}$ is a subgroup of $\mathfrak{G}$ iff the linear transformations representing $\mathfrak{H}$ in the vector space $V$ make up a subset of those representing $\mathfrak{G}$.' The definition of a construction in a kind of structure should, ideally, use only concepts intrinsic to the structure itself.

# 5 Going Relational

That the ontology–structure split may be necessary does not mean that it has to be treated just as explicitly as in concrete category theory.[4] What it seems like we would need to say is that an embedding is a morphism that *has* an inverse, but that the inverse in question does not have to be defined on the whole codomain. Category theory is a generalization of the concept of mapping between mathematical structures, and such mappings are always expected to

[4]Another "explicit" fix would be to add embeddings or subobjects as primitives, for instance through a functor $\mathcal{C} \to \mathbf{Set}$ that gives the hom-set of embeddings into each object of $\mathcal{C}$. A very well-developed explicit fix is the theory of *model categories* of Quillen, which relies on an explicit specification of classes of *fibrations*, *cofibrations*, and *weak equivalences*. The cofibrations in a model category are often, but not always, interpretable as embeddings.

be defined on the whole of their domain. So it may not be very strange that category theory, as it is, cannot handle embeddings.[5]

To get an overview of the available obtions, let us expand our earlier taxonomy of function types. We mentioned structure preservation and structure reflection, as well as identity reflection. But if we consider the the most general link that can hold between two sets—a relation—then there is, of course also identity preservation, i.e. that if $x = x'$, $xRy$, and $x'Ry'$, then $y = y'$. It is just that, as we mentioned before, this is part of the defining characteristics of what it is to be a function.

Somewhat graphically picturing the domain of a binary relation as *left* and the codomain as *right*, relations in general can thus have structure preservation and identity preservation in both left-to-right and right-to-left vairants. But there are also other properties of interest, which become more important as we expand our view from functions to relations. For one thing, we have *left* and *right totality*: the conditions that for all $x \in \mathsf{dom}\, R$ (respectively all $x \in \mathsf{cod}\, R$), there is some $y \in \mathsf{cod}\, R$ (respecively some $y \in \mathsf{dom}\, R$) such that $xRy$. All mappings are left-total, and right-totality is the property usually referred to as surjectivity.

There is also a more general type of property, which I think will be useful to identify. We say, informally, that $R$ is *left substantial* iff the preimage of $R$ makes up a self-subsistent object, and *right substantial* if its image does. Now, what does 'make up a self-subsistent object' mean here? Roughly, that the preimage or image is itself a subobject of the domain or codomain. Because any subset of a set is a set, all relations between sets should be both left and right substantial.

The same does not hold for richer structures. Consider a class of groups, with relations between them. According to the intended interpretation, a relation between groups $\mathfrak{g}$ and $\mathfrak{h}$ is left substantial iff its preimage is a subgroup of $\mathfrak{g}$, and right substantial iff its image is a subgroup of $\mathfrak{h}$. Or, consider the functor $F : \mathcal{C} \to \mathcal{D}$ below:

$$
\begin{tikzcd}
A \arrow[d, "f"'] & & & & F(A) \arrow[d, "F(f)"] \\
B & B' \arrow[d, "g"'] & & {} \arrow[r, Rightarrow, "F"'] & F(B) = F(B') \arrow[d, "F(g)"] \\
& {} & C & & F(C)
\end{tikzcd}
$$

As a functor, $F$ is of course left total and thereby also left substantial. But it is not right substantial, because the arrows in the image of $F$ are not closed under composition, so its image is not a category. As this example shows, not everything that is traditionally seen as a mapping is right-substantial.

[5] To be sure, there are notions of partial function available to a category theorist. However, these are special versions of category-theoretic relations, and, as such, share the problems of these which we identify below.

Since set-theoretic functions are defined on whole sets, they are left substantial, and since the image of any function is a set, they are also right substantial. The case is somewhat more complex for functional predicates in formal set theory; these are reasonably interpretable as neither left nor right substantial, since they are defined on the whole universe, and can take values in the whole universe. On the other hand, the axiom of replacement guarantees that *if* a functional predicate $F$ is left-substantial, then it is also right-substantial. This axiom can then be seen as a kind of left-to-right preservation of substantiality.

Generalizing this example, many of the axioms of various set theories can be seen as conditions on what relations are substantial. The axiom of separation, for instance, can be expressed as the condition that any symmetric and transitive functional relation $F$ on a set $A$ is left- (or right-) substantial: from the conditions of symmetry and transitivity it follows that if $F(a,b)$, then $F(a,a)$, and so by functionality, $a = b$, so $F$ is a subrelation of the identity relation on $A$. Saying that such an identity relation is substantial is, according to our interpretation, the same as saying that its image (or preimage) is a set.

We can also apply the same reasoning to other types of structure. Let us call a (left- or right-)substantial subrelation of the identity relation a *subidentity*. Subidentities of groups correspond one-to-one to subgroups, and likewise for categories, vector spaces, and any other kind of mathematical structure. They are therefore ideally suited to play the role of substructure, and thus also of determining what an embedding might be. The problem with category theory is that the only left-substantial relations that it straightforwardly allows are those that are total, so we would need to consider some other kind of structure; preferably one that abstracts from the notion of relation rather than from that of function.

Category theory itself *does* provide us with a notion of relation [12, p. 38]. We say that $m_1 : A \to B, m_2 : A \to C$ are *joint monics* if, for any two morphisms $f, g : X \to A$ such that $m_1 \circ f = m_1 \circ g$ and $m_2 \circ f = m_2 \circ g$, we have that $f = g$. A *binary relation* on the objects $A, B$, category-theoretically, is an isomorphism class of objects $R$ together with joint monics $m_{\mathsf{dom}} : R \to A$, $m_{\mathsf{cod}} : R \to B$:

$$\begin{tikzcd} & R \arrow[dl, "m_{\mathsf{dom}}"'] \arrow[dr, "m_{\mathsf{cod}}"] & \\ A & & B \end{tikzcd}$$

In a category with products, we can also (and equivalently) say that a relation between $A$ and $B$ is a subobject of of the product $A \times B$. The characterization through subobjects may, however, make us somewhat skeptical. Indeed, the relation construct given here is a straight generalization of that of subobject, and it faces the same difficulties. Consider the topological spaces $\mathcal{R}$, $\mathcal{A}$ and $\mathcal{B}$, with two continuous injective maps $f : \mathcal{R} \to \mathcal{A}$ and $g : \mathcal{R} \to \mathcal{A}$. This does induce a relation on the underlying sets $A, B$, definable by the equivalence that $R(a,b)$ iff there is an $r \in R$ such that $f(r) = a$ and $g(r) = b$. But such a relation pays no respect at all to the *topologies* in question. For *any* two topological spaces $\mathcal{A}$

and $\mathcal{B}$, and any subsets of their underlying sets $A$ and $B$ of the same cardinality, we can find a relation in this sense which is total on $A$ and $B$. Just take $\mathcal{R}$ to have some set of $A$'s and $B$'s cardinality as underlying set, and the discrete topology. If we take homeomorphism to be defined in terms of a one-to-one relations, the result is that all topological spaces with underlying sets of equal cardinality come out as homeomorphic.

Even if the notion of monic as subobject only captures identity reflection, and not structure reflection, it at least captures structure preservation, since a monic is a morphism. In contrast, the categorical definition of relation does not even capture structure preservation: from the structure on the image and the existence of a relation to $B$, we can say nothing at all about the structure of the preimage. Perhaps all structure-preserving or structure-reflecting relations should be relations in the categorical sense, but not every categorical relation is structure-preserving or structure-reflecting.[6]

A way to treat relations primitively, while still being compatible with standard category theory, is Freyd's [12] concept of *allegory.* Just as a category can be seen as a generalization of a monoid of functions on a set, an allegory can be viewed as a generalization of a relation algebra. In an allegory, the objects are domains and codomains of relations, and the morphisms are the relations. Formally, such an allegory is a category $\mathcal{A}$ with two operations:

- a unary operation $^\circ$ on morphisms called *reciprocation*, which gives the converse of any relation, and
- a binary partial operation $\cap$ on morphisms called *intersection*, defined for any two morphisms with the same domains and codomains.

*Composition* of relations is just the regular composition of morphisms in category theory. Freyd gives a set of axioms for the behavior of $^\circ$ and $\cap$, which imply, among other things, that reciprocation is an involution, and that $\cap$ gives rise to a modular lattice through the order $R \leqslant S \equiv_{df.} R = S \cap R$. The list of axioms is, however, rather long, and we will not study it in detail here.

The category **Rel** of sets and set-theoretical relations, with $^\circ$ interpreted as relation converse, and $\cap$ as intersection, is an allegory. So, as Freyd shows, is the category of categorical relations of any *regular*[7] category $\mathcal{C}$, where the composition $RS$ of the relations $(R, f_{\mathsf{dom}}, f_{\mathsf{cod}}), (S, g_{\mathsf{dom}}, g_{\mathsf{cod}})$ is defined as in the diagram

[6] Actually, even the first condition is questionable: why, when describing a relation between $A$ and $B$, does the relation itself have to be an object (such as $R$)? We do not require this for morphisms. It holds when the category has exponentials, but categories with exponentials are a special case.

[7] A regular category is a category in which ($i$) every arrow is an equalizer (i.e. a regular monic), ($ii$) every two arrows that have an equalizer also have a coequalizer, and ($iii$) pullbacks of regular epics exist, and are themselves epic [5, p. 90]. Being a regular category is strictly weaker than being an abelian category, but many categories that interest us still do not fulfill this criterion. Some examples are **Top**, **ModΣ**, and **Cat**. This should not surprise us, since the problems we are concerned with appear mainly in categories in which some monomorphisms are not regular.

$$\begin{tikzcd}
& & RS \arrow[dl, "h_{\mathsf{dom}}"'] \arrow[dr, "h_{\mathsf{cod}}"] & & \\
& R \arrow[dl, "f_{\mathsf{dom}}"'] \arrow[dr, "f_{\mathsf{cod}}"] & & S \arrow[dl, "g_{\mathsf{dom}}"'] \arrow[dr, "g_{\mathsf{cod}}"] & \\
A & & B & & C
\end{tikzcd}$$

with all pairs of shown morphisms from the same objects being jointy monic.

It is certainly the case that allegories give us a nice generalization of relation algebras, and the fact that each allegory *is* a kind of category means that the powerful tools of category theory are applicable in allegories as well. But a category is not *just* a generalization of a monoid of endofunctions, but rather of a monoid of *structure-preserving* endofunctions. Likewise, we want our theory of relations to not presuppose that every set-theoretically definable relation preserves or reflects the kind of structure we are after.

The inability of allegory theory to do this rests on its requirement that the converse of a morphism should always itself be a morphism. This does not hold even for constructs in which the morphisms are one-to-one mappings between the underlying sets. As we have noted several times, the fact that a continuous function has an inverse does not guarantee that this inverse is itself continuous. Just as the usual categorical notion of subobject only works for categories close to **Set**, the allegory concept only works properly for categories very close to **Rel**.

The same holds for the (at least) two other versions of a categorical theory of relations that have been proposed: *bicategories of relations* [8] and *1-categories equipped with relations* [7]. Both of these attempt to capture an unrestricted calculus of relations which allows any relation to have a converse. As such, they have also turned out to be equivalent to the theory of allegories [17]. But the problem with allegories as well as these, in a nutshell, is that they are too liberal: a morphism in an allegory can be properly taken to involve neither of the 8 conditions we mentioned. What we need is a concept that still implies left-to-right structure preservation, and possibly also left or right substantiality. To such a concept we now turn.

# 6 A Quick Sketch of Relator Theory

From a philosophical point of view, allegory theory may also seem to be somewhat lacking as a foundation for structuralism. The concept of allegory depends on that of function, and defines relation, but relations are logically prior to functions. This follows from the fact that we can have predicate logics without function symbols, but there is no such thing as a functional logic without relations: even simple term calculi tend to, at least, need an equality relation in order to be useful. Those that do not still have the relation of a term being of type, and this includes untyped lambda calculi as well, since these are just typed lambda calculi with a single type (cf. [14, pp.146–157]). Furthermore, given relations, all we need is an identity predicate in order to express functions as well. In contrast, defining relations given functions requires the addition of something like a subobject classifier, with all its accompanying structure.

The upshot is that instead of, as allegory theory does, considering a kind of category, we should consider structures of which categories are a kind. Here I will sketch a theory of such structures. As a preliminary, let us generalize the category concept. We say that the disjoint collections $\mathsf{obj}$, $\mathsf{hom}$ together with the mappings $\mathsf{dom} : \mathsf{hom} \to \mathsf{obj}$, $\mathsf{cod} : \mathsf{hom} \to \mathsf{obj}$, $\mathsf{id}\colon \mathsf{obj} \to \mathsf{hom}$ and the partial operation $\circ : \mathsf{hom} \times \mathsf{hom} \to \mathsf{hom}$ form a *categoroid* iff

($i$) $\mathsf{dom}(\mathsf{id}\ (a)) = \mathsf{cod}(\mathsf{id}\ (a)) = a$, and $\mathsf{id}\ (a) \circ f = f$ for all $f$ such that $\mathsf{cod}(f) = a$, and $f \circ \mathsf{id}\ (a) = f$ for all $f$ such that $\mathsf{dom}(f) = a$,

($ii$) if $g \circ f$ is defined, then $\mathsf{cod}\, f = \mathsf{dom}(g)$, and

($iii$) if both $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are defined, they are equal.

A categoroid is thus exactly like a category, except that composition does not have to be defined even when arrows share domains and codomains. The reason for this is that we want to be able to differentiate left substantial arrows from non-substantial ones, and substantiality is not in general conserved in composition. Take, for instance, the following graphical example of a categoroid of *discs in* $\mathbb{R}^2$, where the morphisms are partial functions, and composition is defined as usual for these. Consider the following composition of morphisms:

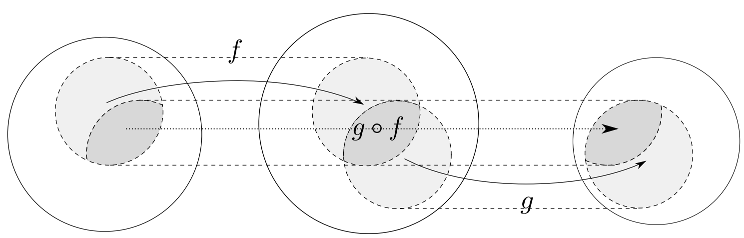


Figure 1: Failure of Composition

Here, both $f$ and $g$ take a subdisc of their domains to a subdisc of their codomains, so they are both left and right substantial. But $g \circ f$ is not substantial, since neither its image nor its preimage is a disc.

We noted in the last section that substantial subrelations of identity correspond one-to-one to substructures of an object, so every left substantial relation should have such a subrelation of identity corresponding to its preimage. The following is intended to capture this fact: let a *preimage operator* on a categoroid be a mapping $(\,\cdot\,)^{\triangleleft} : \mathsf{hom} \to \mathsf{hom}$ which fulfills the following axioms:

| | |
|---|---|
| *Composition*: | $f^{\triangleleft} \circ g$ is defined iff $f \circ g$ is. |
| *Domain*: | $\mathsf{dom}(f^{\triangleleft}) = \mathsf{cod}(f^{\triangleleft}) = \mathsf{dom}(f)$. |
| *Left identity*: | $f \circ f^{\triangleleft}$ is always defined and $f \circ f^{\triangleleft} = f$. |
| *Commutativity*: | $f^{\triangleleft} \circ g^{\triangleleft} = g^{\triangleleft} \circ f^{\triangleleft}$ if $f^{\triangleleft} \circ g^{\triangleleft}$ or $g^{\triangleleft} \circ f^{\triangleleft}$ is defined. |
| *Minimality*: | If $f \circ g^{\triangleleft} = f$, then $f^{\triangleleft} \circ g^{\triangleleft} = f^{\triangleleft}$. |

An *image* operator $(\,\cdot\,)^{\triangleright}$ can be defined symmetrically, but since we will mainly be interested in left substantiality, we will only concern ourselves with $^{\triangleleft}$. Now, left substantiality can be interpreted straightforwardly as the condition that every morphism must have a preimage. Thus we define a *form* to be a categoroid with a preimage operator.[8]

Since we intend to use forms to describe relations rather than mappings, we will now adopt a slightly different symbolism. Let us use the terminology

| | |
|---|---|
| $\mathsf{rel}$ | as a synonym for $\mathsf{hom}$ |
| $R, S, T, \ldots$ | for elements of $\mathsf{rel}$, |
| $a, b, c, \ldots$ | for elements of $\mathsf{obj}$, |
| $RS$ | for $S \circ R$ (note the change of order), and |
| $I_a$ | for $\mathsf{id}\,(a)$. |

We will, indeed, often use this terminology when discussing categories as well, since these are categoroids. We will commonly refer to the elements of $\mathsf{rel}$ as *relations*, and throughout we assume that $^{\triangleleft}$ binds tighter than relation composition. Thus from now on, the word ‘relation’ will refer to an element of $\mathsf{rel}$, and we will use ‘set-theoretic relation’ for the set-theoretic variant, i.e. a subset of the Cartesian product of two sets.

Some further definitions are useful. We say that an endorelation $R : a \to a$ is *commutative* iff for all endorelations $S : a \to a$ such that either $RS$ or $SR$ is defined, $RS = SR$. We say that $R$ is a *left identity* for $S$ iff $RS = S$, and a *right identity* for $S$ iff $SR = S$. We call the relation $I_x$, asserted to exist in the third of the identity axioms, the *identity relation* on $x$. It can be shown to be unique by the same reasoning as is used in category theory.

**Proposition 1.** *The preimage operation in a form has the following properties:*

*(i)* $R^{\triangleleft} R^{\triangleleft} = R^{\triangleleft}$.

*(ii) If* $R = S^{\triangleleft}$ *and* $S = R^{\triangleleft}$*, then* $S = R$.

*(iii)* $R^{\triangleleft\triangleleft} = R^{\triangleleft}$.

*Proof.* (*i*) $R^{\triangleleft} R^{\triangleleft} R = R^{\triangleleft}(R^{\triangleleft} R) = R^{\triangleleft}(R) = R$, so by minimality $R^{\triangleleft} R^{\triangleleft} = R^{\triangleleft}$.

[8] The name seems reasonable, and is adopted in the spirit of MacLane’s “pleasure of purloining words from the philosophers” [15, p. 29]. At least in on the Kantian interpretation of the words, categories are among the forms, and that is what we are aiming for here as well. If we were to be more careful, we would use the word *left-form* instead, to indicate that it is the *preimage* operator which is defined. A *right-form* would then be a categoroid with an image operator, and we could perhaps call a categoroid which is both a left- and a right-form a *biform*. Alas, there is no direct connection to the notion of *differential form* already in use in mathematics. I hope that no confusion will ensue.

(*ii*) By assumtion, we have that $RS = S$ and $SR = R$, so $SRS = RS = S$. But, by minimality, this means that $S(SR) = SR = S$, so $S = R$.

(*iii*) By left identity, $R^{\lhd\lhd}R \lhd= R^{\lhd}$ and $R^{\lhd}R = R$, which implies that $R^{\lhd\lhd}R = R$. By minimality we thus get that $R^{\lhd\lhd}R^{\lhd} = R^{\lhd}$. But we also have $R^{\lhd}R^{\lhd} = R^{\lhd}$, so, again by minimality, we get $R^{\lhd}R^{\lhd\lhd} = R^{\lhd\lhd}$. Commutativity then gives that $R^{\lhd}R^{\lhd\lhd} = R^{\lhd\lhd}R^{\lhd}$, so $R^{\lhd} = R^{\lhd\lhd}$.

□

For the identity relation $I_a$, we have that $I_a^{\lhd} = I_a$. This means that it is reasonable to, more generally, count each relation $R : a \to a$ such that $R^{\lhd} = R$ as a *subidentity relation*, in the sense of section 5. We call the class of $a$'s subidentities $\mathcal{I}(a)$, and among these we write $R \leqslant_a S$ iff $RS = R$. Instead of $R, S, T \ldots$, we will often use $A, B, C, \ldots$ for relations that are subidentities.

**Lemma 1.** *When both $A$ and $B$ are subidentities such that $AB$ (or $BA$) is defined, then $AB$ is a subidentity.*

*Proof.* For $AB$ to be a subidentity, we need to show that $(AB)^{\lhd} = AB$. But by definition $(AB)^{\lhd}AB = AB$, and by symmetry and idempotence $ABAB = AB$, so minimality gives that $(AB)^{\lhd}AB = (AB)^{\lhd}$. □

**Proposition 2.** *$(\mathcal{I}(a), \leqslant_a)$ is a partial order with top $I_a$.*

*Proof.* Reflexivity follows directly from $A^2 = A$. For transitivity, assume $A \leqslant_a B$ and $B \leqslant_a C$. We then have that $ABC = (AB)C = AC$, and $ABC = A(BC) = AB$, but since $AB = A$, we get that $AC = A$, i.e. $A \leqslant_a C$. For antisymmetry, assume that $A \leqslant_a B$ and $B \leqslant_a A$, so that $A = AB$ and $BA = B$. By the commutativity of subidentities, $AB = BA$, so $A = B$.

That $I_a^{\lhd} = I_a$ is trivial, so $I_a$ is a subidentity. Its status as top follows from the fact that it acts as an identity for *all* relations, and thus also all other subidentities. □

Subidentities, or coreflexive relations as Freyd [12, p. 198] calls them, will play the role of substructure. To connect them with embeddings, however, we also need a concept of inverse. As expected, such a concept can be obtained by generalizing from the category-theoretic case: we say that $S$ is a *left subinverse* of $R$ iff $SR = S^{\lhd}$, a *right subinverse* of $R$ iff $RS = R^{\lhd}$, and a *subinverse* if it is both a left and a right subinverse.

**Proposition 3.** *The subinverse of $R$ is uniquely determined whenever it exists.*

*Proof.* Assume that both $S$ and $T$ are inverses for $R$, which means that $RS = R^{\lhd} = RT$, $SR = S^{\lhd}$, and $TR = T^{\lhd}$. We then have that

$$S = S^{\lhd}S = SRS = SR^{\lhd} = SRT = S^{\lhd}T$$

and, by symmetric reasoning, that $T = T^{\lhd}S$. Putting it together, we get

$$S = S^{\lhd}T^{\lhd}S = T^{\lhd}S^{\lhd}S = T^{\lhd}S = T$$

□

In view of this theorem, we often write the inverse of $R$ as $R^{-1}$. A relation having an inverse is called a *subisomorphism*. A subisomorphism $R : a \to b$ is called

- an *embedding* iff $R^{\lhd} = I_a$,
- an *extraction* iff $R^{-1\lhd} = I_b$, and
- an *isomorphism* iff it is both an embedding and an extraction.

We will show why this concept of embedding indeed is the right one in the next section, by looking at its applications. We should note, however, that it only relies on the notions of subinverse and preimage, so we have sofar not made any use of the relational notion of *converse*. And in fact, the question of converses is orthogonal to that of embeddings. If we want to, a converse of $R : a \to b$ can be introduced definitionally as the relation $R^c : b \to a$ such that $R^c \lhd$ is the right identity for $R$, and for all $A \leqslant_a R^{\lhd}$ and all $B \leqslant_b R^{c\lhd}$, we have that

$$A = (RB)^{\lhd} \Leftrightarrow B = (R^c A)^{\lhd}$$

Proving this relation to be unique if it exists is left as an exercise for the reader. It does exemplify an interesting feature of forms, however: in many cases, what would in set theory be expressed as a condition on elements of the domain or codomain can instead be expressed through equations between relations composed with subidentities.

As in the introduction of any mathematical structure, we should also introduce transformations between forms. If we want the class of forms to itself be a form, these should have more in common with relations than functors. In category theory, such an entity is sometimes called a *distributor* (cf. [4, p. 308]), but we will stick to following MacLane's example of purloining a word from Carnap, and call it a *relator*. Formally, a relator $\mathfrak{R} : \mathcal{A} \to \mathcal{B}$ is

($i$) a relation $\mathfrak{R}$ from $\mathsf{obj}(\mathcal{A})$ to $\mathsf{obj}(\mathcal{B})$, together with

($ii$) for every pair of rel-sets $\mathsf{rel}(a, a')$ and $\mathsf{rel}(b, b')$ with $a, a' \in \mathsf{obj}(\mathcal{A})$ and $b, b' \in \mathsf{obj}(\mathcal{B})$ such that $a\mathfrak{R}b$, a relation $\mathbf{R}^{bb'}_{aa'}$ from $\mathsf{rel}(a, a')$ to $\mathsf{rel}(b, b')$, such that, for all $R \in \mathsf{rel}(a, a')$ and all $S \in \mathsf{rel}(b, b)$ for which $R\mathbf{R}^{bb'}_{aa'}S$, we have

  (a) for all $A \in a'$ and $B \in b'$ such that $A\mathbf{R}^{bb}_{aa}B$, $A = \bar{R} \Rightarrow B = \bar{S}$, and

  (b) for all $U \in \mathsf{rel}(a', a'')$ and $V \in \mathsf{rel}(b', b'')$ with $a'' \in \mathsf{obj}(\mathcal{A})$ and $b'' \in \mathsf{obj}(\mathcal{B})$ such that $a''\mathcal{R}b''$ and such that $AU$ is defined, $U\mathbf{R}^{b'b''}_{a'a''}V$ entails that $AU\mathbf{R}^{bb''}_{aa''}BV$.

The definition may perhaps look complicated, but it provides a rather straightforward generalization of the categorical functor concept. A relator between $\mathcal{A}$

and $\mathcal{B}$ is, informally, a relation from $\mathcal{A}$ to $\mathcal{B}$ such that whenever an equation using composition, identity, and preimaging holds on the left-hand side, the same equation holds on the right-hand side. When $\mathcal{A}$ and $\mathcal{B}$ are categories, i.e. when all arrows are left-total and functional (see thm. 3), and $\mathfrak{R}$ is also left-total and functional, it is the same as a functor.

In the interest of brevity, we will not go into how to define relator-theoretic versions of natural transformations and adjoints.

# 7 Embeddings in Specific Forms

In this section, we give a number of theorems correlating the abstract form concept outlined above to specific concrete structures. As the first example, we have

**Theorem 1.** *The categoroid* **Rel***, with* $R^{\triangleleft}$*, for* $R : a \to b$ *given by*

$$R^{\triangleleft} = \{(\xi, \xi) \in a \times a \mid (\exists \zeta \in b) \xi R \zeta\}$$

*is a form. A relation is* $R$ *is an embedding iff it is injective, functional and left total.*

*Proof.* That **Rel** satisfies composition, associativity, and identity follows directly from the fact that it does so as a category, and that all of these are equivalent to or weakenings of the corresponding category-theoretical axioms. For left substantiality, let $R^{\triangleleft}$ be defined as in the statement of the theorem. Only the minimality condition is not trivial to show. Assume that that $S : a \to a$ and $R : a \to b$ are such that $SR = R$, and, for contradiction, that there are elements $\alpha, \alpha' \in a$ such that $\alpha S \alpha'$ and $\alpha \neq \alpha'$. Define the relation $T : a \to a$ to hold only for the pair $(\alpha', \alpha)$. Then $ST = \{(\alpha, \alpha)\}$, but $TS$ is a set of pairs of the form $(\alpha', \beta)$ for some $\beta \in a$. Since $\alpha \neq \beta$, $ST \neq TS$, so $S$ cannot be commutative.

Contrapositively, a commutative relation is always an identity relation. We now show that, given such a relation $S$ such that $SR = R$, $SR^{\triangleleft} = R^{\triangleleft}$. But $SR = R$ can hold only if the preimage of $R$ (and thus of $R^{\triangleleft}$) is contained in the image (and preimage) of $S$, and since, for subrelations of identity relations $S$ and $R^{\triangleleft}$, $SR^{\triangleleft} = S \cap R^{\triangleleft}$, it follows directly that $SR^{\triangleleft} = R^{\triangleleft}$.

Now assume that $R : a \to b$ is injective, functional, and left total. We want to show that there is an inverse relation $R^{-1}$ such that $R^{\triangleleft} = I_a$. But such a relation is given by the converse of $R$, as is quickly checked. Conversely, assume that $R$ is left total and has an inverse. Assume, for contradiction, that there are $\alpha, \alpha'$ such that $\alpha R \beta$ and $\alpha' R \beta$. We know that $RR^{-1} = R^{\triangleleft}$, so if $\alpha \neq \alpha'$, then $\alpha RR^{-1} \alpha'$ cannot hold, which means that $R$ has to be injective. Functionality is proved in a similar way. □

**Theorem 2.** *The category* **pfSet** *of sets with partial functions as arrows is the subform of* **Rel** *containing the same objects, but with only those relations* $R : a \to b$ *for which* $RS = RT$ *entails* $RS^{\triangleleft} = RT^{\triangleleft}$*, for all* $S, T : b \to c$.

*Proof.* Set-theoretically, every partial function is of course a relation. What we need to show is that $R : a \to b$ is a partial function iff $RS = RT \to RS^{\triangleleft} = RT^{\triangleleft}$. For the left-to-right direction, assume that for all $\alpha \in a$ and $\beta, \beta' \in b$, that if $\alpha R \beta$ and $\alpha R \beta'$, then $\beta = \beta'$, and let $S, T$ be arbitrary relations from $b$ to a set $c$. Assume that $\alpha RS^{\triangleleft}\beta$, from which it follows that there is some $\gamma \in c$ such that $\alpha RS\gamma$. Because $RS = RT$, we have that, for all such $\gamma$, there is an element $\beta' \in b$ such that $\alpha R\beta'$ and $\beta' T\gamma$, so $\alpha RS^{\triangleleft}\beta$ entails that $\alpha RT^{\triangleleft}\beta'$. The functionality of $R$ then gives that $\beta = \beta'$, and reversing the argument gives that

$$\alpha RS^{\triangleleft}\beta \Leftrightarrow \alpha RT^{\triangleleft}\beta$$

For the converse, assume that $R$ is *not* functional, i.e. that there are $\alpha, \beta, \beta'$ such that $\alpha R\beta$, $\alpha R\beta'$, and $\beta \neq \beta'$. Let $c$ be a set and $\gamma \in c$, and let the relations $S$, $T$ be defined as

$$S =_{df.} \{(\beta, \gamma)\} \qquad T =_{df.} \{(\beta', \gamma)\}$$

Then we have that $RS = RT$ (both are the relation that hold only between $\alpha$ and $\gamma$), but $RS^{\triangleleft} = \{(\alpha, \beta)\}$, and $RT^{\triangleleft} = \{(\alpha, \beta')\}$. □

**Corollary 1.** *The category* **Set** *is the subform of the category* **pfSet** *such that* $R^{\triangleleft} = I_a$ *for all* $R : a \to b$.

Thm. 2 indicates that we might be able to take the condition

$$(\forall S, T : b \to c)(RS = RT \to RS^{\triangleleft} = RT^{\triangleleft})$$

as a *definition* for what it means for a relation $R$ to be functional. We shall, however, not pursue this line of thought farther in this text.

Equally important as capturing **Rel** and **Set** is our intention to give a *generalization* of the category concept, rather than a specialization of it. This is shown in the next theorem.

**Theorem 3.** *Let* $\mathcal{C}$ *be a category, and for any arrow* $R : a \to b$, *let* $R^{\triangleleft} = 1_a$. *Then* $\mathcal{C}$ *is a form. Conversely, any form* $\mathcal{F}$ *for which this holds becomes a category when we disregard the preimage operator.*

*Proof.* All the categoroid axioms are straightforward weakenings of the category axioms, so they are naturally satisfied. For the axiom of preimages, we need to show that $R^{\triangleleft} = I_{\mathsf{dom}(R)}$ is a preimage operator. The only condition which is not trivial, however, is the minimality one, which follows from the uniqueness of identity arrows in a category.

Conversely, let $\mathcal{F}$ be a form. All we have to do to show that its underlying categoroid is a category is to show that composition is always defined whenever the domains and codomains match up. So consider $R : a \to b$ and $S : b \to c$. By the composition axiom of the preimage operator, $RS$ is defined iff $RS^{\triangleleft}$, but since $S^{\triangleleft} = I_b$, and since $RI_b$ is always defined, so is $RS$. □

The form given from $\mathcal{C}$ in this theorem is often not very interesting, since it explicitly treats all relations as left total, which means that only isomorphisms end up being counted as embeddings. In order to represent some larger class $E$ of arrows in $\mathcal{C}$ as embeddings, we have to add images and subinverses for the arrows in $E$. Identifying $E$ is, moreover, made more difficult by the fact that if we do not assume all arrows to be left total, monicity is not sufficient to guarantee injectivity: we can have that $m \circ f = m \circ g$ even though $f \neq g$, as long as the parts of $\mathsf{dom}\, m$ where $f$ and $g$ differ are not in the preimage of $m$.

On the other hand, we may also take another route, and represent categorical relations in terms of a form, in which case we get that relator theory is a generalization of allegory theory, albeit in a slightly different way than it is a generalisation of category theory. The following theorem, given here without proof, indicates how.

**Theorem 4.** *Let $\mathcal{C}$ be a category. Then there is a form $\mathcal{F}$ with*

*(i)* $\mathsf{obj}(\mathcal{F})$ *consisting of the subobjects in $\mathcal{C}$, i.e. classes of pairs$(A, f)$, where $f$ is a monic with domain $A$, closed under the existence of mutual monomorphisms.*

*(ii)* $\mathsf{rel}(\mathcal{F})$ *consisting of the categorical binary relations of $\mathcal{C}$, i.e classes of triples $(A_k, d_k, c_k)$, where $A_k \in \mathsf{obj}\,\mathcal{C}$, $d_k$ and $c_k$ are jointly monic with domain $A_k$, such that for each $R \in \mathsf{rel}$, if $(A_k, d_k, c_k) \in R$ and there is an isomorphism $\iota : A_k \to A_l$ such that $A_l$ has joint monics $d_l$ and $c_l$ with $d_l \circ \iota = d_k$ and $c_l \circ \iota = c_k$, then $(A_l, d_l, c_l) \in R$.*

*(iii) $RS$ is defined iff $(A_R, d_R, c_R) \in R$, $(A_S, d_S, c_S) \in S$, $D = \mathsf{cod}\, c_R = \mathsf{dom}\, c_S$, and $d_R, c_S$ have a pullback*

$$\begin{tikzcd} P \arrow[r, "c_P"'] \arrow[d, "d_P"'] & A_S \arrow[d, "d_S"'] \\ A_R \arrow[r, "c_R"'] & D \end{tikzcd}$$

*In that case, $RS$ is the class of triples containing $(P, c_R \circ d_P, d_S \circ d_P)$.*

*(iv) $R^{\triangleleft}$, where $R$ contains the triple $(A, d, c)$, is the subobject that contains $(A, d)$.*

*(v) $1_x$, for each subobject $x$ containing $(A, f)$, is the categorical relation that contains the triple $(A, f, f)$.*

It is now time to move on to our philosophically motivating cases: **Top**, **Mod$\boldsymbol{\Sigma}$**, and **Ord**. In all of these, it does not matter if we take the relations to be set-theoretic relations or to be partial functions, but in the interest of generality, we will do the former. This, however, requires us to say which the relations in question are.

Given topological spaces $a = (X, \mathcal{T})$, $b = (Y, \mathcal{T}')$, a *continuous relation* $R : a \to b$ is a relation from $X$ to $Y$ such that $R^{-1}[Z]$ is open for all open $Z \subseteq Y$. A continuous relation which is left total and functional is easily seen to be exactly a continuous function. Let **rTop** be the the category with all topological spaces as objects, and continuous relations as relations.

**Theorem 5.** **rTop**, *with $R^{\triangleleft}$ defined as in thm. 1, is a form. The embeddings are the topological embeddings.*

*Proof.* For the left-to-right direction, we only need to show that the composition of two continuous relations is a continuous relation; the other form axioms then follow directly from thm. 1 But if $x$ is an open subset of $c$, and $R : a \to b$ and $S : b \to c$ are continuous, $S^{-1}[x]$ is open, and therefore also $S^{-1}R^{-1}(x) = (RS)^{-1}(x)$.

Assume now that $R : a \to b$ is a topological embedding, i.e. $R$ is an injective continuous total function with a continuous inverse $S : b' \to a$, where $b'$ is the image of $R$. Let $S'$ be the partial function from $b$ to $a$ that is defined, and equal to $S$, in the whole of $b'$. $S'$ easily checked to be a subinverse of $R$, and since $R^{\triangleleft} = I_a$, $R$ is an embedding.

Conversely, assume that $R$ has a subinverse $R^{-1}$, and that $R^{\triangleleft} = I_a$. We need to show that $R$ is a left total injective function, and that its set-theoretic inverse is its subinverse. First, assume for contradiction that $\alpha \in a$, and that there are $\beta, \beta' \in b$ such that $\beta \neq \beta'$, $\alpha R \beta$, and $\alpha R \beta'$. Let $c$ be a two-element set $\{\gamma, \gamma'\}$ with the indiscrete topology, and let $S : b \to c$ and $T : b \to c$ be the relations

$$S =_{df.} \{(\beta, \gamma)\} \qquad T =_{df.} \{(\beta', \gamma)\}$$

These are trivially continuous, and we also have that $RS = RT$. Applying $R^{-1}$ to the left, we get $R^{-1}RS = R^{-1}RT$, so $R^{\triangleright}S = R^{\triangleright}T$. But, as is easily shown though the method used in theorem 1,

$$R^{\triangleright} = \{(\xi, \xi) \mid \exists \zeta \in a \ (\zeta R \xi)\}$$

so $R^{\triangleright}S$ can hold only between $\beta$ and $\gamma$, and $R^{\triangleright}T$ only between $\beta'$ and $\gamma$. Hence $R^{\triangleright}S \neq R^{\triangleright}T$, and we have our contradiction.

That $R$ is left total is easily shown from $R^{\triangleleft} = I_a$ since $I_a$ is the identity relation on $a$. The identity of $R^{-1}$ with the set-theoretic inverse follows from the uniqueness of subinverses, and its continuity from the definition of relations on **rTop**. □

For $\mathbf{Mod\Sigma}$, let a *homomorphic relation* $R$ between models $\mathfrak{M}$ and $\mathfrak{N}$, with domains $D$ and $E$, be a relation from $D$ to $E$ such that for any $n$-tuples $\sigma = (s_1, \ldots, s_n)$, $\tau = (t_1, \ldots, t_n)$ such that $s_k \in D$ and $t_k \in E$, and all predicates $P^n$ in $\Sigma$, if $\sigma \in (P^n)^{\mathfrak{M}}$, then $\tau \in (P^n)^{\mathfrak{N}}$. In other words, if a certain predicate holds in the preimage of the relation, then the same predicate holds in the image. A *model homomorphism* is a homomorphic relation which is left total and functional. Let $\mathbf{rMod\Sigma}$ be the categoroid with all first-order models of signature $\Sigma$ as objects, homomorphic relations as relations, and composition

$RS$ defined as usual for set-theoretic relations, but undefined whenever $R$'s image and $S$'s preimage are disjoint.

**Theorem 6.** **rMod**$\mathbf{\Sigma}$*, with* $R^{\triangleleft}$ *defined as in thm. 1, is a form. The embeddings are the model embeddings.*

*Proof.* Checking that homomorphic relations are closed under composition, and that the identity relation is homomorphic, is trivial. The only slightly tricky part is that, since a domain cannot be empty, a composition $RS$ is defined only if the image of $R$ and the preimage of $S$ overlap. When they do, however, the result is always a homomorphic relation. Checking that the preimage of a homomorphic relation is a homomorphic relation is also routine.

We now wish to show that a model embedding $R$'s set-theoretic inverse $R^{-1}$ is a subinverse of $R$. But this follows directly from the fact that $R$, as a model embedding, preserves and reflects relations in its preimage, so of course $R^{-1}$ must do so as well, and therefore be a homomorphic relation. That it is the set-theoretic inverse of $R$ is then enough to make it a subinverse.

Now assume that $R : a \to b$ is a homomorphic relation, and that $R^{-1} : b \to a$ is its subinverse. Showing that $R$ is functional can be done the same way as in the last theorem, although we need to consider relations $S, T : b \to c$ with $c$ as a one-element domain with all predicates holding of its element, rather than a two-element topological space with the indiscrete topology. Likewise, showing totality and injectivity proceeds as in the topological case. □

Finally, we come to the form of **rOrd** of preordered sets with monotone relations. This is, however, a rather trivial in light of the previous result, since preordered sets are models of a first-order theory with a single binary predicate $\leqslant$, and monotone relations between them are just homomorphic relations. All that needs to be done is to make sure that the models involved in the proof are all preorders, so that the proof goes through. We leave this as an exercise for the reader.

# 8 Conclusions and Further Work

We have now dwelt in some detail on the problem of defining embeddings category-theoretically, and studied several ways to address it. While we have not *proved* it impossible to do so, strengthening the monic concept in order to capture that of embedding properly seems almost hopeless. Ideally, we would like to have such a proof, but since this would require a deep and careful dive into the model theory of category theory, we leave it as an open research problem.

Concrete categories give the most straightforward solution: split the structure into a basis, and another category "on top of" that. The split, however, incurs a certain amount of arbitrariness. In this, it is reminiscent of philosophical splits between particulars and property instances, forms and substances, or essential and accidental properties. Maybe such arbitrariness, and the split itself, is necessary. But it seems worth the attempt to avoid it, and one way to do so is to found structuralism on a theory of relations rather than a theory of transformations.

The first such theory we looked at was that of allegories. As a foundation for structuralism, however, it has certain drawbacks. The first is that it presupposes category theory, and indeed category theory with quite a lot of extra structure (i.e. regular categories). The second is that, in it, all relations have converses. This is not something we want if we wish to be able to distinguish between structure-preservation and structure-reflection.

The method I advocate is to instead start with a kind of structure which is more primitive than a category, and I give a brief overview of these structures, which I call *forms*. The theory of forms (or *relator theory*) does solve the problem of defining embeddings. As a conjecture, I would guess that it may also solve the dual problem, i.e. that of defining quotient objects in purely structural terms. One of the deep insights of category theory is just this duality, and indeed it seems to be equally difficult to characterize what a quotient is, without giving a separate definiton for each category. For such an application, however, it seems likely that we would need to use right-forms rather than left-forms. A theory of biforms could be used to characterize both embeddings and quotients in parallel.

Much more speculatively, forms could also turn out to be useful for the treatment of weak $\infty$-categories, where the classical theory of identity has sometimes been held to be problematic. As we noted, the notions of *function*, *operator*, and *mapping*, on which category theory is based philosophically, are intimately bound up with identity. In order to preserve the similarities between categories and forms, we have presented the latter in terms of functions such as composition or identity. But it is a rather simple matter to frame relator theory using just relations, and convert identities into coherence axioms. When this is done, generalization to higher forms, i.e. forms in which the rel-sets are themselves object-classes of forms, should be possible. But of course, we will not know if this actually works until the program has been carried out.

There is still one nagging issue, however: given the complications we have encountered here already, one might ask if we have not headed off in the wrong direction already at the start. In specific constructs, both embeddings and quotients are easily identified in set-theoretical terms. Could this be made into an argument that set theory is, after all, necessary in a foundation for structuralism, and that our problems stem from trying to use category theory, or form theory, without a set-theoretical interpretation? For category theory, such an interpretation is given precisely by a faithful functor to the category of sets. A closely related strategy would be to use Bourbaki's theory of structures, which were, after all, designed for this very purpose.

What would certainly *not* be adequate as a theory of structure is the model theory of first-order logic. At the very least, we will need models of higher-order logics in order to capture structures such as topologies, and we also need to disentangle the notion of morphism from that of model homomorphism in order to be able to separate, say, topologies with continuous functions from topologies with open maps as morphisms, or Hilbert spaces with linear maps from Hilbert spaces with isometries as morphisms. But making both these changes more or less lands us in a theory equivalent to Bourbaki's. However, as one of the Bourbakians himself noted, "this notion has since been superseded by that of category and functor, which includes it in a more general and convenient form." [10, quoted in [2]]

There are, however, many reasons beyond convenience and generality for the superiority of category theory to Bourbakian structure theory. One of these is the dualities that emerge—in our case, between embeddings and quotients. Another is the possibility of giving *invariant* descriptions of structures, i.e. without reference to specific instantiations of them, much as group theory gives us a way to study symmetries without reference to a specific representation. In this, the categorical approach is clearly preferable to a set-theoretic one.

The theory of forms, as it has been outlined here, is not intended to be a replacement for category theory, but only a slight generalization which can be employed when category theory on its own gives insufficient descriptive power. Since all categories can be interpreted as forms, it should be fairly straightforward to combine the theories, and use which one is the more convenient for any specific task.

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