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Folkow, Peter D.; Kristensson, Gerhard; Olsson, Peter

1996

CODEN:LUTEDX/(TEAT-7047)/1-19/(1996)

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Peter D. Folkow, Gerhard Kristensson, Peter Olsson

Department of Electroscience
Electromagnetic Theory
Lund Institute of Technology
Sweden

Peter D. Folkow

Division of Mechanics
Chalmers University of Technology
SE-412 96 Göteborg
Sweden

Gerhard Kristensson

Department of Electromagnetic Theory
Lund Institute of Technology
P.O. Box 118
SE-221 00 Lund
Sweden

Peter Olsson

Division of Mechanics
Chalmers University of Technology
SE-412 96 Göteborg
Sweden

Editor: Gerhard Kristensson

Abstract

In this paper, wave splitting technique is applied to a homogeneous Timoshenko beam. The purpose is to obtain a diagonal equation in terms of the split fields. These fields are calculated in the time domain from an appropriate set of boundary conditions. The fields along the beam are represented as a time convolution of Green functions with the excitation. The Green functions do not depend on the wave fields but only on the parameters of the beam. Green functions for a Timoshenko beam are derived, and the exponential behaviour of these functions as well as the split modes are discussed. A transformation that extracts the exponential part is performed. Some numerical examples for various loads are presented and compared with results appearing in the literature.

1 Introduction

The usefulness of time domain techniques to solve direct and inverse scattering problems has been corroborated during the last decade. The mathematical tools to solve these problems are the wave splitting, in conjunction with the invariant imbedding or the Green function technique. The wave splitting concept in the time domain was introduced by Corones, Davison and Krueger [6, 7] in order to solve the inverse scattering problem in acoustics. The major developments in the solution of the inverse scattering problem using time domain techniques are found in electromagnetics, *e.g.* Refs. [19, 23]. Other fields of expansive research are the elastic applications [12, 29], the viscoelastic case [3] and the fluid-saturated porous media [9]. Recently, the three- dimensional inverse scattering problem has been focused [31]. The interested reader is referred to Ref. [8] for additional references and for a general overview of research field.

The Green function approach to solve the inverse scattering problem was first introduced by Krueger and Ochs [23] for a non-dispersive second order wave equation, and by Kristensson [18] in the dispersive case. A number of problems has been investigated ever since, such as scattering problems for a dispersive cylinder [17], non-dispersive anisotropic media [28], dissipative stratified media [14, 15] and the three-dimensional case [32]. During the last years, the Green function technique has been adopted to solve scattering problems for complex media, *e.g.* anisotropic, gyrotropic, bi-isotropic and non- stationary media [1, 13, 16, 20–22].

The application of the theory alluded to above to fourth order hyperbolic systems, *e.g.* the Timoshenko beam equation [30], calls for the development of new and more extended methods. In a first paper [26], the wave splitting of this fourth order system was developed. This paper also contains a discussion and a comparison between the different equations that are frequently used in the literature to model wave propagation in a beam, *i.e.* the Euler-Bernoulli, the Rayleigh, and the Timoshenko equations. As a result, it was found that only the Timoshenko equation is a hyperbolic system, and therefore suitable as a model for propagation of transient waves in a beam.

The aim of this paper is to further develop the Green function formulation as it applies to a fourth order hyperbolic system. In order to illustrate the method, some numerical results are shown. These are compared with results calculated in the Laplace domain, Refs. [4] and [25]. The advantages of performing the calculations in the time domain are discussed.

In Section 2 the basic equations used in this paper as well as conservation of energy are presented. The wave splitting is introduced in Section 3, and the asymptotic behavior of the kernels in the splitting is analyzed in Section 4. The dynamics and the splitting are modified due to the long time behavior of the wave splitting. This analysis is presented in Section 5. The Green function equations are introduced in Section 6, and some numerical results are presented in Section 7. The paper ends with two appendices.

2 Basic equations

In [26] a time domain wave splitting of the Timoshenko equation was derived. The starting point in that case was a formulation of the equation in terms of the dependent variables $\{u, \psi, \partial_z u, \partial_z \psi\}$. Here $u(z,t)$ and $\psi(z,t)$ are the mean vertical displacement and the mean angle of rotation of the cross section, respectively. However, there is reason to modify the starting point of the splitting so as to simplify the application of boundary conditions. Hence, instead of using the variables above, we express the Timoshenko equation in terms of $\{u, \psi, \gamma, \partial_z \psi\}$, where $\gamma(z,t)$ is the mean shear angle, defined as $\gamma(z,t) = \partial_z u(z,t) - \psi(z,t)$. Written as a spatially first order system, the Timoshenko equation reads

$$\frac{\partial}{\partial z}\begin{pmatrix} u \\ \psi \\ \gamma \\ \partial_z \psi \end{pmatrix} = \mathcal{D}\begin{pmatrix} u \\ \psi \\ \gamma \\ \partial_z \psi \end{pmatrix} \tag{2.1}$$

where

$$\mathcal{D} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c_1^{-2}\partial_t^2 & 0 & -\frac{\partial \ln f_1}{\partial z} & 0 \\ 0 & c_2^{-2}\partial_t^2 & -f_1/f_2 & -\frac{\partial \ln f_2}{\partial z} \end{pmatrix}$$

In the present paper, where a homogeneous beam with constant shape of the cross section is studied, the operator $\mathcal{D}$ simplifies to

$$\mathcal{D} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c_1^{-2}\partial_t^2 & 0 & 0 & 0 \\ 0 & c_2^{-2}\partial_t^2 & -f_1/f_2 & 0 \end{pmatrix} \tag{2.2}$$

The two velocities c_1 (effective shear velocity) and c_2 (rod velocity) are defined by

$$c_1 = \sqrt{\frac{k'G}{\rho}} \qquad c_2 = \sqrt{\frac{E}{\rho}}$$

while the shear stiffness f_1 and the bending stiffness f_2 are given by

$$f_1 = k'GA \qquad f_2 = EI \tag{2.3}$$

In these equations E is Young's modulus, G the shear modulus and ρ is the density of the beam. I and A are the moment of inertia and the area of the cross section, respectively. We assume that

$$\nu = \frac{E - 2G}{2G} \qquad 0 \leq \nu < \frac{1}{2} \tag{2.4}$$

which amounts to saying that the material is compressible with a non-negative Poisson's ratio. The shear coefficient k' depends on the shape of the cross section. A relatively simple expression for this quantity is

$$k' = \frac{bI}{SA} \tag{2.5}$$

where b is the width of the beam at $y = 0$ and S is the static area moment of the upper half of the cross section, that is

$$S = \int_{y \geq 0} y \, dA$$

However, a more accurate determination of k' is given by Cowper [11]. These expressions, based on a 3D analysis, reveal that k' is not purely a geometrical factor but does in fact also depend on the Poisson ratio. We refer to [11] for a comparison between various theories. For future needs, we define the radius of gyration r_0 and a characteristic time τ as

$$r_0 = \sqrt{\frac{I}{A}} = \frac{c_1}{c_2}\sqrt{\frac{f_2}{f_1}} \qquad \tau = \frac{1}{2c_1}\left(1 - \frac{c_1^2}{c_2^2}\right)\sqrt{\frac{f_2}{f_1}}$$

We will as well have reason to refer to the non-dimensional ratio

$$q = \frac{c_2^2 + c_1^2}{c_2^2 - c_1^2}$$

which under the assumption in equation (2.4) satisfies

$$\frac{3 + k'}{3 - k'} < q \leq \frac{2 + k'}{2 - k'}$$

2.1 Conservation of energy

The energy in a Timoshenko beam satisfies an equation of continuity. Consider a homogeneous beam. The kinetic and potential energy density are defined as [24]

$$\epsilon_{kin} = \frac{1}{2}\left(\frac{f_1}{c_1^2}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{f_2}{c_2^2}\left(\frac{\partial \psi}{\partial t}\right)^2\right)$$

$$\epsilon_{pot} = \frac{1}{2}\Big(f_1\gamma^2 + f_2\left(\frac{\partial\psi}{\partial z}\right)^2\Big)$$

So, we have the equation of continuity

$$\frac{\partial}{\partial t}\left(\epsilon_{kin} + \epsilon_{pot}\right) + \frac{\partial j}{\partial z} = 0$$

where the power flow is given by

$$j = -\left(f_1\,\gamma\,\frac{\partial u}{\partial t} + f_2\,\frac{\partial\psi}{\partial z}\,\frac{\partial\psi}{\partial t}\right)$$

The Timoshenko equation, while being inherently dispersive, is thus a non-dissipative equation which conserves the energy of the total field.

3 Wave splitting

We wish to perform a transformation that diagonalizes the Timoshenko equation for a homogeneous beam. The present formulation essentially follows [26], apart from the choice of dependent variables. Introduce the transformation of the dependent variables

$$\begin{pmatrix} u_1^+ \\ u_2^+ \\ u_1^- \\ u_2^- \end{pmatrix} = \mathcal{P}\begin{pmatrix} u \\ \psi \\ \gamma \\ \partial_z\psi \end{pmatrix} \tag{3.1}$$

with formal inverse

$$\begin{pmatrix} u \\ \psi \\ \gamma \\ \partial_z\psi \end{pmatrix} = \mathcal{P}^{-1}\begin{pmatrix} u_1^+ \\ u_2^+ \\ u_1^- \\ u_2^- \end{pmatrix} \tag{3.2}$$

The matrix $\mathcal{P}$ has an operator representation

$$\mathcal{P} = \mathcal{Q}\begin{pmatrix} -\left(\lambda_2^2 - c_1^{-2}\partial_t^2\right) & -\lambda_1 & \mathcal{S}\lambda_2 - \lambda_1 & 1 \\ \lambda_1^2 - c_1^{-2}\partial_t^2 & \lambda_2 & -\left(\mathcal{S}\lambda_1 - \lambda_2\right) & -1 \\ -\left(\lambda_2^2 - c_1^{-2}\partial_t^2\right) & \lambda_1 & -\left(\mathcal{S}\lambda_2 - \lambda_1\right) & 1 \\ \lambda_1^2 - c_1^{-2}\partial_t^2 & -\lambda_2 & \mathcal{S}\lambda_1 - \lambda_2 & -1 \end{pmatrix}$$

while the operator $\mathcal{P}^{-1}$ is defined as

$$\mathcal{P}^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -\lambda_1\left(1 - \mathcal{U}\lambda_2^2\right) & -\lambda_2\left(1 - \mathcal{U}\lambda_1^2\right) & \lambda_1\left(1 - \mathcal{U}\lambda_2^2\right) & \lambda_2\left(1 - \mathcal{U}\lambda_1^2\right) \\ -\lambda_1\mathcal{U}\lambda_2^2 & -\lambda_2\mathcal{U}\lambda_1^2 & \lambda_1\mathcal{U}\lambda_2^2 & \lambda_2\mathcal{U}\lambda_1^2 \\ \lambda_1^2 - c_1^{-2}\partial_t^2 & \lambda_2^2 - c_1^{-2}\partial_t^2 & \lambda_1^2 - c_1^{-2}\partial_t^2 & \lambda_2^2 - c_1^{-2}\partial_t^2 \end{pmatrix}$$

The various operators appearing in $\mathcal{P}$ and $\mathcal{P}^{-1}$ can be found in Appendix A. The performed change of basis turns the Timoshenko equation (2.1) into

$$\frac{\partial}{\partial z}\begin{pmatrix} u_1^+ \\ u_2^+ \\ u_1^- \\ u_2^- \end{pmatrix} = *\begin{pmatrix} u_1^+ \\ u_2^+ \\ u_1^- \\ u_2^- \end{pmatrix} \tag{3.3}$$

where

$$* = \begin{pmatrix} -\lambda_1 & 0 & 0 & 0 \\ 0 & -\lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix}$$

The operators λ_i are the eigenvalues of (2.2) and may be represented by

$$\lambda_i f(t) = c_i^{-1} \frac{\partial f(t)}{\partial t} + \big(F_i(\cdot) * f(\cdot)\big)(t) \quad i = 1, 2 \tag{3.4}$$

whereupon the dynamics is written

$$\frac{\partial u_i^{\pm}(z,t)}{\partial z} = \mp\Big(c_i^{-1} \frac{\partial u_i^{\pm}(z,t)}{\partial t} + \big(F_i(\cdot) * u_i^{\pm}(z,\cdot)\big)(t)\Big) \tag{3.5}$$

The u_i^+ are thus the fields which propagate towards increasing z, *i.e.* right-going fields, and u_i^- are the left-going fields. The star $(*)$ denotes time convolution, which throughout this paper is defined by

$$\big(f(\cdot) * g(\cdot)\big)(t) = \int_0^t f(t')g(t-t')\,dt'$$

All fields are assumed quiescent at time $t < 0$. The kernel functions $F_i(t)$ have certain properties that are worth emphasizing. They can be expressed in series representations according to [26]

$$\begin{cases} F_1(t) = \dfrac{H(t)}{c_2\tau^2} \displaystyle\sum_{k=1}^{\infty} \frac{\Gamma(3/2)}{k!\Gamma(3/2-k)} (-1)^k (q+1)^{-k} W_k(t/\tau) \\ F_2(t) = \dfrac{H(t)}{c_2\tau^2} \displaystyle\sum_{k=1}^{\infty} \frac{\Gamma(3/2)}{k!\Gamma(3/2-k)} (q-1)^{-k} W_k(t/\tau) \end{cases} \tag{3.6}$$

where $H(t)$ is the Heaviside step function and $W_k(\xi)$ are integrals of modified Bessel functions

$$W_k(\xi) = \partial_\xi^{-k+1} \frac{kI_k(\xi)}{\xi} \qquad \partial_\xi^{-1} f(\xi) = \int_0^\xi f(\xi')\,d\xi'$$

For small arguments, the kernel functions may readily be expanded in a power series

$$F_i(t) = H(t) \sum_{k=0}^{\infty} a_{i,k} t^{2k}$$

However, for large arguments it is advantageous to represent (3.6) asymptotically, since $W_k(\xi)$ are of exponential order $1/\tau$. We obtain

$$F_i(t) \approx e^{t/\tau} \sum_{k=1}^{\infty} b_{i,k} t^{-(2k+1)/2}$$

Schemes for computing the coefficients $a_{i,k}$ and $b_{i,k}$ are indicated in [26].

4 Long time behaviour

The exponential behaviour of the kernels $F_i(t)$ is made apparent by the use of Laplace transform techniques. Denote the transformation as $\mathcal{L}F_i = \widetilde{F}_i(s)$, whereupon equation (3.4) turns into

$$\widetilde{F}_i(s) = \widetilde{\lambda}_i(s) - {c_i}^{-1}s \tag{4.1}$$

The transformed eigenvalue operators are [25]

$$\widetilde{\lambda}_{1,2}(s) = A\sqrt{s}\left(s \pm q^{-1}\sqrt{s^2 - \tau^{-2}}\right)^{1/2} \qquad A = \sqrt{q/c_2^2(q-1)}$$

When studying the singularities of equation (4.1), we see that there are branch points according to

$$\begin{array}{llll} F_1: & s = 0 & s = \pm 1/\tau & \\ F_2: & s = 0 & s = \pm 1/\tau & s = \pm i c_1/r_0 \end{array}$$

Since the inversion integral is performed along a line in the half- plane of convergence, this line must be to the right of $1/\tau$. Thus, the structure of (4.1) indicates that the kernels F_i actually are of exponential order $1/\tau$, see Ref. [27]. We may now make comments on the long time behaviour of the split modes $u_i^{\pm}$. By analysing the transformations, defined by (3.1) and (3.2), it turns out that the split modes increase exponentially for all feasible combinations of the physical fields. Hence, unless an unbounded amount of energy is injected into the beam, the different modes must coexist in order to cancel each other for large times. It is primarily the low frequency intervals that are affected, since these parts of the wave propagate at the lowest speed.

5 The modified splitting and dynamics

From a numerical point of view the properties of the modes discussed above are of course troublesome. It is somewhat unpleasant to deal with modes which eventually diverge exponentially at every point along the beam. Fortunately, a remedy is immediately suggested by the analysis of the previous section. Since the split fields are of exponential order $1/\tau$, we extract a corresponding factor. Define

$$v_i^{\pm}(z,t) = \exp(\pm z/c_i\tau - t/\tau)u_i^{\pm}(z,t)$$

These new fields are of exponential order 0. The dynamics of the $v_i^{\pm}$ fields are

$$\frac{\partial v_i^{\pm}(z,t)}{\partial z} = \mp\Big(c_i^{-1}\frac{\partial v_i^{\pm}(z,t)}{\partial t} + \big(A_i(\cdot) * v_i^{\pm}(z,\cdot)\big)(t)\Big) \tag{5.1}$$

where the new kernel functions $A_i(t)$ are given as

$$A_i(t) = e^{-t/\tau}F_i(t) \tag{5.2}$$

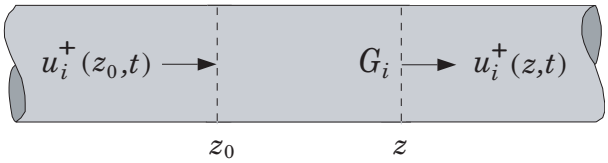


Figure 1: The Green function approach.

Thus, the dynamics according to (3.5) and (5.1) have the same basic structure, except that the functions in the latter are much more well behaved. Note that from the expressions of the $F_i(t)$ we can infer that $A_i \in L^1 \cap L^2$. In conformity with (3.1), a transformation of $\{u, \psi, \gamma, \partial_z \psi\}$ to the modified field $v_i^\pm$ is obtained by means of a simple modification of the operator $\mathcal{P}$. Define the diagonal operator matrix by $\mathcal{E}$

$$\mathcal{E} = e^{-t/\tau} \begin{pmatrix} e^{z/c_1\tau} & 0 & 0 & 0 \\ 0 & e^{z/c_2\tau} & 0 & 0 \\ 0 & 0 & e^{-z/c_1\tau} & 0 \\ 0 & 0 & 0 & e^{-z/c_2\tau} \end{pmatrix}$$

The new operator that corresponds to $\mathcal{P}$ is then given by $\mathcal{R} = \mathcal{E}\mathcal{P}$ with formal inverse $\mathcal{R}^{-1} = \mathcal{P}^{-1}\mathcal{E}^{-1}$.

6 The Green functions

This section contains an analysis of the relationship between the fields in a fixed cross section and the fields in an arbitrary position along the beam, see Figure 1. We solely study a homogeneous beam of infinite extension, implying that multiple reflections at ends are not taken into account. Therefore, it is sufficient to exclusively examine the right-going waves, since the left-going fields can be treated in an analogous manner.

The operator that maps the fields at a certain position to those at another cross section has an integral representation. By placing the origin $z = 0$ at the fixed cross section, we obtain (Appendix B)

$$u_i^+(z, t + z/c_i) = u_i^+(0, t) + \big(G_i(z, \cdot) * u_i^+(0, \cdot)\big)(t) \tag{6.1}$$

The kernels $G_i(z, t)$ are the Green functions. Equations for the Green functions are obtained from the representation above and the dynamics (3.5). The total derivative with respect to z of both expressions yields

$$d_z u_i^+(z, t + z/c_i) = \big(\frac{\partial}{\partial z} G_i(z, \cdot) * u_i^+(0, \cdot)\big)(t) \tag{6.2}$$

$$d_z u_i^+(z, t + z/c_i) = -\big(F_i(\cdot) * u_i^+(z, z/c_i + \cdot)\big)(t) \tag{6.3}$$

where $d_z = d/dz$. In the right-hand side of equation (6.3), we can apply the representation according to equation (6.1). Using this relation together with (6.2), we obtain

$$\frac{\partial}{\partial z} G_i(z, t) = -F_i(t) - \big(F_i(\cdot) * G_i(z, \cdot)\big)(t) \tag{6.4}$$

with boundary and initial conditions

$$G_i(0,t) = 0 \quad t > 0 \qquad G_i(z,t) = 0 \quad t < 0 \qquad G_i(z,0^+) = -zF_i(0^+)$$

This integro-differential equation may be rewritten by applying Laplace transform techniques. The transformed equation turns into an ordinary differential equation

$$\frac{\partial}{\partial z}\widetilde{G}_i(z,s) = -\widetilde{F}_i(s) - \widetilde{F}_i(s)\widetilde{G}_i(z,s)$$

with formal solution

$$\widetilde{G}_i(z,s) = \exp\bigl(-z\widetilde{F}_i(s)\bigr) - 1 \tag{6.5}$$

Consequently, the Green functions can be expressed in a series containing repeated convolutions of $F_i(t)$. However, it is convenient to proceed in another way. By differentiating (6.5) with respect to s, it yields

$$\frac{\partial}{\partial s}\widetilde{G}_i(z,s) = -z\frac{\partial}{\partial s}\widetilde{F}_i(s)\bigl(\widetilde{G}_i(z,s) + 1\bigr)$$

A transformation of this expression to the time domain reads

$$G_i(z,t) = -\frac{z}{t}\int_0^t t'F_i(t')G_i(z,t-t')dt' - zF_i(t) \tag{6.6}$$

These Green functions equations are Volterra equations of the second kind.

The analysis above is easily modified when the fields introduced in Section 5 are concerned. Equation (6.1) turns into

$$v_i^+(z,t+z/c_i) = v_i^+(0,t) + \bigl(B_i(z,\cdot) * v_i^+(0,\cdot)\bigr)(t)$$

The modified Green functions $B_i(z,t)$ are defined as

$$B_i(z,t) = e^{-t/\tau}G_i(z,t)$$

Thus, the integro-differential equation (6.4) is written

$$\frac{\partial}{\partial z}B_i(z,t) = -A_i(t) - \bigl(A_i(\cdot) * B_i(z,\cdot)\bigr)(t)$$

with $A_i(t)$ given by (5.2).

7 Numerical verification

We intend to determine the physical fields along the beam from given boundary conditions. This is accomplished by applying the transformations of variables in conjunction with the Green function technique. Hence, a given set of boundary conditions are transformed to the split fields according to equation (3.1). These fields are mapped to a position along the beam by means of (6.1), whereupon the inverse transformation, (3.2), is performed.

Due to the complexity of the system, the problem is solved numerically. The Green functions are easily calculated from the Volterra equations (6.6). Using the trapezoidal rule to approximate the integrals, we obtain

$$\begin{aligned} G_i(z,jh) = &-\frac{z}{2j} jhF_i(jh)G_i(z,0^+) \\ &-\frac{z}{j}\sum_{k=1}^{j-1} khF_i(kh)G_i\big(z,(j-k)h\big) - zF_i(jh) \qquad j \geq 1 \end{aligned}$$

where the parameter h is the time step. It is then straightforward to tackle equation (6.1). Concerning the convolutions appearing in the operators $\mathcal{P}$ and $\mathcal{P}^{-1}$, we could have used the trapezoidal rule as well. However, a more efficient method is justified by the following arguments. Suppose our interest is focused on the propagating fields in the vicinity of the wave fronts. Since the local time coordinates originates from these wave fronts, the relevant results are obtained from a relatively small interval of time. We may thus expand the functions defined in Appendix A in a power series, see Ref. [2]. When expressed in these series, the convolutions are readily performed. Given functions as

$$f(t) = \sum_{k=0}^{\infty} a_k t^k \qquad g(t) = \sum_{k=0}^{\infty} b_k t^k$$

the convolution reads

$$\big(f(\cdot) * g(\cdot)\big)(t) = t\sum_{k=0}^{\infty} c_k t^k \qquad c_k = \frac{1}{(k+1)!}\sum_{j=0}^{k} a_j b_{k-j} j!(k-j)!$$

Repeated application of this result allows us to compute multiple convolutions in an efficient manner.

As noted in Section 6, it suffices to examine only the right-going waves. Consequently, we may consider a semi-infinite beam with a proper set of boundary conditions at the left end. In order to elucidate the choice of boundary conditions, equation (3.1) is written

$$\begin{cases} u_i^+ = \mathcal{P}_{i1}u + \mathcal{P}_{i2}\psi + \mathcal{P}_{i3}\gamma + \mathcal{P}_{i4}\partial_z\psi \\ u_i^- = \mathcal{P}_{i1}u - \mathcal{P}_{i2}\psi - \mathcal{P}_{i3}\gamma + \mathcal{P}_{i4}\partial_z\psi = 0 \end{cases}$$

Combining the fields as $u_i^+ \pm u_i^- = u_i^+$, we obtain

$$u_i^+ = 2\mathcal{P}_{i1}u + 2\mathcal{P}_{i4}\partial_z\psi \tag{7.1}$$

$$u_i^+ = 2\mathcal{P}_{i2}\psi + 2\mathcal{P}_{i3}\gamma \tag{7.2}$$

In the numerical examples below, the set of boundary conditions are prescribed in terms of the fields appearing in the right-hand sides of (7.1) or (7.2). The interpretation of these fields can be made more tangible, since the fields γ and $\partial_z\psi$ are related to the shear force Q and the bending moment M as [24]

$$\gamma = Q/f_1 \qquad \partial_z\psi = M/f_2$$

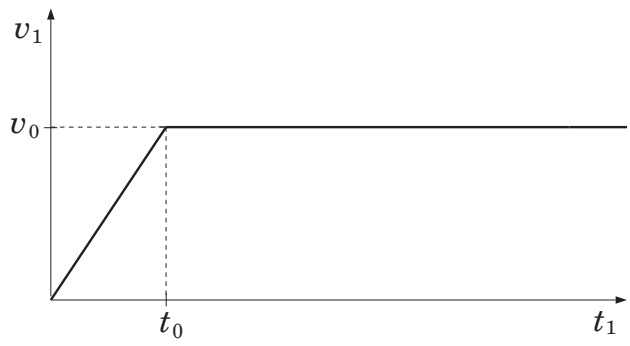


Figure 2: $v_1(0, t_1)$

where f_i are defined in (2.3).

Consider equation (7.1) in the case of zero bending moment, $\partial_z \psi = 0$, at $z = 0$. This problem is treated by Boley and Chao [4], where the Timoshenko equation is solved in the Laplace domain and re-transformed numerically. Here, the Poisson ratio and the shear coefficient are set to $\nu = 0.3$ and $k' = 0.813$. The latter is obtained for a cross section of rectangular shape, using a theory somewhat different from [11]. Introduce as nondimensional variables $z_1 = z/r_0$ and $t_1 = tc_2/r_0$. We hereby have the arrival of the wavefront at $t_1 = z_1$. Moreover, express the vertical displacement and its time derivative in nondimensional form

$$u(z,t) = r_0 u_1(z_1, t_1) \qquad \partial_t u(z,t) = c_2 v_1(z_1, t_1)$$

The vertical displacement at the boundary is prescribed in terms of its time derivative, according to Figure 2. The results are shown in Figures 4 and 5.

In Figure 4, two different positions along the beam are examined

$$\begin{aligned} \text{I}: &\qquad z_1 = 0 \qquad t_0 = 0 \\ \text{III}: &\qquad z_1 = 5 \qquad t_0 = 0 \end{aligned}$$

Figure 4a displays the variation with time of the shear force. This graph may be compared with the results from [4], given in Figure 4b. The two plots are in good agreement. Figure 4c and 4d show the vertical displacement and the bending moment, respectively. The arrival of the shear waves at $z_1 = 5$ is apparent in all the graphs. This occurs when $t_1 \approx 8.94$. In Figure 5, two different loadings are considered

$$\begin{aligned} \text{I}: &\qquad z_1 = 5 \qquad t_0 = 3 \\ \text{III}: &\qquad z_1 = 5 \qquad t_0 = 6 \end{aligned}$$

A comparison with [4] is done in Figure 5a and 5b. Here, the differences between the plots are more apparent than in Figure 4. All edges are sharpened in the solution performed in the time domain, Figure 5a. The arrival of the shear waves, as well as the influence of the change in the vertical displacement at the left end at time $t_1 = t_0$, appear clearly in this plot. Consequently, the solutions performed in the Laplace

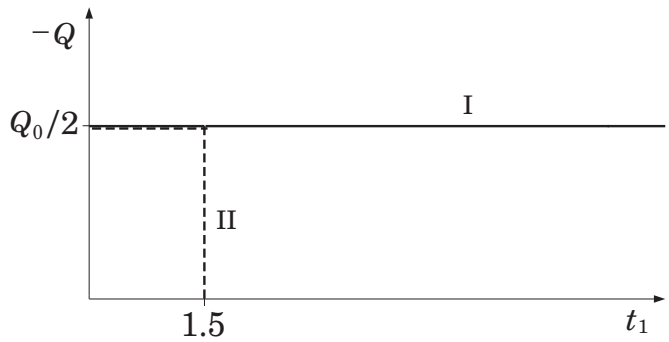


Figure 3: $Q(0, t_1)$

domain using numerical integration show a restriction in reproducing such distinct variations. Neither the vertical displacement nor the bending moment contain any sort of visible edges, Figure 5cd.

In the work of Miklowitz [25], boundary conditions according to equation (7.2) are considered. Here, an infinite beam is excited at its center by a suddenly applied shear force, Q_0. Therefore, the boundary conditions are $\psi = 0$ and $Q = -Q_0/2$. A long-time solution using the method of stationary phase is used in [25]. The Poisson ratio and the shear coefficient are set to $\nu = 0.3$ and $k' = 2/3$. The latter is obtained for a cross section of rectangular shape, using elementary theory (2.5). We introduce a new set of nondimensional variables $z_1 = z/r_0$ and $t_1 = 4tc_2/r_0 - 4z_1$, by which the vertical displacement may be expressed $u_1(z_1, t_1) = r_0 u(z, t)$. This transformation causes the wavefront to arrive at $t_1 = 0$. Both a step and a rectangular pulse in the shear force is prescribed at the boundary, according to Figure 3. Figure 6 shows the fields at $z_1 = 1$.

In Figure 6a, a modified time variable has been introduced in order to simplify comparisons with the results from [25], Figure 6b. This is due to the fact that the shear waves are assumed to have arrived at the time marked by τ_{01} in Figure 6b. Since the shear waves arrive at $t_1 \approx 3.9$, the modified variable is set to $t_1' = t_1 - 3.65$. The two plots do not resemble each other in many aspects. The main reason is the use of a long-time solution in Figure 6b. As can be seen from the definition of the variables, neither the time nor the space variables are very big. Figure 6c shows that the vertical displacement is mainly affected by the shear waves, while the bending moment is being influenced by the faster longitudinal waves as well, Figure 6d.

The numerical examples presented above emphasize some of the advantages when performing calculations in the time domain. At points where the fields change abruptly, the resolutions using time domain solutions are generally superior to the ones where calculations in the Laplace domain have been used. The inversion of the Laplace transform is a delicate task, since it is equivalent to the solution of an integral equation of the first kind [33]. Concerning the numerical implementation, the time domain method yields very fast algorithms. Once the transformation matrices $\mathcal{P}$ and $\mathcal{P}^{-1}$ defined in Section 3 and the Green functions G_i from Section 6 have been calculated, the fields are readily determined along the beam for various boundary

conditions.

8 Conclusions

Wave splitting technique in conjunction with the Green function approach have been utilised to solve the wave propagation problem in a homogeneous Timoshenko beam. The analysis is performed in the time domain, and the pertinent integro-differential equations for the Green functions have been derived. The long time behavior of the Green functions leads to a modification of these equations. This new set of equations is appropriate for numerical implementations. The theory is illustrated in a series of numerical computations. These confirm several pleasant properties that can be expected when working in the time domain. Transient wave propagation is treated in a natural way using time domain techniques and the effects of rapid changes in the propagating fields is readily visible. Since the Green functions are independent of the wave fields, the numerical implementation is performed in an efficient way for various boundary conditions.

Apart from being an efficient tool to solve direct problems, the major interests towards the Green function technique concerns applications to the inverse formulation. Hence, the Green function approach may be extended to cover direct and inverse problems for more complicated structures, such as an inhomogeneous beam resting on a viscoelastic layer. Direct and inverse analysis of corresponding problems are natural extensions of this paper.

Acknowledgement

The present work is supported by the Swedish Research Council for Engineering Sciences (TFR) and this is gratefully acknowledged.

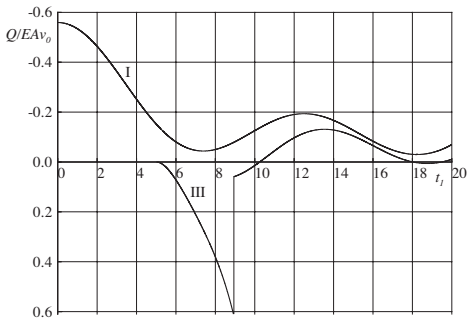
Appendix A Operators appearing in $\mathcal{P}$ and $\mathcal{P}^{-1}$

The operators $\mathcal{Q}, \mathcal{U}$ and $\mathcal{S}$ act as convolution operators

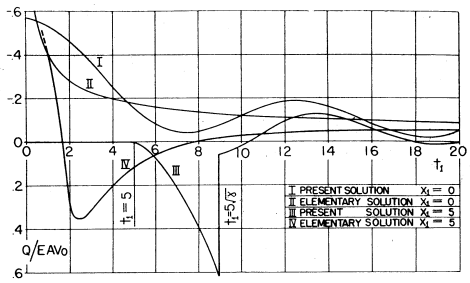
$$\begin{cases} \mathcal{Q}f(t) = \big(Q(\cdot) * f(\cdot)\big)(t) \\ \mathcal{U}f(t) = \big(U(\cdot) * f(\cdot)\big)(t) \\ \mathcal{S}f(t) = \dfrac{c_1}{c_2}\Big(f(t) + \big(S(\cdot) * f(\cdot)\big)(t)\Big) \end{cases}$$

where

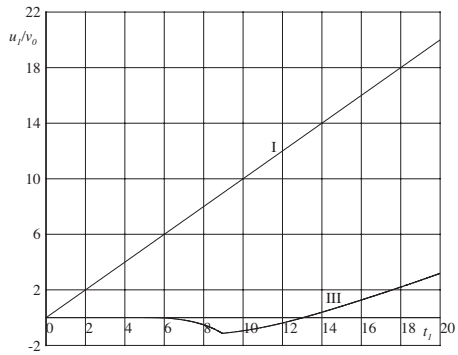
$$\begin{cases} Q(t) = \dfrac{r_0 c_2}{4} H(t) \displaystyle\int_0^{t/\tau} I_0(\xi)\, d\xi \\ U(t) = \dfrac{r_0 c_2^2}{c_1} H(t) \sin\left(\dfrac{c_1 t}{r_0}\right) \\ S(t) = \dfrac{c_1}{r_0} H(t) \displaystyle\int_0^{c_1 t/r_0} \dfrac{J_1(\xi)}{\xi}\, d\xi \end{cases}$$



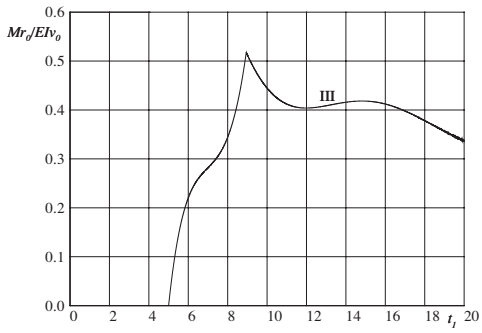
(a) The shear force Q.



(b) The shear force Q according to [4].

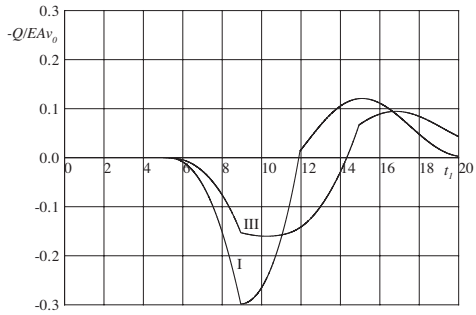


(c) The vertical displacement u_1.

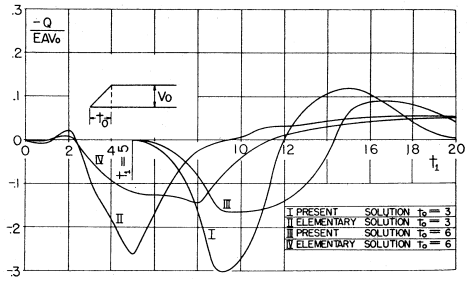


(d) The bending moment M.

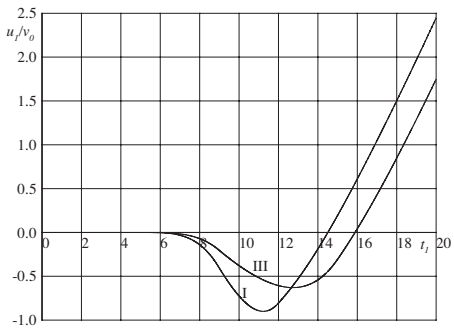
Figure 4: $M(0, t_1) = 0$ and $u_1(0, t_1)$ given. I : $z_1 = 0$ and $t_0 = 0$ III : $z_1 = 5$ and $t_0 = 0$.



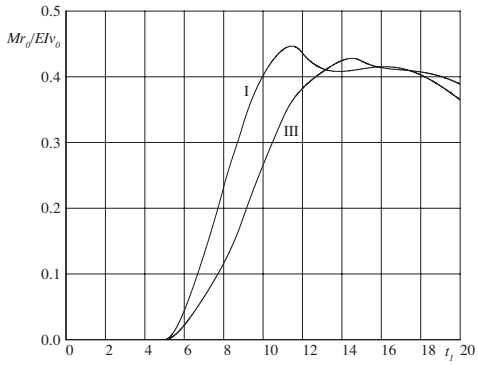
(a) The shear force Q.



(b) The shear force Q according to [4].

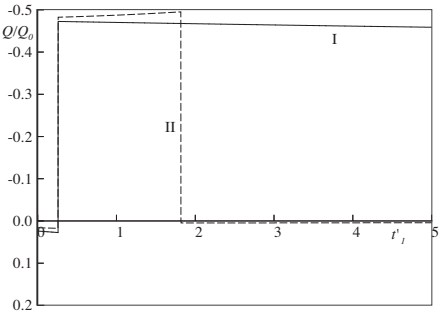


(c) The vertical displacement u_1.

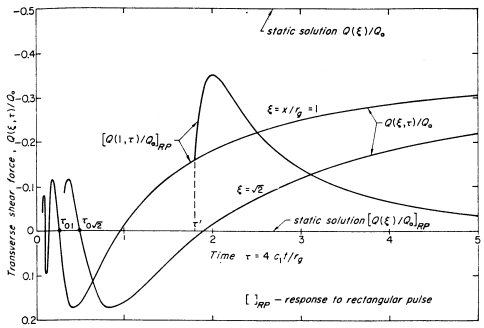


(d) The bending moment M.

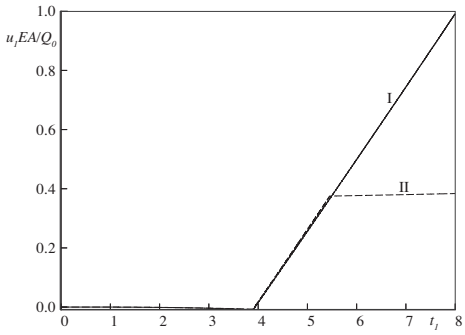
Figure 5: $M(0, t_1) = 0$ and $u_1(0, t_1)$ given. I : $z_1 = 5$ and $t_0 = 3$ III : $z_1 = 5$ and $t_0 = 6$.



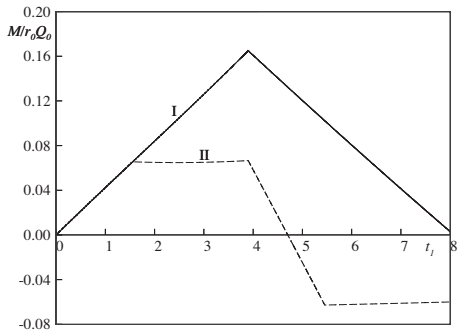
(a) The shear force Q.



(b) The shear force Q according to [25].



(c) The vertical displacement u_1.



(d) The bending moment M.

Figure 6: $\psi(0, t_1) = 0$ and $Q(0, t_1)$ given. I : $z_1 = 1$ and step shear force II : $z_1 = 1$ and rectangular shear force of width $\Delta t_1 = 1.5$.

The operator $\mathcal{Q}$ satisfies

$$2\mathcal{Q}\left(\lambda_1^2-\lambda_2^2\right)=2\left(\lambda_1^2-\lambda_2^2\right)\mathcal{Q}=1$$

Here, λ_1^2 and λ_2^2 can be represented as

$$\begin{cases}\lambda_1^2=\dfrac{1}{c_1^2}\dfrac{\partial^2}{\partial t^2}-\dfrac{1}{2r_0c_2\tau}-V(\cdot)*\\ \lambda_2^2=\dfrac{1}{c_2^2}\dfrac{\partial^2}{\partial t^2}+\dfrac{1}{2r_0c_2\tau}+V(\cdot)*\end{cases}$$

where the function $V(t)$ is defined by

$$V(t)=\frac{1}{r_0c_2\tau t}H(t)I_2(t/\tau)$$

An error in the expressions for λ_i^2 in Ref. [26] has been corrected above.

Appendix B The canonical representation

Since the differential operator in the Timoshenko equation is linear and invariant to time-translations, we may define operators $\mathcal{T}_i(z)$ that map the fields $u_i^+(0,t)$ to the internal fields $u_i^+(z,t)$. Introduce the canonical impulse responses $U_i^+(z,t)$, according to

$$\delta_i(t)\xrightarrow{\mathcal{T}_i(z)}U_i^+(z,t)$$

where $\delta_i(t)$ are the Dirac delta functions. The fields $u_i^+(0,t)$ are thus transformed as

$$u_i^+(0,t)\xrightarrow{\mathcal{T}_i(z)}\int_{-\infty}^{\infty}u_i^+(0,t')U_i^+(z,t-t')dt'=u_i^+(z,t)$$

according to Borel's theorem [5]. If causality is considered, we obtain

$$u_i^+(z,t)=\int_{0^-}^{t^+-z/c_i}u_i^+(0,t')U_i^+(z,t-t')dt'$$

By extracting the distribution and performing a transformation of the time coordinate, the former equation can be written as

$$u_i^+(z,t+z/c_i)=u_i^+(0,t)V_i^+(z,z^+/c_i)+\int_{0^+}^{t^-}u_i^+(0,t')U_i^+(z,t+z/c_i-t')dt' \quad \text{(B.1)}$$

Here, $V_i^+(z,t)$ are the step function responses. The magnitudes of the steps can easily be determined, as $V_i^+(z,t)$ must satisfy the dynamics (3.5). Hence

$$d_zV_i^+(z,z^+/c_i)=0 \implies V_i^+(z,z^+/c_i)=1$$

since $V_i^+(0,0^+) = 1$. Defining the Green functions by $G_i(z,t) = U_i^+(z, t + z/c_i)$, equation (B.1) reads

$$u_i^+(z, t + z/c_i) = u_i^+(0,t) + \int_0^t u_i^+(0,t')G_i(z, t - t')dt'$$

where we have prescribed the integrands to contain no distributions. So, the Green functions are causal with boundary conditions according to

$$G_i(0,t) = 0 \quad t > 0$$

This representation is of course analogous to the results when applying Duhamel's integral [10].

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