New physical limitations in scattering and antenna problems

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NEW PHYSICAL LIMITATIONS IN SCATTERING AND ANTENNA PROBLEMS

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Abstract: The extinction cross section integrated over all frequencies is shown to be related to the electric and magnetic polarizability dyadics by exploiting the analytic properties of the forward scattering dyadic. This identity can be used threefold: 1) in scattering theory to bound the total scattering properties of an arbitrary scatterer, 2) in antenna theory to derive new physical limitations on antennas, and 3) in material modeling. The theory is illustrated by numerical simulations with excellent agreement.

THE INTEGRATED EXTINCTION

Let an incident plane wave impinge on a bounded scatterer of arbitrary shape in the direction $\hat{\boldsymbol{k}}$. The material properties of the scatterer are in general both inhomogeneous and dispersive, and not necessarily isotropic. The dominant contribution of the scattered field at large distance, r, in the forward direction $\hat{\boldsymbol{k}}$ is expressed by the scattering dyadic, $\mathbf{S}(k,\hat{\boldsymbol{k}})$, via

$$\boldsymbol{E}_{\mathrm{s}}(k,r\hat{\boldsymbol{k}}) = \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\mathbf{S}(k,\hat{\boldsymbol{k}})\cdot\boldsymbol{E}_0 \quad \text{as } r\to\infty$$

where k is a complex variable in the upper half plane with $\mathrm{Re}\,k=\omega/\mathrm{c}_0$. Here, $\boldsymbol{E}_0$ is the (Fourier) amplitude of the incident wave evaluated at the origin.

The analytic properties of $\mathbf{S}$ are employed to determine an integral identity for the extinction cross section, σ_{ext}, by the use of the Hilbert transform. For this purpose, introduce the analytic function

$$\kappa(k) = -\frac{\hat{\boldsymbol{p}}_{\mathrm{e}}\cdot\mathbf{S}(k,\hat{\boldsymbol{k}})\cdot\hat{\boldsymbol{p}}_{\mathrm{e}}^{*}}{\mathrm{i}k} + \frac{A(\hat{\boldsymbol{k}})}{2\pi}, \qquad \mathrm{Im}\,k\geq 0$$

where $A(\hat{\boldsymbol{k}})$ is real-valued, and $\hat{\boldsymbol{p}}_{\mathrm{e}}=\boldsymbol{E}_0/|\boldsymbol{E}_0|$ is a complex-valued vector, independent of k, that represents the electric polarization, and, moreover, satisfies $\hat{\boldsymbol{p}}_{\mathrm{e}}\cdot\hat{\boldsymbol{k}}=0$. The function $\kappa(k)$ is assumed to vanish in the limit $|k|\to\infty$ in the upper half plane. This assumption is correct for a large class of scatterers and supported by the Physical Optics approximation for a perfectly conducting scatterer. Furthermore, the function κ is assumed to be square integrable in k on the real axis, and, due to causality of the scattering dyadic in the forward direction, its inverse Fourier transform vanishes on the negative real axis, *i.e.*,

$$\kappa_{\mathrm{t}}(\tau) = \frac{1}{2\pi}\int_{\mathbb{R}}\kappa(k)\mathrm{e}^{-\mathrm{i}k\tau}\ \mathrm{d}k = 0 \quad \text{for } \tau<0$$

Under these assumptions, it follows by Titchmarsh's theorem that κ satisfies the Hilbert transform

$$\kappa(k) = \frac{1}{\pi\mathrm{i}}\mathcal{P}\int_{\mathbb{R}}\frac{\kappa(k')}{k'-k}\ \mathrm{d}k' \tag{1}$$

where k is real-valued and $\mathcal{P}$ denotes Cauchy's principle value. Based on the imaginary part of (1), it is straightforward to derive the integrated extinction ($\lambda=2\pi/k$)

$$2\pi\int_0^\infty\frac{\sigma_{\mathrm{ext}}(k)}{k^2}\ \mathrm{d}k = \int_0^\infty\sigma_{\mathrm{ext}}(\lambda)\ \mathrm{d}\lambda = \pi^2\left(\hat{\boldsymbol{p}}_{\mathrm{e}}^{*}\cdot\boldsymbol{\gamma}_{\mathrm{e}}\cdot\hat{\boldsymbol{p}}_{\mathrm{e}} + \hat{\boldsymbol{p}}_{\mathrm{m}}^{*}\cdot\boldsymbol{\gamma}_{\mathrm{m}}\cdot\hat{\boldsymbol{p}}_{\mathrm{m}}\right) \tag{2}$$

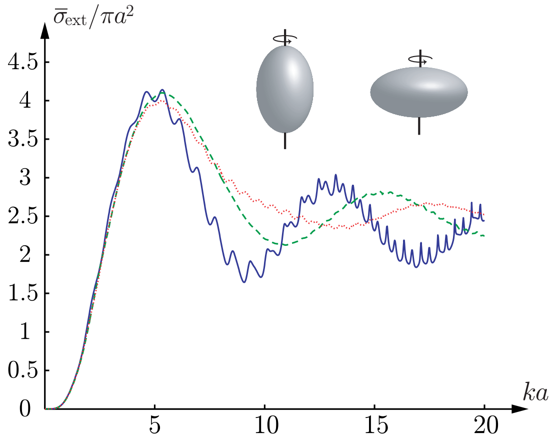

Figure 1: The extinction cross section averaged over all incident directions for a prolate (dashed) and oblate (dotted) non-magnetic spheroid with permittivities $\epsilon = 2$ and semi-axis ratios $\xi = 1/2$. The solid curve depicts the volume-equivalent sphere of radius a.

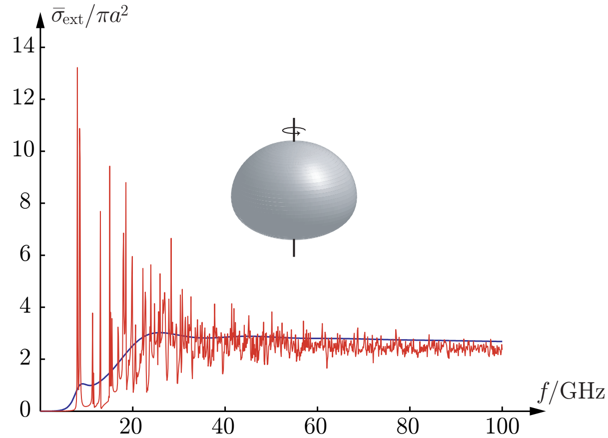

Figure 2: The extinction cross section averaged over all incident directions for a raindrop of volume-equivalent radius $a = 2\,\mathrm{mm}$. The smooth curve shows a dispersive material (Debye-model), while the oscillatory curve presents the corresponding a non-dispersive case ($\epsilon = 80.2$), with identical long wavelength permittivity.

via the optical theorem $\sigma_{\rm ext}(k) = -4\pi \,\mathrm{Re}\, \kappa(k)$, and the low-frequency behavior of the forward scattering dyadic, $\mathbf{S}(k, \hat{\boldsymbol{k}})$, see Kleinman *et al.* [1], *viz.*,

$$\kappa(0) = \left(\hat{\boldsymbol{p}}_{\rm e}^* \cdot \boldsymbol{\gamma}_{\rm e} \cdot \hat{\boldsymbol{p}}_{\rm e} + \hat{\boldsymbol{p}}_{\rm m}^* \cdot \boldsymbol{\gamma}_{\rm m} \cdot \hat{\boldsymbol{p}}_{\rm m}\right)/4\pi$$

Here, $\hat{\boldsymbol{p}}_{\rm m} = \hat{\boldsymbol{k}} \times \hat{\boldsymbol{p}}_{\rm e}$ denotes the magnetic polarization, and $\boldsymbol{\gamma}_{\rm e}$ and $\boldsymbol{\gamma}_{\rm m}$ are the electric and magnetic polarizability dyadics, respectively. These dyadics are real-valued and symmetric, and they scale with the volume of the scatterer. A full derivation of (2) is presented in Sohl *et al.* [2]. The sum of the scattering and absorption cross sections is the extinction cross section, $\sigma_{\rm ext} = \sigma_{\rm s} + \sigma_{\rm a}$. The three cross sections are by definition real-valued and non-negative. A direct consequence of (2) is that any two scatterers with the same polarizability dyadics share the same value of the integral.

APPLICATIONS

Three major applications of the result presented in the previous section are natural, and these topics are briefly commented below. The full analyses are found in Sohl *et al.* [2] and in two forthcoming papers.

Scattering. In this application we concentrate on the scattering cross section. The first, most obvious, application of (2) is to find physical bounds on the scattering properties of a scatterer. To this end, use the estimate

$$\int_{\Lambda} \sigma_{\rm s}(\lambda)\;\mathrm{d}\lambda \leq \int_0^{\infty} \sigma_{\rm ext}(\lambda)\;\mathrm{d}\lambda = \pi^2 \left(\hat{\boldsymbol{p}}_{\rm e}^* \cdot \boldsymbol{\gamma}_{\rm e} \cdot \hat{\boldsymbol{p}}_{\rm e} + \hat{\boldsymbol{p}}_{\rm m}^* \cdot \boldsymbol{\gamma}_{\rm m} \cdot \hat{\boldsymbol{p}}_{\rm m}\right)$$

and we see that the broadband scattering properties of a scatterer in any wavelength interval $\Lambda \in [0, \infty)$, are bounded by the static polarizability properties of the scatterer. We illustrate this identity in Figure 1 for three scatterers with different polarizability dyadics. Using the results presented in Jones [3], it is seen that among all isotropic, homogeneous scatterers of equal volume and susceptibilities, the spherical scatterer minimizes the averaged integrated extinction, and consequently, the solid curve has a lower integrated extinction compared to the other two. Several more numerical examples that verify and illustrate these results are presented in Sohl *et al.* [2], as well as additional lower and upper bounds on the integrated extinction.

Antennas. In the antenna application the absorption problem is in focus. Using similar arguments as in the scattering case, but retaining the absorption cross section, which is related to the partial realized gain G_Λ of the antenna, we obtain

$$G_\Lambda B \leq \frac{4\pi}{\lambda_0^3}\int_\Lambda \sigma_{\mathrm{a}}(\lambda)\ \mathrm{d}\lambda \leq \frac{4\pi}{\lambda_0^3}\int_0^\infty \sigma_{\mathrm{ext}}(\lambda)\ \mathrm{d}\lambda = \frac{4\pi^3}{\lambda_0^3}\left(\hat{\boldsymbol{p}}_{\mathrm{e}}^*\cdot\boldsymbol{\gamma}_{\mathrm{e}}\cdot\hat{\boldsymbol{p}}_{\mathrm{e}} + \hat{\boldsymbol{p}}_{\mathrm{m}}^*\cdot\boldsymbol{\gamma}_{\mathrm{m}}\cdot\hat{\boldsymbol{p}}_{\mathrm{m}}\right)$$

where B denotes the relative bandwidth of the antenna. This inequality bounds the product of the partial realized gain and the relative bandwidth, and it shows several similarities with the classic results by Chu [4], which gives bounds on the directivity and the Q-value of an antenna. However, the results by Chu hold only for spherical inclusions. The antenna result presented here has no such limitations, but is instead valid for an antenna of arbitrary shape and electrical size. Additional advantages are that the present analysis includes a description of the material as well as the polarization properties of the antenna. Moreover, no resonance model is assumed, and no introduction of a quality factor Q is needed.

Material modeling. The integrated extinction (2) relates the scattering properties at all frequencies to the static properties of the scatterer. As a consequence, all materials that have the same static properties, yield the same integrated extinction. These effects are clearly seen in Figure 2, where the areas under the two curves are identical, despite the fact that the material properties of the two objects are different at all non-zero frequencies. As a result of (2), the larger the scattering, the smaller the bandwidth (higher Q-value). It is therefore not possible to obtain arbitrary high scattering at a certain frequency interval without paying the price of more sharp resonances. As an additional consequence, an object that shows bi-isotropic effects has an integrated extinction that is identical to the corresponding isotropic scatterer with the same static properties. Since all chiral effects vanish in the low-frequency limit, it is clear that the introduction of chirality cannot enhance the integrated extinction. Similar arguments apply to materials that show negative permittivity and/or permeability.

CONCLUSIONS

It is shown in this paper that one of the most fundamental physical properties, *viz.*, causality, implies that the extension cross section integrated over all wavelengths is identical to the sum of the electric and magnetic polarizability properties of the scatterer. This identity leads to a number of interesting bounds on scattering, antennas, and material modeling. It is believed that there are more physical quantities that apply to this theory.

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