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LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

An Upper Bound on Decoding Bit-Error Probability with Linear Coding on Extremely Noisy Channels[1]

Marc Handlery, Rolf Johannesson, James L. Massey, and Per Ståhl
Department of Information Technology
Lund University, Box 118, S-221 00 Lund, Sweden
e-mail: {marc,rolf,pers}@it.lth.se, JamesMassey@compuserve.com

Abstract — **When concatenated coding schemes operate near channel capacity their component encoders may operate above capacity. The decoding bit-error performance of binary convolutional codes near and above capacity is investigated.**

Let $G(D)$ be a $b \times c$ generator matrix of a rate $R = b/c$ convolutional code. We define a *tap-minimal right pseudo inverse* of the generator matrix $G(D)$ to be a right pseudo inverse of $G(D)$ with the minimum number of taps among all right pseudo inverses. By the number of "taps" in a right pseudo inverse we mean the total number of nonzero coefficients in the power series that are entries of this $c \times b$ matrix.

We now define the *pseudo-inverse decoder* (π-decoder) for convolutional codes. Assume that we use a convolutional code $\mathcal{C}$ encoded by the generator matrix $G(D)$ for transmission over a binary symmetric channel (BSC) with crossover probability ϵ. The decoding technique is as simple as it gets: The received sequence $\boldsymbol{r}$ is fed directly to a tap-minimal right pseudo inverse of $G(D)$ whose output is the decoded information sequence.

The exact decoding bit-error probability using the π-decoder is $P_b = \frac{1}{b}\sum_{i=1}^{b}\frac{1}{2}(1-(1-2\epsilon)^{n_i})$. Clearly this is an upper bound on the decoding bit-error probability with minimum bit-error probability decoding. We call it the *π-bound.* For probabilities $\epsilon < 0.5$, it suggests that systematic encoders, which have the fewest taps in their tap-minimal right pseudo inverse, give lower bit-error probability than nonsystematic ones. Fig. 1 shows that for large ϵ, the π-bound is very tight.

When we transmit over a binary erasure channel (BEC), either a zero or a one is assigned randomly to the erased digits in the channel output sequence which thereafter is fed to a tap-minimal right pseudo inverse of $G(D)$ whose output is the decoded information sequence. Then, $P_b = \frac{1}{b}\sum_{i=1}^{b}\frac{1}{2}(1-(1-p)^{n_i})$ for this "π-decoder", where p is the erasure probability of the BEC. This P_b is again an upper bound on the bit-error probability with minimum bit-error probability decoding.

Using Ancheta's bound on linear source coding [1], we can show that the minimum bit-error rate that can be achieved with rate R *linear* coding for a BEC is

$$P_b = (1 - C/R)/2\,, \tag{1}$$

where $C = 1 - p$ is the capacity of the BEC.

Shamai *et al.* [2] have given a general formulation for the minimum code rate required to approach a specified bit-error probability, showing that nonsystematic codes are inherently superior to systematic codes. For systematic coding on the BEC, this minimum code rate can be explicitly written as

$$R = \frac{C}{1-(1-C)h\left(\frac{P_b}{1-C}\right)} \tag{2}$$

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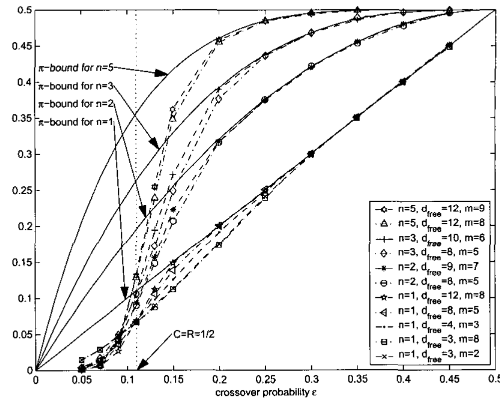

Fig. 1: π-bounds and simulated decoding bit-error probability on a BSC of various rate $R = 1/2$ convolutional codes.

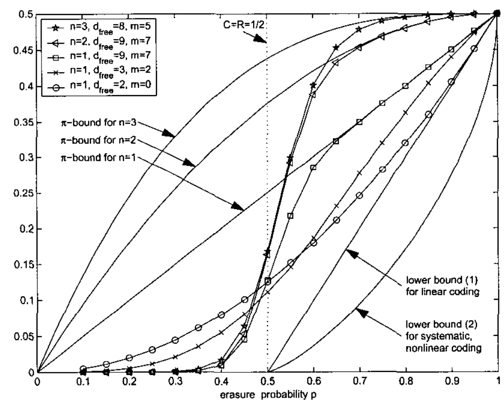

Fig. 2: π-bounds and simulated decoding bit-error probability on a BEC of various rate $R = 1/2$ convolutional codes.

for all bit-error probabilities P_b with $0 < P_b \leq (1-C)/2$. In Fig. 2 the lower bounds (1) and (2) are plotted for $R = 1/2$.

We see from Fig. 2 that it is *impossible* with linear coding to obtain the performance that Shamai *et al.* have shown can be obtained with systematic coding. We conclude that this latter performance necessarily requires the use of nonlinear codes so that the bounds of [2], while interesting in their own right, are not relevant to the analysis of linear coding schemes. The inherent superiority of nonsystematic codes over systematic codes appears to be limited to the case where nonlinear codes are used, which is the atypical case in practice.

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