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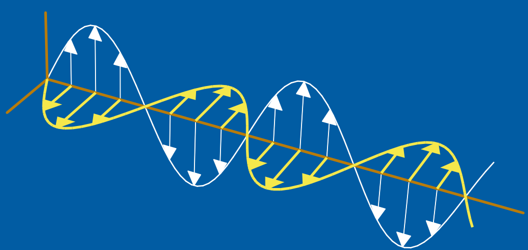

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Abstract

In this paper, we introduce a first order accurate resonance model based on a second order Padé approximation of the reflection coefficient of a narrowband antenna. The resonance model is characterized by its Q factor, given by the frequency derivative of the reflection coefficient. The Bode-Fano matching theory is used to determine the bandwidth of the resonance model and it is shown that it also determines the bandwidth of the antenna for sufficiently narrow bandwidths. The bandwidth is expressed in the Q factor of the resonance model and the threshold limit on the reflection coefficient. Spherical vector modes are used to illustrate the results. Finally, we demonstrate the fundamental difficulty of finding a simple relation between the Q of the resonance model, and the classical Q defined as the quotient between the stored and radiated energies, even though there is usually a close resemblance between these entities for many real antennas.

1 Introduction

The bandwidth of an antenna system can in general only be determined if the impedance is known for all frequencies in the considered frequency range. However, even if the impedance is known, the bandwidth depends on the specified threshold level of the reflection coefficient and the use of matching networks. The Bode-Fano matching theory [4, 11] gives fundamental limitations on the reflection coefficient using any realizable matching networks and hence a powerful definition of the bandwidth for any antenna system. However, as it is an analytical theory it requires explicit expressions of the reflection coefficient for all frequencies.

The quality (Q) factor of an antenna is a common and simple way to quantify the bandwidth of an antenna [2, 7, 14]. The Q of the antenna is defined as the quotient between the power stored in the reactive field and the radiated power. There are several attempts to express the Q factor in the impedance of the antenna, see *e.g.*, [14] with references. In [14], an approximation based on the frequency derivative of the input impedance, $Q \approx \omega|Z'|/(2R)$, is introduced and shown to be very accurate for some antennas.

In this paper, we employ a Padé approximation to show that the Bode-Fano bandwidth of a narrowband antenna is determined by the amplitude of the frequency scaled frequency derivative of the reflection coefficient, $\omega_0|\rho'|$. Moreover, $Q_\rho = \omega_0|\rho'| = \omega|Z'|/(2R)$ is identified as the Q factor of a first order accurate approximating resonance model of the antenna. We observe that the classical Q-factor, defined as the quotient between the stored and radiated energies, of the antenna system is not utilized nor needed in the analysis. However, there is a close resemblance between the Q-factor derived from the differentiated reflection coefficient, Q_ρ, and the classical Q-factor, Q. It is shown that $Q \approx Q_\rho$ for the spherical vector modes if Q is sufficiently large. This is also seen from the approximation of the Q-factor $Q \approx \omega|Z'|/(2R) = Q_\rho$ considered in [14]. However, a simple example is used to demonstrate that there are no simple relation between Q and Q_ρ for general antennas.

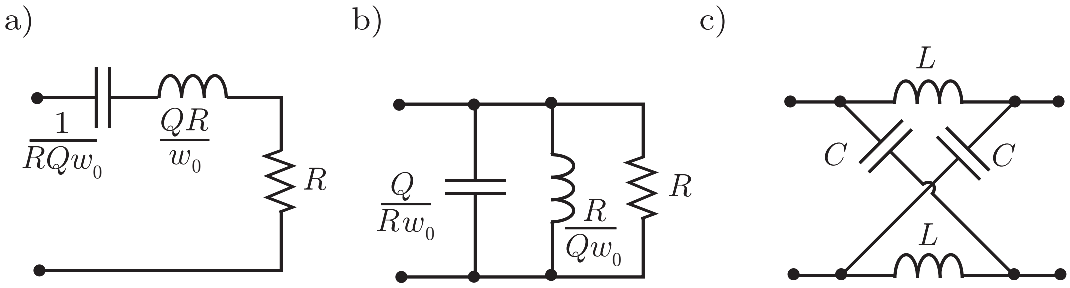

Figure 1: Lumped circuits. a) the series RCL circuit. b) the parallel RCL circuit. c) a lattice network.

The rest of the paper is outlined as follows. In Section 2, the Q factor and lumped RCL circuits are reviewed. The Padé approximation of the reflection coefficient is introduced in Section 3. In Section 4, the Bode-Fano bandwidth of the resonance model and the bandwidth of the corresponding antennas are analyzed. The results are illustrated using spherical vector modes in Section 5. In Section 6, an antenna constructed with a flat reflection coefficient is used to demonstrate the fundamental difficulties of finding a simple relation between Q and Q_ρ for general antennas. Conclusions are given in Section 7.

2 Q factor and resonance circuits

The Q factor (quality factor, antenna Q or radiation Q) is commonly used to get an estimate of the bandwidth of an antenna. Since, there is an extensive literature on the Q factor for antennas, see *e.g.*, [2, 3, 6, 7, 14], only some of the results are given here. The Q factor of the antenna is defined as the quotient between the power stored in the reactive field and the radiated power [2, 7], *i.e.*,

$$Q = \frac{2\omega \max(W_{\mathrm{M}}, W_{\mathrm{E}})}{P}, \tag{2.1}$$

where ω is the angular frequency, W_{M} the stored magnetic energy, W_{E} the stored electric energy, and P the dissipated power. At the resonance frequency, ω_0, there are equal amounts of stored electric energy and stored magnetic energy, *i.e.*, $W_{\mathrm{E}} = W_{\mathrm{M}}$.

The Q factor is also fundamentally related to the lumped resonance circuits [11]. The basic series (parallel) resonance circuit consists of series (parallel) connected inductor, capacitor, and resistor, see Figure 1ab. With a resonance frequency ω_0 and resistance R, we have $L = RQ/\omega_0$ and $C = 1/(RQ\omega_0)$ and $L = R/(Q\omega_0)$ and $C = Q/(R\omega_0)$ in the series and parallel cases, respectively. It is easily seen that the Q factor defined in (2.1) is consistent with the lumped resonance circuits [11].

The transmission coefficient of the resonance circuits in Figure 1ab, is

$$t_{\mathrm{RCL}}(s) = \frac{1}{1 + \frac{Q}{2}\left(\frac{\omega_0}{s} + \frac{s}{\omega_0}\right)}, \tag{2.2}$$

where $s = \sigma + \mathrm{i}\omega$ denote the Laplace parameter. It has one zero at the origin, $s = 0$, and one zero at infinity, $s = \infty$. The corresponding reflection coefficient is

$$\rho_{\mathrm{RCL}}(s) = \frac{Z(s) - R}{Z(s) + R} = \pm \frac{1 + (s/\omega_0)^2}{1 + (s/\omega_0)^2 + 2s/(\omega_0 Q)} \tag{2.3}$$

where the $+$ and $-$ minus signs correspond to series and parallel circuits, respectively. The zeros and poles of the reflection coefficient are

$$\lambda_{\mathrm{o}1,2} = \pm \mathrm{i}\omega_0 \quad \text{and } \lambda_{\mathrm{p}1,2} = \frac{\omega_0}{Q}\left(-1 \pm \mathrm{i}\sqrt{Q^2 - 1}\right), \tag{2.4}$$

respectively. We also observe that differentiation of the reflection coefficient with respect to $\mathrm{i}\omega/\omega_0$ gives Q, *i.e.*,

$$\left.\frac{\partial \rho_{\mathrm{RCL}}}{\partial \omega}\right|_{\omega=\omega_0} = \frac{\pm \mathrm{i}Q}{\omega_0} \tag{2.5}$$

and hence $Q = \omega_0 |\rho'_{\mathrm{RCL}}(\omega_0)|$.

3 Padé approximation of the reflection coefficient

Here, we consider a local approximation of a given reflection coefficient, $\tilde{\rho}$, of an antenna. We assume that the resonance frequency, ω_0, and the frequency derivative of the reflection coefficient, $\tilde{\rho}'(\mathrm{i}\omega_0)$ are known. The model, ρ, is required to be a local approximation to the first order, *i.e.*, it is tuned to the resonance frequency

$$\rho(\mathrm{i}\omega_0) = \tilde{\rho}(\mathrm{i}\omega_0) = 0, \tag{3.1}$$

and its frequency derivative is specified

$$\left.\frac{\partial \rho}{\partial \omega}\right|_{\omega=\omega_0} = \left.\frac{\partial \tilde{\rho}}{\partial \omega}\right|_{\omega=\omega_0} = \tilde{\rho}'. \tag{3.2}$$

We also require that the model is unmatched far from the resonance frequency

$$|\rho(0)| = |\rho(\infty)| = 1. \tag{3.3}$$

The error in the approximation can be estimated with the second order derivative of the reflection coefficient. We assume that the reflection coefficients are continuously differentiable two times. This gives an error of second order in $\beta = 2(\omega - \omega_0)/\omega_0$, *i.e.*,

$$|\rho(\mathrm{i}\omega) - \tilde{\rho}(\mathrm{i}\omega)| = \mathcal{O}(\beta^2). \tag{3.4}$$

Observe that a curve fitting techniques might be more practical for experimental data, see *e.g.*, [10].

We start with a Padé approximation of the reflection coefficient. A general Padé approximation of order 2,2 is

$$\rho(s) = \gamma \frac{1 + a_1 s + a_2 s^2}{1 + b_1 s + b_2 s^2} \tag{3.5}$$

where a_1, a_2, b_1, b_2 are real valued constants. As the reflection coefficient has an arbitrary phase at resonance, it is necessary to consider a complex valued coefficient γ. We interpret this as a slowly varying function $\tilde{\gamma}(s)$ where $\tilde{\gamma}(\mathrm{i}\omega) \approx \gamma$ over the considered frequency interval. The requirement (3.3) gives $|\gamma| = 1$ and $|a_2| = |b_2|$. We also have $\tilde{\gamma}(-\mathrm{i}\omega) \approx \gamma^*$ for any physically realizable model. The resonance frequency imply $a_1 = 0$ and $a_2 = \omega_0^{-2}$. Differentiation with respect to the angular frequency gives

$$\gamma \frac{-\frac{2}{\omega_0}}{1 + b_1 \mathrm{i}\omega_0 - b_2\omega_0^2} = \tilde{\rho}' \tag{3.6}$$

and hence $b_2 = \omega_0^{-2}$ and $b_1 = 2/(\omega_0 Q_\rho)$, where we have introduced the Q factor in the resonance approximation as

$$Q_\rho = |\tilde{\rho}'(\mathrm{i}\omega_0)\omega_0| \tag{3.7}$$

in accordance with (2.5). We observe the resemblance with the approach in [14] showing that the Q factor of some antennas, Q, can be approximated with the frequency derivative of the impedance, *i.e.*,

$$Q \approx \omega_0 \frac{|Z_1'|}{2R} = \omega_0 |\tilde{\rho}'| = Q_\rho. \tag{3.8}$$

The Padé approximation of the reflection coefficient can be written

$$\rho(s) = \frac{-\mathrm{i}\tilde{\rho}'}{|\tilde{\rho}'|} \frac{1 + (s/\omega_0)^2}{1 + (s/\omega_0)^2 + 2s/(\omega_0 Q_\rho)}. \tag{3.9}$$

The special case with $\arg\tilde{\rho}' = \pi/2$ ($\arg\tilde{\rho}' = -\pi/2$) gives the classical lumped series (parallel) RCL circuit approximation. Observe that Q_ρ is the Q factor of the approximating resonance circuit and not the Q factor of the original system.

We can interpret the general cases with $\operatorname{Re}\tilde{\rho}' \neq 0$ as the result with a cascade coupled RCL circuit and a transmission line with characteristic impedance R. A transmission line with length d rotates the reflection coefficient an angle $\phi = -2dk_0 = -2d\omega_0/c_0$ in the complex plane. It is also possible to consider a lattice network that rotates the reflection coefficient [13]. A lattice network with capacitance, C, and inductance, $L = R^2C$, as shown in Figure 1, has reflection coefficient $\rho_{\mathrm{L}}(s) = 0$ and transmission coefficient

$$t_{\mathrm{L}}(s) = \frac{1 - sRC}{1 + sRC} = \frac{1 - \alpha s/\omega_0}{1 + \alpha s/\omega_0}, \tag{3.10}$$

where we have introduced the dimensional free parameter $\alpha = \omega_0 RC$. The reflection coefficient of the cascaded lattice and RCL circuit is

$$\rho(s) = t_{\mathrm{L}}^2(s)\rho_{\mathrm{RCL}}(s) = \pm\left(\frac{1 - \alpha s/\omega_0}{1 + \alpha s/\omega_0}\right)^2 \frac{1 + (s/\omega_0)^2}{1 + (s/\omega_0)^2 + 2s/\omega_0/Q_\rho} \tag{3.11}$$

where it is seen that the lattice network rotates the reflection coefficient an angle $\phi = -4\arctan(\alpha)$. It is easily seen that $\alpha = -\tan(\phi/4)$ and hence $0 < \alpha < 1$ as it is sufficient to consider $-\pi < \phi < 0$. The transmission coefficient of the cascaded system is given by $t = t_{\mathrm{L}} t_{\mathrm{RCL}}$.

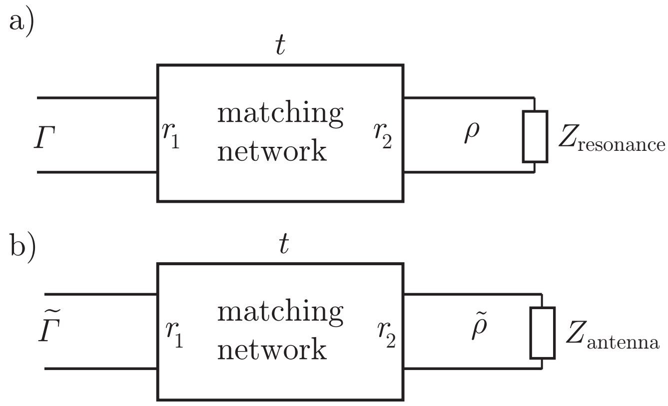

Figure 2: Illustration of the lossless matching networks. The matching network has the reflection coefficients r_1 and r_2 and transmission coefficient t. The same matching network is used in the two cases. a) the resonance circuit with reflection coefficient ρ gives Γ. b) the antenna with reflection coefficient $\tilde{\rho}$ gives $\widetilde{\Gamma}$.

4 Bandwidth and matching

The reflection coefficient (3.11) provides a local approximation of the reflection coefficient of the antenna. Assume that the error of the reflection coefficient of the approximate circuit is of size ϵ, *i.e.*,

$$|\rho(\mathrm{i}\omega) - \tilde{\rho}(\mathrm{i}\omega)| \leq \epsilon \tag{4.1}$$

over the frequency band of interest. We consider a general lossless matching network to determine the bandwidth of the antenna and the approximate resonance circuits as illustrated in Figure 2. The error in the reflection coefficient after matching is estimated as

$$\begin{aligned} |\Gamma - \widetilde{\Gamma}| = |t|^2 \left| \frac{\rho}{1 - r_2\rho} - \frac{\tilde{\rho}}{1 - r_2\tilde{\rho}} \right| = |t|^2 \frac{|\rho - \tilde{\rho}|}{|1 - r_2\rho||1 - r_2\tilde{\rho}|} \\ \leq \frac{1 - |r_2|^2}{(1 - \delta|r_2|)^2}\epsilon \leq \frac{\epsilon}{1 - \delta^2}, \end{aligned} \tag{4.2}$$

where $\delta = \max(|\rho|, |\tilde{\rho}|)$. It is observed that the approximate circuit can be used in the matching analysis as long as the error, ϵ, is sufficiently small and the reflection coefficients are less than unity. The reflection coefficient of the matched antenna is estimated by the triangle inequality as

$$\big||\widetilde{\Gamma}| - |\Gamma|\big| \leq |\Gamma - \widetilde{\Gamma}| \leq \frac{\epsilon}{1 - \delta^2} = \mathcal{O}(\beta^2), \tag{4.3}$$

where we used (3.4).

The Bode-Fano theory is used to get fundamental limitations on the matching network [4, 12]. The Bode-Fano theory uses Taylor expansions of the reflection coefficient around the zeros of the transmission coefficient to get a set of integral

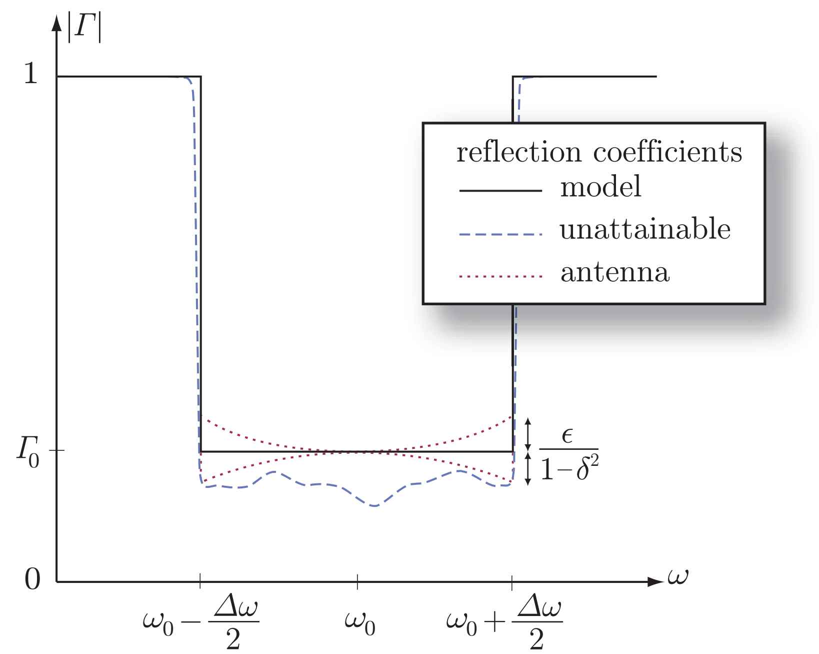

Figure 3: Illustration of the Bode-Fano limits. The model gives the threshold Γ_0. The threshold level of the corresponding antenna is estimated with (4.3). The dashed curve illustrates an unattainable reflection coefficient.

relations for the reflection coefficient. We start with the lumped RCL circuit. The transmission coefficient (2.2) of the RCL circuit has a single zero at the origin and a single zero at infinity. The Bode-Fano theory gives the integral relations

$$\frac{2}{\pi}\int_0^\infty \frac{1}{\omega^2}\ln\frac{1}{|\Gamma(\mathrm{i}\omega)|}\,\mathrm{d}\omega = \sum_i \lambda_{\mathrm{o}i}^{-1} - \lambda_{\mathrm{p}i}^{-1} - 2\lambda_{\mathrm{r}i}^{-1} = \frac{2}{\omega_0 Q} - 2\sum_i \lambda_{\mathrm{r}i}^{-1} \tag{4.4}$$

and

$$\frac{2}{\pi}\int_0^\infty \ln\frac{1}{|\Gamma(\mathrm{i}\omega)|}\,\mathrm{d}\omega = \sum_i \lambda_{\mathrm{o}i} - \lambda_{\mathrm{p}i} - 2\lambda_{\mathrm{r}i} = 2\frac{\omega_0}{Q} - 2\sum_i \lambda_{\mathrm{r}i} \tag{4.5}$$

by a Taylor expansion around $s=0$ and $s=\infty$, respectively. Here, $\lambda_{\mathrm{o}i}$, $\lambda_{\mathrm{p}i}$, and $\lambda_{\mathrm{r}i}$ denote the zeros (2.4) of ρ_{RCL}, the poles (2.4) of ρ_{RCL}, and arbitrary complex valued numbers with positive real part, respectively. We assume that the matching is symmetric around the resonance frequency, *i.e.*, the frequency range $\omega_0 - \Delta\omega/2 \leq \omega \leq \omega_0 + \Delta\omega/2$ is considered. The relative bandwidth, B, is given by $B = \Delta\omega/\omega_0$. Set

$$K = \inf_{|\frac{\omega}{\omega_0}-1|\leq\frac{B}{2}} \frac{2}{\pi}\ln\frac{1}{|\Gamma(\mathrm{i}\omega)|} = \frac{2}{\pi}\ln\frac{1}{\sup_{|\frac{\omega}{\omega_0}-1|\leq\frac{B}{2}}|\Gamma(\mathrm{i}\omega)|} \tag{4.6}$$

to simplify the notation [4].

The integrals in (4.4) and (4.5) are estimated from below giving

$$\frac{B}{1-B^2/4}K \leq \frac{2}{Q} - 2\sum_i \frac{\omega_0}{\lambda_{\mathrm{r}i}} \quad \text{and } BK \leq \frac{2}{Q} - 2\sum_i \frac{\lambda_{\mathrm{r}i}}{\omega_0}, \tag{4.7}$$

where the coefficients $\lambda_{\mathrm{r}i}$ have a positive real-valued part. Both inequalities can be satisfied with a complex conjugated par, $\lambda_{\mathrm{r}1}=\lambda_{\mathrm{r}2}^*$. This reduces the inequalities to

$$K\leq\frac{2}{BQ}\left(1-\frac{B^2}{4}\right). \tag{4.8}$$

Hence, the reflection coefficient is bounded as

$$\sup|\Gamma(\mathrm{i}\omega)|\geq\Gamma_0=\mathrm{e}^{-\frac{\pi}{QB}(1-B^2/4)}=\mathrm{e}^{-\frac{\pi}{QB}}+\mathcal{O}(B/Q) \tag{4.9}$$

for any realizable Γ where we introduced the Bode-Fano threshold limit, Γ_0, on the reflection coefficient. The inequality (4.9) states that it is not possible to construct a lossless matching network such that $|\Gamma|$ is strictly smaller than Γ_0 over the considered frequency range. The Bode-Fano threshold limit, Γ_0, and an unattainable reflection coefficient are illustrated in Figure 3. The corresponding wideband and narrowband Bode-Fano bandwidths are given by

$$B=\sqrt{Q^2K_0^2+4}-QK_0\sim\frac{\pi}{Q\ln\Gamma_0^{-1}}+\mathcal{O}(Q^{-3}) \tag{4.10}$$

where $K_0=2\ln\Gamma_0^{-1}/\pi$. The decibel scale of the reflection coefficient, $\Gamma_{\mathrm{dB}}=20\log\Gamma_0$, simplifies the narrowband bandwidth to

$$B\approx\frac{27}{Q|\Gamma_{\mathrm{dB}}|}. \tag{4.11}$$

The reflection coefficient, ρ_{RCL}, together with its Bode-Fano limits, Γ_0, are illustrated in Figure 4. The frequency scaling $\beta=2(\omega-\omega_0)/\omega_0$ is used to emphasize the character of the reflection for different values of Q. The parameter β can be interpreted as the relative bandwidth, *i.e.*,

$$B=\frac{\Delta\omega}{\omega_0}=2\frac{\omega-\omega_0}{\omega_0}=\beta, \tag{4.12}$$

if ω is considered to be the upper frequency limit. The Bode-Fano limits (4.10) are shown for the maximal reflection coefficient $\Gamma_0=-10,-20,-30\,\mathrm{dB}$ and Q factors $2,4,10,\infty$. It is observed that the curves are indistinguishable for $Q>10$.

In the general case of a RCL circuit and the lattice network, the transmission coefficient has an additional zero at $\sigma=\omega_0/\alpha$. Observe that the appropriate reflection coefficient in the Bode-Fano theory is given by (2.3) since the reflection coefficient of the lattice network is zero for all frequencies. This gives the additional integral relation

$$\int_0^\infty\frac{\sigma}{\sigma^2+\omega^2}\ln\frac{1}{|\Gamma|}\,\mathrm{d}\omega=\frac{\pi}{2}A_0^{\sigma_\nu}-\frac{\pi}{2}\operatorname{Re}\sum_i\frac{-\lambda_{\mathrm{r}i}^*-\sigma}{-\lambda_{\mathrm{r}i}-\sigma} \tag{4.13}$$

where $A_0^{\sigma_\nu}=\ln|\rho_{\mathrm{RCL}}(\sigma)|^{-1}$. We solve these equations in a similar way as for the RCL circuit. For simplicity, we start with a complex conjugated pair of zeros in the right half plane $\lambda_{\mathrm{r}i}/\omega_0=x\pm\mathrm{i}y$. This gives the inequality

$$K\arctan\frac{\alpha B}{1+\alpha^2(1-B^4)}\leq\ln\frac{1+\alpha^2+2\beta/Q}{1+\alpha^2}-2\frac{(\alpha^{-1}+x)^2-y^2}{(\alpha^{-1}+x)^2+y^2}. \tag{4.14}$$

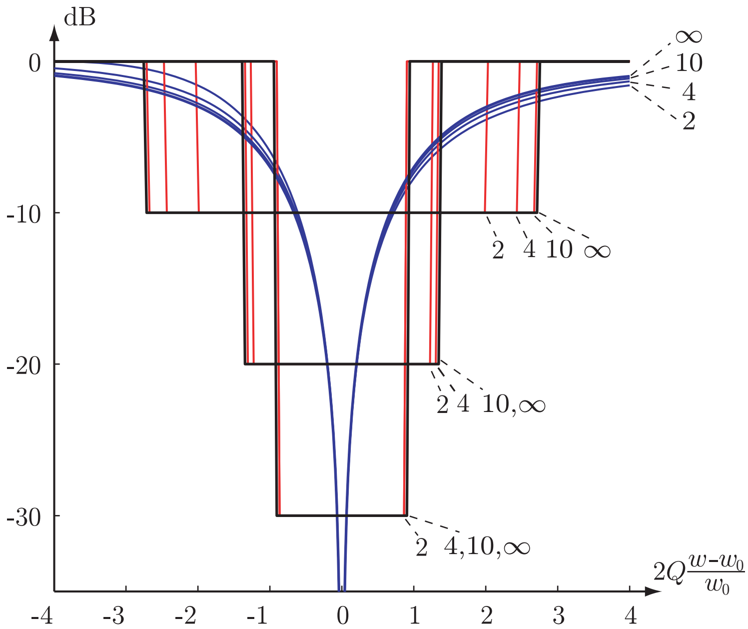

Figure 4: Reflection coefficient of a resonance circuit for different Q factors and Bode-Fano matching networks. The Q factors $Q = 2, 4, 10, \infty$ and the Bode-Fano limits corresponding to $-10, -20, -30\,\mathrm{dB}$ are shown.

A narrow band assumption $B \ll 1$ and $Q \gg 1$ gives

$$KB \leq \frac{2}{Q} - \frac{4\alpha}{Q^2(1+\alpha^2)} - 2\,(\alpha + 1/\alpha)\,\frac{(\alpha^{-1}+x)^2 - y^2}{(\alpha^{-1}+x)^2 + y^2} + \mathcal{O}(B^3) + \mathcal{O}(Q^{-3}). \quad (4.15)$$

We observe that the second order correction, $-4Q^{-2}\alpha/(1+\alpha^2)$, can be compensated with a large imaginary part, y, of the zeros in the right half plane. It gives the result $KB \leq 2/Q$ as for the case of the narrowband RCL circuit. The effect of the rotation is hence negligible for large Q factors.

The Bode-Fano limits give fundamental limitations on the relation between the magnitude of the reflection coefficient and the bandwidth for the resonance models considered here. The relations can be extended to the antenna with estimates (4.1) and (4.3). The reflection coefficient of the antenna after matching is estimated by (4.3) as

$$\sup|\widetilde{\Gamma}| = \widetilde{\Gamma}_0 \geq \Gamma_0 - \frac{\epsilon}{1-\delta^2} = \mathrm{e}^{-\frac{\pi}{QB}} + \mathcal{O}(\epsilon). \quad (4.16)$$

Invert to get an estimate of the bandwidth

$$B \leq \frac{\pi}{Q_\rho \ln(\widetilde{\Gamma}_0 + \epsilon/(1-\delta^2))^{-1}} \approx \frac{\pi}{Q_\rho \ln \widetilde{\Gamma}_0^{-1}}\left(1 + \frac{\epsilon}{\widetilde{\Gamma}_0 \ln \widetilde{\Gamma}_0^{-1}(1-\delta^2)}\right)$$
$$= \frac{\pi}{Q_\rho \ln \widetilde{\Gamma}_0^{-1}} + \mathcal{O}(\epsilon) = \frac{\pi}{Q_\rho \ln \widetilde{\Gamma}_0^{-1}} + \mathcal{O}(B^2), \quad (4.17)$$

where we used the estimate (4.3). Hence, the bandwidth of the antenna can be estimated by the Q factor, $Q_\rho = \omega_0|\tilde{\rho}'|$, of the approximating resonance model as long as the bandwidth is sufficiently narrow giving

$$B \sim \frac{\pi}{Q_\rho \ln \widetilde{\Gamma}_0^{-1}}, \quad \text{for } B \ll 1. \tag{4.18}$$

5 Approximation of spherical vector waves

An arbitrary electromagnetic field can be expanded in spherical vector waves [1, 8, 9]

$$\boldsymbol{E}(\boldsymbol{r}) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \sum_{\tau=1}^{2} a_{\tau ml}\, \mathbf{v}_{\tau ml}(k\boldsymbol{r}) + f_{\tau ml}\, \mathbf{u}_{\tau ml}(k\boldsymbol{r}) \tag{5.1}$$

$$\boldsymbol{H}(\boldsymbol{r}) = \frac{\mathrm{i}}{\eta_0} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \sum_{\tau=1}^{2} a_{\tau ml}\, \mathbf{v}_{\bar{\tau} ml}(k\boldsymbol{r}) + f_{\tau ml}\, \mathbf{u}_{\bar{\tau} ml}(k\boldsymbol{r}) \tag{5.2}$$

The terms labeled by $\tau = 1$, l, and m identify magnetic 2^l-poles and the terms labeled by $\tau = 2$, l, and m identify electric 2^l-poles. The outgoing spherical vector waves $\mathbf{u}$ are given by

$$\mathbf{u}_{1ml}(k\boldsymbol{r}) = \mathrm{h}_l^{(2)}(kr)\, \mathbf{A}_{1ml}(\hat{\boldsymbol{r}}) \tag{5.3}$$

$$\mathbf{u}_{2ml}(k\boldsymbol{r}) = \frac{1}{k} \nabla \times \left(\mathrm{h}_l^{(2)}(kr)\, \mathbf{A}_{1ml}(\hat{\boldsymbol{r}})\right) \tag{5.4}$$

where $\mathrm{h}_l^{(2)}$ denotes the spherical Hankel function and $\mathbf{A}$ denote the spherical vector harmonics. There are several common definitions of the spherical vector harmonics [1, 8, 9]. For $\tau = 1, 2$, we use

$$\mathbf{A}_{1ml}(\hat{\boldsymbol{r}}) = \frac{1}{\sqrt{l(l+1)}} \nabla \times (\boldsymbol{r}\, \mathrm{Y}_l^m(\hat{\boldsymbol{r}})) \tag{5.5}$$

$$\mathbf{A}_{2ml}(\hat{\boldsymbol{r}}) = \hat{\boldsymbol{r}} \times \mathbf{A}_{1ml}(\hat{\boldsymbol{r}}), \tag{5.6}$$

where Y_l^m denotes the spherical harmonics [1, 8, 9].

The impedance of a TM mode normalized with the intrinsic impedance, η_0, is ($\xi = ka = \omega a/c_0$)

$$Z = R + \mathrm{i}X = \mathrm{i}\frac{(\xi \mathrm{h}_l^{(2)}(\xi))'}{\xi \mathrm{h}_l^{(2)}(\xi)} = \frac{1}{|\xi \mathrm{h}_l^{(2)}|^2} + \mathrm{i}\,\mathrm{Re}\,\frac{(\xi \mathrm{h}_l^{(2)})'}{\xi \mathrm{h}_l^{(2)}} \tag{5.7}$$

where we used the Wronskian $\mathrm{h}_l^{(2)} \mathrm{h}_l^{(2)\prime *} - \mathrm{h}_l^{(2)*} \mathrm{h}_l^{(2)\prime} = 2\mathrm{i}\xi^{-2}$. The series expansions of the Hankel functions [1] gives the expansions

$$R(\xi) \sim \xi^{2l} \frac{l! 2^l}{(2l)!} \quad \text{and } X \sim -\frac{l}{\xi} \tag{5.8}$$

for small ξ. Tune the impedance with a series inductor, *i.e.*, $\omega_0 L = -X$. This gives the impedance $Z_1 = Z + \mathrm{i}\omega L$. Differentiate the impedance with respect to the angular frequency ω

$$\begin{aligned} Z_1' &= -2R\frac{a}{c_0}\,\mathrm{Re}\,\frac{(\xi \mathrm{h}_l^{(2)})'}{\xi \mathrm{h}_l^{(2)}} + \mathrm{i}\frac{a}{c_0}\left(\frac{n(n+1)}{\xi^2} - 1 - \mathrm{Re}(\frac{\mathrm{h}_l^{(2)\prime}}{\mathrm{h}_l^{(2)}} + \frac{1}{\xi})^2 + \frac{c_0}{a}L\right) \\ &= -2\alpha RX + \mathrm{i}\alpha\left(\frac{n(n+1)}{\xi^2} - 1 + R^2 - X^2 - \frac{X}{\xi}\right) \end{aligned} \tag{5.9}$$

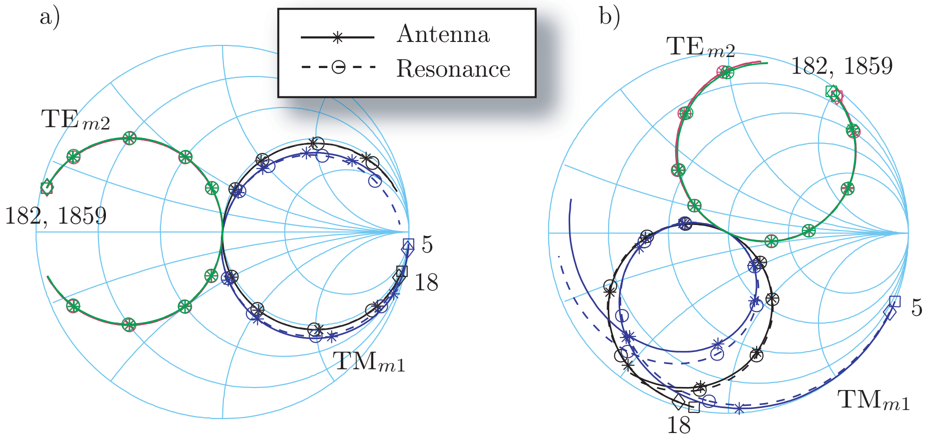

Figure 5: Illustration of the resonance circuit approximations. The frequencies corresponding to $Q_\rho\beta = -4, -2, -0.5, 0.5, 1, 2, 4$ are indicated with a star and a circle for the modes and the resonance models, respectively. TM cases with $Q_\rho = 5$ and $Q_\rho = 18$ and TE cases with $Q_\rho = 182$ and $Q_\rho = 1859$ are shown. a) without transmission line. b) with a $\lambda_0/(2\pi)$, *i.e.*, $k_0 d = 1$, long transmission line.

The frequency derivative of reflection coefficient, $\tilde{\rho} = (Z_1 - R)/(Z_1 + R)$, is given by

$$\omega\frac{\partial\tilde{\rho}}{\partial\omega} = \omega\tilde{\rho}' = \omega\frac{Z_1'}{2R} = -kaX + \frac{\mathrm{i}ka}{2R}\left(\frac{n(n+1)}{k^2a^2} - \frac{X}{ka} - X^2 - 1 + R^2\right) \tag{5.10}$$

The derivatives (5.10) is used to get resonance models of the TE and TM reflection coefficients, $Q_\rho = \omega|\tilde{\rho}'(\omega)|$. The TM (TE) case gives series (parallel) circuits combined with lattice networks. In Figure 5a, the reflection coefficient, $\tilde{\rho}$, together with their resonance models (3.11) are depicted for spheres with radius $k_0a = 0.4$ and $k_0a = 0.65$. The TM and TE cases are shown for $l = 1$ and $l = 2$, respectively. The Q factors in the resonance model are $Q_\rho = 5, 18, 182, 1859$. The frequencies $Q_\rho\beta = -4, -2, -0.5, 0.5, 1, 2, 4$ are indicated with a star and a circle for the modes and the resonance models, respectively. It is only for the lower values of Q_ρ, we can observe a small discrepancy between modes and their models. The curves are indistinguishable for the higher values of Q_ρ. This is also seen in Figure 6 where the error $\|\rho(\mathrm{i}\omega) - \tilde{\rho}(\mathrm{i}\omega)\| = \sup_\omega |\rho(\mathrm{i}\omega) - \tilde{\rho}(\mathrm{i}\omega)|$ is depicted. The error is of second order in B, *i.e.*, 40 dB for each decade in B, in accordance with (3.4).

We also consider the case where the TM and TE modes are connected to a transmission line with length $\lambda_0/(2\pi)$. The transmission line rotates the reflection coefficients as seen in Figure 5b. This require a larger compensation with the lattice network in the model. We observe that the differences between the model and the rotated modes increases. However, the error is still very small for the larger values of Q_ρ as seen in Figure 6.

As the error can increase by the matching network we consider the error of the matched reflection coefficient, *i.e.*, $\|\Gamma - \widetilde{\Gamma}\|$. The error is estimated by (4.2). It is observed that the error increases as the magnitude of the unmatched reflection

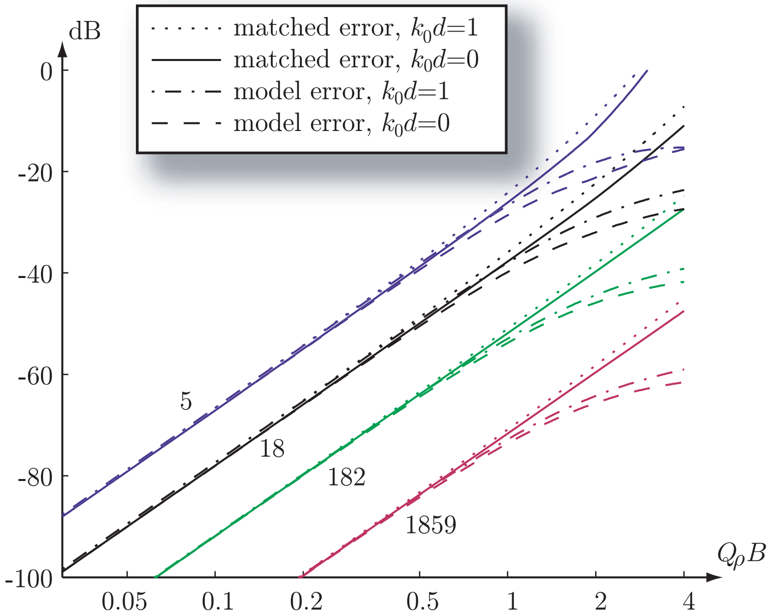

Figure 6: Errors in the resonance models corresponding to Figure 5. The model error is given by $|\rho-\tilde{\rho}|$ and (4.2) is used to estimate the error, $|\Gamma-\tilde{\Gamma}|$, after matching.

coefficient increases. This is also illustrated by the solid and dashed curved in Figure 6. The additional error by the matching, $1/(1-\delta^2)$, are negligible for $Q_\rho B \ll 1$ and increases to approximately 2 dB for $Q_\rho B = 1$ and 14 dB for $Q_\rho B = 4$.

It is also illustrative to compare the Q factor of the resonance model with the Q factor of the radiating system. The Q factor of the TE and TM modes can either be determined by the equivalent circuits [2, 7] or by an analytic expression functions [3]. The Q of the TM_{lm} or TE_{lm} mode is given by

$$Q = \xi + \frac{\xi}{2R_l}\left(\frac{l(l+1)}{\xi^2} - \frac{X_l}{\xi} - X_l^2 - R_l^2\right). \tag{5.11}$$

The Q factor depends only on the l-index and there are $2(2l+1)$ modes for each l index. The six lowest order modes have $Q = (ka)^{-3} + (ka)^{-1}$. By combination of one TE_{m1} mode and one TM_{m1} mode the Q factor is reduced to $Q = (ka)^{-3}/2 + (ka)^{-1}$. The Q factor has the asymptotic expansion $Q \sim (2l)!l/(\xi^{(2l+1)}l!2^l)$. The resonance circuit approximation has a Q factor, $Q = |\omega_0\tilde{\rho}'|$. We get

$$\frac{\omega_0\tilde{\rho}'}{\mathrm{i}Q} = 1 + \frac{\xi(Z-1)}{Q} \sim 1 + (-\xi + l\mathrm{i})\frac{\xi^{(2l+1)}l!2^l}{(2l)!l} \tag{5.12}$$

where we see that the resonance circuit approximation of the Q factor is very good for small ξ or equivalently large Q-values.

We consider the Bode-Fano fractional bandwidth of the TM_{m1} and TE_{m1} modes to determine the errors in the Q-factor approximations [5]. The transmission coefficient of the TM_{m1} and TE_{m1} modes has a double zero at $s = 0$. The corresponding

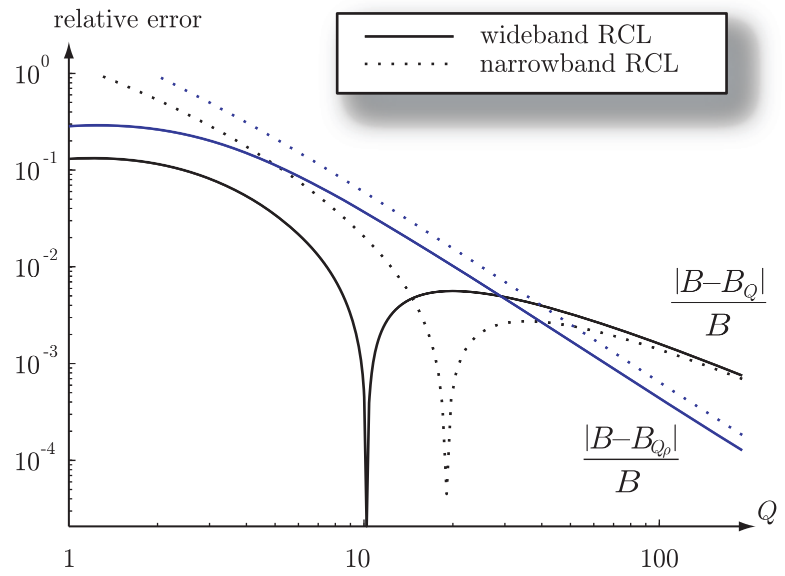

Figure 7: Relative errors, $|B - B_\text{Q}|/B$, of the bandwidth in the Q-factor approximations of the TM_{m1} and TE_{m1} modes.

reflection coefficient is

$$\Gamma_1(s) = \frac{1}{1 + \frac{2sa}{c_0} + \frac{2s^2a^2}{c_0^2}} \tag{5.13}$$

without zeros $\lambda_{\text{o}i}$ but with the two poles $\lambda_{\text{p}1,2} = (-1 \pm \text{i})c_0/(2a)$. The coefficients of the Taylor series around $s = 0$ give the two integral relations

$$\frac{2}{\pi}\int_0^\infty \omega^{-2} \ln \frac{1}{|\Gamma(\text{i}\omega)|}\,\text{d}\omega = \sum_i \lambda_{\text{o}i}^{-1} - \lambda_{\text{p}i}^{-1} - 2\lambda_{\text{r}i}^{-1} = \left(\frac{2a}{c_0} - 2\sum_i \lambda_{\text{r}i}^{-1}\right) \tag{5.14}$$

and

$$\frac{2}{\pi}\int_0^\infty \omega^{-4} \ln \frac{1}{|\Gamma(\text{i}\omega)|}\,\text{d}\omega = \frac{-1}{3}\sum_i \frac{1}{\lambda_{\text{o}i}^3} - \frac{1}{\lambda_{\text{p}i}^3} - \frac{2}{\lambda_{\text{r}i}^3} = \left(\frac{4a^3}{3c_0^3} + \frac{2}{3}\sum_i \lambda_{\text{r}i}^{-3}\right), \tag{5.15}$$

where the coefficients $\lambda_{\text{r}i}$ have a positive real-valued part. Assuming a bandwidth and K as in (4.6) gives

$$K\frac{B}{1 - B^2/4} \leq 2k_0a - 2\sum_i \frac{\omega_0}{\lambda_\text{r}} \tag{5.16}$$

and

$$K\frac{B + B^3/12}{(1 - B^2/4)^3} \leq \frac{4(k_0a)^3}{3} + \frac{2}{3}\sum_i \frac{\omega_0^3}{\lambda_\text{r}^3} \tag{5.17}$$

where $k_0 = \omega_0/c_0$. It is noted that it is enough to consider one coefficient λ_r or a complex conjugated pair. These equations can be solved numerically with respect to B and λ_r.

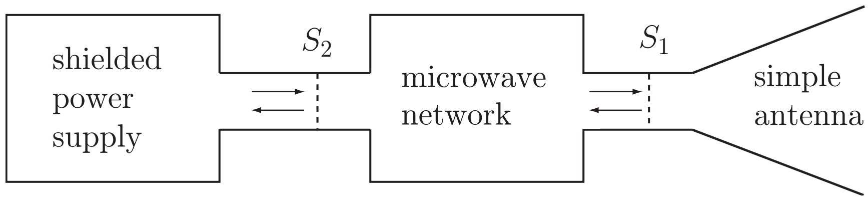

Figure 8: Illustration of the antenna prototype. A transmission line is used to connect the shielded power supply, the microwave network, and the simple antenna.

The fractional bandwidth, B, given by (5.16) and (5.17) is compared with the fractional bandwidth, B_{Q}, determined by the resonance approximation (4.10) to determine the error in the resonance approximation. We consider the Q factors determined by the stored and radiated fields (2.1), *i.e.*, (5.11), and by the resonance approximation (3.7), *i.e.*, (5.10). The relative error $|B - B_{\mathrm{Q}}|/B$ is depicted in Figure 7 for the threshold reflection coefficient $\Gamma_0 = 1/3$. It is observed that the errors are small for large Q factors and that they approach 0 as $Q \to \infty$ as known from the asymptotic expansions. We also observe that the narrowband approximation in (4.10) is good for large Q factors. The error of the resonance approximation, Q_ρ, decays faster than the error in the Q-approximation as Q increases. This is in accordance with the construction of the resonance approximation as a local approximation of the reflection coefficient.

6 Q factor of general antennas

There have been several attempts to express the Q factor of a general antenna in the impedance of the antenna, see [14] and references there in. Common versions are

$$Q \approx \frac{\omega_0}{2R(\omega_0)}|X'(\omega_0)| \tag{6.1}$$

and

$$Q \approx \frac{\omega_0}{2R(\omega_0)}|Z'(\omega_0)| = \omega_0|\rho'(\omega_0)| = Q_\rho \tag{6.2}$$

where the antenna is assumed to be tuned to resonance at ω_0. We observe that (6.2) reduces to (6.1) for the special case of $R'(\omega_0) = 0$. We consider the more general approximation (6.2) as it is invariant to shifts in the reference plane in the feed line. This approximation has been extensively tested and it is confirmed that it is a good approximation for many antennas. However, this does not mean that it is a good approximation for a general antenna.

To better understand the requirements on the antennas where (6.2) is good and at the same time, why it is difficult to prove these types of approximations for general antennas we consider an antenna model as depicted in Figure 8. The antenna model is composed by a shielded power supply, a microwave network, and a simple antenna. With the simple antenna we mean an antenna with known characteristic, *e.g.*, dipole, spherical vector mode, or resonance model. A transmission line with a propagating TEM mode is used to connect the different components. We consider

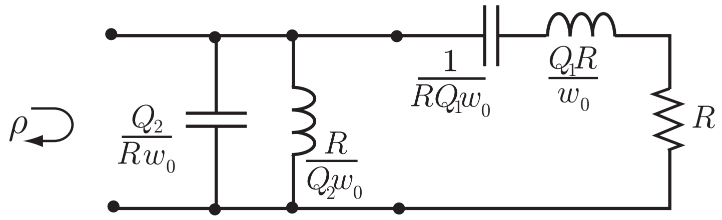

Figure 9: Circuit model of the antenna with an arbitrary small $\rho'(\omega_0)$.

two possible reference planes denoted by S_1 and S_2. The impedance properties of the simple antenna are defined in the reference plane S_1, here modeled with the reflection coefficient ρ_1. We use the reference plane, S_2, to define a more complex antenna characterized with the reflection coefficient ρ_2. Observe that, although it might be more practical to consider this as an antenna together with a matching network, it also possible to consider it as a single antenna. The Maxwell equations on the region outside the reference planes can be used to determine the properties of both antennas. The only difference is that, in reality, it might be more practical to use simpler equations and approximations to determine the properties of the microwave network.

For simplicity, we assume that the simple antenna can be approximated with a resonance model (3.11) around the resonance frequency, specifically we assume a series RCL circuit with Q factor Q_1. Let the microwave network be modeled with a parallel LC circuit, as seen in Figure 9. The reflection coefficient at S_2 is given by

$$\rho_2 = \rho_{Q_2} + \frac{t_{Q_2}^2 \rho_{Q_1}}{1 - \rho_{Q_1}\rho_{Q_2}}, \tag{6.3}$$

where ρ_{Q_i} is defined by (2.3) and $t_{Q_2}(\omega_0) = 1$. The frequency derivative of ρ_2 at the resonance frequency is

$$\rho_2'(\omega_0) = \rho_{Q_2}'(\omega_0) + \rho_{Q_1}'(\omega_0) = \frac{\mathrm{i}}{\omega_0}(Q_1 - Q_2). \tag{6.4}$$

Here, it is observed that it is possible to construct antennas with an arbitrary small frequency derivative of the reflection coefficient. Obviously, this is just an example of a matching network giving a flat reflection coefficient [11]. The Q-factor of the antenna is on the contrary increasing. The Q-factor of the circuit model is $Q = Q_1 + Q_2$. This simple example indicates that it is very difficult to find a simple relation between the frequency derivative of the reflection coefficient (or equivalently the impedance) and the Q-factor of general antennas. However, as shown with the Padé approximation in this paper and the results in [14], the approximation is very accurate for many common antennas.

7 Conclusions

In this paper, the Q factor of antennas are analyzed from an approximation theory point of view. The reflection coefficient of an antenna is approximated with a second order Padé approximation around the resonance frequency. This resonance model is first order accurate, and hence good for narrow bandwidths. The resonance model is characterized by a Q-factor of an underlying RCL circuit, defined as $Q_\rho = \omega|\rho'(\omega)|$. The Bode-Fano matching theory is used to determine the bandwidth of the approximate model. Moreover, it is shown that the original antenna has the same bandwidth for sufficiently narrow bandwidths.

Even if the Q-factor, defined by the stored and radiated energies, of the antenna system is not used in the analysis, there is a close resemblance between the Q-factor derived from the differentiated reflection coefficient, Q_ρ, and the classical Q-factor, Q. It is shown that $Q \approx Q_\rho$ for each spherical vector mode if Q is sufficiently large. This is also seen for many antennas from the approximation of the Q-factor $Q \approx \omega|Z'|/(2R) = Q_\rho$ considered in [14]. However, a simple example is used to illustrate that the there is not a simple relation between Q and Q_ρ for every antenna.

Acknowledgments

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