Higher Order Calculations for Low Energy Precision Physics

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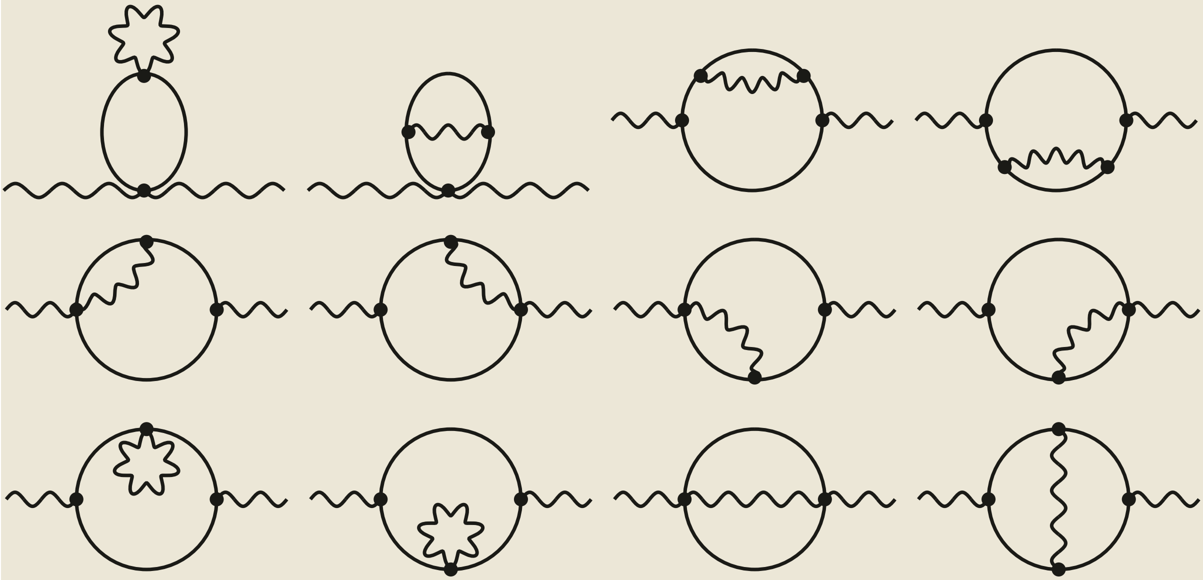

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by Nils Hermansson Truedsson

Thesis for the degree of Doctor of Philosophy
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Abstract

This thesis concerns higher order calculations needed for precision physics in the low energy region of particle physics. Of the four papers it contains, the first two introduce calculations at order p^8 in the power counting of chiral perturbation theory, which is an effective field theory of QCD at low energies. The remaining two papers concern the hadronic contributions to the muon anomalous magnetic moment, or muon $g-2$, which are responsible for the main uncertainty in the theoretical prediction of the quantity.

Paper I. The pion mass and decay constant are calculated at order p^8 within two-flavour chiral perturbation theory. A small numerical study of the quark mass dependence is performed, and there is good agreement with lower order results at the physical point.

Paper II. The order p^8 mesonic chiral Lagrangian is derived for two, three as well as a general number of flavours. This is done by explicitly creating all operators allowed by the relevant symmetries, and finding a minimal basis of operators. Special cases where some of the external fields are set to zero are also considered.

Paper III. The finite volume effects from the next-to-leading order electromagnetic corrections to the hadronic vacuum polarisation are here calculated in QED_L. This is needed for precision calculations of the muon $g-2$ on the lattice. The analytic results are compared to lattice simulations as well as numerical lattice perturbation theory. There is good agreement between the methods, and it is found that the electromagnetic corrections are suppressed to such an extent that they for moderately sized lattices and pion masses in principle can be neglected for the currently sought precision on the hadronic vacuum polarisation.

Paper IV. Short-distance constraints on the hadronic light-by-light contribution to the muon $g-2$ are here derived. Such constraints are useful for the matching of hadronic models valid at low energies to the high energy region. In particular, the 4-point function entering into the hadronic light-by-light piece is calculated as a 3-point function in the presence of an external electromagnetic field. We show that the quark loop is the first term in an operator product expansion, and also consider the next term containing the condensate $\langle \overline{q}\, \sigma_{\alpha\beta}\, q \rangle$ which is related to the magnetic susceptibility of the QCD vacuum. This latter contribution is found to be negligible due to the suppression in quark masses and sizes of the condensates.

Key words
Effective Field Theory, Chiral Perturbation Theory, The Muon Anomalous Magnetic Moment, Lattice Gauge Theory, Finite Volume Effects

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A doctoral thesis at a university in Sweden takes either the form of a single, cohesive research study (monograph) or a summary of research papers (compilation thesis), which the doctoral student has written alone or together with one or several other author(s).

In the latter case the thesis consists of two parts. An introductory text puts the research work into context and summarizes the main points of the papers. Then, the research publications themselves are reproduced, together with a description of the individual contributions of the authors. The research papers may either have been already published or are manuscripts at various stages (in press, submitted, or in draft).

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Till mor och far

Doktorn meddelar sig åter med professorn som nu inte inlåter sig på teorier utan kort ordinerar:
– Gå på djupet!
Och på djupet går man under en förväntansfull halvtimma. Till sist stöter man på ett hårt föremål, spänningen blir olidlig. Föremålet blottas med all försiktighet, och pang: det är ett cementrör!

Utdrag ur novellen Arkeologi i samlingen Vänner emellan (Fritiof Nilsson Piraten, 1955)

Populärvetenskaplig sammanfattning

Att studera fysik är ett sätt att försöka förklara och förstå universum. Inom partikelfysik undersöks mikrokosmos, d.v.s. universum på minsta möjliga längdskalor, och de partiklar som finns där. I dagens partikelfysikexperiment är det möjligt att nå 10^{-19} meter, alltså tio miljarder gånger mindre än en nanometer, vilket i sin tur är en en miljard gånger mindre än längden av en människa. Naturen fungerar helt annorlunda på dessa längdskalor, bland annat på grund av de kvantmekaniska effekter som blir viktiga någonstans runt atomära och molekylära storlekar på någon nanometer.

De krafter som styr partiklarna i mikrokosmos är den starka, den svaga och den elektromagnetiska kraften. Det är utifrån dessa som partikelfysikens mycket framgångsrika Standardmodell är formulerad. Denna teoretiska modell både beskriver och förutsäger många av de fenomen som kan observeras i experiment, men det finns ett antal exempel där den inte verkar fungera riktigt lika bra. Nyckelordet här är verkar, detta eftersom både mätningar och beräkningar har en begränsad noggrannhet. Genom att minska dessa ofrånkomliga fel i så stor utsträckning som möjligt går det alltså att säga huruvida teori och experiment överensstämmer eller inte.

Ett av de i skrivande stund mest intressanta problemen är hur en så kallad myon påverkas av ett magnetiskt fält. Denna partikelegenskap påverkas av de andra partiklarna inom Standardmodellen och att räkna ut hur kräver speciella tekniker vid låga energier. Detta beror på att den starka kraften får Standardmodellens kvarkar att bilda bundna tillstånd som kallas hadroner. De kanske mest kända hadronerna är protonen och neutronen som tillsammans bildar atomkärnor.

En av de vanligaste metoderna för att göra beräkningar vid låga energier är att använda effektiva modeller som bara inkluderar hadroner. Man kan likna det vid hur en rullande boll beskrivs som just ett objekt med en viss friktion mot underlaget snarare än en mängd molekyler vars individuella rörelser och interaktioner med underlaget måste bokföras. Noggrannheten i en precisionsberäkning kvantifieras av dess ordning i något bestämt mått, och ju högre ordning desto mindre är felet.

Det är precis sådana högre ordningars beräkningar som denna doktorsavhandling berör. I de första två av avhandlingens artiklar introduceras en ny ordning för en av de mest framgångsrika effektiva lågenergiteorierna, så kallad kiral störningsräkning. I de sista två görs beräkningar som är viktiga för att minska de teoretiska felen i Standardmodellens förutsägelse av just myonens ovannämnda magnetiska egenskap. Även om Standardmodellen visar sig inte kunna förklara vissa fenomen, så är precisionsberäkningar i regel aldrig gjorda förgäves. De kan användas för att leda vägen till en ny, förbättrad teoretisk partikelfysikmodell som förhoppningsvis ger fler insikter och ytterligare intressanta förutsägelser.

Contents

Higher Order Calculations for Low Energy Precision Physics

During the past century tremendous effort has been put into the understanding of Nature at smaller and smaller scales. The underlying idea behind this is to see what the most fundamental building blocks of the Universe are, and to obtain a predictive theory describing their interactions and properties. The building blocks in question are called fundamental particles and the best theoretical model that is currently known is the Standard Model (SM) of particle physics. The length scale of elementary particle physics today is 10^{-19} m, and to get a sense of how small this really is, one may compare to some more well-known length scales. Macroscopic life exists on scales on the order of 1 m and the biological cells in any organism typically have a radius of 10^{-5} m. These cells in turn consist of molecules that are ten thousand times smaller, i.e. roughly 10^{-9} m, but a molecule is itself built up of atoms with radii on the order of 10^{-10} m. The size of the atomic nucleus consisting of protons and neutrons is roughly $10^{-14} - 10^{-15}$ m. The protons and neutrons are not fundamental either, since they consist of quarks bound together by the so-called strong force of Nature. Quarks are examples of the particles in the SM and are at present thought to be pointlike, a conception in agreement with existing experimental data. Experiments continue to probe deeper and deeper into the microscopic world, but in order to reach deeper into nuclear matter an increasing amount of energy is required. In other words, there is an inverse relation between distance and energy. Particle physics is therefore also commonly known as high energy physics.

The physics on these scales is fundamentally different from the classical physics governing the macroscopic world, primarily for two reasons. First of all, quantum mechanics (QM) becomes important around atomic and molecular length scales. In QM energies are quantised and wavefunctions introduced, whose squared amplitude represents a probability density, and quantum mechanical states can be put in superposition. One consequence of this is that in calculations of a process such as the propagation of a quantum mechanical particle between two points in spacetime one has to consider all possible ways it can do so, where each possible way is associated to a probability. A fundamental constant appearing in QM is Planck's reduced constant $\hbar \approx 1.06 \cdot 10^{-34}$ Js.

In addition to QM, also special relativity (SR) is required when describing quantum mechanical particles moving at high velocities. The main postulates of SR are that physical laws are the same for all observers and that the speed of light $c \approx 3 \cdot 10^{8}$ m/s is the same in all frames of reference. A fundamental relation is the equivalence between mass and energy,

$$E = mc^2 \,, \tag{1.1}$$

which implies that particles can be created from energy alone. It can thus be expected that annihilation of particles in collider experiments, the main tools to experimentally probe the microcosmos, will produce a plethora of particles. This is indeed true, and the most advanced accelerator where this is done today is the Large Hadron Collider (LHC) in Switzerland.

Particle physics thus joins together two of the cornerstones in physics that were both discovered in the twentieth century. However, it is not possible to simply merge the two into a theory of quantised relativistic particles since that leads to a violation of causality. One way to cure this is to instead introduce a quantum field theory (QFT) where the particles arise as excitations in the fields. An additional argument to introduce QFT, which is perfectly valid albeit perhaps not so theoretically satisfying, is that it works. It gives excellent predictions of observables such as cross sections that can be measured in collider experiments.

This thesis contains four papers, henceforth referred to as Papers I–IV, for which the fundamental ideas and topics are introduced and explained in sections 1–8 below. First of all, an introductory overview of the SM is given in section 1 together with the main ideas motivating the work in the papers and this thesis. After that section the discussion will be more technical and it is assumed that the reader is familiar with at least some QFT. The technical discussion in section 2 starts with the most important concepts in QFT needed for an understanding of the papers as well as some generalities about loop integral calculations. Then focus is put on the QFT of the strong force (quantum chromodynamics, or, QCD in short) in section 3. This is followed by an introduction to effective field theory in section 4. Papers I and II concern chiral perturbation theory (ChPT). This low energy effective field theory is introduced in section 5. Following that, lattice gauge theory is discussed in section 6. This section is particularly important for Paper III which concerns one of the two kinds of hadronic contributions to the anomalous magnetic moment of the muon, and this latter quantity is presented in section 7. In section 8, the operator product expansion used in Paper IV for the second kind of hadronic contribution is introduced.

1 An Overview of the Standard Model

The Standard Model of particle physics describes the elementary particles that are known at present and how they interact via three of the four fundamental forces in Nature, namely the electromagnetic, the weak and the strong forces. It makes no attempt at explaining gravity, but the effective field theory (EFT) techniques presented in section 4 can be used to write down an effective theory of quantum gravity. On the mathematical level, the SM is governed by its Lagrangian, $\mathcal{L}_{SM}$, which is written in terms of the fields of the respective particles. The mathematical structure will be further elaborated in section 2 below. In this section, a brief and elementary overview of the SM is given. Focus is put on the quantum numbers such as spin and colour charge that serve to distinguish particle properties.

1.1 The particle content

Each of the fundamental forces is mediated through the exhange of bosonic particles, i.e. with integer spin, known as gauge bosons. These are the photon for the electomagnetic force, the $W^{\pm}$ and Z bosons for the weak force and the gluon for the strong force. In addition to these bosons there is also the Higgs boson, a scalar particle whose perhaps most renowned effect is that it allows for a mechanism through which particles can obtain masses. The remaining particles all have half-integer spin and are known as fermions. The fermionic content of the SM is divided into the two classes quarks and leptons. There are six quarks, namely up (u), down (d), charm (c), strange (s), top (t) and bottom (b), as well as six leptons, i.e. the electron (e), muon (μ) and tau (τ), each with its corresponding neutrino. The masses of the SM particles range from zero for the massless particles up to roughly 170 GeV for the top quark, and it is not known why e.g. the top quark is so much heavier than an up quark with a mass on the order of 2 MeV.

In addition to spin, each particle has other specific properties which are defined in terms quantum numbers. For this thesis, the electric charge and colour charge are of importance. The number of colours is three and they are respectively labelled red, green and blue. It is due to the fact that quarks carry colour and electrons do not that the former take part in the strong interaction whereas the latter cannot. It should further be noted that for every particle there is an antiparticle of the same mass as the particle but with opposite quantum numbers, e.g. with an electric charge of flipped sign, and some particles are their own antiparticles such as the photon.

1.2 The low energy region

Quarks are never observed freely in Nature since the strong force binds them together into composite colour neutral objects known as hadrons. Therefore, in experiments one always detects hadrons or decay products thereof. There are two kinds of hadrons. These are known as baryons and mesons, respectively. The baryons consist of three quarks and the most obvious examples of such are the proton and neutron with the quark contents *uud* and *udd*, respectively. The mesons contain one quark and one antiquark. The lightest pseudoscalar mesons are π^0, $\pi^\pm$, K^0, $\overline{K}^0$ and η. These latter particles are very important for this thesis as they are used as effective degrees of freedom in Papers I, II and III, and such effective models also motivate the short-distance constraints derived in Paper IV, as will be explained in sections 7 and 8. The connection between the chiral symmetry breaking and the lightest pseudoscalar mesons will be further elaborated in sections 3 and 5.

Both the strong and electromagnetic coupling constants, $\alpha_s(Q)$ and $\alpha(Q)$, respectively, depend on energy Q, a concept commonly referred to as running, but behave very differently. At high energies $\alpha(Q)$ increases whereas $\alpha_s(Q)$ decreases, something which leads to the notion of asymptotic freedom for strongly interacting particles. On the other hand, the converse is true for low energies and $\alpha_s(Q)$ increases to such an extent that below energies of roughly 1 GeV it is not possible to use perturbation theory where an expansion is made in $\alpha_s(Q)$ under the assumption that it is small. Therefore, the region $Q \lesssim 1$ GeV is known as the non-perturbative region, or low energy region, of the strong force and to study particle physics there requires the use of special tools such as EFT techniques, see sections 4 and 5, or lattice gauge theory, see section 6. Another approach is the use of QCD sum rules which is related to the operator product expansion (OPE) defined in section 8.

1.3 The validity of the Standard Model

A natural question to ask now, is how well the SM actually works. It turns out that it works remarkably well. One thing that it is not able to explain is the abundance of matter over antimatter. This has been a known problem for a long time, and for any model to explain such an asymmetry three conditions have to be fulfilled [1]. The SM fails with respect to the amount of *CP* violation [2], and many extensions of the SM have been proposed to explain this. These extensions contain a varying number of new particles, and although much experimental effort has been made to observe such particles none have been directly detected so far.

In the absence of experimental evidence for such particles, one may test the SM by doing high precision calculations which later can be compared with experiments. Given the success of the SM, any new physics model must naturally reproduce all that the SM describes well. Therefore, with more precision tests it is possible to pinpoint precisely where the SM fails, which in turn hopefully aids in the construction of new models.

An observable that has received much attention is the anomalous magnetic moment of the muon, see section 7, whose SM value is in tension with the Brookhaven National Laboratory experimental value at approximately 3.5σ [3–5]. As a consequence, there is much effort to improve both the theoretical and the experimental precision. On the experimental side there is the E989 experiment at Fermilab [6] and also E34 at J-PARC [7]. The Fermilab experiment is already running and results are expected in the coming year, whereas the J-PARC experiment expects to start running in the early 2020s. The biggest theoretical uncertainty at the moment comes from hadronic contributions, i.e. the low energy degrees of freedom, and in order to reduce this uncertainty higher order calculations are needed. This issue motivates the calculations in Papers III and IV herein. There are of course many other observables that can be considered, e.g. using the very successful low energy effective field theory ChPT. In Papers I and II, ChPT is considered at higher orders in the chiral expansion, which in principle can lead to improved precision calculations in the future as long as the unknown low energy constants showing up are estimated in some way (see section 5 for further details about this issue).

Finally, it should be mentioned that a review of the current status of high energy physics can be found in Ref. [8]. For lattice specific results, as well as some from ChPT, there is also Ref. [9]. Having explained the basic and underlying notions and goals of particle physics in the low energy region, attention must be turned to the more mathematical aspects of field theory and how to do computations. In the remainder of this thesis, natural units where $c = \hbar = 1$ will be used.

2 Quantum Field Theory

Quantum field theory is the foundation on which theoretical particle physics relies. The modern approach to QFT is the path integral formulation and it is assumed that the reader has seen this formulation before. However, those aspects of the path integral and QFT most needed for introducing the papers are still reviewed in this section. In addition, certain technicalities regarding loop integrals are discussed in detail as well.

2.1 Path integral approach

In order to motivate the path integral formulation of QFT it is important to discuss briefly what the goals of any calculation are. In general, the interesting quantities to calculate are transition probabilities from some given initial state, $|i\rangle$, say, to some final state $|f\rangle$.

The labels refer to initial and final, respectively, and contain all the information about the quantum numbers, momenta and so on. The transition between these two states is given via an object called the S matrix, such that the transition amplitude is

$$\langle f | S | i \rangle \, . \tag{1.2}$$

An important property of the S matrix is its unitarity, i.e. $SS^\dagger = S^\dagger S = 1$, which leads to the optical theorem relating the imaginary part of the scattering amplitude to the total cross section. The theorem in question is particularly important for the dispersive approach to the theoretical calculations of the muon anomalous magnetic moment, this as it offers a way to use experimentally measured cross sections in derived dispersion relations. This will be further elaborated on in section 7.

Consider now a scalar theory with field ϕ and the intital and final states $|i\rangle = |\mathbf{k}_1, \ldots, \mathbf{k}_m\rangle$ and $|f\rangle = |\mathbf{p}_1, \ldots, \mathbf{p}_n\rangle$, i.e. an $m \to n$ process. Define further the n-point function as the Fourier transformed correlation function with n fields, i.e.

$$\prod_{i=1}^{n} \int d^4x_i \, e^{ip_i \cdot x} \langle 0 | \, T \left\{ \prod_{i=1}^{n} \phi(x_i) \right\} |0\rangle \, , \tag{1.3}$$

where T denotes time ordering. The transition amplitude in (1.2) is then related to these correlation functions via

$$\begin{aligned} &\left(\prod_{i=1}^{n} \int d^4x_i \, e^{ip_i \cdot x} \right) \left(\prod_{j=1}^{m} \int d^4y_j \, e^{-ik_j \cdot y_j} \right) \langle 0 | \, T \left\{ \prod_{i=1}^{n} \phi(x_i) \prod_{j=1}^{m} \phi(y_j) \right\} |0\rangle \\ &\sim \left(\prod_{i=1}^{n} \frac{i\sqrt{Z}}{p_i^2 - m^2 + i\varepsilon} \right) \left(\prod_{j=1}^{m} \frac{i\sqrt{Z}}{k_j^2 - m^2 + i\varepsilon} \right) \langle \mathbf{p}_1, \ldots, \mathbf{p}_n | \, S \, | \mathbf{k}_1, \ldots, \mathbf{k}_m \rangle \, , \end{aligned} \tag{1.4}$$

where Z is the field-strength renormalisation. By looking at the Källén-Lehmann spectral representation of the 2-point function it can be shown that Z is the residue of the single-particle pole. The relation in (1.4) is known as the LSZ theorem and gives a prescription for how to calculate S matrix elements in terms of correlation functions.

Having related the transition amplitudes to n-point functions, these latter functions need to be defined such that they can be calculated. The basic object needed is the path integral, and the relation between the two is

$$\langle 0 | \, T \left\{ \prod_{i=1}^{n} \phi(x_i) \right\} |0\rangle = \frac{\int \mathcal{D}\phi \, e^{iS} \prod_{i=1}^{n} \phi(x_i)}{\int \mathcal{D}\phi \, e^{iS}} \, , \tag{1.5}$$

where $S = \int d^4x \, \mathcal{L}$ is the action defined in terms of the Lagrangian $\mathcal{L}$ which satisfies all imposed symmetries and depends on the fields of the theory as well as their respective derivatives. The integration over the measure $\mathcal{D}\phi$ is understood simply as the sum over all possible field configurations, i.e. all possible field configurations have to be taken into account when calculating an amplitude in QFT. This is completely analogous to Young's double slit experiment where a given particle is emitted from a source and in order to reach a detector can go through one of two slits. In calculating the probability, i.e. the square of the amplitude, of where it will end up in the detector, one has to sum over all possible paths. Of course, for field theories with more field content than a single field ϕ generalisations of the n-point functions can be defined. These are called Green's functions or simply correlation functions.

There are two main ways to calculate the right-hand side of (1.5). The first is the perturbative approach where one assumes that all interaction strengths are small such that a Taylor expansion can be made. This gives the diagrammatic expansion in terms of Feyman graphs. These diagrams can be calculated using derived Feynman rules. A discussion on perturbation theory can be found in any QFT textbook, and it is assumed that the reader already knows the fundamentals behind this. So, focus will instead be put on how to handle the loop integrals arising in perturbative calculations such as those in Papers I, III and IV. The second way to calculate the n-point function is by discretising space-time on a Euclidean lattice and Monte Carlo sampling field configurations [10]. This is the non-perturbative approach referred to as lattice calculations and is discussed further in section 6.

2.2 Symmetry considerations

Before going on to loop integral techniques, some remarks about the Lagrangian and symmetry considerations must be made. The Lagrangian is built from the fields of interest in such a way that the theory remains invariant under certain transformations of the fields. For instance, the Lagrangian of quantum electrodynamics has a $U(1)$ gauge symmetry, i.e. it remains invariant under transformations $\psi(x) \to \exp\{i\alpha(x)\}\,\psi(x)$ for some local parameter $\alpha(x)$ and a charged fermionic field $\psi(x)$. Gauge invariance is realised by introducing a covariant derivative, D_μ, containing both the ordinary partial derivative ∂_μ as well as a number of gauge fields. These gauge fields correspond to the gauge bosons.

Symmetries have important consequences as they put constraints on the Green's functions of the theory. The first step toward realising this is to consider the expectation value of some operator $\mathcal{O}[\phi]$, here depending on only one field ϕ for simplicity. The expectation value is in the path integral approach given by

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}\phi \, e^{iS[\phi]} \mathcal{O}[\phi] \,, \tag{1.6}$$

where the normalisation factor $Z = \int \mathcal{D}\phi \, e^{iS}$ was introduced. The expectation value must be invariant under under a change of variables $\phi \to \phi + \delta\phi$. Assuming that the integration

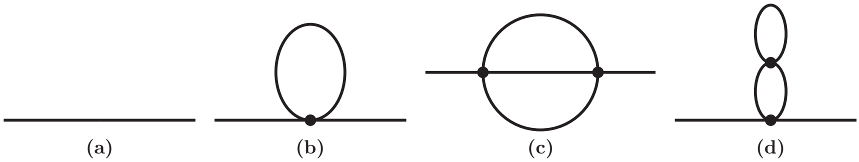


Figure 1.1: The one-particle-irreducible (1PI) diagrams that contribute to the 2-point function up to $\mathcal{O}(\lambda^2)$ in ϕ^4-theory.

measure remains invariant and expanding to first order in $\delta\phi$ one finds that

$$i\langle \delta S \mathcal{O} \rangle = -\langle \delta \mathcal{O} \rangle \,. \tag{1.7}$$

This is the general non-anomalous Ward identity from which both conserved currents as well as relations between the Green's functions can be obtained. In scalar quantum electrodynamics (sQED), which will be defined later, it is easy to derive the Ward identity from the symmetry $\delta S = 0$, namely

$$k_\mu \Gamma^\mu(p,k) = \Delta^{-1}(p+k) - \Delta^{-1}(p)\,, \tag{1.8}$$

where $\Gamma^\mu(p,k)$ is the interaction vertex between a charged scalar of incoming momentum p and a photon of incoming momentum k, and $\Delta^{-1}(p)$ is the full scalar inverse propagator with momentum p.

2.3 Loops, regularisation and renormalisation

In an interacting field theory there is an infinite series of diagrams that in principle need to be calculated for a given time ordered correlation function. However, this is usually not possible and calculations are performed at fixed order in the perturbative expansion. As an example consider ϕ^4-theory where the interaction term in the Lagrangian is $\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}\phi^4$ and λ is a small coupling. The 2-point function receives contributions from the diagrams in Fig. 1.1 up to second order in λ. The first diagram is the tree level contribution, and the rest are loop diagrams. Using the ϕ^4-theory Feynman rules easily obtained from the path integral method, the loop integral of the tadpole diagram in Fig. 1.1(b) is given by

$$I_{\text{tadpole}} \sim (-i\lambda) \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon}\,, \tag{1.9}$$

where k is the loop momentum. The divergence in the large k limit is an ultraviolet (UV) divergence and needs to be handled in some way. Had the loop integral had an infrared (IR) divergence, i.e. in the limit of soft k, this would also have required some consideration. IR divergences can be found in e.g. sQED due to the massless photon whose propagator goes as

k^{-2}. UV and IR divergences are handled by means of regularisation and there are several methods for doing so, some methods more common than others. For the UV case, the basic idea is to isolate divergent pieces and then renormalising the theory by realising that the parameters of the Lagrangian not necessarily are physical. Going between the physical and unphysical, so-called bare, parameters yields diagrammatic contributions containing so-called counterterms cancelling the infinities exactly. For IR divergences in sQED as an example, one possibility is to introduce a photon mass everywhere in the calculation and whose contribution in the end should vanish.

Regularisation

One of the most common regularisation techniques for divergences such as that in (1.9) is dimensional regularisation, where the fact that the degree of divergence of the diagram depends on the dimension of the integration measure is exploited. By thus letting the number of dimensions be $d = 4 - 2\varepsilon$ in a calculation and in the end making an expansion in the small parameter ε, the divergences should appear as poles in ε in the limit $\varepsilon \to 0$.

In order to reach such an isolation of the poles, it is instructive to start from the Feynman parameter technique. This builds on the observation that for a general loop diagram with n different propagators P_i,

$$\frac{1}{\prod_{i=1}^{n} P_i^{m_i}} = \int_0^1 dx_1 \cdots dx_n \, \delta\left(1 - \sum_i x_i\right) \frac{\prod_i x_i^{m_i - 1}}{\left(\sum_i x_i P_i\right)^{\sum_i m_i}} \frac{\Gamma\left(\sum_i m_i\right)}{\prod_i \Gamma(m_i)} \,. \tag{1.10}$$

Any loop integral can thus be written on the form

$$I \sim \int dx_1 \cdots dx_n \, \mu^{2\varepsilon} \int \frac{d^d k}{(2\pi)^d} \frac{f(k)}{(k^2 - \Delta)^n} \,, \tag{1.11}$$

where the renormalisation scale $\mu^{2\varepsilon}$, to be discussed later, was introduced on dimensional grounds and Δ is a function of external momentum scales, mass scales and $x_1, \dots, x_n$. The function $f(k)$ can in general have any kind of indices such as e.g. Lorentz indices. Common examples of f are $f(k) = k^\mu$, $f(k) = k^\mu k^\nu$ and $f(k) = k^\mu k^\nu k^\alpha k^\beta$. The first observation to make regarding these functions is that any function with an odd number of loop momenta vanishes under integration by symmetry. Furthermore, products of momenta with free Lorentz indices can be expressed in terms of the metric. For the examples above it follows that

$$\begin{aligned} k^\mu k^\nu &\longleftrightarrow \frac{1}{d} k^2 g^{\mu\nu} \,, \\ k^\mu k^\nu k^\alpha k^\beta &\longleftrightarrow \frac{1}{d(d+2)} k^2 \left(g^{\mu\nu} g^{\alpha\beta} + g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha}\right) \,. \end{aligned} \tag{1.12}$$

Thus, the tensorial structure can be projected out and one is left with scalar integrals.

After a Wick rotation in momentum space so that $k^0 \to ik_E^0$, where E denotes Euclidean, the integration measure can be written in terms of d dimensional spherical coordinates according to

$$\int \frac{d^d k_E}{(2\pi)^d} = \int \frac{d\Omega_d}{(2\pi)^d} \int_0^\infty dk_E \, k_E^{d-1} \,. \tag{1.13}$$

Here, $d\Omega_d$ is the solid angle element of the unit sphere in d dimensions. Using the definition of the Γ function the following two formulae can be derived,

$$\begin{aligned} \int \frac{d^d k_E}{(2\pi)^d} \frac{1}{\left(k_E^2 + \Delta\right)^n} &= \frac{1}{(4\pi)^{d/2}} \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \,, \\ \int \frac{d^d k_E}{(2\pi)^d} \frac{k_E^2}{\left(k_E^2 + \Delta\right)^n} &= \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma\left(n - \frac{d}{2} - 1\right)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \,. \end{aligned} \tag{1.14}$$

Note that I_{tadpole} in (1.9) is given by the first equation for $\Delta = m^2$ and $n = 1$. Having found expressions for the loop integrals in terms of Γ, the right-hand sides may then be expanded in ε. For instance,

$$\int \frac{d^d k_E}{(2\pi)^d} \frac{1}{\left(k_E^2 + \Delta\right)^2} \longrightarrow \frac{1}{16\pi^2} \left(\frac{1}{\varepsilon} + \log \frac{4\pi}{\Delta} - \gamma_E\right) + \mathcal{O}(\varepsilon) \,, \tag{1.15}$$

where γ_E is the Euler-Mascheroni constant. The divergence in terms of a pole in ε is now clearly visible and the calculation has been regularised. All that remains is to renormalise the theory, i.e. to get rid of the divergence all together. Before discussing that, there is reason to stop and think about how to do higher order loop computations using dimensional regularisation.

Dimensional regularisation for higher order loop diagrams

For the 1–loop case discussed in the previous section, dimensional regularisation boiled down to the two equations in (1.14), which could then be expanded in ε. In going beyond 1–loop order, additional complications appear. As a first example, one can get overlapping divergences, i.e. divergences from both the large and small momentum limits multiplied. Such divergences are called non-local divergences and in general have the structure

$$\frac{1}{\varepsilon} \log q^2 \,, \tag{1.16}$$

where q^2 is some external momentum scale. This is a non-polynomial divergence multiplied by the usual pole in ε. However, in the renormalisation procedure described below, such divergences always cancel as well.

Secondly, and perhaps more severely restricting, one has to be able to take the set of loop integrals from the diagrams calculated and reduce them to a set of known integrals commonly referred to as master integrals. The major limitation here is to actually have a set of master integrals that can be expanded in ε. Very often such integrals are only known when the masses in the propagators are degenerate, and since these calculations are very involved they are not discussed further here. The interested reader is advised to have a look at e.g. Refs. [11–14].

Another limitation is how to be able to reduce the integrals in question to the basis of master integrals, if such a basis even exists. There is software available such as Reduze [15], which employs a Laporta algorithm and uses integration-by-parts (IBP) as well as Lorentz invariance to solve a system of linear equations of the loop integrals. The output is a linear combination of master integrals that have to be known.

In Papers I and III the calculations contain 3- and 2-loop integrals which are handled in precisely the above way. The master integrals showing up have been known for some time and can be found in Refs. [13, 14].

Renormalisation

Once a field theory has a regularisation prescription, the separated infinities can be eliminated by means of renormalisation. The fundamental observation to make is that when writing down a Lagrangian, the parameters such as masses or coupling constants need not be the ones physically measurable. The same is true for the fields in the Lagrangian. By introducing field strength renormalisations (recall the appearance of these in the LSZ formula in (1.4)), one can rewrite the Lagrangian in terms of the physical parameters and counterterms containing the divergent pieces. In dimensional regularisation these are the poles in ε. The additional terms yield additional Feynman rules, and once these counterterms are known and included in a calculation, i.e. by adding all diagrams of a given order where the counterterms can be inserted, the result will be finite [16–18].

The counterterms can be determined via a set of conditions related to the n-point functions of the theory. These conditions are imposed at the renormalisation scale μ^2 and are known as renormalisation conditions. One should further note that only the poles are needed to get rid of the divergences. As a consequence, there is a choice of renormalisation scheme, i.e. whether or not to include also finite pieces in the counterterms (these always cancel in the end either way). A particularly common renormalisation scheme is the modified minimal subtraction scheme $\overline{\text{MS}}$ where only the combination

$$\frac{1}{\varepsilon} - \gamma_E + \log 4\pi \tag{1.17}$$

is kept. Renormalisation in ChPT will be discussed further in section 5. For explicit examples of renormalisation in QED and QCD, see e.g. Ref. [19] for a nice review.

Before proceeding to introduce QCD, the renowned quantum field theory of the strong interaction, some remarks will be made on the ambiguity of choosing a renormalisation scale and its consequences.

The renormalisation scale

The renormalisation conditions can be imposed at any scale μ^2. This ambiguity implies that for a physical observable the scale dependence must cancel at any given order in perturbation theory and the renormalised parameters of the theory change. For a generic theory with a coupling parameter λ, this can be summarised in the so-called β function

$$\frac{d\lambda}{d\log\mu} = \beta(\lambda)\,. \tag{1.18}$$

One of the main implications of the above equation is that depending on what $\beta(\lambda)$ actually is (particularly its sign), the field theory in question can be e.g. confining at low or high energies. It is precisely this that makes QED and QCD fundamentally different – the strong coupling constant α_s increases at low energies so that perturbation theory breaks down below energies on the order of 1 GeV and for high energies there is asymptotic freedom, whereas QED behaves in the opposite way. The phenomenological implications are many, and as has been remarked upon earlier the main subject of this thesis is how to deal with the strong force at low energies. In the next section, QCD, its symmetries and β function are introduced in more detail.

3 Quantum Chromodynamics

The strong force governs the interactions between the colour charged quarks and gluons. It turns out that quarks naturally fall into two categories, light and heavy quarks such that $m_q \ll 1$ GeV and $m_q \geq 1$ GeV, respectively. This can be seen in table 1.1 where the masses as well as quantum numbers of the quarks are shown. This distinction is interesting from symmetry considerations and forms the basis for chiral perturbation theory discussed further in section 5. In order to see this, the quantum field theory describing the strong force, i.e. quantum chromodynamics, is introduced from a mathematical point of view

Table 1.1: The quarks in the Standard Model. The masses are defined in $\overline{\text{MS}}$, and no uncertainties have been included here. Isospin here refers to the third component, i.e. I_3. For further information see for instance Ref. [8].

	Light quarks			Heavy quarks		
	u	d	s	c	b	t
Mass	2.2 MeV	4.7 MeV	93 MeV	1.3 GeV	4.2 GeV	160 GeV
Charge	2/3	−1/3	−1/3	2/3	−1/3	2/3
Isospin	1/2	−1/2	0	0	0	0
Strangeness	0	0	−1	0	0	0

below. Special emphasis is put on the symmetries of the Lagrangian and the discussion is limited to properties relevant for the low energy sector and papers herein.

3.1 Construction and symmetries

Quantum chromodynamics is a non-Abelian gauge theory constructed from an $SU(N_c)$ symmetry where N_c is the number of colours. The actual number of colours is $N_c = 3$ and the QCD Lagrangian must therefore be invariant under local $SU(3)_c$ rotations in colour space. As will be seen below, also other $SU(3)$ symmetries can be found in the QCD Lagrangian. The underlying mathematics is the same for all of them so first consider a general $SU(3)$ symmetry. The eight generators t^a of $SU(3)$ are given in terms of the Gell-Mann matrices λ^a according to

$$t^a = \frac{1}{2}\lambda^a\,,\; a \in \{1, \dots, 8\}\,. \tag{1.19}$$

The Gell-Mann matrices are traceless and in addition satisfy

$$\begin{aligned} \mathrm{Tr}\left[\lambda^a\lambda^b\right] &= 2\delta^{ab}\,, \\ \left[\frac{1}{2}\lambda^a, \frac{1}{2}\lambda^b\right] &= if^{abc}\,\frac{1}{2}\lambda^c\,, \end{aligned} \tag{1.20}$$

where the structure constants f^{abc} are given by $4if^{abc} = \mathrm{Tr}\left(\left[\lambda^a, \lambda^b\right]\lambda^c\right)$. From this one sees that the structure constants are completely antisymmetric. There is also an additional matrix $\lambda^0 = \sqrt{2/3}\,I$, where I is the identity, that can be added to the set of λ^a without changing the properties in (1.20). As this matrix is not traceless the generators t^a for $a = 0, \dots, 8$ instead correspond to $SU(3) \times U(1)$. Note finally that any object with $SU(3)$ indices transforms with the group elements

$$U = e^{-i\sum_a t^a\alpha^a} \in SU(3)\,, \tag{1.21}$$

where α^a are parameters that are local for gauge symmetries.

Denoting the quark field of flavour f and mass m_f by q_f, the QCD Lagrangian is given by

$$\mathcal{L}_{QCD} = \sum_f \bar{q}_f \left(i\not{D} - m_f \right) q_f - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a \,, \tag{1.22}$$

where D_μ is the covariant derivative containing the strong coupling $g = \sqrt{\alpha_s/(4\pi)}$ and the gauge fields A^a_μ, i.e.

$$D_\mu = \partial_\mu - i g \, t^a_c \, A^a_\mu \,. \tag{1.23}$$

A subscript c was included on the generators t^a_c to indicate that they correspond to $SU(3)_c$. The field strength tensor is given by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \,. \tag{1.24}$$

Note that the non-vanishing of f^{abc} for a non-Abelian gauge theory such as $SU(3)$ implies self-interactions among the gauge bosons of the theory. For the abelian gauge theory QED there can be no such interactions and unlike gluons it thus follows that photons do not interact with each other.

Before proceeding to the other symmetries of the QCD Lagrangian, consider first the running of α_s in terms of the β function of QCD. This differential equation can be written (cf. (1.18))

$$\beta(\alpha_s(\mu)) = \frac{d\,\alpha_s(\mu)}{d \log \mu} \,. \tag{1.25}$$

Perturbatively calculating $\beta(\alpha_s(\mu))$ to one-loop order then yields the differential equation

$$-2\beta_0 \frac{\alpha_s^2(\mu)}{4\pi} = \frac{d\,\alpha_s(\mu)}{d \log \mu} \,, \tag{1.26}$$

where $\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_F \, n_q$ is defined in terms of the colour factors $C_A = 3$ and $T_F = 1/2$, and the number of quarks with masses below the scale μ is n_q. First of all note that n_q changes with μ, and a discontinuous jump will occur whenever a new mass scale is passed in the evolution. This requires matching in the running, but this will not be discussed further here. Next note that $\beta_0 > 0$ in QCD. If one now solves the one-loop renormalisation group equation above, the result is

$$\alpha_s(\mu) = \frac{4\pi \alpha_s(Q)}{4\pi + \alpha_s(Q) \beta_0 \log \mu^2/Q^2} \,, \tag{1.27}$$

where Q is the boundary value, or reference scale, used in the solution of the differential equation. It is now trivial to see that $\alpha_s(\mu)$ grows at low energies so that perturbation theory fails below some characteristic scale $\Lambda_{QCD} \sim 340$ MeV [8]. This behaviour arises due to a positive β_0. In QED the corresponding constant is negative, and thus has the opposite behaviour, i.e. it diverges at high energies instead.

In order to study the low energy region of QCD, one may thus restrict the discussion to degrees of freedom relevant on scales lighter or on the order of Λ_{QCD}. We therefore restrict to an upper scale of 1 GeV and note that only the light quarks with masses much smaller than 1 GeV need to be included. There is now a choice of whether or not to include the strange quark as it is much heavier than the up and down quarks. In the following, we therefore consider N_f light quarks which phenomenologically corresponds to either two or three. A first step in finding additional symmetries in the QCD Lagrangian is to exclude the quark mass terms, so that the Lagrangian is given by

$$\mathcal{L}_{QCD} = \bar{q} i \not{D} q - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a \,. \tag{1.28}$$

The flavour sum has now been replaced by scalar product in flavour space for the vector q containing the N_f lightest quark flavours. Clearly, this has a global $SU(N_f)_F$ symmetry, where the subscript now stands for flavour, and in addition to this also has a global $U(1)$ symmetry. The generators of $SU(N_f)$ are defined analogously to those in $SU(3)$. For instance, in $SU(2)$ the generators are defined via the Pauli matrices τ_i rather than the Gell-Mann matrices. It can further be checked that $\mathcal{L}_{QCD}$ remains invariant under the discrete transformations parity, charge conjugation, time reversal and hermitian conjugation.

In the chiral basis obtained from the projection operators $P_L = (1 - \gamma_5)/2$ and $P_R = (1 + \gamma_5)/2$ the Lagrangian can be written

$$\mathcal{L}_{QCD} = \bar{q}_L \, i \not{D} \, q_L + \bar{q}_R \, i \not{D} \, q_R - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a \,. \tag{1.29}$$

This is symmetric under $SU(N_f)_L \times SU(N_f)_R \times U(1)_L \times U(1)_R$. The excluded quark mass terms in the chiral basis are of the form

$$\mathcal{L}_{mass} = -\bar{q}_L \, M \, q_R - \bar{q}_R \, M \, q_L \,, \tag{1.30}$$

where M is the quark mass matrix for the N_f lightest quarks. Clearly, these terms break the above symmetries, but since the quark masses are light we can still consider the symmetries to be approximately preserved. In the following, the goal is to reach chiral perturbation theory, so focus is put on the chiral symmetry group $SU(N_f)_L \times SU(N_f)_R$.

3.2 Conserved currents

Now that the classical symmetries of the Lagrangian have been found, it is possible to look at the conserved currents. From Noether's theorem one knows that for every continuous symmetry there is a conserved current. Thus, one expects $2(N_f^2 - 1)$ conserved currents for the chiral symmetry group, and this is indeed true. By doing a chiral transformation

$$q_{L,R} \to g_{L,R}\, q_{L,R}\,, \tag{1.31}$$

it can be shown that the currents

$$J^{\mu,a}_{L,R} = \bar{q}_{L,R}\,\gamma^\mu\, t^a_{L,R}\, q_{L,R}\,, \tag{1.32}$$

are conserved [20]. Here the $t^a_{L,R}$ are generators of $SU(N_f)_{L,R}$. When quark masses are included, the current divergences have the form

$$\partial_\mu J^{\mu,a}_{L,R} = -i\Big[\bar{q}_{L,R}\, t^a_{L,R}\, M\, q_{R,L} - \bar{q}_{R,L}\, M\, t^a_{L,R}\, q_{L,R}\Big]\,, \tag{1.33}$$

which clearly vanish in the massless limit. It is also conventional to use the linear combinations $J^{\mu,a}_V = J^{\mu,a}_R + J^{\mu,a}_L$ and $J^{\mu,a}_A = J^{\mu,a}_R - J^{\mu,a}_L$ known as the vector and axial currents, respectively. In an analogous way, the $U(1)_L \times U(1)_R$ symmetry found before also corresponds to conserved currents (singlets under the chiral symmetry group, however). Again, one has currents of the form

$$J^\mu_{L,R} = \bar{q}_{L,R}\,\gamma^\mu\, q_{L,R}\,, \tag{1.34}$$

with corresponding singlet vector and axial currents J^μ_V and J^μ_A. It is interesting to note that the singlet axial current has an anomaly in the quantum theory, but this is not discussed further here.

It has so far been shown that massless QCD is invariant under the chiral symmetry group, but that this symmetry is explicitly broken by non-vanishing quark masses. In addition, it turns out that the chiral symmetry group can be spontaneously broken by a non-vanishing quark vacuum expectation value, i.e. for $\langle \bar{q}q \rangle \neq 0$. The spontaneous breakdown is of the form

$$SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V\,, \tag{1.35}$$

where the subscript V means vectorial and corresponds to $g_L = g_R$ where $g_{L,R} \in SU(N_f)_{L,R}$. This will be discussed in more detail in the next section, as well as the identification of the $N_f^2 - 1$ Goldstone bosons with the equally many lightest pseudoscalar mesons in terms of the CCWZ formalism [21].

3.3 The spontaneously broken chiral symmetry

Before simply accepting that the chiral symmetry of QCD is broken, one may of course ask what the consequences would be if it were not. The hadronic spectrum would have additional states due to a parity doubling, but this is not observed in experiments and so chiral symmetry must be broken. The quark masses do break the chiral symmetry explicitly, but since it is realised approximately for small quark masses it should still be of relevance. In particular, Goldstone's theorem states that for every broken generator there is an associated massless Goldstone boson. An interesting idea to explore is thus whether or not an approximately realised symmetry can have light Goldstone bosons, or, pseudo-Goldstone bosons. For the chiral symmetry which is exact in the massless limit, it can be guessed that the corresponding pseudo-Goldstone bosons should have masses related to the light quark masses. This is precisely the case for the chiral symmetry breaking and the $N_f^2 - 1$ lightest pseudoscalar mesons as will be shown below. An interesting consequence is that it implies a mass gap between the pseudo-Goldstone modes and the baryons such as the proton. The proton consists of uud and has a mass of roughly 1 GeV. However, the neutral pion, which is the lightest mesonic state and consists of $\bar{u}u$ and $\bar{d}d$, has a mass ~ 140 MeV, i.e. much smaller than the proton mass even though the quark content is very similar. The spontaneous breakdown of the chiral symmetry thus offers both theoretically as well as phenomenologically satisfying explanations of the low energy spectrum of QCD.

The CCWZ formalism

In order to reach this spontaneous symmetry breaking, consider a non-zero scalar quark-antiquark vacuum expectation value

$$0 \neq \langle \bar{q}q \rangle \,, \tag{1.36}$$

where q is a vector in flavour space. Now, if this is rewritten in the chiral basis one has

$$0 \neq \langle \bar{q}_L\, q_R + \bar{q}_R\, q_L \rangle \,. \tag{1.37}$$

Performing a chiral rotation $(g_L, g_R) \in G$, where the chiral symmetry group $G = SU(N_f)_L \times SU(N_f)_R$, then gives

$$0 \neq \langle \bar{q}_L\, q_R + \bar{q}_R\, q_L \rangle \longrightarrow \langle \bar{q}_L\, g_L^\dagger\, g_R\, q_R + \bar{q}_R\, g_R^\dagger g_L\, q_L \rangle \,. \tag{1.38}$$

From this it is obvious that only for the special rotations $g_L = g_R$ will the vacuum expectation value $\langle \bar{q}q \rangle$ remain invariant. In other words, only the group $H = SU(N_f)_V \subset G$, where V means precisely $L = R$, leaves the vacuum invariant and so we have the spontaneous symmetry breaking

$$SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V \,. \tag{1.39}$$

Denote the corresponding $N_f^2 - 1$ Goldstone bosons associated to this symmetry breaking as $\phi(x)$ and the vacuum configuration as ϕ_V. First of all note that ϕ_V is invariant under H, i.e. $\phi_V \xrightarrow{H} \phi_V$. The field configuration $\phi(x)$ is an excitation around the vacuum and can be obtained from the vacuum via

$$\phi(x) = \Xi_g(x)\,\phi_V\,, \tag{1.40}$$

where Ξ_g is a mapping for $g \in G$. Due to the invariance of the vacuum under H, one also has

$$\phi(x) = \Xi_{gh}(x)\,\phi_V\,, \tag{1.41}$$

for any $h \in H$. It should be noted that for each g this defines a mapping between a field configuration $\phi(x)$ and the coset $gH = \{gh\,|\,h \in H\}$. The consequence is that the Goldstone bosons are associated with the elements of the set of cosets $G/H = \{gH\}$. By choosing a $g' \in G$ one obtains a coset representative of $\phi(x)$. Denote this coset representative by $\Xi(x) = \Xi_{g'}(x)$. The CCWZ choice [21] (in the notation of Ref. [22]) is to parametrise $\Xi(x)$ in terms of the *broken* generators X^a so that

$$\Xi(x) = e^{iX^a\,\pi^a(x)}\,, \tag{1.42}$$

where $\pi^a(x)$ is a basis of the Goldstone bosons. Now, since $\phi(x) \to g\phi(x)$ for any $g \in G$ it follows that

$$\Xi(x) \to g\,\Xi(x)\,. \tag{1.43}$$

Note that g in general includes also the unbroken generators so that $g\Xi(x)$ does not have the CCWZ form in (1.42). However, one may use a compensating field $h \in H$ such that another representative of $\phi(x)$, $\Xi'(x)$, say, is related to $\Xi(x)$ via

$$\Xi'(x) = g\,\Xi(x)\,h^{-1}\,. \tag{1.44}$$

It thus follows that $\Xi(x)$ transforms as

$$\Xi(x) \longrightarrow g\,\Xi(x)\,h^{-1}\,. \tag{1.45}$$

The above analysis was completely general and did not rely on the specific groups being $G = SU(N_f)_L \times SU(N_f)_R$ and $H = SU(N_f)_V$. With the goal of reaching a low energy theory described in terms of the Goldstone bosons of the chiral symmetry breaking, i.e. the $N_f^2 - 1$ lightest pseudoscalar mesons, we must now consider the specific groups in question. With the generators of G denoted as $t^a_{L,R}$ and the broken generators $X^a = t^a_L - t^a_R$ one has

$$\Xi(x) = e^{i(t^a_L - t^a_R)\,\pi^a(x)}\,. \tag{1.46}$$

By writing any element $g \in G$ as a block diagonal matrix, i.e. $g = \mathrm{diag}\,(g_L, g_R)$ where $g_{L,R} \in SU(N_f)_{L,R}$, it follows that $\Xi(x)$ can be written as

$$\Xi(x) = \begin{pmatrix} \xi(x) & 0 \\ 0 & \xi^\dagger(x) \end{pmatrix}, \tag{1.47}$$

for

$$\xi(x) = e^{iT^a \pi^a(x)}\,. \tag{1.48}$$

With $h = \mathrm{diag}\,(g_V, g_V) \in H$ for some g_V, it follows that ξ transforms as

$$\xi \longrightarrow g_V \xi\, g_R^\dagger = g_L\, \xi\, g_V^\dagger\,. \tag{1.49}$$

In a similar fashion, one could have chosen another set of broken generators X^a and obtained objects equivalent to $\xi(x)$. The second most common choice of broken generators is $X^a = T_L^a$, and from that choice $\Sigma(x) = \xi^2(x)$ can be defined. This transforms as

$$\Sigma \longrightarrow g_L \Sigma\, g_R^\dagger = g_R\, \xi\, g_L^\dagger\,. \tag{1.50}$$

The objects ξ and Σ serve as building blocks when constructing a Lagrangian containing the Goldstone bosons, and in section 5 these building blocks will be discussed further. In that setting one usually denotes $\xi = u$ and $\Sigma = U$. For some more details and examples regarding the chiral symmetry breaking, see e.g. Refs. [22, 23].

3.4 QCD with external fields

As a final remark on QCD let us consider the so-called external field method. This is one of the cornerstones of ChPT and was introduced in Refs. [24, 25]. The idea is to add the external fields v, a, s and p (vector, axial vector, scalar and pseudoscalar, respectively) to the massless QCD Lagrangian $\mathcal{L}_{QCD}$, i.e.

$$\mathcal{L} = \mathcal{L}_{QCD} + \mathcal{L}_{ext} = \mathcal{L}_{QCD} + \bar{q}\,\gamma^\mu (v_\mu + \gamma_5\, a_\mu) q - \bar{q}\,(s - i\,\gamma_5\, p) q\,. \tag{1.51}$$

The external fields are hermitian and colour neutral flavour matrices. The motivation behind doing this is the possibility of including interactions with particles other than those already included in QCD. Green's functions can be obtained via functional differentiation of the generating functional

$$e^{iZ[v,a,s,p]} = \langle 0 | T e^{i \int d^4x\, \mathcal{L}_{ext}} | 0 \rangle\,. \tag{1.52}$$

In the chiral basis the Lagrangian of the external fields is

$$\mathcal{L}_{ext} = \bar{q}_L \gamma^\mu \ell_\mu q_L + \bar{q}_R \gamma^\mu r_\mu q_R - \bar{q}_R (s + i p) q_L - \bar{q}_L (s - i p) q_R\,, \tag{1.53}$$

where the left and right handed external fields $\ell_\mu = v_\mu - a_\mu$ and $r_\mu = v_\mu + a_\mu$ were defined. Note that the full Lagrangian $\mathcal{L}$ is invariant under the local $SU(N_f)_L \times SU(N_f)_R$ transformation $q_{L,R}(x) \to g_{L,R}(x)\, q_{L,R}(x)$ if the external fields transform according to

$$
\begin{aligned}
\ell_\mu &\longrightarrow g_L \ell_\mu g_L^\dagger + i g_L \partial_\mu g_L^\dagger \,, \\
r_\mu &\longrightarrow g_R\, r_\mu\, g_R^\dagger + i g_R\, \partial_\mu\, g_R^\dagger \,, \\
s + ip &\longrightarrow g_R (s + ip) g_L^\dagger \,.
\end{aligned} \tag{1.54}
$$

Some remarks are in place. As was mentioned earlier, the external field method allows for inclusion of particles beyond QCD such as the photon. Quark masses can be included by setting $s = M$, where again M is the quark mass matrix. Furthermore, if one imposes the local gauge invariance obtained in the external field method for QCD at high energies it must also be true in any low energy theory of QCD. Building such a theory in terms of the Goldstone bosons of the chiral symmetry breaking and the external fields above results in chiral perturbation theory, a low energy effective field theory of QCD. In the next section the most fundamental aspects of effective field theories are discussed. Once this has been done, all the tools needed for the construction of chiral perturbation theory have been introduced.

4 Effective Field Theory

Effective field theory is a powerful tool that is used to make predictions valid only below some certain energy scale Λ, above which the theory becomes invalid. The concept of an effective theory is perhaps the most natural way to do physics, and it has certainly been the historical way to do it as well. Consider for instance Newtonian gravity which breaks down in sufficiently strong gravitational fields due to effects neglected from the more fundamental theory general relativity. Another example is the quantum mechanical system of the hydrogen atom, for which the Schrödinger equation can be used to find energy levels to a reasonable accuracy even though effects of the quarks interacting inside the proton and so on are disregarded. However, the numerical parameters of an effective theory of course depend on whatever is left out. The proton mass depends on the quark masses, so changing the quark masses would give a different numerical value of the proton mass to use in the Schrödinger equation, but the functional form is the same. It is precisely this idea that comes into the construction of effective field theories in particle physics.

The most important question to ask when constructing an EFT is what the relevant degrees of freedom are. For QCD, at low energies below some scale $\Lambda_\chi \sim 1$ GeV these are the lightest pseudoscalar mesons instead of the quarks and gluons. From this realisation one can build chiral perturbation theory. On the other hand, at energies well below the W

mass scale m_W, the W boson may be integrated out of the electroweak theory to yield Fermi theory. This is done by writing down the amplitude for the process $us \to du$, which from the electroweak SM Lagrangian is

$$\left(-\frac{ig}{\sqrt{2}}\right)^2 V_{us}V_{ud}^* \left[\overline{u}\,\gamma_\mu P_L\, s\right] \left[\overline{d}\,\gamma^\mu P_L\, u\right] \frac{-i}{p^2 - m_W^2}\,, \tag{1.55}$$

where V_{ij} are CKM matrix elements. Note that for $p^2/m_W^2 \ll 1$, the propagator may be expanded to yield

$$\left(-\frac{ig}{\sqrt{2}}\right)^2 V_{us}V_{ud}^* \left[\overline{u}\,\gamma_\mu P_L\, s\right] \left[\overline{d}\,\gamma^\mu P_L\, u\right] \frac{i}{m_W^2} + \mathcal{O}\left(m_W^{-4}\right)\,. \tag{1.56}$$

This could in principle have been obtained from the effective Fermi theory Lagrangian

$$\mathcal{L}_{\text{Fermi}} = -\frac{4G_F}{\sqrt{2}} V_{us}V_{ud}^* \left[\overline{u}\,\gamma_\mu P_L\, s\right] \left[\overline{d}\,\gamma^\mu P_L\, u\right]\,, \tag{1.57}$$

where the Fermi constant $G_F = g^2/(4\sqrt{2}\,m_W^2)$ was introduced. This explicitly shows that the non-relevant degrees of freedom, here the W boson, have direct impact on the parameters of the effective theory. For this straightforward example, one can directly relate the parameters with parameters from the full theory, so-called matching, but this is not possible in general. For such theories, one can only obtain values for the parameters by e.g. fitting to experimental data.

The higher orders in $1/m_W^2$ that were left out in the expansion of the W boson propagator put a restriction on the precision of predictions made from the effective theory. For increased precision also higher order terms must be considered. For any EFT the effective Lagrangian may in principle contain an infinite number of terms and for a given precision it must be truncated at some certain order. A general Lagrangian can be written

$$\mathcal{L}_{\text{EFT}} = \mathcal{L}_{D\leq 4} + \sum_{d=1}^{\infty} \frac{1}{\Lambda^d} \sum_{i_d} c_{i_d} \mathcal{O}_{i_d}\,, \tag{1.58}$$

where Λ is the scale at which the EFT breaks down, the c_{i_d} are dimensionless parameters known as Wilson coefficients containing the information of the physics above Λ, and the $\mathcal{O}_{i_d}$ are dimension $d + D$ operators built from the relevant degrees of freedom. That it is possible to build an EFT in this way is asserted by Weinberg's "theorem" in Ref. [26]. This theorem states that by using the most general Lagrangian, i.e. by including all possible operators in terms of the relevant degrees of freedom, such that the imposed symmetries of the EFT are satisfied, the most general S matrix consistent with the assumptions of quantum field theory is automatically obtained.

Note that having an infinite number of terms in the effective Lagrangian prohibits any practical application. This is true for two reasons. Of course, it is impossible to calculate an infinite number of diagrams. Moreover, with an infinite number of terms the in general divergent and infinitely many Wilson coefficients need to be renormalised with infinitely many counterterms, i.e. the theory is non-renormalizable. However, if one in some way characterises how important the different contributions are in terms of some counting (cf. an expansion in the coupling constant in ordinary perturbation theory within the Standard Model), then to each order in this counting only a finite number of diagrams and counterterms are needed. The characterisation of importance is known as power counting, and is essential in any EFT. The conclusion is that by writing down the most general effective Lagrangian to a certain order in the power counting and renormalising the Wilson coefficients a finite result is obtained. The renormalisation is done order by order, and a scheme has to be chosen for this. A common choice is dimensional regularisation and $\overline{\text{MS}}$, and the procedure of renormalisation in an EFT is essentially a generalisation of that discussed in section 2.3. For further details about renormalisation in EFTs see Ref. [19].

It is interesting to note that the Standard Model is expected to work only up to some certain energy scale as well, and can therefore be thought of as an effective field theory with only the dimension 4 operators included, i.e. it is renormalisable. Over the years there has been a lot of work on Standard Moedel EFT (or SMEFT, in short), where also higher dimensional operators are taken into account. See Ref. [27] for a review.

5 Chiral Perturbation Theory

Chiral perturbation theory is an EFT of QCD at low energies. It is built from the spontaneous symmetry breaking of the chiral symmetry

$$SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V \,. \tag{1.59}$$

The construction relies on writing down an effective chirally invariant Lagrangian $\mathcal{L}_\chi$ satisfying also Lorentz invariance and the discrete symmetries of the QCD Lagrangian. This is built in terms of the $N_f^2 - 1$ lightest pseudoscalar mesons, which for $N_f = 2$ are the pions and for $N_f = 3$ are the pions, the kaons and the eta meson. The quark contents, masses and quantum numbers of these states are given in table 1.2. As was discussed in section 3.4, the external fields ℓ_μ, r_μ, s and p are also included. Such a Lagrangian can then be used for perturbative calculations, where each order is defined by a power counting in terms of the momentum p^2. The Lagrangian is expanded order by order according to

$$\mathcal{L}_\chi = \sum_{n=1}^{\infty} \mathcal{L}_{2n} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \mathcal{L}_8 + \dots \,, \tag{1.60}$$

Table 1.2: The masses and quantum numbers of the lightest pseudoscalar mesons composed of u, d and s quarks. Isospin here refers to the third component, i.e. I_3. For further information see for instance Ref. [8].

	π^0	$\pi^+\ (\pi^-)$	$K^0\ \left(\overline{K}^0\right)$	$K^+\ (K^-)$	η
Quark content	$\left(u\overline{u}-d\overline{d}\right)/2$	$u\overline{d}\,(d\overline{u})$	$d\overline{s}\left(s\overline{d}\right)$	$u\overline{s}\,(s\overline{u})$	$\left(u\overline{u}+d\overline{d}-2s\overline{s}\right)/\sqrt{6}$
Mass	135 MeV	140 MeV	498 MeV	494 MeV	548 MeV
Charge	0	$+1(-1)$	0	$+1(-1)$	0
Isospin	0	$+1(-1)$	$-1/2(+1/2)$	$+1/2(-1/2)$	0
Strangeness	0	0	$+1(-1)$	$+1(-1)$	0

The four displayed terms are referred to as leading order (LO), next-to leading order (NLO), next-to-next-to leading order (NNLO) and next-to-next-to-next-to leading order (NNNLO) respectively. These are the only known Lagrangians of ChPT. The LO piece is easily obtained but the higher order ones require some more work. NLO was introduced in Refs. [24, 25], NNLO in Ref. [28] and NNNLO in Ref. [29], i.e. Paper II here. This is a non-renormalisable EFT, but calculating to a given order in the expansion always gives a finite result if all contributions to that order are taken into account. This will be discussed in section 5.3 in more detail. It should furthermore be noted that the expansion really is an expansion in p^2/Λ_χ^2, where $\Lambda_\chi \sim 4\pi F \sim 1$ GeV is the scale at which the EFT is expected to break down, where $F \approx 93$ MeV is the pion decay constant. However, the σ and ρ have masses below 1 GeV and should set the real limit of the applicability of ChPT. The pion decay constant F naturally arises in ChPT from the LO Lagrangian, as illustrated later.

Each of the Lagrangians in $\mathcal{L}_\chi$ has the form

$$\mathcal{L}_{2n} = \sum_i c_i^{(2n)}\, \mathcal{O}_i^{(2n)}\,, \tag{1.61}$$

where the operators $\mathcal{O}_i^{(2n)}$ are local operators containing the meson fields, and the $c_i^{(2n)}$ are low energy constants (LECs) whose renormalised numerical values determine the predictivity of the EFT. In the following, the foundations of ChPT will be discussed in more detail.

5.1 Construction of the Lagrangian

In order to construct ChPT, recall the CCWZ formalism and the external field method in sections 3.3 and 3.4. As was mentioned there, it is conventional to redefine the two Goldstone boson bases denoted by ξ and $\Sigma = \xi^2$ as u and $U = u^2$, respectively. For an arbitrary chiral transformation $(g_L, g_R) \in SU(N_f)_L \times SU(N_f)_R$ and a compensator field

$h \in SU(N_f)_V$ these objects transform as

$$\begin{aligned} u &\longrightarrow g_R\, u\, h = h\, u\, g_L\,, \\ U &\longrightarrow g_R U g_L^\dagger\,. \end{aligned} \tag{1.62}$$

We will here consider the U basis as that is the canonical choice. The external fields, which are $N_f \times N_f$ matrices in flavour space, transform according to

$$\begin{aligned} \chi &\equiv 2B\,(s + i\,p) \longrightarrow g_R\, \chi\, g_L^\dagger\,, \\ \ell_\mu &\equiv v_\mu - a_\mu \longrightarrow g_L \ell_\mu\, g_L^\dagger - i\,\partial_\mu\, g_L\, g_L^\dagger\,, \\ r_\mu &\equiv v_\mu + a_\mu \longrightarrow g_R\, r_\mu\, g_R^\dagger - i\,\partial_\mu\, g_R\, g_R^\dagger\,, \end{aligned} \tag{1.63}$$

where the field χ was introduced and a constant B included in its definition. The parameter B is related to the vacuum expectation value $\langle \bar{q}q \rangle$, as well as the leading order pion decay constant, F, via $B = -\langle \bar{q}q \rangle / N_f^2 F$. The definition of $U(x)$ in terms of the $SU(N_f)$ generators T^a and a parametrisation of the meson fields $\pi^a(x)$ is (cf. the CCWZ choice in (1.46))

$$U(x) = \exp\, iT^a \pi^a(x)/F\,, \tag{1.64}$$

where the pion decay constant has been introduced into the exponential on dimensional grounds as will be seen shortly. For $N_f = 2$, i.e. when only up and down quarks are taken into account, the matrix $\pi = T^a \pi^a$ (with suppressed spacetime dependence) in the exponent takes the form

$$N_f = 2\ : \quad \pi = \begin{pmatrix} \pi^0 & \sqrt{2}\,\pi^+ \\ \sqrt{2}\,\pi^- & -\pi^0 \end{pmatrix}. \tag{1.65}$$

In the $SU(3)$ theory where also the strange quark is included one has

$$N_f = 3\ : \quad \pi = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\,\pi^+ & \sqrt{2}\,K^+ \\ \sqrt{2}\,\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\,K^0 \\ \sqrt{2}\,K^- & \sqrt{2}\,\overline{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}. \tag{1.66}$$

The matrix $U(x)$ is therefore completely defined in terms of meson fields and the pion decay constant. Before being able to write down a Lagrangian, we also need to consider the symmetries of the theory.

In the external field method the chiral symmetry was promoted to a local one and in order that the theory be gauge invariant, the external traceless fields ℓ_μ and r_μ must be included in

Table 1.3: Discrete transformation properties of the chiral building blocks. Here, $\varepsilon(0) = -\varepsilon(i = 1,2,3) = 1$.

	P	C	h.c.
U	$U^\dagger$	U^T	$U^\dagger$
χ	$\chi^\dagger$	χ^T	$\chi^\dagger$
$F_L^{\mu\nu}$	$\varepsilon(\mu)\varepsilon(\nu)F_R^{\mu\nu}$	$-\left(F_R^{\mu\nu}\right)^T$	$F_L^{\mu\nu}$
$F_R^{\mu\nu}$	$\varepsilon(\mu)\varepsilon(\nu)F_L^{\mu\nu}$	$-\left(F_L^{\mu\nu}\right)^T$	$F_R^{\mu\nu}$

a covariant derivative D_μ transforming as the operator O it acts on. One therefore defines

$$D_\mu O = \begin{cases} \partial_\mu O - ir_\mu O + iO\ell_\mu, & O \longrightarrow g_R\, O g_L^\dagger\,, \\ \partial_\mu O - i\ell_\mu O + iOr_\mu, & O \longrightarrow g_L\, O g_R^\dagger\,, \\ \partial_\mu O - ir_\mu O + iOr_\mu, & O \longrightarrow g_R\, O g_R^\dagger\,, \\ \partial_\mu O - i\ell_\mu O + iO\ell_\mu, & O \longrightarrow g_L\, O g_L^\dagger\,. \end{cases} \tag{1.67}$$

From a Lagrangian builder's perspective, this transformation property is cumbersome to keep track of at higher orders in the chiral counting. The u basis such that $u^2 = U$ on the other hand allows for a much simpler covariant derivative, and is the one used in Paper II. However, this basis is not considered further here. Now, the inclusion of ℓ_μ and r_μ in the covariant derivative introduces the field strength tensors

$$\begin{aligned} F_L^{\mu\nu} &\equiv \partial^\mu \ell^\nu - \partial^\nu \ell^\mu - i\,[\ell^\mu, \ell^\nu] \longrightarrow g_L F_L^{\mu\nu} g_L^\dagger\,, \\ F_R^{\mu\nu} &\equiv \partial^\mu r^\nu - \partial^\nu r^\mu - i\,[r^\mu, r^\nu] \longrightarrow g_R F_R^{\mu\nu} g_R^\dagger\,. \end{aligned} \tag{1.68}$$

All the needed building blocks for the chiral Lagrangians have thus been defined. For a Lagrangian at order p^{2n}, the building blocks must now be combined into Lorentz invariant operators of order p^{2n} satisfying chiral symmetry as well as the discrete symmetries of QCD. The chiral orders of the building blocks and the covariant derivative are given by

$$U \sim \mathcal{O}(1)\,, \quad D_\mu \sim \mathcal{O}(p)\,, \quad \chi \sim \mathcal{O}\left(p^2\right)\,, \quad F_{L,R}^{\mu\nu} \sim \mathcal{O}\left(p^2\right)\,. \tag{1.69}$$

The discrete symmetry transformations of the building blocks are given in table 1.3.

There can be no non-trivial Lagrangian at $\mathcal{O}(p^0)$. The only possibility would be $\langle U^\dagger U\rangle$, but since U is an $SU(N_f)$ matrix such a term would just give an overall constant. There is no Lorentz invariant term to write down at $\mathcal{O}(p)$. For the Lagrangian at order p^2, i.e. at LO, it is easy to see that no non-vanishing Lorentz invariant combinations can be written down with the field strength tensors. Therefore only U, D_μ and χ can be used. Note further that U and χ are $SU(N_f)$ matrices, and so again the respective unitary properties limit the possible structures even more. Bearing in mind the discrete symmetries, we are left with the possible combinations

$$\left\langle \left(D_\mu U\right)^\dagger D^\mu U\right\rangle , \quad \left\langle U^\dagger D^2 U + U\left(D^2 U\right)^\dagger \right\rangle , \quad \left\langle \chi^\dagger U + \chi U^\dagger \right\rangle . \tag{1.70}$$

However, since the action is invariant under the addition of a total derivative it is possible to do partial integration. Then it follows that the structure with D^2 can be rewritten as the first one, and the most general chiral mesonic Lagrangian of order p^2 is

$$\mathcal{L} = \frac{F^2}{4}\left\langle \left(D_\mu U\right)^\dagger D^\mu U\right\rangle + \frac{F^2}{4}\left\langle \chi^\dagger U + \chi U^\dagger \right\rangle . \tag{1.71}$$

Note that by including F^2 as an overall factor, the mass dimension of the Lagrangian is 4. Also, the inclusion of F in U gives the kinetic terms of the meson fields the canoncial form. This Lagrangian contains two LECs, first of all F but also B in the definition of χ.

Some further remarks about the above derivation are in place. The symmetries imposed important constraints to minimise the number of terms in the Lagrangian. Then, partial integration was used to reduce the number of a priori possible terms even further. At higher orders in ChPT, several additional identities come into play, where examples are Bianchi identities, identities following from the Cayley-Hamilton theorem (for a fixed N_f) and field redefinitions. In fact, field redefinitions are equivalent to using the LO equation of motion. The mentioned identities are further elaborated on in Paper II and with the exception of those from the Cayley-Hamilton theorem will not be discussed here. What is interesting to point out, however, is the computational problem of how to derive the linear relations from the identities and minimise the basis. For this purpose, consider an arbitrary $N_f \times N_f$ matrix A. The characteristic polynomial A is given by

$$0 = p(\lambda) = \det\left(\lambda I - A\right) = \lambda^{N_f} \det\left(I - \frac{A}{\lambda}\right) , \tag{1.72}$$

where λ is an eigenvalue of A and I is the unit matrix. This is equivalent to

$$p(\lambda) = \lambda^{N_f} \exp\left\langle \ln\left(I - \frac{A}{\lambda}\right)\right\rangle . \tag{1.73}$$

Expanding this in $1/\lambda$ gives the polynomial coefficients in $p(\lambda)$. The Cayley-Hamilton theorem states that A must satisfy its own characteristic equation, i.e. the matrix equation $p(A) = 0$. For $N_f = 2$ and $N_f = 3$ one obtains

$$N_f = 2 : \quad A^2 - A\langle A\rangle - \frac{1}{2}\langle A^2\rangle + \frac{1}{2}\langle A\rangle^2 = 0 ,$$

$$N_f = 3: \quad A^3 - A^2\langle A\rangle - \frac{1}{2}A\langle A^2\rangle + \frac{1}{2}A\langle A\rangle^2 - \frac{1}{3}\langle A^3\rangle + \frac{1}{2}\langle A\rangle\langle A^2\rangle - \frac{1}{6}\langle A\rangle^3 = 0\,. \tag{1.74}$$

Taking the two-flavour case as an example and letting $A = B + C$ for the 2×2 matrices B and C then yields

$$N_f = 2: \quad \{B, C\} - B\langle C\rangle - C\langle B\rangle - \langle BC\rangle + \langle B\rangle\langle C\rangle = 0\,. \tag{1.75}$$

This can now be traced with some 2×2 matrix D to yield

$$N_f = 2: \quad \langle\{B, C\}\,D\rangle - \langle BD\rangle\langle C\rangle - \langle CD\rangle\langle B\rangle - \langle D\rangle\langle BC\rangle + \langle D\rangle\langle B\rangle\langle C\rangle = 0\,, \tag{1.76}$$

which in turn can be used to find relations. To achieve this, suppose that all N^0_{2n} possible operators of order p^{2n} allowed by symmetry have been constructed from the chiral building blocks. These are (products of) traces of products of the building blocks. Assuming that one of them can be written

$$\mathcal{O}_i = \langle A_1 A_2 A_3 A_4\rangle\,, \tag{1.77}$$

for some 2×2 matrices $A_{1,2,3,4}$, then (1.76) can be used to give relations for this operator. The way to do this is by partitioning the product $A_1A_2A_3A_4$ as BCD in every way possible, and for every such choice a relation will be obtained from (1.76). Terms of the form $\langle A_1A_2\rangle\langle A_3A_4\rangle$ will correspond to some other operator in the list of N^0_{2n} operators allowed by symmetry. Proceeding in a similar fashion for all the different identities yields a set of N_{rel} linear relations which can be written as

$$R_{kj}\,\mathcal{O}_j = 0\,. \tag{1.78}$$

Here, R_{kj} is an $N_{\text{rel}} \times N^0_{2n}$ matrix. The rank of the matrix R_{kj} gives the number of linearly independent relations, and the minimal number of operators in the Lagrangian is thus $N^0_{2n} - \text{rank}\,(R)$. Knowing this, one can construct a minimal operator basis from the set of linear relations. This is precisely what is done in Paper II.

Note further the possibility, starting at NLO, of having operators depend on only external fields. Such terms are called contact terms and are unphysical but important for renormalisation purposes. Secondly, the minimal number of operators and thus the number of LECs increase with the chiral order, as can be seen in table 1.4. Therefore also the predictivity of the theory decreases unless the values of the LECs can be fixed. In section 5.3 some of these issues will be addressed.

Before proceeding, an important point should be made. The connection between QCD and ChPT is provided via the equality of the generating functionals of the respective theories, i.e.

$$e^{iZ[\ell,r,s,p]} = \frac{1}{Z_{ext}}\int_{\text{fields}} e^{i\int d^4x\,\mathcal{L}_{ext}} = \frac{1}{Z_\chi}\int_{\text{fields}} e^{i\int d^4x\,\mathcal{L}_\chi}\,, \tag{1.79}$$

Table 1.4: The minimal number of terms in the Lagrangians at the indicated orders and numbers of flavour. Also the number of contact terms is listed for the respective orders.

	N_f		$N_f = 3$		$N_f = 2$	
	Total	Contact	Total	Contact	Total	Contact
p^2	2	0	2	0	2	0
p^4	13	2	12	2	10	3
p^6	115	3	94	4	56	4
p^8	1862	22	1254	21	475	23

where the integral operators with subscript are over all field configurations possible and $Z_{ext/\chi}$ are given by the integrals in the respective numerators with the external fields put to zero [23]. It is from this correspondence that one can show that the constant F in the $\mathcal{L}_2$ really is the pion decay constant, and that B is related to $\langle \bar{q}q \rangle$.

5.2 Some remarks on the meson masses

From the LO Lagrangian one can obtain first estimates of the meson masses. These are obtained by setting $s = M$, where again M is the quark mass matrix in the N_f theory. Choosing $N_f = 3$ the results are [22, 23]

$$\begin{aligned} M_{\pi^\pm}^2 &= B(m_u + m_d)\,, \quad M_{\pi^0}^2 = B(m_u + m_d) - \Delta M^2\,, \\ M_{K^\pm}^2 &= B(m_u + m_s)\,, \quad M_{K^0}^2 = (m_d + m_s)B\,, \\ M_\eta^2 &= \frac{2}{3}\left(\frac{1}{2}(m_u + m_d) + 2m_s\right) B + \Delta M^2\,, \end{aligned} \tag{1.80}$$

where ΔM^2 is a measure of isospin breaking and is to first order given by

$$\Delta M^2 = \frac{B}{4}\frac{(m_u - m_d)^2}{m_s - \frac{1}{2}(m_u + m_d)}\,. \tag{1.81}$$

This leads to estimates of the quark mass ratios,

$$\begin{aligned} \frac{m_u}{m_d} &= \frac{M_{K^\pm}^2 - M_{K^0}^2 + 2M_{\pi^0}^2 - M_{\pi^\pm}^2}{M_{K^0}^2 - M_{K^\pm}^2 + M_{\pi^\pm}^2}\,, \\ \frac{m_s}{m_d} &= \frac{M_{K^0}^2 + M_{K^\pm}^2 - M_{\pi^\pm}^2}{M_{K^0}^2 - M_{K^\pm}^2 + M_{\pi^\pm}^2}\,. \end{aligned} \tag{1.82}$$

The prediction is a mass ratio

$$m_u : m_d : m_s = 0.55 : 1 : 20.3\,. \tag{1.83}$$

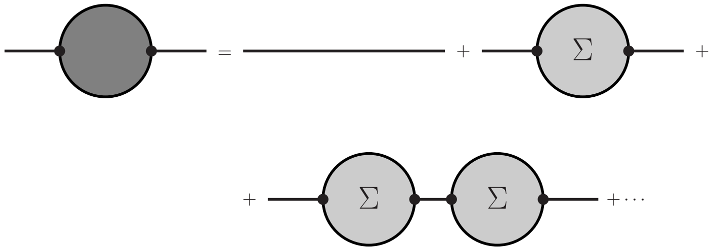


Figure 1.2: The full propagator as a series of propagators with Σ insertions.

The isospin breaking effects which are on the order of $(m_d - m_u)/m_s \sim 0.025$ are therefore very small. There is yet another well-known relation to be obtained from (1.80), namely the Gell-Mann-Okubo relation [30, 31]

$$M_{\pi^0}^2 + 3M_\eta^2 = 4M_{K^0}^2\,, \tag{1.84}$$

which holds for the isospin limit.

The formulae above were obtained from the LO Lagrangian. However, these masses are shifted at higher orders and this is considered in isospin symmetric two-flavour ChPT at NNNLO in Paper I. The mass shift is obtained from higher order corrections to the 2–point function which serve to move the pole of the propagator, as will be seen below.

In order to be completely general, consider a propagating meson with LO mass M and external momentum p. The full propagator is equal to the scalar 2-point function as it is the sum of all possible diagrams with two external scalar legs, both external legs being the meson in question. This correlation function can be constructed from repeated insertions of the scalar self energy Σ as in Fig. 1.2. The full propagator can thus be resummed as a geometric series according to

$$\frac{i}{p^2 - M^2 + i\varepsilon}\sum_{n=0}^{\infty}\left(-i\Sigma\left(p^2\right)\frac{i}{p^2 - M^2 + i\varepsilon}\right)^n = \frac{i}{p^2 - M^2 - \Sigma\left(p^2\right) + i\varepsilon}\,. \tag{1.85}$$

Defining the physical mass as the pole of this full propagator shows that

$$M_\phi^2 = M_0^2 + \Sigma\left(M_\phi^2\right)\,. \tag{1.86}$$

Moreover, expanding around the physical mass and combining this with the LSZ theorem in (1.4) shows that the field strength renormalisation is

$$Z_\phi = \left(1 - \frac{\partial\Sigma}{\partial p^2}\right)^{-1}\Bigg|_{p^2 = M_\phi^2}\,. \tag{1.87}$$

In order to do the calculation of M_ϕ^2 at fixed order, one has to consider the contributions to $\Sigma\left(p^2\right)$ to that very order. The self energy thus has the form

$$\Sigma\left(p^2\right) = \Sigma_4\left(p^2\right) + \Sigma_6\left(p^2\right) + \Sigma_8\left(p^2\right) + \dots\,, \tag{1.88}$$

where the subscripts refer to orders in the chiral expansion. This allows the mass in (1.86) to be written as an expansion according to

$$M_\phi^2 = M^2\left(1 + M_4^2 + M_6^2 + M_8^2 + \dots\right)\,. \tag{1.89}$$

The NLO and NNLO contributions for the pion mass in the two-flavour theory have been known for some time, see Ref. [24] for the NLO result and Refs. [32,33] for that at NNLO. In Paper I the NNNLO is calculated for the first time. Note that at every order new LECs appear. The pion decay constant F_π defined through

$$\langle 0|A_\mu(0)|\pi(p)\rangle = i\sqrt{2}\,p_\mu F_\pi\,, \tag{1.90}$$

where $A_{\mu(0)}$ is the axial current, can be calculated at a given order in the chiral expansion as well. Then the outgoing pion is replaced by the axial current. The NLO and NNLO results can be found in the same references as for the mass, and the NNNLO contribution is calculated in Paper I for the first time as well.

5.3 Renormalisability and low energy constants

As was shown in the previous section, a number of LECs show up at every order in the chiral expansion. These must in general be renormalised, and in calculations of processes such as e.g. pion-pion scattering to a fixed order, all contributions must be taken into account for the final result to be finite. The power counting of a diagram is formalised in terms of Weinberg's power counting formula, where the dimensionality D is given by [26]

$$D = 2 + 2L + \sum_n (2n-2)N_{2n}\,. \tag{1.91}$$

The number of loops in the diagram is here given by L and N_{2n} is the number of vertices from $\mathcal{L}_{2n}$. So, when calculating at, say, order p^6, one must find all values of L and N_{2n} such that $D = 6$. The above formula is easily motivated using a naive scaling of a generic diagram. Since each term in Weinberg's power counting formula is positive, no calculation at order p^{2n} will receive contributions from $\mathcal{L}_{>2n}$. If this were not true, the EFT approach would be useless and non-renormalisability something to be avoided.

As an example of renormalisation of the LECs, consider the NLO Lagrangian in the three-flavour case. It contains ten parameters commonly denoted as L_i as well as two contact terms H_i. The renormalised versions of the physical LECs are below denoted L_i^r. The renormalisation is done in $\overline{\text{MS}}$, but the convention in ChPT is to include an extra term according to

$$L_i = L_i^r(\mu) - \left\{ \frac{1}{\varepsilon} + \log 4\pi - \gamma_E + 1 \right\} \frac{\Gamma_i \, \mu^{-2\varepsilon}}{32\pi^2} \,, \tag{1.92}$$

where μ is the renormalisation scale, ε is defined in terms of $d = 4 - 2\varepsilon$ and Γ_i are constants (whose values will not be repeated here). Similar expressions exist for the contact terms, and once all coefficients have been determined any process is rendered finite by re-expressing all LECs in terms of their renormalised counterparts. As a final remark on the renormalisation, the results for the NNLO LECs can be found in Ref. [34], and the renormalisation at NNNLO of the specific combinations of LECs showing up for the pion mass and decay constant can be found in Paper I.

The renormalised values of the LECs that show up at every order in the chiral expansion must have known numerical values in order to have a predictive EFT. The values in question can be obtained in several ways. One natural way is to fit to experimental data. As an example, for pion-pion and pion-kaon scattering one can obtain the parameters L_1^r, L_2^r and L_3^r. Another valuable tool to use is the assumption that the lowest lying states beyond those included in ChPT should saturate the values of the LECs, and an example is so-called vector meson dominance which is discussed further in e.g. Refs. [33, 35, 36]. Also large N_c arguments can be used [23]. A final and very important way to estimate LECs is to use lattice data. On the lattice the quark masses are parameters that can be varied, and the same is true in ChPT. By fitting ChPT results to lattice data for varying quark masses thus allows for unique extractions of LECs that cannot be made in experiments. For further details, see e.g. Ref. [9].

6 Lattice Gauge Theory

Lattice gauge theory, first proposed in Ref. [37], allows one to non-perturbatively calculate quantities using Monte Carlo simulations on discretised spacetimes of finite size. This is particularly important for QCD which is non-perturbative for energies below 1 GeV. Also other gauge theories can be formulated on the lattice, but due to the relevance of QCD in Nature this is what will be focused on below.

In Minkowski space, QCD is defined in terms of the partition function

$$\mathcal{Z}_{\text{QCD}} = \int \mathcal{D}U \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, e^{iS_{\text{QCD}}[\psi, \bar{\psi}, U]} \,, \tag{1.93}$$

where U now corresponds to the gluons and ψ is the quark field. Performing a Wick rotation to Euclidean space through $x_0 \to ix_0^E$ changes $S_{\text{QCD}}[\psi, \bar{\psi}, U] \to iS^E_{\text{QCD}}[\psi, \bar{\psi}, U]$ and

$$\mathcal{Z}_{\text{QCD}} \to \int \mathcal{D}U\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\, e^{-S^E_{\text{QCD}}[\psi,\bar{\psi},U]}\,. \tag{1.94}$$

The expectation value of an operator $\mathcal{O} \equiv \mathcal{O}[\psi, \bar{\psi}, U]$ is then

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}_{\text{QCD}}} \int \mathcal{D}U\mathcal{D}\psi\,\mathcal{D}\bar{\psi}\,\mathcal{O}\, e^{-S^E_{\text{QCD}}[\psi,\bar{\psi},U]}\,. \tag{1.95}$$

However, one may further integrate out the quark fields from the path integral by using that the QCD action can be written

$$S^E_{\text{QCD}}[\psi, \bar{\psi}, U] = S^E_g[U] + S^E_D[\psi, \bar{\psi}, U]\,, \tag{1.96}$$

where $S^E_g[U]$ is the pure gauge action and $S^E_D[\psi, \bar{\psi}, U] = \sum_f \bar{q}_f\big(D + m_f\big)\, q_f$ is the Dirac action for the Dirac operator D and a sum over flavours f. This yields, if $\mathcal{O}$ only depends on the gauge fields,

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}_{\text{QCD}}} \int \mathcal{D}U\,\mathcal{O}\, e^{-S^E_g[U]} \prod_f \det\Big(D + m_f\Big)\,. \tag{1.97}$$

This looks just like a classical statistical system with Boltzmann weights, i.e. with a probability density function

$$\frac{1}{\mathcal{Z}_{\text{QCD}}} e^{-S^E_g[U]} \prod_f \det\Big(D + m_f\Big)\,. \tag{1.98}$$

By thus generating N field configurations according to this distribution through importance sampling the expectation value can then be estimated as

$$\langle \mathcal{O} \rangle \approx \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}[U_i]\,. \tag{1.99}$$

Lattice gauge theory therefore builds upon Markov Chain Monte Carlo methods and introduces statistical uncertainty. Also, due to the complexity of the systems involved a considerable amount of computing power is required. It should further be noted that to do this on a computer, spacetime has to be put on a discretised lattice of finite size $V = L_0L_1L_2L_3$ and lattice spacing a. This naturally has several effects, which will be discussed later. Regarding the computational costs, the use of supercomputers and parallel programming is essential in most calculations, so there is important use of concepts such as domain decomposition methods to solve large systems of linear equations. These linear systems arise for various reasons. For instance, if the operator $\mathcal{O}$ would have depended on the quark and antiquark fields, then the process of integrating out the fields in question would have given rise to factors of

$$\left(D + m_f\right)^{-1} . \tag{1.100}$$

This is a complicated object to compute due to its size, and inverting it means solving a set of linear equations. From the computational perspective, it is important to keep in mind that the condition number of doing this is given by the ratio of the largest to the smallest eigenvalue. The smallest eigenvalue is related to the quark masses, so the condition number $\kappa \sim m_f^{-1}$. This means that the physically relevant case where the light quark masses are very small requires much more computational power. It has therefore been customary to simulate at several unphysical quark masses and later extrapolate to the physical point using e.g. chiral perturbation theory, but in recent years physical point calculations have been possible for some processes. Also the determinant present in the path integral requires consideration. A common approach to simplify the computation of this part is to let the sea quarks, i.e. quarks not occuring in the operator $\mathcal{O}$, have different masses from the valence quarks present in $\mathcal{O}$. This is the so-called quenched approach. For more details on the computational tools needed in modern simulations, see e.g. Ref. [10].

6.1 Consequences of formulating quantum field theory on the lattice

A natural question to ask is what effect the discretisation as well as finite size of the lattice has. The discretisation acts as a regulator since it defines a finite resolution of the lattice and short-distance effects are thus ignored. In momentum space this means that lattice momenta are constrained in magnitude by a^{-1}. As will be seen in the next section, this has consequences for the construction of the action. Since the continuum is given by the limit $a \to 0$, calculations are often made at several lattice spacings from which a continuum extrapolation then can be made.

The finite size of the box also has effects. One way to reduce these is to use periodic boundary conditions, but the finite size effects must still be calculated and subtracted from any quantity calculated on the lattice. For a lattice with $L_i = L$ and $L_0 = T$ it follows that for any lattice site $x_\mu = a\, n_\mu$, where n_μ is a vector of integers, a field $\phi(x_\mu + L_\mu) = \phi(x_\mu)$ for any direction μ with periodic bondary conditions. If one assumes that all dimensions have periodicity, then it is easy to see that the momentum is discretised according to

$$p_\mu = \frac{2\pi}{L_\mu} n_\mu . \tag{1.101}$$

Using this fact allows one to access the finite volume effects, which can be seen by considering the relation (in one dimension)

$$\sum_{n=-\infty}^{\infty} \delta(x-n) = \sum_{n=-\infty}^{\infty} e^{i2\pi nx}, \tag{1.102}$$

for any variable x. Choosing $x = pL/2\pi$ then gives

$$\frac{2\pi}{L} \sum_{n=-\infty}^{\infty} \delta(p - 2\pi n/L) = \sum_{n=-\infty}^{\infty} e^{inpL}. \tag{1.103}$$

Multiplying by a function $f(p^2)$ and integrating over p results in

$$\frac{1}{L} \sum_{n=-\infty}^{\infty} f\left(p_n^2\right) = \int \frac{dp}{2\pi} f\left(p^2\right) + \sum_{n\neq 0} \int \frac{dp}{2\pi} f\left(p^2\right) e^{inpL}, \tag{1.104}$$

which is the so-called Poisson summation formula relating the sum over allowed momenta to integrals. This is used heavily together with effective field theory techniques in Paper III to derive finite volume effects. There is choice in what effective field theory to use when calculating such effects, where chiral perturbation theory is one that has been used extensively (see for instance Ref. [38] and references therein). Also non-relativistic effective field theories can be used as in Ref. [39].

The Poisson summation formula shows the difference between the finite size effects from massive and massless particles. To see this, consider for the moment a continuous 4-dimensional Euclidean space. Then, a typical calculation including a particle of mass m will involve the sum

$$\sum_{n} \frac{1}{p_n^2 + m^2} = \int \frac{dp}{2\pi} \frac{1}{p^2 + m^2} + \sum_{\mathbf{n}\neq 0} \int \frac{dp}{2\pi} \frac{1}{p^2 + m^2} e^{inpL}, \tag{1.105}$$

where the Poisson summation formula was used. For any n in the second integral on the right the residue theorem can be used to yield

$$\int \frac{dp}{2\pi} \frac{1}{p^2 + m^2} e^{inpL} = \frac{1}{2m} e^{-nmL}, \tag{1.106}$$

that is, the sum over allowed momenta in (1.105) is equal to the infinite volume integral up to exponentially suppressed effects for a massive particle. It turns out that for massless particles such as the photon, the finite volume effects scale as powers of the inverse lattice size instead. This is shown explicitly in Paper III. Thus, QED will potentially have much larger finite size effects than QCD which is problematic for precision calculations on the lattice. This is relevant for the muon anomalous magnetic moment (see section 7). In addition, there is also the problem of infrared divergences from zero modes. Regularisation techniques for handling such modes will be discussed in section 6.3.

6.2 Formulation of the lattice action

For any simulation the relevant action has to be defined. The choice of action is ambiguous in the sense that any term vanishing in the continuum limit can be added as long as it satisfies the symmetries of the theory. It is therefore the symmetries of the theory that are the guiding principle in its construction. For instance, in QCD the $SU(3)_c$ gauge symmetry requires that $S \to S$ when $\psi(x) \to V(x)\psi(x)$ and $\bar{\psi}(x) \to V^\dagger(x)\bar{\psi}(x)$ for $V(x) \in SU(3)_c$. In QCD the gluons enter through the covariant derivative which on a discretised lattice is a non-local object. The gluons are thus defined to live on the links between sites and represented by the link variable $U_\mu(x) = \exp ia A_\mu(x)$ for the gauge field $A_\mu(x)$. This has to transform as $U_\mu(x) \to V(x)U_\mu(x)V^\dagger(x + a\hat{\mu})$ for the action to be gauge invariant. The off-set in x in the gauge transformation of $U_\mu(x)$ indicates that gluons are link variables, i.e. fields connecting lattice points. For instance, this means that combinations such as

$$W^{1\times 1}_{\mu\nu} = \left\{ U_\mu(x) U_\nu(x + a\hat{\mu}) U^\dagger_\mu(x + a\hat{\mu}) U^\dagger_\nu(x) \right\} , \tag{1.107}$$

are gauge invariant. From the definition of U_μ as a link variable, $W^{1\times 1}_{\mu\nu}$ is a loop of size 1×1 on the lattice and is known as a Wilson loop or plaquette. The pure gauge action can then be written as

$$S_g[U] = \beta a^4 \sum_x \sum_{\mu<\nu} \left(1 - \frac{1}{3} \mathrm{Re}\ \mathrm{Tr}\ W^{1\times 1}_{\mu\nu} \right) , \tag{1.108}$$

where $\beta = 6/g^2_{\mathrm{lat}}$ is the bare lattice coupling and the trace is in colour space. Note that in the continuum limit the link variable may be expanded as

$$U_\mu(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(ia A_\mu(x) \right)^n , \tag{1.109}$$

which when inserted in the plaquette action yields

$$S_g[U] \longrightarrow \int d^4x \frac{1}{4\, g^2_{\mathrm{lat}}} \mathrm{Tr} \left\{ F^2_{\mu\nu}(x) \right\} + \dots . \tag{1.110}$$

This is the continuum pure gauge action of QCD where the field strength tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g_{\mathrm{lat}} \left[A_\mu, A_\nu \right]$. The omitted terms above depend on higher orders of the lattice spacing a. Thus, for faster convergence to the continuum limit one may add appropriate terms to the action cancelling a certain number of the higher order terms. This is the so-called Symanzik improvement [40, 41]. For the fermionic part of the action, one has to generalise the derivative present in the regular continuum theory. This can again be done with the help of the link variable, and the simplest choice is

$$S_g[U] = -a^4 \sum_x \sum_\mu \frac{1}{2a} \Bigg\{ \bar{\psi}(x)(r - \gamma_\mu) U_\mu(x) \psi(x + a\hat{\mu})$$

$$+ \bar{\psi}(x + a\hat{\mu})(r + \gamma_\mu) U_\mu^\dagger(x)\psi(x)$$
$$+ \bar{\psi}(x)\left(m + \frac{4r}{a}\right)\psi(x)\Bigg\}, \tag{1.111}$$

where r is a parameter between 0 and 1. From this action one can derive the propagator which is of the form

$$\frac{1}{a^{-1}\sin ap_\mu} \,. \tag{1.112}$$

Unlike the propagator in usual field theory this has 2^d poles in d dimensions. This is known as the fermion doubling problem and there are several ways to attempt to solve it. However, these remedies usually break chiral symmetry which in itself causes new problems. No further details will be provided on this as it is not of much relevance to the papers herein. As a final remark, it is convenient to define the lattice momentum

$$\hat{p}_\mu = \frac{2}{a}\sin\frac{a}{2}p_\mu \,. \tag{1.113}$$

6.3 Isospin breaking corrections

Up and down quarks have until recently been treated as mass degenerate electrically neutral particles in most lattice calculations. This is known as the isospin symmetric limit and is easily motivated due to the sought precision. First of all, the very small mass difference on the order of $m_u - m_d \sim 2$ MeV determined at 2 GeV in $\overline{\text{MS}}$ [42], has effects starting at $\mathcal{O}\Big((m_u - m_d)/\Lambda_{\text{QCD}}\Big) \sim 10^{-3} - 10^{-2}$. Moreover, the first electromagnetic corrections occur at $\mathcal{O}(\alpha) \sim 10^{-2}$. Thus, as precision is improved on the lattice also isospin breaking corrections need to be taken into account. The different isospin breaking corrections are separated into two categories, namely those coming from the strong and electromagnetic sectors. The strong isospin breaking corrections can be included by using $m_u \neq m_d$, either in a quenched or non-quenched scenario. The electromagnetic corrections, however, require inclusion of the electromagnetic Maxwell action. As will be shown below, this involves some new considerations that lead to certain difficulties. In order to solve these, various lattice formulations of QED have been introduced. The strong isospin breaking effects are omitted from the discussions below.

The total action for studying isospin breaking effects from both QED and QCD is given by

$$S_{\text{QED+QCD}}\left[\psi, \bar{\psi}, U, A\right] = S_f\left[\psi, \bar{\psi}, U, A\right] + S_A\left[A\right] + S_G\left[U\right] \,, \tag{1.114}$$

where A is the photon field, U the gluon field and ψ the quark fields. The three terms on the RHS are the fermionic action, the Maxwell gauge action and the pure gluon gauge action, respectively. The Maxwell gauge action corresponds to a $U(1)$ symmetry, i.e. an Abelian symmetry. The full path integral, or, partition function, for QCD and QED combined, is thus

$$\mathcal{Z} = \int \mathcal{D}U \mathcal{D}A \, \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, e^{-S_{\mathrm{QED+QCD}}[\psi,\bar{\psi},U,A]} \,. \tag{1.115}$$

Any expectation value of an operator can now be taken with respect to this partition function, either by means of stochastically generating the gauge fields [43, 44] or perturbatively as an expansion in the small electric charge [44–46].

QED on the lattice

Although it may seem trivial to introduce QED on the lattice, subtleties arise due to the zero modes of photons. One possible solution to this is to subtract the zero modes in some way, but this results in non-local theories. Two different non-local formulations with zero mode subtractions are QED$_L$ and QED$_{TL}$. Other approaches are to introduce charge conjugation boundary conditions (QED$_C$) or a ficticious photon mass (QED$_M$). In Paper III, QED$_L$ is used and this framework is therefore discussed below. It was first introduced in Ref. [47], but another, more recent, definition can be found in Ref. [48].

In infinite volume the Maxwell gauge action is written

$$S_A[A] = \int d^4x \left[\frac{1}{4} \Big(F_{\mu\nu}(x) \Big)^2 + \frac{1}{2} \Big(\partial_\mu A_\mu(x) \Big)^2 \right] \,. \tag{1.116}$$

Appropriate summation over indices has been left out for simplicity. Using the definition of the field strength tensor in terms of the gauge field and partially integrating yields

$$S_A[A] = -\frac{1}{2} \int d^4x \, A_\mu(x) \partial^2 A_\mu(x) \,. \tag{1.117}$$

The propagator obtained from this is

$$D_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{\delta_{\mu\nu}}{k^2} e^{ik\cdot(x-y)} \,. \tag{1.118}$$

The finite volume equivalent can be obtained by introducing the Fourier transform

$$A_\mu(x) = \frac{1}{V} \sum_k \widetilde{A}_\mu(k) e^{ik\cdot x} \,,$$

$$\widetilde{A}_\mu(k) = \int d^4x \, A_\mu(x) e^{-ik\cdot x} \,, \tag{1.119}$$

where $k_\mu = \left(\frac{2\pi n_\mu}{L_\mu}\right)$. The Maxwell action is then given by

$$S_A\left[\widetilde{A}\right] = \frac{1}{2V}\sum_k k^2 \widetilde{A}_\mu(k) \,. \tag{1.120}$$

The propagator is therefore

$$D_{\mu\nu}(x-y) = \sum_k \frac{\delta_{\mu\nu}}{k^2} e^{ik\cdot(x-y)} \,, \tag{1.121}$$

where the problematic zero modes $k^2 = 0$ corresponding to IR divergences can be seen. One solution to this problem is to simply disregard the $k = 0$ modes altogether. This is the approach in QED_{TL}. In QED_L, on the other hand, one throws away modes with $\mathbf{k} = 0$ on every time slice. The advantage of QED_L is that it is reflection positive [49], and it is the QED formulation used in Paper III. As a final remark on QED_L, the photon propagator can be written

$$D^{\mathrm{QED}_L}_{\mu\nu}(x-y) = \sum_k{}' \, \frac{\delta_{\mu\nu}}{k^2} e^{ik\cdot(x-y)} \,, \tag{1.122}$$

where the primed sum indicates that $\mathbf{k} = 0$ is excluded. Further note that in the definition of QED_L only the gauge part was considered. This means that it can be applied both to regular QED as well as sQED.

6.4 Lattice perturbation theory

The lattice is a regulator in the sense that it provides a momentum cut-off in terms of the lattice spacing a through

$$\int_{-\infty}^{\infty} \frac{d^4p}{(2\pi)^4} \longrightarrow \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4} \,. \tag{1.123}$$

Moreover, momentum is discretised when the lattice has a finite volume and momentum integrals become sums over the allowed momenta instead. Lattice perturbation theory (LPT) is a tool that allows for perturbative calculations on a discretised spacetime that can be either infinite or of finite size. In a weak coupling regime, LPT can be used as a check of numerical lattice calculations, just like in Paper III herein. In that case the weak coupling is the electric charge occurring in the sQED Lagrangian. However, LPT can also be used for renormalisation of operators on the lattice, especially in going to the continuum limit to compare lattice results to physical results. An example is the operator product expansion introduced in section 8, where the matrix element of two operators $A(x)$ and $B(0)$ approaching each other can be written as a sum over matrix elements of some operators $\mathcal{O}_i(0)$ weighted with perturbative coefficients c_i (see (1.150)). The perturbative coefficients are often calculated in a scheme such as $\overline{\text{MS}}$, whereas the long distance matrix elements are calculated in some non-perturbative renormalisation scheme such as RI/MOM or RI/SMOM (introduced and discussed in e.g. Refs. [50–52]).

For a good review on LPT, see Ref. [53]. Below, the Euclidean discretised action of sQED is introduced as well as the associated Feynman rules obtained in LPT.

Scalar QED on the lattice

Scalar QED is used in Paper III to calculate the hadronic vacuum polarisation function introduced in section 7. This field theory is here introduced in order to illustrate how LPT is used and the aim is to highlight some of the aspects left out in Paper III.

First of all, the action can be decomposed into one part containing the scalar field ϕ and gauge field A as well as a pure gauge part containing only A according to

$$S[\phi, A] = S_\phi[\phi, A] + S_A[A] \,. \tag{1.124}$$

In the following the pure gauge term $S_A[A]$ is neglected. The scalar part $S_\phi[\phi, A]$ is given by

$$S_\phi[\phi, A] = \frac{a^4}{2} \sum_x \phi^*(x) \Delta \phi(x) \,, \tag{1.125}$$

where $\Delta = m^2 - \sum_\mu D^*_\mu D_\mu$ and the covariant derivative is defined in terms of the electric charge e and gauge link $U_\mu(x) = \exp iea A_\mu(x)$ as

$$\begin{aligned} D_\mu \phi(x) &= \frac{1}{a}\Big[U_\mu(x)\phi(x + a\hat{\mu}) - \phi(x)\Big] \,, \\ D^*_\mu \phi(x) &= \frac{1}{a}\Big[\phi(x) - U^\dagger_\mu(x - a\hat{\mu})\phi(x - a\hat{\mu})\Big] \,. \end{aligned} \tag{1.126}$$

Note that by defining a translation operator τ_μ such that $\tau_\mu f(x) = f(x + a\hat{\mu})$ one finds that

$$\Delta = \frac{1}{a^2} \sum_\mu \Big(2 - e^{ieaA_\mu}\tau_\mu - \tau_{-\mu}\, e^{-ieaA_\mu}\Big) + m^2 \,. \tag{1.127}$$

In the weak coupling regime this can be expanded in powers of the electric charge according to

$$\Delta = \sum_{i=0}^{\infty} \Delta_i \, e^i \,, \tag{1.128}$$

where the Δ_i are functions. The first four terms are given by

$$\begin{aligned}
\Delta_0 &= m^2 - \frac{1}{a^2} \sum_\mu \left(\tau_\mu + \tau_{-\mu} - 2 \right) \,, \\
\Delta_1 &= -\frac{i}{a} \sum_\mu \left(A_\mu \tau_\mu - \tau_{-\mu} A_\mu \right) \,, \\
\Delta_2 &= \frac{1}{2} \sum_\mu \left(A_\mu^2 \tau_\mu + \tau_{-\mu} A_\mu^2 \right) \,, \\
\Delta_3 &= \frac{ia}{3!} \sum_\mu \left(A_\mu^3 \tau_\mu - \tau_{-\mu} A_\mu^3 \right) \,.
\end{aligned} \tag{1.129}$$

Several things can be noted here. The gauge link was introduced from the requirement of gauge invariance under $U(1)$ rotations on the lattice. The effect of this, as can be seen above, is an infinite number of terms in the action, i.e. there are infinitely many interaction vertices. However, these are ordered in e so only a finite number of them contribute at a fixed order and, as expected, the unphysical ones vanish in the continuum limit $a \to 0$. From this expansion one can derive momentum space Feynman rules, and as an example consider Δ_1. First define the Fourier transforms

$$\begin{aligned}
A_\mu(x) &= \int \frac{d^4 k}{(2\pi)^4} \, e^{ik\cdot(x + a\hat{\mu}/2)} \, \widetilde{A}_\mu(k) \,, \\
\phi(x) &= \int \frac{d^4 k}{(2\pi)^4} \, e^{ik\cdot x} \, \tilde{\phi}(k) \,, \\
\phi^*(x) &= \int \frac{d^4 k}{(2\pi)^4} \, e^{-ik\cdot x} \, \tilde{\phi}^*(k) \,.
\end{aligned} \tag{1.130}$$

From the scalar action one then obtains

$$\begin{aligned}
&- \frac{ia^4}{a} \sum_{x,\mu} \left(\phi^*(x) A_\mu(x) \phi(x + a\hat{\mu}) - \phi^*(x) A_\mu(x - a\hat{\mu}) \phi(x - a\hat{\mu}) \right) \\
&= - \frac{ia^4}{a} \sum_{x,\mu} \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \frac{d^4 p'}{(2\pi)^4} \, \tilde{\phi}^*(p) \, \widetilde{A}_\mu(q) \, \tilde{\phi}\left(p'\right) \, e^{i(q-p+p')\cdot x} e^{iaq\cdot\hat{\mu}/2} \\
&\qquad \times \left(e^{iap'\cdot\hat{\mu}} - e^{-ia(q+p')\cdot\hat{\mu}} \right)
\end{aligned}$$

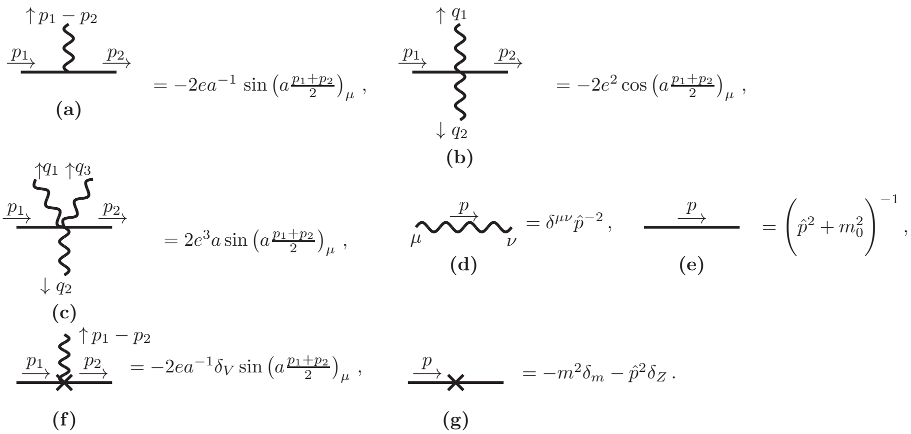


Figure 1.3: The LPT sQED Feynman rules relevant for Paper III. The crossed vertices correspond to counterterm insertions.

$$= \sum_\mu \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\, \tilde{\phi}^*(p)\, \widetilde{A}_\mu(q)\, \tilde{\phi}(p-q)\frac{2}{a}\sin\left(\frac{a}{2}\left(p+p'\right)_\mu\right) , \qquad (1.131)$$

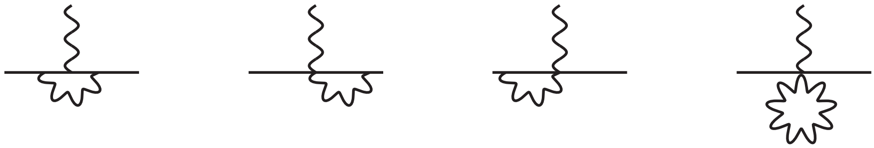
where the momentum conservation $p' = p - q$ was used. This yields the scalar-scalar-photon interaction vertex as $-2\,a^{-1}e\,\sin\left(\frac{a}{2}\left(p+p'\right)_\mu\right)$, or, if recognising the lattice momentum, $-e\,\widehat{p+p'}_\mu$. The other Feynman rules can be obtained in a similar fashion and in Fig. 1.3 the ones relevant to the calculation in Paper III are given. The counterterm rules for δ_m, δ_V and δ_Z are also included (their definitions are given in Paper III and will not be repeated here).

Having found LPT Feynman rules with no continuum analogues it is easy to see that the main consequence of these extra terms for a non-vanishing lattice spacing is the existence of pure lattice diagrams. One of the simplest examples that can be encountered is an additional diagram to the electromagnetic vertex function at 1-loop order. The 1-loop contributions are shown in Fig. 1.4, and the pure lattice diagram is the tadpole diagram in (d).

7 The Muon Anomalous Magnetic Moment

The charged leptonic sector of the Standard Model allows for important precision tests that can help constrain new physics. The muon has since its discovery in 1936 [54, 55] attracted much attention, in particular with respect to its magnetic properties. Being a lepton, it has spin 1/2 and thus, in classical quantum mechanics, a magnetic moment

$$\mathbf{M} = g_\mu \frac{e}{2m_\mu}\mathbf{S} , \qquad (1.132)$$



(a) Vertex 1-loop (b) Right sail (c) Left sail (d) Tadpole

Figure 1.4: The 1-loop corrections to the electromagnetic vertex function.

where $\mathbf{M}$ is its magnetic moment, $\mathbf{S}$ its spin, $m_\mu \approx 105.7$ MeV its mass and g_μ the so-called gyromagnetic ratio. The gyromagnetic ratio is in the classical theory exactly equal to 2, which is known as the Dirac value. However, in quantum field theory the interaction with an external electromagnetic field is given by the complete electromagnetic vertex function, i.e. it includes the sum of all virtual corrections to the electromagnetic vertex. This in practice means that g_μ will deviate from the Dirac prediction, i.e. it is a direct probe of particle physics beyond classical quantum mechanics. It is therefore conventional to measure the deviation from the Dirac prediction, i.e. the so-called anomalous magnetic moment

$$a_\mu = \frac{g_\mu - 2}{2} \,. \tag{1.133}$$

This is one of the most precisely measured quantities in particle physics today. The reason it has attracted so much attention is due to the current discrepancy of roughly 3.5σ between the experimentally measured value at Brookhaven National Laboratory (BNL) [56] and the theoretical value calculated from the Standard Model [57]. Clearly, there is much need for high precision calculations as well as measurments to clarify whether or not there really is a discrepancy.

7.1 On the experimental side

The BNL value is

$$a_\mu = 116\,592\,091(54)_{\text{stat}}(33)_{\text{sys}} \times 10^{-11}\,[0.54\,\text{ppm}] \,. \tag{1.134}$$

At Fermilab an experiment with the aim of decreasing the total relative error to 0.1 ppm is currently running. An updated value is expected within the coming year. Complementary to this, an experiment at J-PARC is right now being constructed. The latter relies on a new technique compared to BNL and Fermilab, namely using ultracold muons, which will offer an additional test of the measured value. The main idea behind the measurments at BNL and Fermilab is to exploit the Larmor precession of the muon spin due to a_μ as the particle goes in a circular orbit. The anomalous magnetic moment can thus be accessed via the angular frequency $\omega_a = \omega_s - \omega_c$, which is the difference between the spin precession angular frequency ω_s and the angular frequency of the muon momentum ω_c. The relation between a_μ and the vectorial quantity $\vec{\omega}_a$ is

$$\vec{\omega}_a = \frac{e}{m_\mu}\left(a_\mu \vec{B} - \left[a_\mu - (\gamma^2 - 1)^{-1}\right]\vec{v} \times \vec{E}\right), \tag{1.135}$$

where $\vec{B}$, $\vec{E}$ and $\vec{v}$ are the magnetic field, electric field and velocity respectively. From this relation one sees that for a cleverly chosen Lorentz factor γ, the term proportional to the electric field vanishes and

$$\omega_a = a_\mu \frac{eB}{m_\mu}. \tag{1.136}$$

By thus having good control of both the magnetic field and muon mass as well as measuring ω_a one can obtain a value of a_μ.

7.2 On the theoretical side

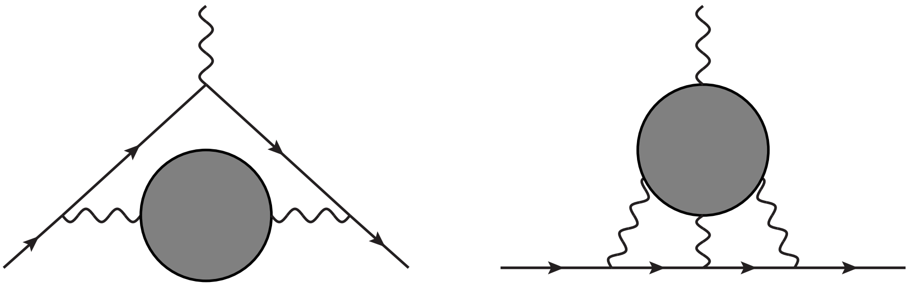
In order to understand the theoretical challenges in obtaining an improved value, one has to first consider the various contributions to the electromagnetic vertex function that exist. The anomalous magnetic moment can be decomposed according to

$$a_\mu = a_\mu^{\text{QED}} + a_\mu^{\text{EW}} + a_\mu^{\text{Hadronic}}, \tag{1.137}$$

where a_μ^{QED} contains the contributions from the QED sector, a_μ^{EW} those from the electroweak sector and a_μ^{Hadronic} contributions from the hadronic sector. The at present most problematic quantity here is the hadronic one, which in turn can be divided into two classes of diagrams according to

$$a_\mu^{\text{Hadronic}} = a_\mu^{\text{HVP}} + a_\mu^{\text{HLbL}}. \tag{1.138}$$

Here, HVP and HLbL are short for hadronic vacuum polarisation and hadronic light-by-light, respectively. The general structures of these two contributions are shown in Fig. 1.5. The shaded blobs contain all possible hadrons, and together with the connecting photons they are respectively referred to as the HVP and HLbL. There are both leading order (LO) and higher order (HO) hadronic contributions to a_μ^{HVP}, but in the following only the LO contribution will be considered. Since the hadronic contributions are relevant for Papers III and IV, they will be properly defined below. However, as a concluding remark before that, it is of importance to quantify the actual discrepancy between theory and experiment. In table 1.5 a summary of the current status of the experimental and theoretical values is given. As can be seen, with an absolute discrepancy of only $3 \cdot 10^{-9}$, which corresponds to a relative discrepancy on the order of $2 \cdot 10^{-6}$, the agreement is very impressive.



(a) Hadronic vacuum polarisation (b) Hadronic light-by-light

Figure 1.5: The two hadronic classes of contributions to the muon anomalous magnetic moment.

The hadronic contributions

The vacuum polarisation tensor, or, vector 2-point function, is in Minkowski space given by

$$\Pi_{\mu\nu}(q) = i \int d^4x\, e^{iq\cdot x} \langle 0| T\left\{ j_\mu(x) j_\nu^\dagger(0) \right\} |0\rangle\,, \tag{1.139}$$

for the vector current j_μ. The above equation with hadronic vector currents is the definition of the HVP. From the Ward-Takahashi identities

$$q^\mu\, \Pi_{\mu\nu}(q) = 0\,, \quad q^\nu\, \Pi_{\mu\nu}(q) = 0\,, \tag{1.140}$$

it follows that

$$\Pi_{\mu\nu}(q) = \left(q_\mu q_\nu - q^2 g_{\mu\nu}\right) \Pi\left(q^2\right)\,, \tag{1.141}$$

where $\Pi\left(q^2\right)$ is a scalar function. This function is UV divergent so it is conventional to calculate

$$\hat{\Pi}\left(q^2\right) = \Pi\left(q^2\right) - \Pi(0)\,. \tag{1.142}$$

In perturbation theory this may be expanded order by order, as in Paper III where $\mathcal{O}(\alpha)$ corrections are included on the LO pion loop. The calculation in Paper III is thus a 2-loop calculation, as can be seen from the contributing diagrams in Fig. 1.6.

The HLbL function is defined in an analogous way, this according to

$$\Pi_{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = -i \int d^4x\, d^4y\, d^4z\, e^{-iq_1\cdot x - iq_2\cdot y - iq_3\cdot z} \langle 0| T\left\{ j_\mu(x) j_\nu(y) j_\lambda(z) j_\sigma(0) \right\} |0\rangle\,. \tag{1.143}$$

Table 1.5: The current experimental and SM values of the $g-2$, all taken from Ref. [8]. The difference $\Delta a_\mu = a_\mu^{\text{Exp}} - a_\mu^{\text{SM}}$ is also quoted. The SM contributions are presented separately in the lower half of the table. The errors are of various kinds, and the interested reader is encouraged to look at Ref. [8] for further details.

	Value ($\times 10^{-11}$)	Error ($\times 10^{-11}$)
a_μ^{Exp}	116 592 091	$(54)_{\text{stat}}(33)_{\text{sys}}$
a_μ^{SM}	116 591 823	$(1)_{\text{EW}}(34)_{\text{Had, LO}}(26)_{\text{Had, HO}}$
Δa_μ	268	$(63)_{\text{Exp}}(43)_{\text{Theory}}$
a_μ^{QED}	116 584 718.95	$(0.08)_{\text{Theory}}$
a_μ^{EW}	153.6	$(1.0)_{\text{Theory}}$
$a_\mu^{\text{HVP, LO}}$	6 931	$(33)_{\text{Exp}}(7)_{\text{Theory}}$
$a_\mu^{\text{HVP, HO}}$	-86.3	$(0.9)_{\text{Theory}}$
a_μ^{HLbL}	105	$(26)_{\text{Theory}}$

In the quark picture, the above currents are of the form $j_\mu = \bar{q}\, Q_q \gamma_\mu q$ for the quark vector q and Q_q being the corresponding quark charge matrix. For three flavours, $Q_q = \text{diag}\,(2/3, -1/3, -1/3)$. The momenta are defined so that q_1, q_2 and q_3 are incoming and correspond to the virtual momenta to be integrated over in a_μ^{HLbL}. The external photon therefore has momentum q_4, which is defined as outgoing so that $q_1 + q_2 + q_3 = q_4$. Note that the $g-2$ kinematics is given by the static limit $q_4 = 0$. This tensor satisfies the Ward identities

$$\left\{ q_1^\mu, q_2^\nu, q_3^\lambda, q_4^\sigma \right\} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = 0\,. \tag{1.144}$$

The HLbL tensor can be decomposed into 138 Lorentz structures [58–60], but from the Ward identities above it can be shown that only 43 structures are independent. However, only 19 structures contribute to the loop integral in a_μ^{HLbL} in the end, but due to crossing symmetries one can write the final integral as a sum of only twelve terms [61].

There are various ways to calculate the two quantities above. In the dispersive approach one relies on analyticity and unitarity to derive dispersion relations. Such dispersion relations can thus give a connection to experimental data, and an important example for the HVP is its connection to the cross section of the process $e^+e^- \to$ hadrons. This is the reason the theoretical value $a_\mu^{\text{HVP, LO}}$ has experimental error in table 1.5. For a discussion on dispersion relations for the HLbL, see e.g. Refs. [61, 62]. Complementary to this, one can use lattice gauge theory. For instance, in state of the art calculations of the HVP electromagnetic corrections are included in order to reach per cent level accuracy, so it is important to have the systematic effects such as those from the finite volume approximation under control. This is the starting point for Paper III and will be discussed below. Yet another option is to use low energy models, an approach extensively used for the HLbL. However, there are several complications arising due to the kinematic structure in the integral of a_μ^{HLbL}. In

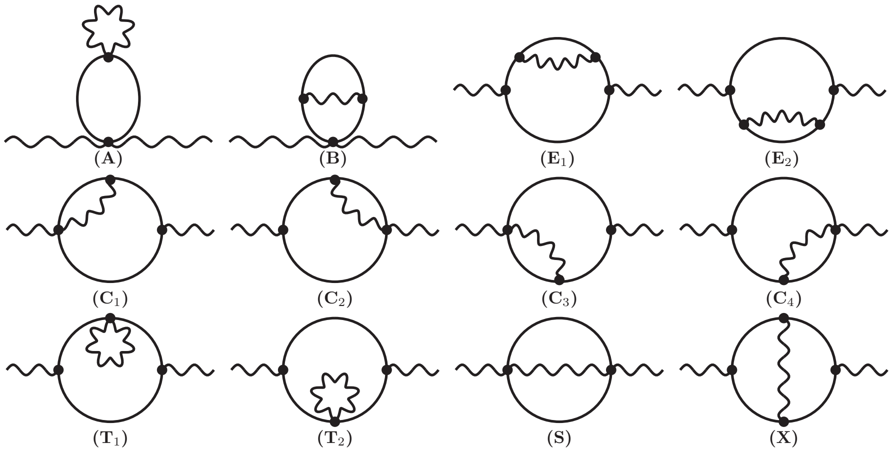


Figure 1.6: The $\mathcal{O}(\alpha)$ corrected pion loop contributions to the HVP from Paper III.

particular, the momentum integrals over q_1, q_2 and q_3 in (1.143) have several regions where the models cease to be valid. An example is the mixed region known as the Melnikov–Vainshtein limit [63], where two momenta are large and one small ($Q_1^2 \approx Q_2^2 \gg Q_3^2$ where $Q_i^2 = -q_i^2$). In such a limit perturbative QCD can clearly not be used either. An alternative approach is to use the OPE (see section 8). Some models might even have the wrong short-distance behaviour, so it is important to have constraints from QCD as a check of how good the models really are. Such short-distance constraints are derived in Paper IV for the HLbL. For further details, see Ref. [64] for an excellent review.

Finite volume effects to the HVP at $\mathcal{O}(\alpha)$

The 2-loop diagrams in Paper III (see Fig. 1.6) contain two loop momenta. Denoting the pion loop momentum as ℓ and that of the photon as k, the contribution from a diagram U, $\hat{\Pi}_U$, in infinite volume has the form

$$\hat{\Pi}_U(q_0^2) = \int \frac{dk_0}{2\pi}\frac{d\ell_0}{2\pi}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\boldsymbol{\ell}}{(2\pi)^3}\,\hat{\pi}_U(k,\ell,q_0)\,, \tag{1.145}$$

where $q = (q_0, 0)$ is the external photon momentum and $\hat{\pi}_U$ is the loop integrand obtained from sQED Feynman rules. Performing the k_0 and ℓ_0 integrals then yields

$$\hat{\Pi}_U(q_0^2) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d^3\boldsymbol{\ell}}{(2\pi)^3}\,\hat{\rho}_U(\mathbf{k},\boldsymbol{\ell},q_0)\,, \tag{1.146}$$

where $\hat{\rho}_U$ is the resulting integrand. The finite volume counterpart is obtained by exchanging each integral by a sum over the corresponding discretised momentum components multiplied by a factor of L. The finite volume effects are thus given by

$$\Delta\hat{\Pi}_U(q_0^2) = \left(\frac{1}{L^6} \sum_{\mathbf{k}}{}' \sum_{\boldsymbol{\ell}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{d^3\boldsymbol{\ell}}{(2\pi)^3} \right) \hat{\rho}_U(\mathbf{k}, \boldsymbol{\ell}, q_0) \,. \tag{1.147}$$

The primed sum indicates the choice of QED_L. Using the Poisson summation formula it is possible to exhange the pion sum for an integral up to exponentially suppressed effects so that

$$\Delta\hat{\Pi}_U(q_0^2) = \left(\frac{1}{L^3} \sum_{\mathbf{k}}{}' - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \right) \int \frac{d^3\boldsymbol{\ell}}{(2\pi)^3} \hat{\rho}_U(\mathbf{k}, \boldsymbol{\ell}, q_0) + \dots \,. \tag{1.148}$$

The omitted terms are the exponentially suppressed terms from the pion loop. Next, knowing that the finite volume effects are dominated by the soft photon modes one may use that $\mathbf{k} = 2\pi\,\mathbf{n}/L$ and expand the integrand above in powers of $1/L$. This is the approach taken in Paper III. The first possible term that can show up is a $1/(M_\pi L)^2$ term, but it is shown that this vanishes identically. Such a cancellation can be understood from the underlying physics, since a soft photon is not able to resolve the charged pions corresponding to the neutral current, i.e. the term of order $1/(M_\pi L)^2$ has to vanish. The implication for lattice calculations is that for moderately sized $M_\pi L$, given the currently sought precision, these electromagnetic effects are in principle negligible.

8 Operator Product Expansion

The operator product expansion is a very useful tool when looking at products of operators in two different spacetime points x and y that approach each other, i.e. when $x \to y$, and singularities appear. This was studied in e.g. Refs. [65, 66], and when used together with dispersion relations provides so-called sum rules. These will not be further elaborated upon here.

Now, when $y = 0$ the OPE can be formulated as

$$\int d^4k\, e^{-ikx} T\{A(x)B(0)\} = \sum_i c_i(k)\mathcal{O}_i(0) \,, \tag{1.149}$$

for some operators A, B and $\mathcal{O}_i$ with respective dimensions d_A, d_B and $d_{\mathcal{O}_i}$, and the divergent Wilson coefficients c_i. The degree of divergence of the Wilson coefficients decreases with the dimensionality of the operator $\mathcal{O}_i$, and is given by $c_i \sim k^{d_A+d_B-d_{\mathcal{O}_i}}$, a result which makes the OPE particularly useful in the sense that the terms in the series become less and less important. The c_i can be computed perturbatively, and since the relation in (1.149) is true on the operator level, one obtains the following relation for the matrix element

$$\int d^4k\, e^{-ik\cdot x}\, \langle 0|\, T\{A(x)B(0)\}\, |0\rangle = \sum_i c_i(k)\, \langle 0|\, \mathcal{O}_i(0)\, |0\rangle \ . \tag{1.150}$$

The matrix elements $\langle 0|\, \mathcal{O}_i(0)\, |0\rangle$ contain the low energy dynamics and can sometimes be evaluated on the lattice. This is one of the ways that the lattice and perturbative calculations can be used together, as mentioned earlier, and highlights the need for connecting the lattice renormalisation schemes and continuum schemes such as $\overline{\text{MS}}$. Another example of when the OPE can be used is for the calculation in Paper IV. There it is used for the HLbL contribution to the muon anomalous magnetic moment, as discussed below.

8.1 An OPE for the HLbL

There are many difficulties in calculating the HLbL contribution to the $g-2$. These arise from the various regions in the loop integration of the three momenta q_1, q_2 and q_3 in (1.143). In addition to this, the external photon with momentum q_4 is soft for the $g-2$ kinematics. Particular problems arise already for the strictly hard, i.e. short-distance, limit when the (Euclidean) loop momenta are all hard. This corresponds to

$$-q_i^2 = Q_i^2 \gg \Lambda_{\text{QCD}}\,. \tag{1.151}$$

The low energy models with hadronic degrees of freedom commonly used are in this limit not valid anymore, but for matching purposes such short-distance constraints obtained from the fundamental theory are very useful.

In the above kinematic setting, the perturbative quark loop is the leading contribution to the HLbL tensor. If one tries to do an OPE in (1.143) by leaving fields uncontracted in the expansion of the time ordered product, one obtains condensates with the same quantum numbers as the vacuum which do not vanish, such as e.g. $\langle \overline{q}q\rangle$. This can be diagramatically represented as in Fig. 1.7, but the problem then is that a quark propagator $i\,S(q_4)$ appears at next-to leading order which is badly divergent for the $g-2$ kinematics. It is therefore not possible to do such an OPE for the HLbL tensor.

Although the situation might appear hopeless, one can consider exchanging the external soft photon leg for an external electromagnetic field in the static limit. The idea behind this stems from Ref. [67]. In this approach one must trade $\Pi^{\mu\nu\lambda\sigma}$ for the 3-point function

$$\Pi^{\mu\nu\lambda}(q_1,q_2) = -i\int d^4x\, d^4y\, e^{-i(q_1\cdot x+q_2\cdot y)}\, \langle 0|\, T\left\{j^\mu(x) j^\nu(y) j^\lambda(0)\right\}\, |0\rangle_F\,, \tag{1.152}$$

where the subscript F indicates that there is an external field. The first contribution in an OPE here is a quark loop with three external hard photons. At higher orders there are now

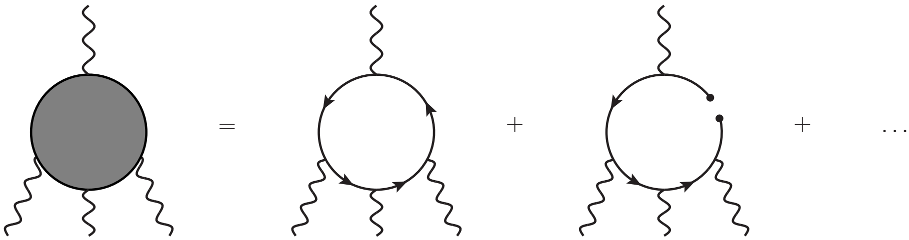


Figure 1.7: A diagrammatic representation of the short-distance OPE of $\Pi^{\mu\nu\lambda\sigma}$. The black vertices on the quark lines in the last diagram represent the insertion of a vacuum condensate.

additional condensates that survive, with the same quantum numbers as the external field, where an example is $\langle \overline{q}\, \sigma_{\alpha\beta} q \rangle \equiv Q_q F_{\alpha\beta} X_q$. The first two contributions are diagrammatically represented in Fig. IV.3. The quantity X_q is related to the magnetic susceptibility of the QCD vacuum and has been estimated using lattice QCD in e.g. Ref. [68]. At higher orders many other condensates appear, and hopefully these can be estimated in some way.

In Paper IV it is shown that the leading contribution in the OPE in an external electromagnetic field gives the same prediction as the quark loop does in the usual approach starting from $\Pi^{\mu\nu\lambda\sigma}$. The next-to leading term proportional to $\langle \overline{q}\, \sigma_{\alpha\beta} q \rangle$ is also calculated and shown to be suppressed due to the small overall factors $m_q X_q$. Although these corrections are negligible, higher order condensates are not necessarily suppressed in that fashion and are therefore of interest to know in the future.

9 Some Concluding Remarks and an Outlook

In the absence of direct experimental evidence of new physics particles, it is very interesting to do precision tests of the Standard Model. The low energy sector of the Standard Model is particularly complicated and thus requires special attention. The phenomenology and high precision physics in this low energy regime motivate this thesis and Papers I–IV.

The discussion in this introduction started from some general properties of quantum field theories, but then specialised to QCD and in particular its low energy regime. It was shown that effective field theory techniques and lattice gauge theory are valuable tools to use in calculations for low energy precision physics. A very timely example of an observable in need of high precision is the anomalous magnetic moment of the muon and the Standard Model prediction of it. This constituted the final part of the introduction. From these considerations the reader should hopefully now have sufficient knowledge to understand Papers I–IV, and the underlying reasons behind what is done therein. However, it would be unnatural to simply stop here without some discussion of the implications of the results

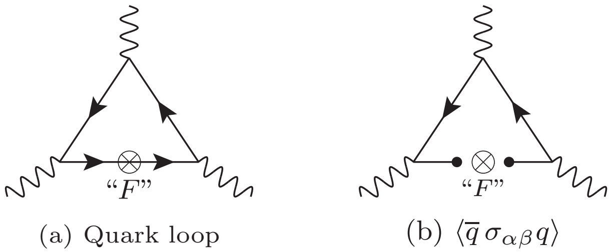


Figure 1.8: The two leading terms in the external field short-distance OPE for $\Pi^{\mu\nu\lambda}$. The presence of the external field F is here depicted with a crossed vertex.

in the papers as well as what lies in the future of high precision physics at low energies.

In Papers I and II chiral perturbation theory was studied at NNNLO. The pion mass and decay constant were calculated at 3-loop order within the two-flavour theory and the mesonic chiral Lagrangian $\mathcal{L}_8$ derived. In order to do precision calculations at this order one would need to know the low energy constants from $\mathcal{L}_8$. This can, in principle, be done, but seeing that not even the NNLO low energy constants are well-known some 20 years after the derivation of $\mathcal{L}_6$ it is unclear how long it will take to obtain values for the NNNLO constants. One should keep in mind, though, that the analytic expressions of the pion mass and decay constant and the knowledge of the number of low energy constants in $\mathcal{L}_8$ have intrinsic interest as well. Moreover, the techniques used for the derivation of the Lagrangian are general and can be applied also to other effective field theories needed for precision physics.

In Papers III and IV higher order calculations needed for reducing the theoretical uncertainty on the Standard Model prediction of the anomalous magnetic moment of the muon were presented. In Paper III it was shown that for the currently sought precision the potentially dangerous finite volume effects from adding QED on the lattice in principle can be neglected. This allows for a lattice gauge theory prediction of the hadronic vacuum polarisation contribution to the muon $g-2$ including the isospin breaking effects required for per cent level accuracy. Note that the method used to derive finite volume effects in Paper III also can be applied for other isospin breaking precision calculations. A notable example is the lattice determination of CKM matrix elements from leptonic and semi-leptonic decays. In Paper IV short-distance constraints relevant for the hadronic light-by-light contribution were derived. It was shown that there is an equivalence between using an external soft photon and doing the calculation in an external electromagnetic field, in particular by showing that the first term in the external field operator product expansion, namely, the 3-point quark loop, agrees with its 4-point vacuum counterpart. In the external field approach, it was shown further that the next-to leading contributions coming from a condensate related to the magnetic susceptibility of the QCD vacuum are negligible as they are suppressed by the smallness of the quark masses and the condensates themselves. However, the higher order condensates are not necessarily suppressed in such a way and for the future it is of interest to derive just how many condensates show up. One would also need to come up with some way of numerically estimating them in order to make predictions.

To conclude, in this era of high precision physics there is still a lot of work to be done in the low energy sector of the Standard Model. The interplay between the effective field theory and lattice gauge theory communities will undoubtedly lead to many interesting findings, in particular in combination with future experimental measurements. What the conclusions will be from these results is impossible to say, except that there will be a better understanding of the most fundamental building blocks of the Universe.

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10 Publications

Paper I

Johan Bijnens, Nils Hermansson-Truedsson: *The Pion Mass and Decay Constant at Three Loops in Two-Flavour Chiral Perturbation Theory*, e-print: arXiv:1710.01901 [hep-ph]. JHEP **11** (2017) 181.

This publication extends previous results for the pion mass and decay constant by one order in the chiral counting, i.e. order p^8 or next-to-next-to-next-to leading order. All calculations were performed independently by J. Bijnens and me. I wrote the first draft of the article, which J. Bijnens then edited.

Paper II

Johan Bijnens, Nils Hermansson-Truedsson, Si Wang: *The order p^8 mesonic chiral Lagrangian*, e-print: arXiv:1810.06834 [hep-ph]. JHEP **01** (2019) 102.

In Paper I the order p^8 tree level contribution was simply parametrised as an unknown constant, this as the corresponding mesonic chiral Lagrangian was unknown. Here, in Paper II, we derive this Lagrangian, partly due to its intrinsic interest in chiral perturbation theory and partly to see just how many low energy constants actually exist. All calculations and checks were done independently by J. Bijnens and me. S. Wang visited Lund as a bachelor student from Shanghai University. The work she did together with J. Bijnens and me resulted in a bachelor thesis on the construction of the mesonic chiral Lagrangian when masses and external fields are excluded. I wrote the first draft of Paper II, which J. Bijnens then edited. I created a table with an example of an explicit basis, which can be found in the supplementary material on the arXiv page.

Paper III

Johan Bijnens, James Harrison, Nils Hermansson-Truedsson, Tadeusz Janowski, Andreas Jüttner, Antonin Portelli: *Electromagnetic finite-size effects to the hadronic vacuum polarization*, e-print: arXiv:1903.10591 [hep-lat]. Phys. Rev. **D100** (2019), 1, 014508.

The motivation for this paper was to understand the finite volume dependence of the leading electromagnetic corrections to the hadronic vacuum polarisation (HVP) in QED_L. This is relevant for the lattice determination of the HVP, which is one of the main sources of uncertainty in the Standard Model value of the muon anomalous magnetic moment. J.

Bijnens, T. Janowski, A. Portelli and I all independently calculated the HVP, both in the finite volume continuum as well as on lattices of finite and infinite volumes using lattice perturbation theory. This was done both analytically and numerically. J. Bijnens and I also calculated the HVP in continuum Minkowski space in infinite volume for completeness. A. Portelli, J. Harrison and A. Jüttner did the lattice calculations. As for the paper, I wrote the first draft together with T. Janowski and J. Harrison, which then was edited by J. Bijnens, A. Jüttner and A. Portelli.

Paper IV

Johan Bijnens, Nils Hermansson-Truedsson, Antonio Rodríguez Sánchez: *Short-distance constraints for the hadronic light-by-light contribution to the muon anomalous magnetic moment*, e-print: arXiv:1908.03331 [hep-ph]. Submitted to Phys. Lett. B.

In this paper we use the operator product expansion to obtain short-distance constraints on the hadronic light-by-light (HLbL) contribution to the anomalous magnetic moment of the muon. The HLbL is one of the two main sources of uncertainty in the theoretical value of the magnetic moment. At low energies the HLbL is often calculated using models. One issue with using low energy models is that they only can be applied in the strict low momentum regions in the final $g-2$ integral, and perturbative QCD on the other hand only works in the high energy region. Using the operator product expansion one can access the other regions of interest, such as the Melnikov-Vainshtein limit, as well. J. Bijnens, A. Rodríguez Sánchez and I individually calculated the short-distance constraints for the region with strictly large (Euclidean) loop momenta using an operator product expansion where the soft external photon is included via an external electromagnetic field. A. Rodríguez Sánchez and I together wrote the first draft which was finished by all three authors together.

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