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SIMPLIFIED APRIORI ESTIMATE FOR THE TIME PERIODIC BURGERS EQUATION

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Abstract. We present here a simplified version of the proof the existence and uniqueness of time-periodic solutions for the Burger's equation published in [5]. This work was an improvement of the proof of [8] using completely different techniques, partly based on [6]. We will expose the main steps of the proof and present a simplified version of the a priori estimate which turns out to be of central importance in the proof.

Introduction

The study of the Burgers equation has a long history starting with the seminal papers by Burgers [1], Cole [2] and Hopf [7] where the Cole-Hopf transformation was introduced. The Cole-Hopf transformation transforms the homogeneous Burgers equation into the heat equation.

More recently there have been several articles dealing with the forced Burgers equation:

$$u_t - \nu u_{xx} + u u_x = f \tag{1}$$

The vast majority treats the initial value problem in time with homogeneous Dirichlet or periodic space boundary conditions (see for instance [9]).

Only recently has the question of the time-periodic forced Burgers equation been tackled ([8, 3, 10, 4]). In most cases [8, 3] the authors are chiefly interested in the inviscid limit (the limit when the viscosity ν tends to zero).

The closest related work to ours is that of Jauslin, Kreiss and Moser [8] in which the authors show existence and uniqueness of a space and time periodic solution of the Burgers equation for a space and time periodic forcing term which is smooth.

1. Definitions

In this section we recall some well known facts and fix some general notations.

1.1. **Fractional Derivatives.** For any positive real number s we may define the fractional derivative of order s in the following way on $\mathcal{D}'(\mathbb{T}, H^*)$:

$$\mathrm{D}^s u = \sum_{k\in\mathbb{Z}} (2\pi \mathrm{i} k)^s u_k \mathrm{e}^{\mathrm{i}2\pi k t} = \sum_{k\in\mathbb{Z}} |2\pi \mathrm{i} k|^s \, \mathrm{e}^{\mathrm{i}\,\mathrm{sgn}(k) s \frac{\pi}{2}} u_k \mathrm{e}^{\mathrm{i}2\pi k t}$$

where we have used the principal branch of the logarithm. The sign function is defined as follows:

$$\mathrm{sgn}(k) := \begin{cases} \frac{k}{|k|} & \text{if } k \neq 0 \\ 0 & \text{if } k = 0 \end{cases}$$

For $s = 0$ we define $\mathrm{D}^0 = \mathrm{Id}$. D^1 coincides with the usual differentiation operator on $\mathcal{D}'(\mathbb{T}, H^*)$. The familiar composition property also holds: $\mathrm{D}^s \circ \mathrm{D}^t = \mathrm{D}^{s+t}$ for any $t, s \geq 0$.

The *adjoint operator* of D^s is defined by using the conjugate of the multiplier of D^s:

$$\mathrm{D}_*^s u = \sum_{k\in\mathbb{Z}} |2\pi \mathrm{i} k|^s \, \mathrm{e}^{-\mathrm{i}\,\mathrm{sgn}(k) s \frac{\pi}{2}} u_k \mathrm{e}^{\mathrm{i}2\pi k t}$$

D^s and D_*^s are adjoints in the sense that for any $u \in \mathcal{D}'(\mathbb{T}, H^*)$ and $\varphi \in \mathcal{D}(\mathbb{T}, H)$:

$$\langle \mathrm{D}^s u, \varphi \rangle = \langle u, \mathrm{D}_*^s \varphi \rangle$$

and similarly:

$$\langle \mathrm{D}_*^s u, \varphi \rangle = \langle u, \mathrm{D}^s \varphi \rangle$$

1.2. **Hilbert Transform.** The *Hilbert transform* $\mathcal{H}$ is defined using the multiplier $-\mathrm{i}\,\mathrm{sgn}\,k$. For $u \in \mathcal{D}'(\mathbb{T}, H^*)$ let

$$\mathcal{H} u = \sum_{k\in\mathbb{Z}} -\mathrm{i}\,\mathrm{sgn}\,k \, u_k \, \mathrm{e}^{\mathrm{i}2\pi k t}$$

For convenience we will denote in the sequel

$$\widetilde{u} := \mathcal{H} u$$

Simple computations then give:

$$\mathrm{D}_*^{\frac{1}{2}} = \mathrm{D}^{\frac{1}{2}} \circ \mathcal{H} = \mathcal{H} \circ \mathrm{D}^{\frac{1}{2}}$$

Notice that if H is a function space then $\mathcal{H}$ maps real functions to real functions. The following properties will be useful in the sequel:

$$\forall u \in \mathrm{H}^{(\frac{1}{2})}(\mathbb{T}, H) \quad \Big(\mathrm{D}^{\frac{1}{2}} u, \mathrm{D}_*^{\frac{1}{2}} \mathcal{H} u\Big)_{\mathrm{L}^2(\mathbb{T},H)} = -\left\|\mathrm{D}^{\frac{1}{2}} u\right\|_{\mathrm{L}^2(\mathbb{T},H)}^2 \tag{2}$$

$$\forall u \in \mathrm{L}^2(\mathbb{T}\times I) \quad \Re\big((u, \mathcal{H}(u))_{\mathrm{L}^2(\mathbb{T}\times I)}\big) = 0 \tag{3}$$

where $\Re$ denotes the real part of the expression.

1.3. **Fractional Sobolev Spaces.** We define fractional Sobolev spaces in the following manner, for any $s \in \mathbb{R}$:

$$\mathrm{H}^{(s)}(\mathbb{T}, H) = \Big\{ u \in \mathcal{D}'(\mathbb{T}, H^*); \quad \sum_{k\in\mathbb{Z}} \left|1 + k^2\right|^s \|u_k\|_H^2 < \infty \Big\}$$

Of course $\mathrm{H}^{(0)}(\mathbb{T}, H) = \mathrm{L}^2(\mathbb{T}, H)$. When $s \geq 0$ then for an $u \in \mathrm{L}^2(\mathbb{T}, H)$: $u \in \mathrm{H}^{(s)}(\mathbb{T}, H) \iff \mathrm{D}^s u \in \mathrm{L}^2(\mathbb{T}, H)$. Moreover $\mathrm{H}^{(s)}(\mathbb{T}, H)$ is then a Hilbert space with the following scalar product:

$$(u, v) := (u, v)_{\mathrm{L}^2(\mathbb{T},H)} + (\mathrm{D}^s u, \mathrm{D}^s v)_{\mathrm{L}^2(\mathbb{T},H)}$$

The following classical result holds: $\big(\mathrm{H}^{(s)}(\mathbb{T}, H)\big)^* = \mathrm{H}^{(-s)}(\mathbb{T}, H^*)$.

1.4. **Anisotropic Fractional Sobolev Spaces.** Let I be an interval in $\mathbb{R}$ and $s \geq 0$. Let $\mathrm{H}^{(s)}(I)$ denote the usual fractional Sobolev space of real-valued s-times differentiable functions on I. $\mathrm{H}_0^{(s)}(I)$ is the closure of $\mathcal{D}(I)$ in $\mathrm{H}^{(s)}(I)$. In that case we have $\big(\mathrm{H}_0^{(s)}(I)\big)^* = \mathrm{H}^{(-s)}(I)$. We will also use the following notations, for α, β nonnegative real numbers:

$$\mathrm{H}^{(\alpha)(\beta)}(\mathbb{T}\times I) = \mathrm{H}^{(\alpha)}(\mathbb{T}, \mathrm{H}^{(\beta)}(I))$$

and

$$\mathrm{H}^{(\alpha,\beta)}(\mathbb{T}\times I) = \mathrm{H}^{(\alpha)(0)}(\mathbb{T}\times I) \cap \mathrm{H}^{(0)(\beta)}(\mathbb{T}\times I)$$

We also introduce $\mathrm{H}_0^{(\alpha,\beta)}(\mathbb{T}\times I)$ as the closure of $\mathcal{D}(\mathbb{T}\times I)$ in $\mathrm{H}^{(\alpha,\beta)}(\mathbb{T}\times I)$. It is clear that $\mathrm{H}_0^{(\alpha,\beta)}(\mathbb{T}\times I) = \mathrm{H}^{(\alpha)(0)}(\mathbb{T}\times I) \cap \mathrm{L}^2(\mathbb{T}, \mathrm{H}_0^{(\beta)}(I))$. Duals of such spaces are denoted as:

$$\begin{aligned}\mathrm{H}^{[-\alpha,-\beta]}(\mathbb{T}\times I) := \left(\mathrm{H}_0^{(\alpha,\beta)}(\mathbb{T}\times I)\right)^* &= \mathrm{H}^{(-\alpha)}(\mathbb{T}, \mathrm{L}^2(I)) + \mathrm{L}^2(\mathbb{T}, \mathrm{H}^{(-\beta)}(I)) \\ &= \mathrm{H}^{(-\alpha)(0)}(\mathbb{T}\times I) + \mathrm{H}^{(0)(-\beta)}(\mathbb{T}\times I)\end{aligned}$$

2. Interpolation and regularity

If $s_k(\xi)$ is the Fourier transform $s_k(\xi) = \hat{u}(k,\xi)$ of a distribution u defined on $\mathbb{T}\times\mathbb{R}$, we have the following Hölder inequality for any $\theta \in [0,1]$:

$$\int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} |k|^{2\alpha(1-\theta)} |\xi|^{2\beta\theta} |s_k(\xi)|^2 \,\mathrm{d}\xi \leq \left(\int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} |k|^{2\alpha} |s_k(\xi)|^2 \,\mathrm{d}\xi\right)^{1-\theta} \left(\int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} |\xi|^{2\beta} |s_k(\xi)|^2 \,\mathrm{d}\xi\right)^{\theta}$$

From this Hölder inequality we deduce

$$\mathrm{H}^{(\alpha,\beta)}(\mathbb{T}\times\mathbb{R}) \hookrightarrow \mathrm{H}^{((1-\theta)\alpha)}(\mathbb{T}, \mathrm{H}^{(\theta\beta)}(\mathbb{R}))$$

So using an extension operator from $\mathrm{H}^{(\theta\beta)}(I)$ to $\mathrm{H}^{(\theta\beta)}(\mathbb{R})$ one can prove the corresponding inclusion:

$$\mathrm{H}^{(\alpha,\beta)}(\mathbb{T}\times I) \hookrightarrow \mathrm{H}^{((1-\theta)\alpha)(\theta\beta)}(\mathbb{T}\times I) \tag{4}$$

For $\alpha = 1/2$ and $\beta = 1$ and $\theta = \frac{1}{3}$ we get:

$$\mathrm{H}_0^{\left(\frac{1}{2},1\right)}(\mathbb{T}\times I) \subset \mathrm{H}^{\left(\frac{1}{2},1\right)}(\mathbb{T}\times I) \subset \mathrm{H}^{(1/3)(1/3)}(\mathbb{T}\times I)$$

Then the vectorial Sobolev inequalities yield:

$$\mathrm{H}_0^{\left(\frac{1}{2},1\right)}(\mathbb{T}\times I) \subset \mathrm{H}^{(1/3)(1/3)}(\mathbb{T}\times I) \hookrightarrow \mathrm{L}^4(\mathbb{T}, \mathrm{H}^{\left(\frac{1}{3}\right)}(I)) \hookrightarrow \mathrm{L}^4(\mathbb{T}, \mathrm{L}^4(I)) = \mathrm{L}^4(\mathbb{T}\times I) \tag{5}$$

Here the injection $\mathrm{H}^{(1/3)(1/3)}(\mathbb{T}\times I) \hookrightarrow \mathrm{L}^4(\mathbb{T}, \mathrm{H}^{(1/3)})$ is compact and thus the injection $\mathrm{H}_0^{\left(\frac{1}{2},1\right)}(\mathbb{T}\times I) \hookrightarrow \mathrm{L}^4(\mathbb{T}\times I)$ is *compact*.

3. Main Result

We define the Burgers Operator by:

$$\mathrm{T} = \mathcal{L} + S$$

where $\mathcal{L}$ and S are defined in the familiar weak form, the bracket being the *duality bracket* between $\mathrm{H}_0^{\left(\frac{1}{2},1\right)}$ and $\mathrm{H}^{\left(-\frac{1}{2},-1\right)}$:

$$\forall v \in \mathrm{H}_0^{\left(\frac{1}{2},1\right)} \quad \langle \mathcal{L}\, u, v\rangle := \left(u_{\sqrt{t}}, v_{\sqrt{t}*}\right) + \mu\,(u_x, v_x)$$

and

$$\forall v \in \mathrm{H}_0^{\left(\frac{1}{2},1\right)} \quad \langle S(u), v\rangle := -\frac{1}{2}\left(u^2, v_x\right)$$

It turns out that the second definition makes sense because of the embedding $\mathrm{H}_0^{\left(\frac{1}{2},1\right)} \subset \mathrm{L}^4$ (see Figure 1).

A weaker result of the main result proved in [5] is

Theorem 1. *For $f \in \mathrm{H}^{(0)(-1)}$ there exists a unique solution $u \in \mathrm{H}_0^{\left(\frac{1}{2},1\right)}$ of*

$$\mathrm{T}\, u = f$$

We will now briefly sketch the proof of that Theorem.

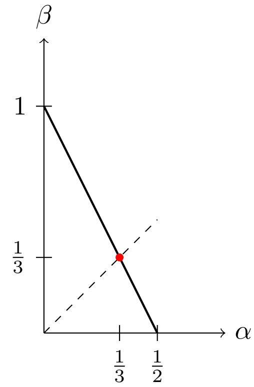

FIGURE 1. $\mathrm{H}_0^{\left(\frac{1}{2},1\right)}$ is included in $\mathrm{H}^{\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)}$ which is included in L^6 by the usual Sobolev inclusion theorem. In particular, $\mathrm{H}_0^{\left(\frac{1}{2},1\right)}$ is included in L^4, so $u \in \mathrm{H}_0^{\left(\frac{1}{2},1\right)} \implies u^2 \in \mathrm{L}^2$. As a result the non-linear term of the Burgers equation may be written as $-(u^2, v_x)$ for a test function $v \in \mathrm{H}_0^{\left(\frac{1}{2},1\right)}$ since $v \in \mathrm{H}_0^{\left(\frac{1}{2},1\right)} \implies v_x \in \mathrm{L}^2$ by definition.

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Theorem 2. *Let $f \in \mathrm{H}^{(0,-1)}$. The set*

$$\bigcup_{\lambda\in[0,1]} (\mathcal{L}+\lambda S)^{-1}\big(\{f\}\big)$$

is bounded in $\mathrm{H}_0^{\left(\frac{1}{2},1\right)}$.

We will need the following Lemma which may be proved using a scaling argument.

Lemma 4.1. *There exists a constant $\mathcal{C} \in \mathbb{R}$ such that for any $u \in \mathrm{H}_0^{\left(\frac{1}{2},1\right)}(Q)$:*

$$\int_Q |u(t,x)|^4 \,\mathrm{d}t\,\mathrm{d}x \leq \mathcal{C}^2 \left(\int_Q |u|^2 \,\mathrm{d}t\,\mathrm{d}x + \int_Q \left|u_{\sqrt{t}}\right|^2 \mathrm{d}t\,\mathrm{d}x\right) \cdot \left(\int_Q |u_x|^2 \,\mathrm{d}t\,\mathrm{d}x\right)$$

which implies that:

$$\left|u^2\right| \leq \mathcal{C}\, \|u\|\, |u_x| \tag{6}$$

Proof of Theorem 2. By definition $\mathcal{L}\,u + \lambda S(u) = f$ means:

$$\forall v \in \mathrm{H}_0^{\left(\frac{1}{2},1\right)} \quad \left(u_{\sqrt{t}}, v_{\sqrt{t}*}\right) + \mu\,(u_x, v_x) - \frac{1}{2}\lambda\left(u^2, v_x\right) = \langle f, v\rangle \tag{7}$$

(1) We notice that for smooth u:

$$\begin{aligned}\left(u^2, u_x\right) &= \int_Q u^2 u_x \\ &= \frac{1}{3}\int_Q (u^3)_x \\ &= 0\end{aligned}$$

and then by density and continuity this holds for all $u \in \mathrm{H}_0^{\left(\frac{1}{2},1\right)}$.

(2) With $v = u$ in (7) we get:

$$\underbrace{\left(u_{\sqrt{t}}, u_{\sqrt{t}*}\right)}_{=0} + \mu\,(u_x, u_x) + \frac{1}{2}\lambda\underbrace{\left(u^2, u_x\right)}_{=0} = \langle f, u\rangle$$

which gives:

$$|u_x|^2 = \frac{\langle f, u\rangle}{\mu} \le \frac{\|f\|\,|u_x|}{\mu}$$

From this we deduce that

$$|u_x| \le \frac{\|f\|}{\mu} \tag{8}$$

(3) Pairing in (7) with the Hilbert transform of u, $v = \widetilde{u}$ we get:

$$\left(u_{\sqrt{t}}, \widetilde{u}_{\sqrt{t}*}\right) + \mu \underbrace{\left(u_x, \widetilde{u}_x\right)}_{=0} + \frac{1}{2}\lambda \left(u^2, \widetilde{u}_x\right) = \langle f, \widetilde{u}\rangle$$

Using the identity (2), the fact that $\left|\widetilde{u_x}\right| = |u_x|$ and that $\lambda \le 1$ we get:

$$\left|u_{\sqrt{t}}\right|^2 \le \frac{1}{2}\left|\left(u^2, \widetilde{u}_x\right)\right| + \|f\|\,|u_x| \tag{9}$$

(4) We estimate $\left|\left(u^2, \widetilde{u}_x\right)\right|$ using Lemma 4.1:

$$\begin{aligned}\left|\left(u^2, \widetilde{u}_x\right)\right| &\le \left|u^2\right| |u_x| \\ &\le \mathcal{C}\,\|u\|\,|u_x|^2\end{aligned} \tag{10}$$

(5) Using the estimate (8) inside (10) we obtain:

$$\begin{aligned}\left|u_{\sqrt{t}}\right|^2 &\le \frac{\mathcal{C}}{2}\|f\|\,|u_x|^2 + \|f\|\,|u_x| \\ &\le \frac{\|f\|^2}{\mu}\left(\frac{\mathcal{C}}{2\mu}\|u\| + 1\right)\end{aligned} \tag{11}$$

Since that estimate does not depend on λ the theorem is proved.

□

The a priori estimate above may now be used to prove existence of solutions by a (nonlinear, compact) degree argument using the Leray-Schauder Theorem (cf. [5]).

5. Cole-Hopf Transformation

The Cole-Hopf transformation is defined by

$$u = \frac{\varphi_x}{\varphi}$$

In our case there are complications due to the fact that $u \in \mathrm{H}_0^{\left(\frac{1}{2},1\right)}$ and u is periodic. This change of variable will transform the periodicity problem into an eigenvalue problem (because the Cole-Hopf transformation linearises the Burger's equation). After working out the details one shows that the uniqueness problem is equivalent to the uniqueness of the *ground state eigenvalue problem*:

Proposition 5.1. *Given $v \in \mathrm{H}_0^{\left(\frac{1}{2},1\right)}$ the solution set of the following equation in K and φ*

$$\begin{cases}\varphi_t - \mu\varphi_{xx} + v\varphi_x + K\varphi = 0 \\ \varphi > 0 \\ \varphi_x|_{\partial Q} = 0 \\ \varphi \in \mathrm{H}^{(1,2)} \\ K \in \mathbb{R}\end{cases} \tag{12}$$

is $K = 0$ and $\varphi = 1$ if and only if $\mathrm{T}\,u = \mathrm{T}\,v$ *implies $u = v$ (that is, the solution to the original Burger's equation is unique).*

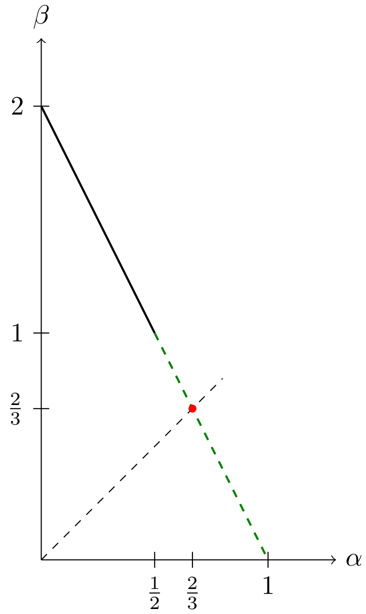

Figure 2. The first step of the Cole-Hopf Transformation is an integration in x. This function U obtained thus ends up in $\mathrm{H}^{(0)(1)} \cap \mathrm{H}^{\left(\frac{1}{2}\right)(1)}$, which delimits the plain line on the graph above. But it follows from $\mathrm{T}\, u \in \mathrm{H}^{(0)(-1)}$ that u is actually also in $\mathrm{H}^{(1)(-1)}$ so U ends up in $\mathrm{H}^{(1)(2)}$ and we have an inclusion in $\mathrm{H}^{\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)}$ which is embedded in continuous Hölder functions.

The proof of that proposition essentially hinges on the embedding properties exposed in section 2 (see Figure 2).

The remaining part of the proof is concerned with the eigenvalue problem of the Proposition above. One first shows that the eigenvalue is zero using a weaker version of the Perron-Frobenius theorem. The second step is to show that the remaining eigenvalue problem is *non degenerate*, namely that the dimension of the eigenspace must be one. This last step makes use of the a priori estimate proved in Theorem 2.

The details of that part of the proof are too lengthy to be exposed here in depth so the interested reader is referred to [5].

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