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Abstract. We present here a weaker version of the existence and uniqueness result in [5]. The weaker result in this note is proved by using an apriori estimate with an easier proof than the stronger key apriori estimate necessary in [5]. The result in [5] was an improvement of the existence and uniqueness result in [8] using completely different techniques.

Introduction

The study of the Burgers equation has a long history starting with the seminal papers by Burgers [1], Cole [2] and Hopf [7] where the Cole-Hopf transformation was introduced. The Cole-Hopf transformation transforms the homogeneous Burgers equation into the heat equation.

More recently there have been several articles dealing with the forced Burgers equation:

$$u_t - \nu u_{xx} + u u_x = f \tag{1}$$

The vast majority treats the initial value problem in time with homogeneous Dirichlet or periodic space boundary conditions (see for instance [9]).

Only recently has the question of the time-periodic forced Burgers equation been tackled ([8, 3, 10, 4]). In most cases [8, 3] the authors are chiefly interested in the inviscid limit (the limit when the viscosity ν tends to zero).

The closest related work to ours is that of Jauslin, Kreiss and Moser [8] in which the authors show existence and uniqueness of a space and time periodic solution of the Burgers equation for a space and time periodic forcing term which is smooth.

1. Definitions

In this section we recall some well known facts and fix some general notations.

1.1.   Fractional Derivatives

For any positive real number s we may define the fractional derivative of order s in the following way on $\mathcal{D}'(\mathbb{T}, H^*)$:

$$\mathrm{D}^s u = \sum_{k \in \mathbb{Z}} (2\pi \mathrm{i} k)^s u_k \mathrm{e}^{\mathrm{i}2\pi kt} = \sum_{k \in \mathbb{Z}} |2\pi \mathrm{i} k|^s \, \mathrm{e}^{\mathrm{i}\,\mathrm{sgn}(k)s\frac{\pi}{2}} u_k \mathrm{e}^{\mathrm{i}2\pi kt}$$

where we have used the principal branch of the logarithm. The sign function is defined as follows:

$$\mathrm{sgn}(k) := \begin{cases} \frac{k}{|k|} & \text{if } k \neq 0 \\ 0 & \text{if } k = 0 \end{cases}$$

For $s = 0$ we define $\mathrm{D}^0 = \mathrm{Id}$. D^1 coincides with the usual differentiation operator on $\mathcal{D}'(\mathbb{T}, H^*)$. The familiar composition property also holds: $\mathrm{D}^s \circ \mathrm{D}^t = \mathrm{D}^{s+t}$ for any $t, s \geq 0$.

The *adjoint operator* of D^s is defined by using the conjugate of the multiplier of D^s:

$$\mathrm{D}^s_* u = \sum_{k \in \mathbb{Z}} |2\pi \mathrm{i} k|^s \, \mathrm{e}^{-\mathrm{i}\,\mathrm{sgn}(k)s\frac{\pi}{2}} u_k \mathrm{e}^{\mathrm{i}2\pi kt}$$

D^s and D^s_* are adjoints in the sense that for any $u \in \mathcal{D}'(\mathbb{T}, H^*)$ and $\varphi \in \mathcal{D}(\mathbb{T}, H)$:

$$\langle \mathrm{D}^s u, \varphi \rangle = \langle u, \mathrm{D}^s_* \varphi \rangle$$

and similarly:

$$\langle \mathrm{D}^s_* u, \varphi \rangle = \langle u, \mathrm{D}^s \varphi \rangle$$

1.2.   Hilbert Transform

The *Hilbert transform* $\mathcal{H}$ is defined using the multiplier $-\mathrm{i}\,\mathrm{sgn}\,k$. For $u \in \mathcal{D}'(\mathbb{T}, H^*)$ let

$$\mathcal{H} u = \sum_{k \in \mathbb{Z}} -\mathrm{i}\,\mathrm{sgn}\,k\, u_k \mathrm{e}^{\mathrm{i}2\pi kt}$$

For convenience we will denote in the sequel

$$\widetilde{u} := \mathcal{H} u$$

Simple computations then give:

$$\mathrm{D}^{\frac{1}{2}}_* = \mathrm{D}^{\frac{1}{2}} \circ \mathcal{H} = \mathcal{H} \circ \mathrm{D}^{\frac{1}{2}}$$

Notice that if H is a function space then $\mathcal{H}$ maps real functions to real functions. The following properties will be useful in the sequel:

$$\forall u \in \mathrm{H}^{\left(\frac{1}{2}\right)}(\mathbb{T}, H) \quad \left(\mathrm{D}^{\frac{1}{2}} u, \mathrm{D}^{\frac{1}{2}}_* \mathcal{H} u\right)_{\mathrm{L}^2(\mathbb{T}, H)} = -\left\|\mathrm{D}^{\frac{1}{2}} u\right\|^2_{\mathrm{L}^2(\mathbb{T}, H)} \tag{2}$$

$$\forall u \in \mathrm{L}^2(\mathbb{T} \times I) \quad \Re((u, \mathcal{H}(u))_{\mathrm{L}^2(\mathbb{T} \times I)}) = 0 \tag{3}$$

where $\Re$ denotes the real part of the expression.

1.3. Fractional Sobolev Spaces

We define fractional Sobolev spaces in the following manner, for any $s \in \mathbb{R}$:

$$\mathrm{H}^{(s)}(\mathbb{T}, H) = \left\{ u \in \mathcal{D}'(\mathbb{T}, H^*); \quad \sum_{k \in \mathbb{Z}} \left|1 + k^2\right|^s \|u_k\|_H^2 < \infty \right\}$$

Of course $\mathrm{H}^{(0)}(\mathbb{T}, H) = \mathrm{L}^2(\mathbb{T}, H)$. When $s \geq 0$ then for an $u \in \mathrm{L}^2(\mathbb{T}, H)$: $u \in \mathrm{H}^{(s)}(\mathbb{T}, H) \iff \mathrm{D}^s u \in \mathrm{L}^2(\mathbb{T}, H)$. Moreover $\mathrm{H}^{(s)}(\mathbb{T}, H)$ is then a Hilbert space with the following scalar product:

$$(u, v) := (u, v)_{\mathrm{L}^2(\mathbb{T}, H)} + (\mathrm{D}^s u, \mathrm{D}^s v)_{\mathrm{L}^2(\mathbb{T}, H)}$$

The following classical result holds: $\left(\mathrm{H}^{(s)}(\mathbb{T}, H)\right)^* = \mathrm{H}^{(-s)}(\mathbb{T}, H^*)$.

1.4. Anisotropic Fractional Sobolev Spaces

Let I be an interval in $\mathbb{R}$ and $s \geq 0$. Let $\mathrm{H}^{(s)}(I)$ denote the usual fractional Sobolev space of real-valued s-times differentiable functions on I. $\mathrm{H}_0^{(s)}(I)$ is the closure of $\mathcal{D}(I)$ in $\mathrm{H}^{(s)}(I)$. In that case we have $\left(\mathrm{H}_0^{(s)}(I)\right)^* = \mathrm{H}^{(-s)}(I)$. We will also use the following notations, for α, β nonnegative real numbers:

$$\mathrm{H}^{(\alpha)(\beta)}(\mathbb{T} \times I) = \mathrm{H}^{(\alpha)}(\mathbb{T}, \mathrm{H}^{(\beta)}(I))$$

and

$$\mathrm{H}^{(\alpha, \beta)}(\mathbb{T} \times I) = \mathrm{H}^{(\alpha)(0)}(\mathbb{T} \times I) \cap \mathrm{H}^{(0)(\beta)}(\mathbb{T} \times I)$$

We also introduce $\mathrm{H}_0^{(\alpha, \beta)}(\mathbb{T} \times I)$ as the closure of $\mathcal{D}(\mathbb{T} \times I)$ in $\mathrm{H}^{(\alpha, \beta)}(\mathbb{T} \times I)$. It is clear that $\mathrm{H}_0^{(\alpha, \beta)}(\mathbb{T} \times I) = \mathrm{H}^{(\alpha)(0)}(\mathbb{T} \times I) \cap \mathrm{L}^2(\mathbb{T}, \mathrm{H}_0^{(\beta)}(I))$. Duals of such spaces are denoted as:

$$\mathrm{H}^{[-\alpha, -\beta]}(\mathbb{T} \times I) := \left(\mathrm{H}_0^{(\alpha, \beta)}(\mathbb{T} \times I)\right)^* = \mathrm{H}^{(-\alpha)}(\mathbb{T}, \mathrm{L}^2(I)) + \mathrm{L}^2(\mathbb{T}, \mathrm{H}^{(-\beta)}(I))$$

$$= \mathrm{H}^{(-\alpha)(0)}(\mathbb{T} \times I) + \mathrm{H}^{(0)(-\beta)}(\mathbb{T} \times I)$$

2. Interpolation and regularity

If $s_k(\xi)$ is the Fourier transform $s_k(\xi) = \hat{u}(k, \xi)$ of a distribution u defined on $\mathbb{T} \times \mathbb{R}$, we have the following Hölder inequality for any $\theta \in [0, 1]$:

$$\int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} |k|^{2\alpha(1-\theta)} |\xi|^{2\beta\theta} |s_k(\xi)|^2 \, d\xi \leq$$

$$\left(\int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} |k|^{2\alpha} |s_k(\xi)|^2 \, d\xi\right)^{1-\theta} \left(\int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} |\xi|^{2\beta} |s_k(\xi)|^2 \, d\xi\right)^{\theta}$$

From this Hölder inequality we deduce

$$\mathrm{H}^{(\alpha, \beta)}(\mathbb{T} \times \mathbb{R}) \hookrightarrow \mathrm{H}^{((1-\theta)\alpha)}(\mathbb{T}, \mathrm{H}^{(\theta\beta)}(\mathbb{R}))$$

So using an extension operator from $\mathrm{H}^{(\theta\beta)}(I)$ to $\mathrm{H}^{(\theta\beta)}(\mathbb{R})$ one can prove the corresponding inclusion:

$$\mathrm{H}^{(\alpha, \beta)}(\mathbb{T} \times I) \hookrightarrow \mathrm{H}^{((1-\theta)\alpha)(\theta\beta)}(\mathbb{T} \times I) \tag{4}$$
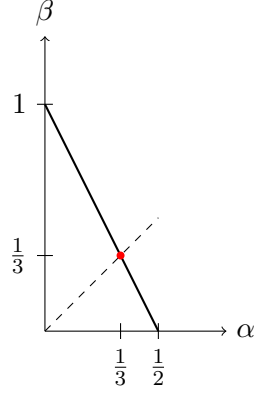
Fig. 1. $H_0^{\left(\frac{1}{2},1\right)}$ is included in $H^{\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)}$ which is included in L^6 by the usual Sobolev inclusion theorem. In particular, $H_0^{\left(\frac{1}{2},1\right)}$ is included in L^4, so $u \in H_0^{\left(\frac{1}{2},1\right)} \implies u^2 \in L^2$. As a result the non-linear term of the Burgers equation may be written as $-(u^2, v_x)$ for a test function $v \in H_0^{\left(\frac{1}{2},1\right)}$ since $v \in H_0^{\left(\frac{1}{2},1\right)} \implies v_x \in L^2$ by definition.

For $\alpha = 1/2$ and $\beta = 1$ and $\theta = \frac{1}{3}$ we get:

$$H_0^{\left(\frac{1}{2},1\right)}(\mathbb{T} \times I) \subset H^{\left(\frac{1}{2},1\right)}(\mathbb{T} \times I) \subset H^{(1/3)(1/3)}(\mathbb{T} \times I)$$

Then the vectorial Sobolev inequalities yield:

$$H_0^{\left(\frac{1}{2},1\right)}(\mathbb{T} \times I) \subset H^{(1/3)(1/3)}(\mathbb{T} \times I) \hookrightarrow L^4(\mathbb{T}, H^{\left(\frac{1}{3}\right)}(I)) \hookrightarrow L^4(\mathbb{T}, L^4(I)) = L^4(\mathbb{T} \times I) \qquad (5)$$

Here the injection $H^{(1/3)(1/3)}(\mathbb{T} \times I) \hookrightarrow L^4(\mathbb{T}, H^{(1/3)})$ is compact and thus the injection $H_0^{\left(\frac{1}{2},1\right)}(\mathbb{T} \times I) \hookrightarrow L^4(\mathbb{T} \times I)$ is *compact*.

3.   Main Result

We define the Burgers Operator by:
$$T = \mathcal{L} + S$$

where $\mathcal{L}$ and S are defined in the familiar weak form, the bracket being the *duality bracket* between $H_0^{\left(\frac{1}{2},1\right)}$ and $H^{\left(-\frac{1}{2},-1\right)}$:

$$\forall v \in H_0^{\left(\frac{1}{2},1\right)} \quad \langle \mathcal{L} u, v \rangle := \left(u_{\sqrt{t}}, v_{\sqrt{t}*}\right) + \mu\left(u_x, v_x\right)$$

and

$$\forall v \in H_0^{\left(\frac{1}{2},1\right)} \quad \langle S(u), v \rangle := -\frac{1}{2}\left(u^2, v_x\right)$$

It turns out that the second definition makes sense because of the embedding $H_0^{\left(\frac{1}{2},1\right)} \subset L^4$ (see Figure 1).

A weaker result of the main result proved in [5] is

Theorem 1. *For $f \in H^{(0)(-1)}$ there exists a unique solution $u \in H_0^{\left(\frac{1}{2},1\right)}$ of*

$$T u = f$$

We will now briefly sketch the proof of that Theorem.

4. A priori estimate

Theorem 2. *Let $f \in \mathrm{H}^{(0,-1)}$. The set*

$$\bigcup_{\lambda \in [0,1]} (\mathcal{L} + \lambda S)^{-1}(\{f\})$$

is bounded in $\mathrm{H}_0^{\left(\frac{1}{2},1\right)}$.

We will need the following Lemma which may be proved using a scaling argument.

Lemma 4..1. *There exists a constant $\mathcal{C} \in \mathbb{R}$ such that for any $u \in \mathrm{H}_0^{\left(\frac{1}{2},1\right)}(Q)$:*

$$\int_Q |u(t,x)|^4 \, \mathrm{d}t \, \mathrm{d}x \leq \mathcal{C}^2 \left(\int_Q |u|^2 \, \mathrm{d}t \, \mathrm{d}x + \int_Q \left| u_{\sqrt{t}} \right|^2 \, \mathrm{d}t \, \mathrm{d}x \right) \cdot \left(\int_Q |u_x|^2 \, \mathrm{d}t \, \mathrm{d}x \right)$$

which implies that:

$$\left| u^2 \right| \leq \mathcal{C} \left\| u \right\| |u_x| \tag{6}$$

Proof of Theorem 2. By definition $\mathcal{L} u + \lambda S(u) = f$ means:

$$\forall v \in \mathrm{H}_0^{\left(\frac{1}{2},1\right)} \quad \left(u_{\sqrt{t}}, v_{\sqrt{t*}} \right) + \mu \left(u_x, v_x \right) - \frac{1}{2}\lambda \left(u^2, v_x \right) = \langle f, v \rangle \tag{7}$$

1. We notice that for smooth u:

$$\begin{aligned} \left(u^2, u_x \right) &= \int_Q u^2 u_x \\ &= \frac{1}{3} \int_Q (u^3)_x \\ &= 0 \end{aligned}$$

and then by density and continuity this holds for all $u \in \mathrm{H}_0^{\left(\frac{1}{2},1\right)}$.

2. With $v = u$ in (7) we get:

$$\underbrace{\left(u_{\sqrt{t}}, u_{\sqrt{t*}} \right)}_{=0} + \mu \left(u_x, u_x \right) + \frac{1}{2}\lambda \underbrace{\left(u^2, u_x \right)}_{=0} = \langle f, u \rangle$$

which gives:

$$|u_x|^2 = \frac{\langle f, u \rangle}{\mu} \leq \frac{\|f\| |u_x|}{\mu}$$

From this we deduce that

$$|u_x| \leq \frac{\|f\|}{\mu} \tag{8}$$

3. Pairing in (7) with the Hilbert transform of u, $v = \widetilde{u}$ we get:

$$\left(u_{\sqrt{t}}, \widetilde{u}_{\sqrt{t*}} \right) + \mu \underbrace{\left(u_x, \widetilde{u}_x \right)}_{=0} + \frac{1}{2}\lambda \left(u^2, \widetilde{u}_x \right) = \langle f, \widetilde{u} \rangle$$

Using the identity (2), the fact that $\left| \widetilde{u_x} \right| = |u_x|$ and that $\lambda \leq 1$ we get:

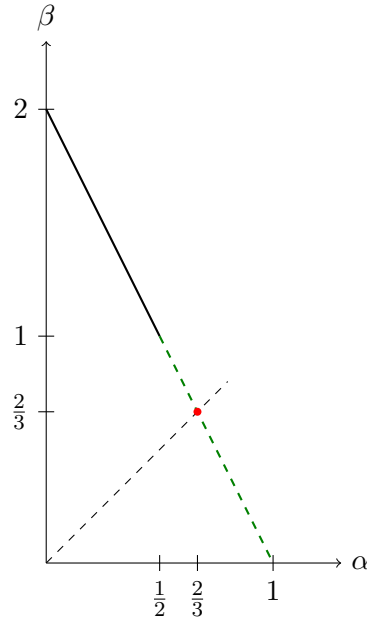$$\left| u_{\sqrt{t}} \right|^2 \leq \frac{1}{2} \left| \left(u^2, \widetilde{u}_x \right) \right| + \|f\| |u_x| \tag{9}$$

Fig. 2. The first step of the Cole-Hopf Transformation is an integration in x. This function U obtained thus ends up in $H^{(0)(1)} \cap H^{\left(\frac{1}{2}\right)(1)}$, which delimits the plain line on the graph above. But it follows from $T\,u \in H^{(0)(-1)}$ that u is actually also in $H^{(1)(-1)}$ so U ends up in $H^{(1)(2)}$ and we have an inclusion in $H^{\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)}$ which is embedded in continuous Hölder functions.

4. We estimate $\left|\left(u^2, \widetilde{u}_x\right)\right|$ using Lemma 4..1:

$$
\begin{aligned}
\left|\left(u^2, \widetilde{u}_x\right)\right| &\leq \left|u^2\right|\left|u_x\right| \\
&\leq \mathcal{C}\,\|u\|\,\left|u_x\right|^2
\end{aligned}
\tag{10}
$$

5. Using the estimate (8) inside (10) we obtain:

$$
\begin{aligned}
\left|u_{\sqrt{t}}\right|^2 &\leq \frac{\mathcal{C}}{2}\,\|f\|\,\left|u_x\right|^2 + \|f\|\,\left|u_x\right| \\
&\leq \frac{\|f\|^2}{\mu}\left(\frac{\mathcal{C}}{2\mu}\,\|u\| + 1\right)
\end{aligned}
\tag{11}
$$

Since that estimate does not depend on λ the theorem is proved.

$\square$

The a priori estimate above may now be used to prove existence of solutions by a (nonlinear, compact) degree argument using the Leray-Schauder Theorem (cf. [5]).

5.   Cole-Hopf Transformation

The Cole-Hopf transformation is defined by

$$
u = \frac{\varphi_x}{\varphi}
$$

In our case there are complications due to the fact that $u \in \mathrm{H}_0^{\left(\frac{1}{2},1\right)}$ and u is periodic. This change of variable will transform the periodicity problem into an eigenvalue problem (because the Cole-Hopf transformation linearises the Burger's equation). After working out the details one shows that the uniqueness problem is equivalent to the uniqueness of the *ground state eigenvalue problem*:

Proposition 5..1. *Given $v \in \mathrm{H}_0^{\left(\frac{1}{2},1\right)}$ the solution set of the following equation in K and φ*

$$\begin{cases} \varphi_t - \mu\varphi_{xx} + v\varphi_x + K\varphi = 0 \\ \varphi > 0 \\ \varphi_x|_{\partial Q} = 0 \\ \varphi \in \mathrm{H}^{(1,2)} \\ K \in \mathbb{R} \end{cases} \tag{12}$$

is $K = 0$ and $\varphi = 1$ if and only if $\mathrm{T}u = \mathrm{T}v$ implies $u = v$ (that is, the solution to the original Burger's equation is unique).

The proof of that proposition essentially hinges on the embedding properties exposed in section 2. (see Figure 2).

The remaining part of the proof is concerned with the eigenvalue problem of the Proposition above. One first shows that the eigenvalue is zero using a weaker version of the Perron-Frobenius theorem. The second step is to show that the remaining eigenvalue problem is *non degenerate*, namely that the dimension of the eigenspace must be one. This last step makes use of the a priori estimate proved in Theorem 2.

The details of that part of the proof are too lengthy to be exposed here in depth so the interested reader is referred to [5].

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