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# ON SELF TUNING REGULATORS

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# ON SELF TUNING REGULATORS

K J Åström B Wittenmark

Abstract

It has been shown in several cases that linear stochastic control theory can be used successfully to design regulators for the steady state control of industrial processes. To use this theory it is necessary to have mathematical models of the system dynamics and the disturbances. In practice it is thus necessary to go through the steps of identification computation of control strategies and implementation. It might also be necessary to repeat the identification if the system dynamics or the disturbances are changing.

Algorithms that combine the steps of identification and control will be discussed in the lecture. The special case of discrete time single-input single-output systems is considered. It is assumed that the disturbances can be characterized as filtered white noise. The main result is a theorem which states that if the algorith converges it will in fact converge to a minimum variance regulator. The behaviour of the algorithm is illustrated with several examples which indicate that it has nice convergence properties.

The result has several practical implications. It can be used to construct self-adjusting regulators for direct digital control. It can be interpreted as a real-time maximum likelihood identification scheme. If the control problem discussed is reformulated as a stochastic control problem the theorem implies that asymptotically there is a finite dimensional sufficient statistic.

The implementation of the algorithm on process computers is also discussed. It is shown that it is feasible to implement it on small computers.

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# 1. INTRODUCTION

It has been shown in several cases that linear stochastic control theory can be used successfully to design regulators for the steady state control of industrial processes. See e.g. [1]. To use this theory it is necessary to have mathematical models of the system dynamics and the disturbances. In practice it is thus necessary to go through the steps of identification, computation of control strategies and implementation. This procedure can be quite time consuming in particular if the identification is done off-line. It might also be necessary to repeat the procedure if the system dynamics or the characteristics of the disturbances are changing, as is often the case for industrial processes.

From a practical point of view it is thus meaningful to consider the control of constant but unknown systems. It is thus desired to find control algorithms for an unknown system which converge to the optimal control algorithms that could be derived if the system characteristics were known. A solution to the problem would thus provide a self-tuning or self-adjusting regulator. The word adaptive is not used in this context since adaptive, although never rigorously defined, usually implies that the charcteristics of the process is changing. The problem to be discussed is thus simpler than the adaptive problem in the sense that the system characteristics is assumed constant. The formulated problem can be solved using nonlinear stochastic control theory. This will, however, even in very simple cases lead to exorbitant computational requirements far exceeding the capabilities of existing process control computers.

The purpose of this paper is to discuss one class of self-adjusting regulators and to analyse their properties. The special case of discrete time single-input single-output systems is considered. It is assumed that the disturbances can be characterized as filtered white noise. A class of algorithms is derived based on the hypothesis of separation of identification and control for a class of linear systems having a structure such that a least squares identification procedure can be used. It is then shown that the algorithms obtained have nice properties. The main result is a characterization of the closed loop systems obtained when the algorithm is applied to a general class of linear systems. It is shown in Theorem 5.1 that if the algorithm converges the closed loop system obtained will be such that certain covariances of the inputs and the outputs of the closed loop system are zero. This is shown under very weak assumptions on the system to be controlled. If it is assumed that the system to be controlled is a sampled finite dimensional linear stochastic system with a time delay in the control signal it is furthermore demonstrated in Theorem 5.2 that if the algorithm converges it will actually converge to the minimum variance regulator.

The major assumptions are that the system is nonminimum phase, that the time delay is known and that a bound can be given to the order of the system. The first two assumptions can be removed at the prize of a more complicated algorithm. The behaviour of the algorithm is illustrated with several examples which indicate that it has nice convergence properties.

The paper is organized as follows: Sections 2, 3 and 4 provide motivation and preliminaries. Control strategies for systems with known parameters are given in Section 2. Section 3 outlines the version of the least squares identification scheme that is used. The problem of identifiability is covered in Section 4. The main results are given in Section 5. In Section 6 it is shown that a modified version of the algorithm converges for a first order system. The convergence properties of the algorithm are further illustrated by the examples in Section 7. Some practical aspects on the algorithm as well as some problems wich remain to solve are given in Section 8. In particular it is shown that the algorithm is easily implemented on a minicomputer.

## 2. MINIMUM VARIANCE CONTROL

Consider a system described by the difference equation

$$y(t) + a_1y(t-1) + \dots + a_ny(t-n) = b_1u(t-k-1)+\dots+$$

$$+ b_nu(t-k-n) + \lambda\left[e(t) + c_1e(t-1)+\dots+c_ne(t-n)\right] ,$$

$$t = 0, \pm 1, \pm 2,\dots \qquad (2.1)$$

where u is the control variable, y is the output and $\{e(t), t = 0, \pm 1, \pm 2,\dots\}$ is a sequence of independent normal (0,1) random variables. If the forward shift operator q, defined by

$$qy(t) = y(t+1)$$

and the polynomials

$$A(z) = z^n + a_1z^{n-1} + \dots + a_n$$

$$B(z) = b_1z^{n-1} + \dots + b_n \qquad b_1 \neq 0$$

$$C(z) = z^n + c_1z^{n-1} + \dots + c_n$$

are introduced the equation (2.1) describing the system can be written in the following compact form:

$$A(q)y(t) = B(q)u(t-k) + \lambda C(q)e(t) \qquad (2.2)$$

It is wellknown that (2.1) or (2.2) is a canonical representation of a sampled finite dimensional single-input single-output dynamical system with time delays in the output whose disturbances are gaussian random processes with rational spectral densities.

If (2.1) is obtained by sampling a finite dimensional system (A,B,C,D) with D = 0 then k = 0. The model (2.1) also admits a time delay τ in the system input which must not be a multiple of the sampling interval.

The number k corresponds to the integral part of τ/h, where h is the sampling interval. It is also wellknown that under certain assumptions the control

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The number k corresponds to the integral part of $\tau/h$, where h is the sampling interval. It is also wellknown that under certain assumptions the control strategy

$$u(t) = - \frac{q^k G(q)}{B(q)F(q)} y(t) \tag{2.3}$$

where F and G are polynomials

$$F(z) = z^k + f_1 z^{k-1} + \dots + f_k \tag{2.4}$$

$$G(z) = g_0 z^{n-1} + g_1 z^{n-2} + \dots + g_{n-1} \tag{2.5}$$

determined from the identity

$$q^k C(q) = A(q)F(q) + G(q) \tag{2.6}$$

is optimal in the sense that the criteria

$$V_1 = Ey^2(t) \tag{2.7}$$

and

$$V_2 = E \frac{1}{N} \sum_1^N y^2(t) \tag{2.8}$$

are minimal. Proofs of these statements are given in [2]. The following conditions are necessary:

- The polynomial B has all zeroes inside the unit circle. (The system (2.1) is minimum phase).
- The polynomial C has all zeroes inside the unit circle.

These conditions are discussed at length in [1]. Let it suffice to mention here that if the system (2.1) is nonminimum phase the control strategy (2.3) will still be a minimum variance strategy. This strategy will, however, be so sensitive that the slightest variation in the parameters will result in an unstable closed loop system. Suboptimal strategies which are less sensitive to parameter variations are also well known. This paper will, however, be limited to minimum phase system.

For a system described by (2.1) it is thus a straight forward task to obtain the minimum variance regulator, if the parameters of the model are known. If the parameters are not known it might be a possibility to try to determine the parameters of (2.1) using some identification scheme and then use the control law (2.3) with the true parameters substituted by their estimates. A suitable identification algorithm is the maximum likelihood method which will give unbiased estimates of the coefficients of the A, B and C polynomials. The maximum likelihood estimates of the parameters of (2.1) are, however, strongly nonlinear functions of the inputs and the outputs. Since finite dimensional sufficient statistics are not known it is not possible to compute the maximum likelihood estimate of the parameters of (2.1) recursively as the process develops. Simpler identification schemes are therefore first considered.

## 3. THE LEAST SQUARES STRUCTURE.

Identification

The problem of determining the parameters of the model (2.1) is significantly simplified if it is assumed that $c_i = 0$ for $i = 1, 2, \ldots, n$. The model is then given by

$$A(q)y(t) = B(q)u(t-k) + \lambda e(t) \tag{3.1}$$

The parameters of this model can be determined simply by the least squares method [3]. The model (3.1) is therefore referred to as a least squares model .

The least squares estimate has several attractive properties. It can easily be evaluated recursively. The estimate can be modified to take different model structures, e.g. known parameters, into account. It can be shown that the least squares estimate will converge to the true parameters e.g. under the following conditions. See [3].

- The output $\{y(t)\}$ is actually generated from a model (3.1)

- The residuals $\{e(t)\}$ are independent.

- The input is persistantly exciting of order greater than 2n. See [4].

- The input sequence $\{u(t)\}$ is independent of the sequence of disturbance sequence $\{e(t)\}$.

These conditions are important. If the residuals are correlated the least squares estimate will be biased. If the input is not persistantly exciting of order 2n or greater or if the input sequence $\{u(t)\}$ depends on $\{e(t)\}$ it may not be possible to determine the parameters at all. (The system is not identifiable). When the inputs u are generated by a feedback they will obviously depend on $\{e(t)\}$. The last condition above will thus require special consideration. This is done in Section 4.

## Control

If $k = 0$ and $B$ has all its zeroes inside the unit circle the minimum variance strategy (2.3) for the least squares model (3.1) reduces to

$$u(t) = \frac{1}{b_1}\Big[a_1 y(t) + \ldots + a_n y(t-n+1) - b_2 u(t-1) -$$

$$- \ldots - b_n u(t-n+1)\Big] \qquad (3.2)$$

If there are time delays in the system i.e. $k \neq 0$ the computation of the control strategy becomes more involved since the identity (2.6) must be resolved. The problem is simplified if it is observed that the computation of the minimum variance regulator for the model (3.1) is equivalent to reduce (3.1) to a model having the form

$$y(t+k+1) + \alpha_1 y(t) + \ldots + \alpha_m y(t-m+1)$$

$$= b_1' u(t) + \ldots + b_{\ell+1}' u(t-\ell) + v(t) \qquad (3.3)$$

where $m = n$ and $\ell = n+k$. The coefficients $\alpha_i$ and $b_i'$ are related to the coefficients $a_i$ and $b_i$ through simple algebraic equations. The disturbance $v$ is a moving average of $e$.
Assuming that $v(t)$ and $v(s)$ are uncorrelated the minimum variance control strategy for (3.3) becomes

$$u(t) = \frac{1}{b_1'}\Big[\alpha_1 y(t) + \ldots + \alpha_m y(t-m+1) -$$

$$- b_2' u(t-1) - \ldots - b_{\ell+1}' u(t-\ell)\Big] \qquad (3.4)$$

In order to obtain a simple computation of the control strategy in the case that there are time delays in the model it could thus be attempted to use the model structure (3.3) which also admits a least squares identification.

## Identification and Control

The problem of controlling the system (3.1) in the case when $k$ and $n$ are known but the parameters $a_i$ and $b_i$ are unknown can now be attempted using the following algorithm

Step 1 Determine the parameters of the least squares model (3.1) or (3.3)

Step 2 Introduce the parameters obtained in step 1 into the control law (3.2) or (3.4) and evaluate the control signal

Since the least squares estimate is easily computed recursively the steps 1 and 2 can easily be performed at each sampling interval. An alternative could be to keep the parameters of the control law constant for a number of sampling intervals. This procedure or slight variations thereof has been suggested by several authors, e.g. Kalman [5] Peterka [6], Wieslander and Wittenmark [7] . In Åström-Wittenmark [2] it is shown that a modification of the algorithm which takes the uncertainties in the parameter estimates into account is optimal with respect to the criterion (2.7) but not with respect to the criterion (2.8). Several fundamental problems related to the algorithm are also discussed in [2] . In section 5 of this paper it is shown that the algorithm given above with a few modifications will actually converge to a minimum variance regulator if it converges at all. This is true when the system to be controlled is governed by a least squares model (3.1) and surprisingly also when the system is governed by the general linear model (2.1).

# 4. IDENTIFIABILITY

When the parameters of the model (3.1) or (3.3) are determined using the least squares estimate and the inputs are generated by a feedback (2.3), (3.2) or (3.4) the inputs are correlated with the disturbances $\{e(t)\}$. Hence it is not obvious that the parameters can be determined. Neither is it obvious that the input generated in this way persistantly exciting of sufficiently high order. It will in fact be shown that the model (3.1) with the feedback (3.2) is not identifiable in the sense that all parameters can be determined [8]. A simple example will be analysed.

Example 4.1

Consider the first order model

$$y(t) + ay(t-1) = bu(t-1) + e(t) \tag{4.1}$$

Assume that a linear regulator

$$u(t) = ky(t) \tag{4.2}$$

is used. If the parameters a and b are known the gain $k = a/b$ would obviously correspond to a minimum variance regulator. If the parameters are not known the gain $k = \hat{a}/\hat{b}$ where $\hat{a}$ and $\hat{b}$ are the least squares estimates of a and b could be attempted. The least squares parameter estimates are determined in such a way that the loss function

$$V = \sum_{1}^{N} \left[y(t+1) + ay(t) - bu(t)\right]^2 \tag{4.3}$$

is minimal with respect to a and b. If the feedback control (4.2) is used the inputs and outputs are linearly related through

$$u(t) - ky(t) = 0 \tag{4.4}$$

Multiply (4.4) by $-\alpha$ and add to the expression within brackets in (4.3). Hence

$$V(a,b) = \sum_{1}^{N} \left[y(t+1) + (a+\alpha k)y(t) - (b+\alpha)u(t)\right]^2$$

The loss function will thus assume the same value for all estimates $\hat{a}$ and $\hat{b}$ such that

$$\hat{a} = a + \alpha . k$$

$$\hat{b} = a + \alpha$$

It is thus not possible to determine the parameters a and b of the model (4.1) when the feedback (4.2) is used.

To avoid the difficulty illustrated in the example one of the parameters could be set to a fixed value, e.g. $\hat{b} = b_0$. This means that in the model (4.1) it is only attempted to estimate the parameter a. The estimated gain then becomes

$$\hat{k} = \frac{\hat{a}}{b_0} = k + \frac{a - bk}{b_0}$$

which equals the correct value only in the case k = a/b.

Hence if a feedback (4.2) is used and the gain k is chosen in such a way that it corresponds to a minimum variance strategy then a least squares estimates of the parameter a, when the value of b is assumed given, corresponds to a minimum variance strategy. This holds independently of the value of $b_0$.

The simple example shows that it is in general not possible to estimate all the parameters of the model (3.1) when the input is generated by a feedback like (3.2). Notice, however, that all parameters can be estimated if the control law is changed. In the particular example it is e.g. easy to show that if the control law (4.2) is replaced by

$$u(t) = ky(t-1)$$

or

$$u(t) = k_1y(t) + k_2y(t-1)$$

it is possible to estimate both parameters of the model.

## 5. THE MAIN RESULT

The properties of the algorithm discussed in Section 3 will now be analysed. The example of Section 4 shows that all the parameters of the model cannot be determined and it is therefore assumed that the parameter $b_1' = \beta_o$ is given. It will be shown in Section 6 that the choice of $\beta_o$ is not crucial. When $b_1'$ is assumed given the form of the equations is if the coefficients are renamed as follows

$$\beta_i = b_{i+1}'/b_1' \qquad i = 1,2,\ldots,\ell$$

The algorithm which combines the steps of identification and control is then composed of the steps

Step 1 Determine the parameters $\alpha_1 \ldots \alpha_m$, $\beta_1 \ldots \beta_\ell$ of the model:

$$y(t+k+1) + \alpha_1 y(t) + \ldots \ldots + \alpha_m y(t-m+1) =$$

$$= \beta_o \left[u(t) + \beta_1 u(t-1) + \ldots + \beta_\ell u(t-\ell)\right] + e(t) \tag{5.1}$$

The parameter $\beta_o$ is assumed known.

Step 2 Choose the control law

$$u(t) = \frac{1}{\beta_o}\left[\alpha_1 y(t) + \ldots + \alpha_m y(t-m+1)\right] -$$

$$- \beta_1 u(t-1) - \ldots - \beta_\ell u(t-\ell) \tag{5.2}$$

which also can be written as

$$u(t) = \frac{\alpha_1 + \alpha_2 q^{-1} + \ldots + \alpha_m q^{-m+1}}{\beta_o\left[1 + \beta_1 q^{-1} + \ldots + \beta_\ell q^{-\ell}\right]}\, y(t) = \frac{q^{\ell-m+1}\mathcal{A}(q)}{\beta_o\,\mathcal{B}(q)}\, y(t) \tag{5.2'}$$

This algorithm is called a <u>self-tuning algorithm</u>.

The model structure (5.1) is chosen because the regulator (5.2) derived from it has the same structure as the minimum variance regulator for (2.1) and (3.1).

The properties of the closed loop system obtainded when the above control algorithm is used will now be analysed.

We have

<u>Theorem 5.1</u>

Assume that the identification algorithm converges and that the closed loop system is such that the output is <u>ergodic</u> (in the second moments). Then the closed loop system has the properties

$$E \ y(t+\tau)y(t) = 0 \qquad \tau = k+1,\ldots,k+m \tag{5.3}$$

$$E \ y(t+\tau)u(t) = 0 \qquad \tau = k+1,\ldots,k+\ell+1 \tag{5.4}$$

<u>Proof</u>

The least squares estimate of the parameters $\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_\ell$ is given by [3]

$$
\left[\begin{array}{cccc|ccc}
-\Sigma y^2(t) & -\Sigma y(t)y(t-1) & \cdots & -\Sigma y(t)y(t-m+1) & \beta_o\Sigma y(t)u(t-1) & \cdots & \beta_o\Sigma y(t)u(t-\ell+1) \\
-\Sigma y(t)y(t-1) & -\Sigma y^2(t-1) & & -\Sigma y(t-1)y(t-m+1) & \beta_o\Sigma y(t-1)u(t-1) & \cdots & \beta_o\Sigma y(t-1)u(t-\ell+1) \\
\vdots & & & & & & \\
-\Sigma y(t)y(t-m+1) & & & -\Sigma y^2(t-m+1) & \beta_o\Sigma y(t-m+1)u(t-1) & \cdots & \beta_o\Sigma y(t-m+1)u(t-\ell) \\
\hline
 & & & & \beta_o^2\Sigma u^2(t-1) & & \beta_o^2\Sigma u(t-1)u(t-\ell) \\
 & & & & & & \beta_o^2\Sigma u^2(t-\ell)
\end{array}\right]
\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_m \\ \beta_1 \\ \vdots \\ \beta_\ell \end{bmatrix} =
$$

$$
= \begin{bmatrix}
\Sigma y(t+k+1)y(t) - \beta_o \Sigma u(t)y(t) \\
\Sigma y(t+k+1)y(t-1) - \beta_o \; u(t)y(t-1) \\
\vdots \\
\Sigma y(t+k+1)y(t-m+1) - \beta_o \Sigma u(t)y(t-m+1) \\
\beta_o \Sigma y(t+k+1)u(t-1) - \beta_o^2 \Sigma u(t)u(t-1) \\
\vdots \\
\beta_o \Sigma y(t+k+1)u(t-\ell) - \beta_o^2 \Sigma u(t)u(t-\ell)
\end{bmatrix} \qquad (5.5)
$$

where the sums are taken over $N_0$ values.

Assume that the identification converges. For sufficiently large $N_0$ the coefficients of control law (5.2) will thus converge to constant values. Introduction of (5.2) into (5.5) gives

$$
\begin{aligned}
&\Sigma y(t+k+1)y(t) = 0 \\
&\Sigma y(t+k+1)y(t-1) = 0 \\
&\vdots \\
&\Sigma y(t+k+1)y(t-m+1) = 0 \\
&\Sigma y(t+k+1)u(t-1) = 0 \\
&\vdots \\
&\Sigma y(t+k+1)u(t-\ell) = 0
\end{aligned}
$$

Using the control law (5.2) it also follows that

$$\Sigma y(t+k+1)u(t) = 0$$

Under the ergodicity assumption the sums can furthermore be replaced by mathematical expectations and the theorem is proven. □

Remark 1

It is sufficient for ergodicity that the system to be controlled is governed by a difference equation of finite order, e.g. like (2.1), and that the closed loop system obtained by introducing the feedback law (5.2) into (2.1) gives a stable closed loop system.

Notice that it is not necessary to assume that the system to be controlled is governed by an equation like (2.1) or (3.1). If such an assumption is made it is possible to obtain the following stronger result.

Theorem 5.2

Let the system to be controlled be governed by (2.1), let the model (5.1) be such that $m \geq n$ and $\ell \geq n+k-1$. Assume that the self-tuning algorithm converges. The self-tuning algorithm then converges to a minimum variance regulator.

Proof

The proof is straightforward but tedious. The idea is to use Theorem 5.1, the descriptions of the system (2.1) and of the regulator (5.2) to show that

$$r_y(\tau) = E\, y(t+\tau)\cdot y(t) = 0, \quad |\tau| \geq k+1 \tag{5.6}$$

The result then follows from the uniqueness of the minimum variance regulator. For the proof there is no loss in generality to assume that $\beta_o = 1$. This can always be achieved through suitable choices of units for u and y.

When the algorithm has converged it might happen that there is a common factor in the denominator and nominator of the control law. Assume that the common factor is of order p. Let

$$u(t) = \frac{q^{\ell-m+1}\mathcal{A}(q)}{\mathcal{B}(q)}\; y(t) \tag{5.7}$$

denote the control law obtained in the limit and after the concellation of the common factor. This implies that $\mathcal{A}$ and $\mathcal{B}$ are of order m-p and $\ell$-p respectively.

The closed loop system then becomes

$$[A\mathcal{B} - q^{\ell-m-k} B\mathcal{A}]\,y(t) = C\mathcal{B}\,e(t)$$

It is of order [illegible] of order

$n = n+ -p$

$r = n+\ell-p \geq 2n+k-p-1$. Let the characteristic equation be

$$z^r + \gamma_1 z^{r-1} + \ldots + \gamma_r = 0$$

It then follows that

$$r_y(\tau) + \gamma_1 r_y(\tau-1) + \ldots + \gamma_r r_y(\tau-r) = 0 \qquad \tau \geq r+1 \tag{5.8}$$

$$r_{yu}(\tau) + \gamma_1 r_{yu}(\tau-1) + \ldots + \gamma_r r_{yu}(\tau-r) = 0 \quad \tau \geq r+1 \tag{5.9}$$

where $r_y(\tau)$ is defined by (5.6) and

$$r_{yu}(\tau) = Ey(t+\tau)u(t)$$

See [1 p.50 and p 98].

It follows from Theorem 5.1 that $r_y(\tau)$ vanishes for $\tau = k+1,\ldots,k+m$. If it could be shown that $r_y(\tau)$ also is zero for $\tau = k+m+1,\ldots,k+r$ the equation (5.6) would then follow from the Yule-Walker equation (5.8).

$$R_y = \begin{bmatrix} \Gamma_{r-m-1} r_y(k+r) \\ \Gamma_{r-m-2} r_y(k+r-1) \\ \vdots \\ \Gamma_1 r_y(k+m+2) \\ r_y(k+m+1) \end{bmatrix}$$

and

$$R_{yu} = \begin{bmatrix} \Gamma_{r-\ell-2} r_{yu}(k+r) \\ \Gamma_{r-\ell-3} r_{yu}(k+r-1) \\ \vdots \\ \Gamma_1 r_{yu}(k+\ell+3) \\ r_{yu}(k+\ell+2) \end{bmatrix}$$

of orders $r-m$ and $r-\ell-1$ respectively where

$$\Gamma_i = 1+\gamma_1 q^{-1} + \ldots + \gamma_i q^{-i}$$

For simplicity it is also assumed that $\ell+1 \geq m$. (If this does not hold it is necessary to increase the order of the $R_{yu}$ vector to $r-m\times 1$ and to make some small modifications in the following).

Using the control law (5.7) and the equations (5.8) and (5.9) it is straightforward but tedious to get $2r-m-\ell-1$ equations to solve for the $2r-m-\ell-1$ unknowns in $R_y$ and $R_{yu}$. Using matrix notations the system will be

$$\begin{bmatrix} \alpha_1 & 0 & 0 & 1 & 0 & 0 \\ \alpha_2 & \alpha_1 & & \beta_1 & 1 & \\ \cdot & \alpha_2 & 0 & & 0 & 0 \\ & & \alpha_1 & & & 1 \\ \alpha_{m-r} & & & & & \beta_1 \\ 0 & & & \beta_{\ell-r} & & \\ \cdot & & & 0 & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & & \alpha_{m-r} & & 0 & 0 \;\; \beta_{\ell-r} \end{bmatrix} \begin{bmatrix} R_y \\ R_{yu} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ \\ \\ \\ \\ \\ \\ 0 \end{bmatrix} \tag{5.10}$$

The inverse of the matrix on the left hand side exists if the polynomials $\mathcal{A}(z)$ and $\mathcal{B}(z)$ have no common factor[9] . This now implies that all elements in $R_y$ are equal to zero and that (5.6) holds.

□

<u>Remark</u>

Since the system to be controlled is governed by (2.1) which has correlated residuals the least squares estimates will be biased. The control law (5.2) obtained under the assumption of the least squares structure with the biased parameters will, however, correspond to the optimal control law.

## 6. CONVERGENCE OF THE ALGORITHM

It would be highly desirable to have general results giving conditions for convergence of the self-tuning system. Since the system (2.1) with the regulator (5.2) and the least squares estimator is described by a set of nonlinear time dependent stochastic difference equations the problem of a general convergence proof is difficult. So far we have not been able to obtain a general result. It has, however, been verified by extensive numerical simulations that the algorithm does in fact converge in many cases. The numerical simulations as well as analysis of simple examples have given insight into some of the conditions that must be imposed in order to ensure that the algorithm will converge.

A significant simplification of the analysis is obtained if the algorithm is modified in such a way that the parameter estimates are kept constant over long periods of time. To be specific a simple example is considered.

<u>Example 6.1</u>

Let the system be described by

$$y(t) + ay(t-1) = bu(t-1) + e(t) + ce(t-1) \qquad |c| \leq 1 \tag{6.1}$$

Assume that the control law

$$u(t) = \hat{\alpha}_n y(t) \tag{6.2}$$

is used in the time interval $t_n < t < t_{n+1}$ where the parameter $\alpha_n$ is determined by fitting the least squares model

$$y(t+1) + \alpha y(t) = u(t) + \varepsilon(t+1) \tag{6.3}$$

to the data $\{u(t), y(t), t = t_{n-1}, \ldots, t_n-1\}$.

The least squares estimate is given by

$$\hat{\alpha}_n = - \frac{\sum_{t=t_{n-1}}^{t_n - 2} y(t)\left[y(t+1) - u(t)\right]}{\sum_{t=t_{n-1}}^{t_n - 2} y^2(t)} =$$

$$= \hat{\alpha}_{n-1} - \frac{\sum_{t=t_{n-1}}^{t_n - 2} y(t+1)y(t)}{\sum_{t=t_{n-1}}^{t_n - 2} y^2(t)} \tag{6.4}$$

where the last equality follows from (6.2). If it is now assumed that $t_n - t_{n-1} \to \infty$ and that

$$|a - b\alpha_{n-1}| < 1 \tag{6.5}$$

which means that the closed loop system used during the time interval $t_{n-1} < t < t_n$ is stable we find that

$$\alpha_n = \alpha_{n-1} - \frac{r_y(1)}{r_y(0)} \tag{6.6}$$

where $r_y(\tau)$ is the covariance function of the stochastic process $\{y(t)\}$ defined by

$$y(t) + (a - b\alpha_{n-1})y(t-1) = e(t) + ce(t-1) \tag{6.7}$$

Straightforward algebraic manipulations now give

$$\alpha_n = \alpha_{n-1} - \frac{(c-a+b\alpha_{n-1})(1-ac+bc\alpha_{n-1})}{1 + c^2 - 2ac + 2cb\alpha_{n-1}} \tag{6.8}$$

In the particular case the analysis of the convergence of the self-tuning regulator thus reduces to the analysis of the nonlinear difference equation given by (6.8).

Introduce

$$x_n = \alpha_n - \frac{a - c}{b} \tag{6.9}$$

the equation (6.8) then reduces to

$$x_{n+1} = g(x_n) = (1-b)x_n + \frac{b^2 c x_n^2}{1-c^2+2bcx_n} \tag{6.10}$$

The point $x = 0$ is a fixpoint of the mapping g which corresponds to the optimal value of gain of the feedback loop, i.e $\alpha = (a-c)/b$. The problem is thus to find if the fixpoint is stable. Since the closed loop system is stable only if

$$-1 < a-b\alpha_{n-1} < 1 \tag{6.11}$$

it is sufficient to consider

$$\frac{c - 1}{b} < x < \frac{c + 1}{b} \tag{6.12}$$

For small x is $g(x) \approx (1-b)x$ which implies that $x = 0$ is a stable fixpoint if $0 < b < 2$.

There is essentially three cases that have to be investigated.

1. $c = 0$
2. $c > 0$ , $0 < b \leq 1$
3. $c > 0$ , $1 < b < 2$

If $c < 0$ it is the same structure as in case 2 and 3 respectively.

Case 1

The equation (6.10) is now reduced to

$$x_{n+1} = (1-b)x_n$$

and the fix point $x = 0$ is stable if $|1-b| < 1$ i.e. $0 < b < 2$.

Case 2

A graph of g for this case is shown in fig.6 .1.
It is straightforward to verify that all initial values in the stability region $\frac{c-1}{b} < x < \frac{c+1}{b}$ will give solutions which converge to zero.

Case 3

A graph of g is shown in fig. 6.2.

It can be shown that $g(g(x)) < x$ which implies that $x_n \to 0$.

Summary

From the analysis above we can conclude that $x = 0$ is a stable fix point if

$-1 < c < 1$
and
$0 < b < 2$

The example shows that under the condition (6.13) the version of the self-tuning algorithm where the parameters of the control law are kept constant over long intervals will in fact converge. The condition (6.13) implies that it is necessary to pick the parameter $\beta_o$ in a correct manner. In the particular example $\beta_o = 1$ was chosen while the correct value was b. The condition (6.13) indicates that in the particular example the choice of $\beta_o$ is not critical. The algorithm will always converge if $\beta_o$ is greater than b. Under-estimation may be serious and the value $\beta_o < 0.5$ b gives an unstable algorithm.

The analysis presented in the simple example can be extended to give stability conditions for the modified algorithm in more complex cases. The analysis required is tedious.

## 7. SIMULATIONS

This section presents a number of simulated examples which illustrate the properties of the self-tuning algorithm.

Example 7.1

Let the system be

$$y(t) + ay(t-1) = bu(t-1) + e(t) + ce(t-1) \tag{7.1}$$

with $a = -0.5$, $b = 3$ and $c = 0.7$. The minimum variance regulator for the system is

$$u(t) = \frac{a-c}{b}\, y(t) = -0.4y(t) \tag{7.2}$$

A regulator with this structure can be obtained by using the self-tuning algorithm based on the model

$$y(t+1) + \alpha y(t) = \beta_0 u(t) + e(t) \tag{7.3}$$

Fig 7.1 shows for the case $\beta_0 = 1$ how the parameter estimate converges to the value $\alpha = -0.4$ which corresponds to the minimum variance strategy.

In Fig 7.2 is shown the expected variance of the output if the present value of $\alpha$ should be used for all future steps of time. Notice that the algorithm has practically adjusted after 50 steps.

The analysis of Example 6.1 shows that, since $b > 2$, and $\beta_0 = 1$ the modified self-tuning algorithm obtained when the parameters of the controller are kept constant over long intervals is unstable. The simulation in Example 7.1 shows that at least in the special case a conservative estimate of the convergence region is obtained by analysing the modified algorithm.

If the value of b is increased further it has been shown that the algorithm is unstable. Unstable realizations have been found for $b = 5$. In such cases it is of course easy to obtain a stable gorithm by increasing algorithm by increasing $\beta_o$. This requires, however, a knowledge of the magnitude of b. □

The system of Example 7.1 is very simple. For example if no control is used the variance will still be reasonably small. The next example is more realistic in this aspect.

Example 7.2

Consider the system

$$y(t) - 1.5y(t-1) + 0.5y(t-2) =$$

$$= u(t-2) - u(t-3) + e(t) + 0.8e(t-1) \tag{7.4}$$

If no control is used the variance of the output is infinite. Also notice that B(z) = z-1. The assumption that B has all zeroes inside the unit circle is thus violated. The minimum variance strategy for the system is

$$u(t) = -2.95y(t) + 1.15y(t-1) - 1.30u(t-1) + 2.30u(t-2) \tag{7.5}$$

A regulator with this structure is obtained by using the self-tuning algorithm with the model

$$y(t+2) + \alpha_1 y(t) + \alpha_2 y(t-1) =$$

$$= u(t) + \beta_1 u(t-1) + \beta_2 u(t-2) + e(t) \tag{7.6}$$

The convergence of the parameters is rather slow as can be seen in Fig 7.3 But the control is fairly good already after about 10 steps of time. The actual loss is shown in Fig.7.4. □

Both examples that have been considered are such that the model used in the self-tuning algorithm is such that the minimum variance regulator can always be obtained. The next example shows what happens when this is not the case.

Example 7.3.

Consider the system

$$y(t) - 1.60y(t-1) + 1.61y(t-2) - 0.776y(t-3) =$$

$$= 1.2u(t-1) - 0.95u(t-2) + 0.2u(t-3) + e(t) +$$

$$+ 0.1e(t-1) + 0.25e(t-2) + 0.873e(t-3) \qquad (7.8)$$

The polynomial A(z) has two complex zeroes near the unit circle ($+0.4 \pm 0.9i$) and one real zero equal to 0.8.

If a self-tuning regulator is determined based on a model with $m = 3$ and $\ell = 2$ it will converge to the minimum variance regulator as expected. Fig7.5 shows the sample covariance of the control error together with a plot of part of the output.

If the self-tuning algorithm instead is based on a model with $m = 2$ and $\ell = 1$ it is no longer possible to obtain the minimum variance regulator for the system since there are not parameters enough in the self-tuning regulator. Theorem 5.1 indicates, however, that if the self-tuning regulator converges its parameters will be such that the covariances $r_y(1)$, $r_y(2)$, $r_{yu}(1)$ and $r_{yu}(2)$ are all zero. The simulation shows that the algorithm does in fact converge ($\beta_o = 1.0$). The covariance function of the output is shown in fig. 7.5. It is seen that the sample covariance $\hat{r}_y(1)$ and $\hat{r}_y(2)$ are within the 5 % confidence interval while $\hat{r}_y(3)$ is not or would be expected from Theorem 5.1.

If a self-tuning algorithm is designed based on a model with $m = 1$ $\ell = 0$ then Theorem 5.1 indicates that $\hat{r}_y(1)$ should vanish. Again the simulation shows that the algorithm does in fact converge and that the sample covariance $\hat{r}_y(1)$ does not differ significantly from zero. See fig. 7.5.

When using regulators of lower order than the optimal minimum variance regulator the parameters in the controller will not converge to values which for the given structure gives minimum variance of the output. In Table 1 is shown the variance of the output for the system above when using different regulators.

The loss when using the self-adjusting regulator is obtained through simulations. The optimal regulator is found by minimizing $r_y(0)$ with respect to the parameters in the controller.

| | | Loss $\frac{1}{N}\sum_{t=1}^{N} y^2(t)$ | |
|---|---|---|---|
| m | ℓ | Self-adjust. | Optimal |
| 3 | 2 | 1.0 | 1.0 |
| 2 | 1 | 2.5 | 1.9 |
| 1 | 0 | 4.8 | 3.4 |

Table 1

□

The previous examples are all designed to illustrate various properties of the algorithm. The algorithm has not yet been applied to an industrial process, however, the following example is a summary of a feasibility study which indicates the practicality of the algorithm for application to basis weight control of a paper machine.

Example 7.4

The applicability of minimum variance strategies to basis weight control on a paper machine was demonstrated in [10]. In this application the control loop is a feedback from a wet basis weight signal to thick stock flow. The models used in [10] were obtained by estimating the parameters of (2.1) using the maximum likelihood method. In one particular case the following model was obtained.

$$y(t) = \frac{b_1q^{-1} + b_2q^{-2}}{1 + a_1q^{-1} + a_2q^{-2}} u(t-2) + v(t) \tag{7.9}$$

where the output y is basis weight in $g/m^2$ and the control variable is thick stock flow in normalized units. The disturbance $\{v(t)\}$ was a drifting stochastic process which could be modelled as

$$v(t) = \lambda \frac{1 + c_1q^{-1} + c_2q^{-2}}{(1 + a_1q^{-1} + a_2q^{-2})(1 - q^{-1})} e(t) \tag{7.10}$$

where $\{e(t)\}$ is white noise. The sampling interval was 36 seconds and the numerical values of the parameters obtained through identification were as follows

$a_1 = -1.283$
$a_2 = 0.495$
$b_1 = 4.614$
$b_2 = -4.050$
$c_1 = -1.438$
$c_2 = 0.550$
$\lambda = 0.382$

To investigate the feasibility of the self-tuning algorithm for basis weight control the algorithm was simulated using the model (7.9) where the disturbance v was the actual realization obtained from measurements on the paper machine. The parameters of the regulator were chosen as $\ell = 3$, $m = 4$, $\beta_0 = 5$ and the initial estimates were set to zero. The algorithm is thus tuning 7 parameters.

The results of the simulation are shown in Fig. 7.6 to 7.8. Fig. 7.6 compares the output obtained when using the self-tuning algorithm with the result obtained when using the minimum variance regulator computed from the process model (7.9) with the disturbance given by (7.10). In the worst case the self-tuning regualtor gives a control error which is about 1 $g/m^2$ greater than the minimum variance regulator. This happens only at two sampling intervals.

After about 100 sampling intervals (1 hour) the output of the systems is very close.

Fig. 7.7 compares the accumulated losses

$$V(t) = \sum_{n=0}^{t} y^2(n)$$

obtained with the minimum variance regulator and the self-tuning regulator. Notice that in the time interval (21, 24) minutes there is a rapid increase in the accumulated loss of the self-tuning regulator of about 17 units. The largest control error during this interval is 2.7 $g/m^2$ while the largest error of the minimum variance regulator is 1 $g/m^2$. The accumulated losses over the last hour is 60 units for the self-tuning regulator and 59 units for the minimum variance regulator.

The control signal generated by the self-tuning algorithm is compared with that of the minimum variance regulator in Fig. 7.8. There are differences in the generated control signals. The minimum variance regulator generates an output which has more rapid variations than the output of the self-tuning regulator.

The parameter estimates obtained have not converged in 100 sampling intervals. In spite of this the regulator obtained will have good performance as has just been illustrated. The example thus indicates that the self-tuning algorithm could be feasible as a basis weight regulator. □

## 8. PRACTICAL ASPECTS

The self-tuning algorithm has not yet been tried on a real industrial process. It has, however, been extensively studied in simulation and it has also been tried on a laboratory process. In this section a few practical aspects on the algorithm will be given.

### A Priori Knowledge

The only parameters that must be known à priori are k, $\ell$, m and $\beta_o$. If the algorithm converges it is easy to find out if the à priori guesses of the parameters are correct simply by analyzing the sample covariance of the output. Compare Example 7.3. The parameter $\beta_o$ should be an estimate of the corresponding parameter of the system to be controlled. The choice of $\beta_o$ is not critical as was shown in the Example 6.1 and 7.1. In the special cases studied in the examples an under-estimate led to a diverging algorithm while an over-estimate was safe.

### Improved Identifiers and Control Algorithms

The control algorithm can be modified to take into account that the estimates are inaccurate. This is discussed in [2],[7], where it is shown that a modification of (5.2) will give strategies that are optimal for the criterion (2.7) for each t. If initial estimates are poor it could also be advisable to select an initial input which is designed to give good estimates and not use the feedback during the first steps. It has been verified by simulation that the region of convergence can be improved by introducing a bound on u.

### Drifting Parameters

In the case where the parameters are drifting the least squares identifier can be replaced by an identifier which discounts old data or by a Kalman filter. See [11],[12]. Simulations have shown that the algorithm can be used to control systems with slowly drifting parameters. It has been demonstrated in [7] that the algorithm will have difficulties if there are rapid parametervariations, in particular if the gain is permitted to change sign. An example in[7] shows that the algorithm in such a case may exhibit a very strange behaviour. The reason is that the self-tuning regulator is not a dual control strategy.

## Nonminimum Phase Systems

Difficulties have been found by a straightforward application of the algorithm to nonminimum phase systems, i.e. systems where the polynomial B has zeros outside the unit circle. Such a case will arise when the system to be controlled actually is nonminimum phase. Underestimating k can also lead to a nonminimum phase model.

Several ways to get around the difficulty have been found. By using a model with $B(z) = \beta_0$ it has in many cases been possible to obtain stable algorithms at the sacrifice of variance.

It is well-known that the minimum variance regulators are extremely sensitive to parameter variations for nonminimum phase systems [1].

This is usually overcome by using suboptimal strategies which are less sensitive [1]. The same idea can be used for the self-tuning algorithms as well. The drawback is that the computations increase because the polynomials F and G of an identity simular to (2.6) must be determined at each step of the iteration. An alternative is to solve a Riccati-equation at each step.

## Implementation on Process Computers

It is our belief that the self-tuning algorithm can be conventiently used in process control application. There are many possibilities. The algorithm can be used as a tool to tune regulators when they are installed. It can be installed among the systems programs and cycled through different control loops repetitively to ensure that the regulators are always properly tuned. For critical loops where the parameters are changing it is also possible to use a dedicated version which allows slowly drifting parameters.

A general self-tuning algorithm requires about 40 FORTRAN statements. When compiled using the standard PDP 15/30 FORTRAN compiler the code consists of 450 memory locations. The number of memory locations required to store the data is $(\ell-1+m)^2 + 3(\ell-1+m) + 2k + 4$. Execution times on a typical process computer (PDP 15) without floating point hardware are given in the table below. The major part of the computing is to update the least squares estimate.

| Number of parameters | Execution time ms |
|---|---|
| 1 | 5 |
| 3 | 16 |
| 5 | 34 |
| 8 | 69 |

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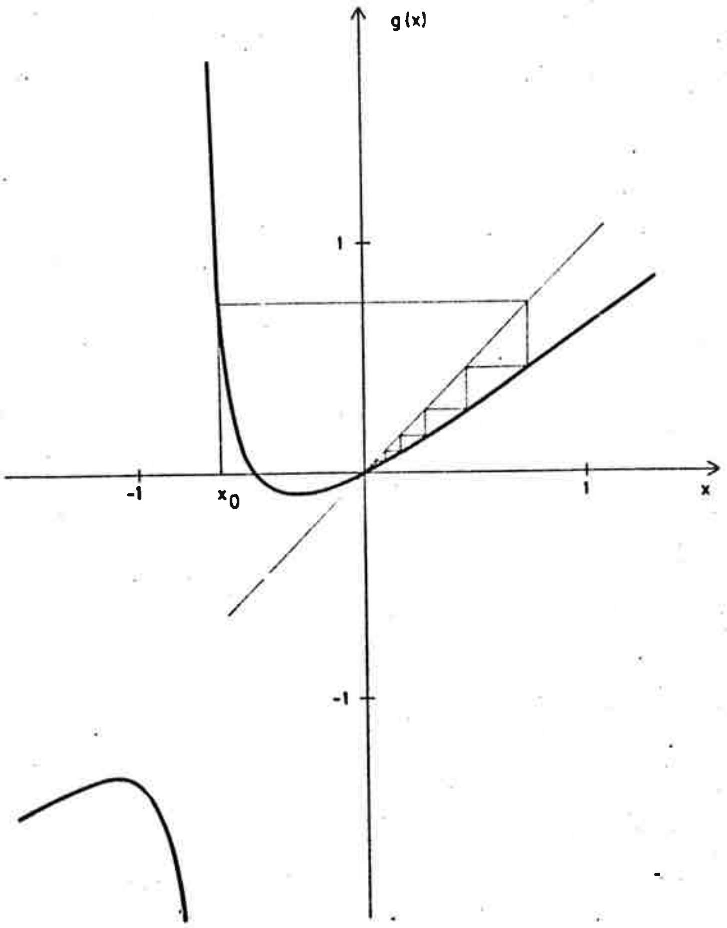


Fig 6.1 Principal shape of g(x) when $0 < b \leq 1$
(a=-0.5, b=0.5 and c=0.7)

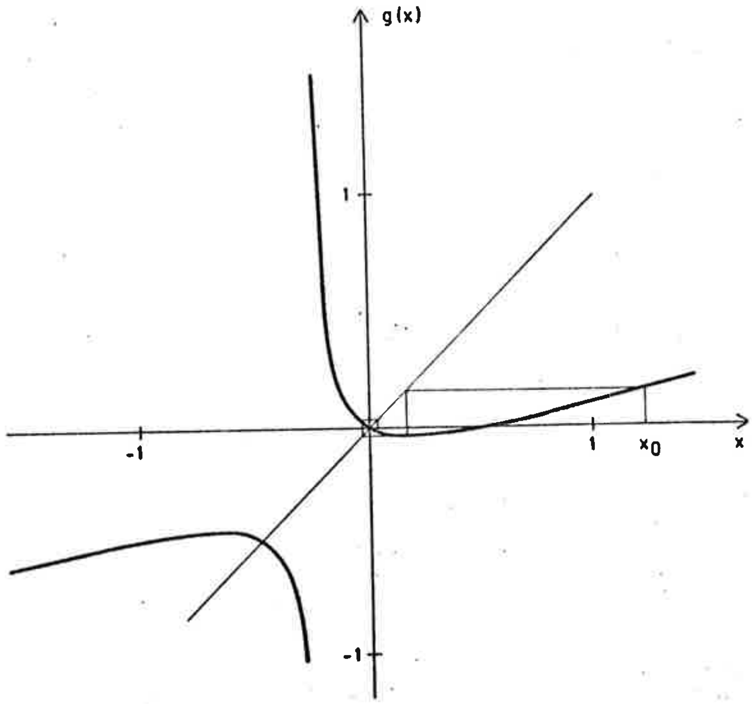


Fig 6.2 Principal shape of g(x) when $1 < b < 2$.
(a=-0.5,b=1.5 and c=0.7)

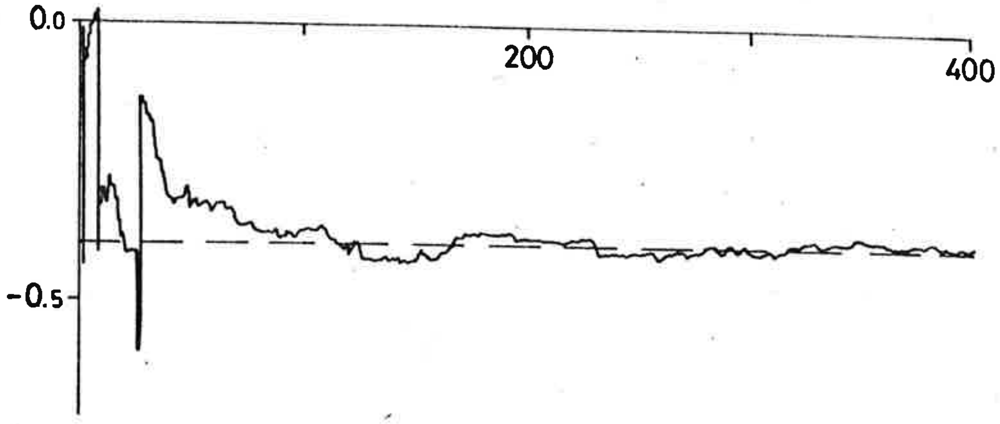


Fig 7.1 Parameter estimate α obtained when the self tuning algorithm based on the model (7.3) is applied to the system given by (7.1). The minimum variance regulator corresponds to α = -0.4 indicated by the dashed line

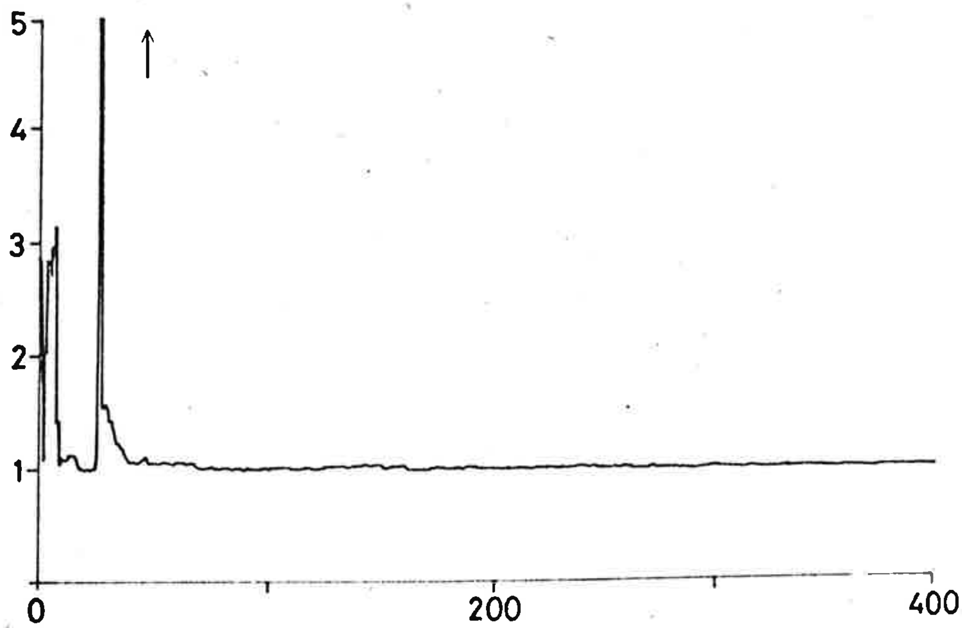


Fig 7.2 Expected variance of the output of Example 7.1 when the control law obtained at time t is kept constant for all future times. Notice that α(26) corresponds to an unstable system.

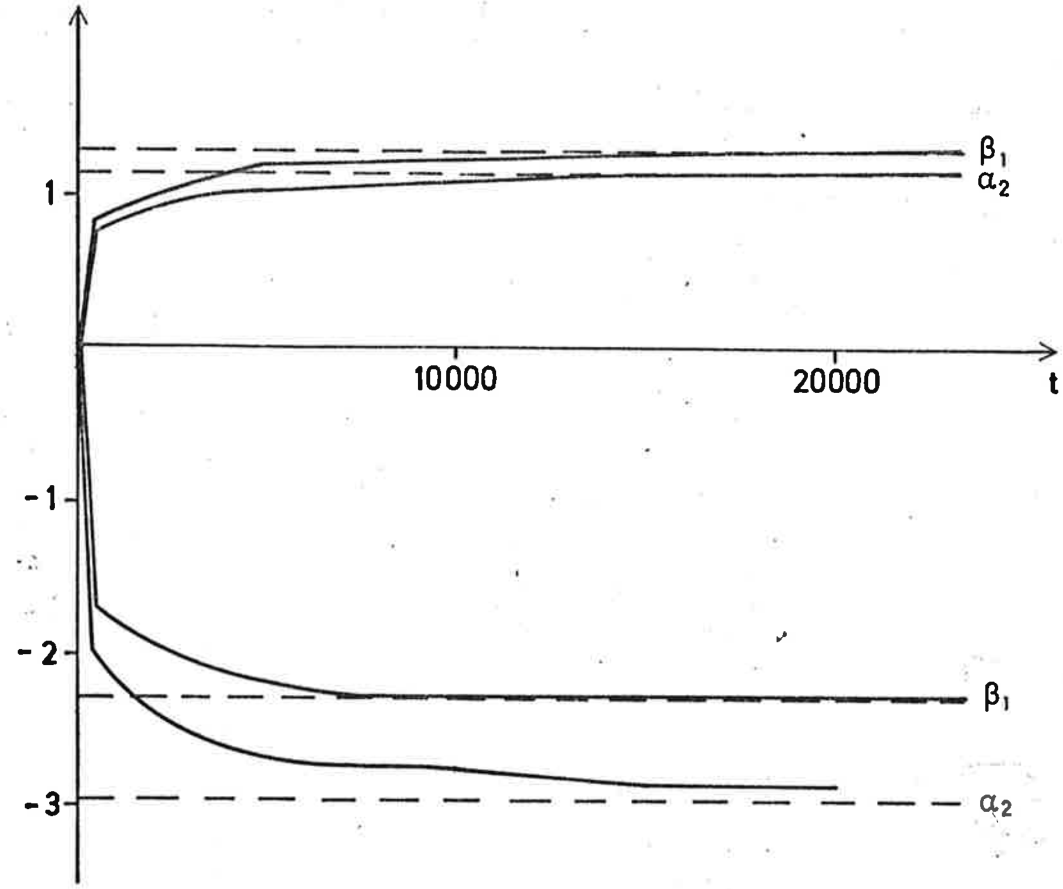


Fig 7.3 Parameter estimates $\alpha_1, \alpha_2, \beta_1$ and $\beta_2$ obtained when the self tuning algorithm based on (7.6) is applied to the system given by (7.4). The dashed lines correspond to the minimum variance strategy parameters.

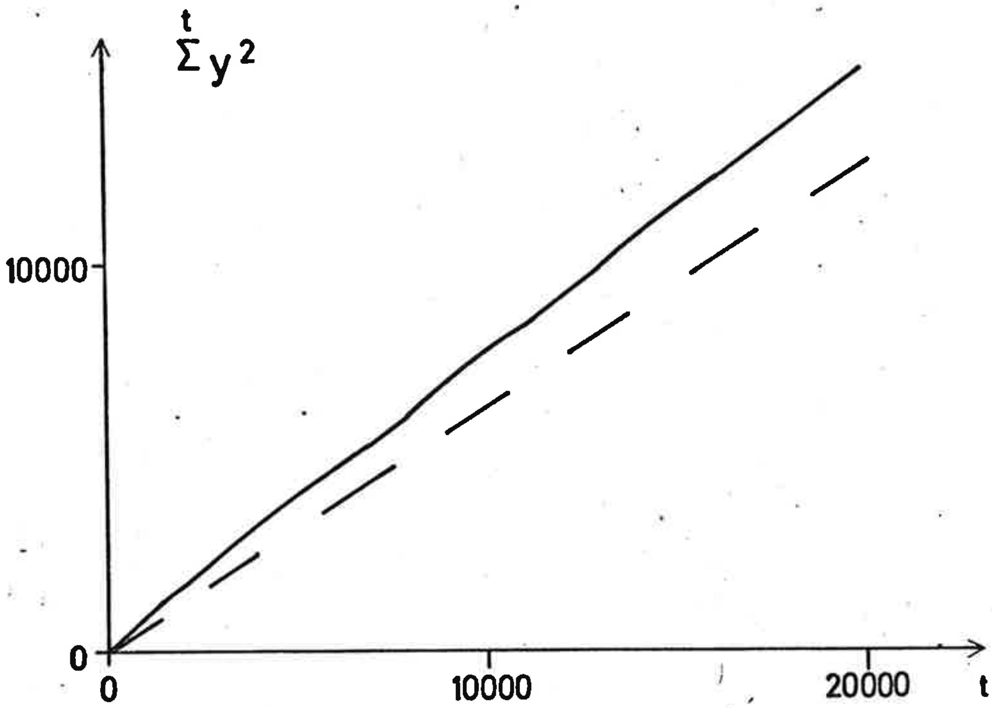


Fig 7.4 Actual loss $V(t) = \sum^{t} y^2(s)$ for a simulation of the system (7.4) with the self tuning algorithm. The dashed line shows expected loss for minimum variance control.

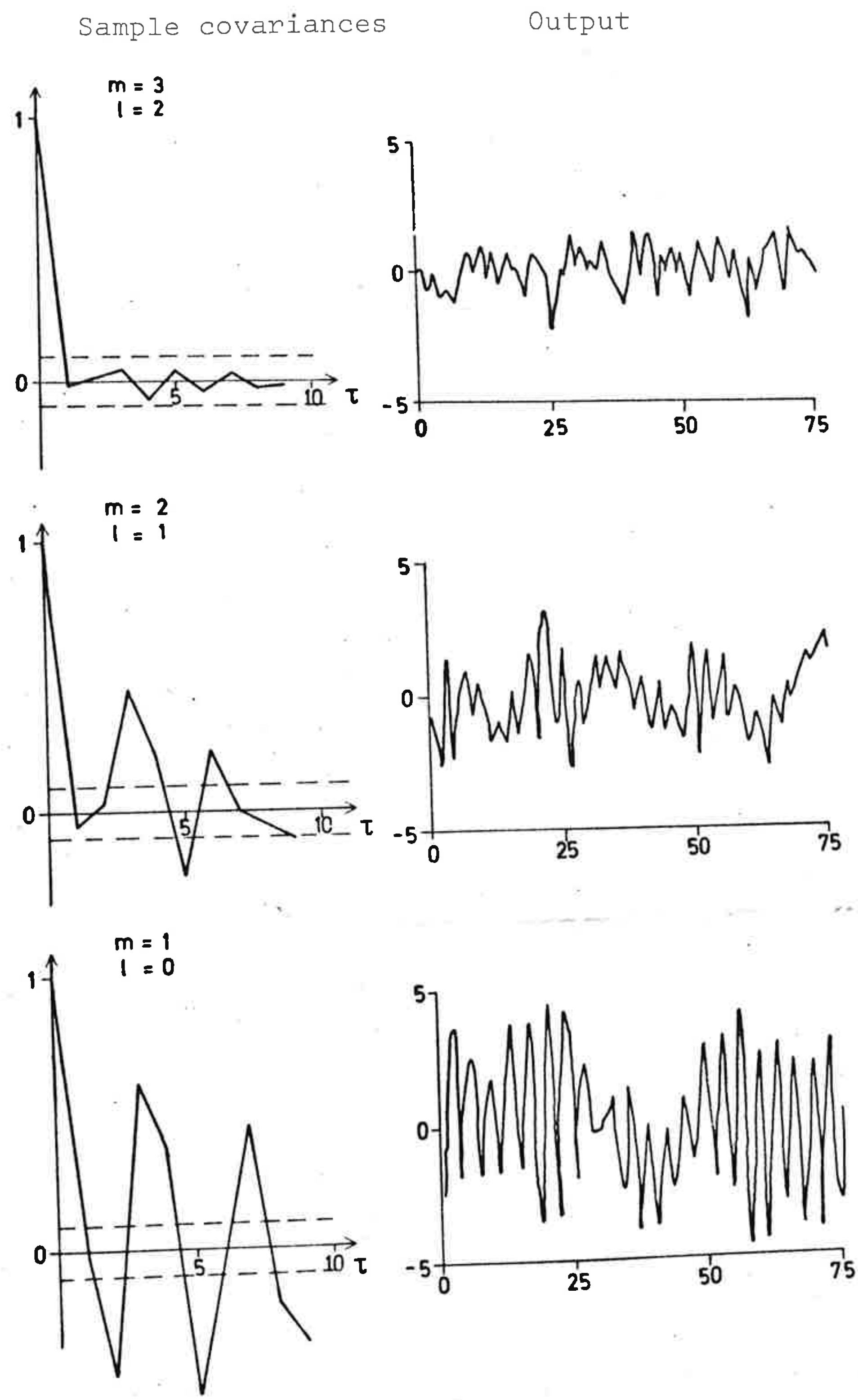


Fig 7.5 Sample covariances $\hat{r}_y(\tau)$ for the output of the system (7.8) when controlling with self tuning regulators of different order. The dashed lines show the 5% confidence intervals for $\tau \neq 0$. Small sections of the outputs are also shown.

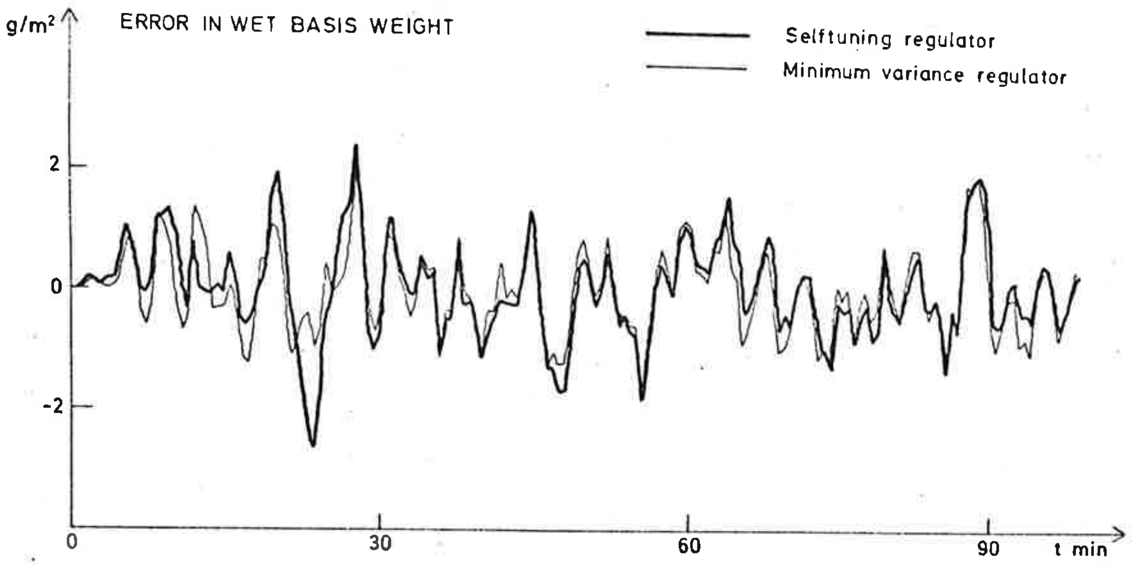


Fig 7.6 Error in simulated wet basis weight for the self tuning and the minimum variance regulator based on maximum likelihood identification

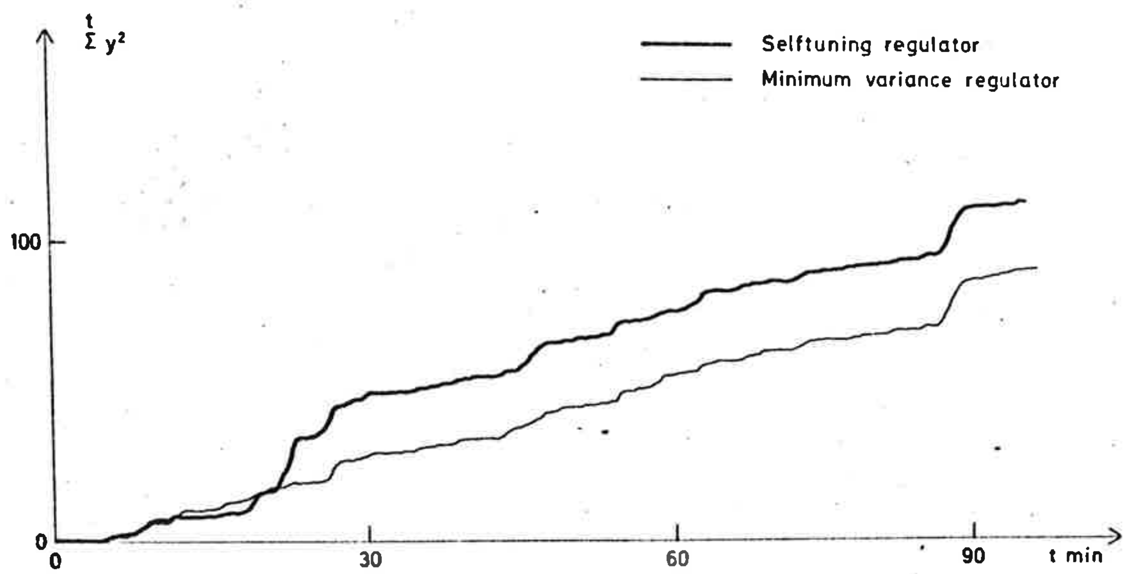


Fig 7.7 Accumulated loss for Example 7.4

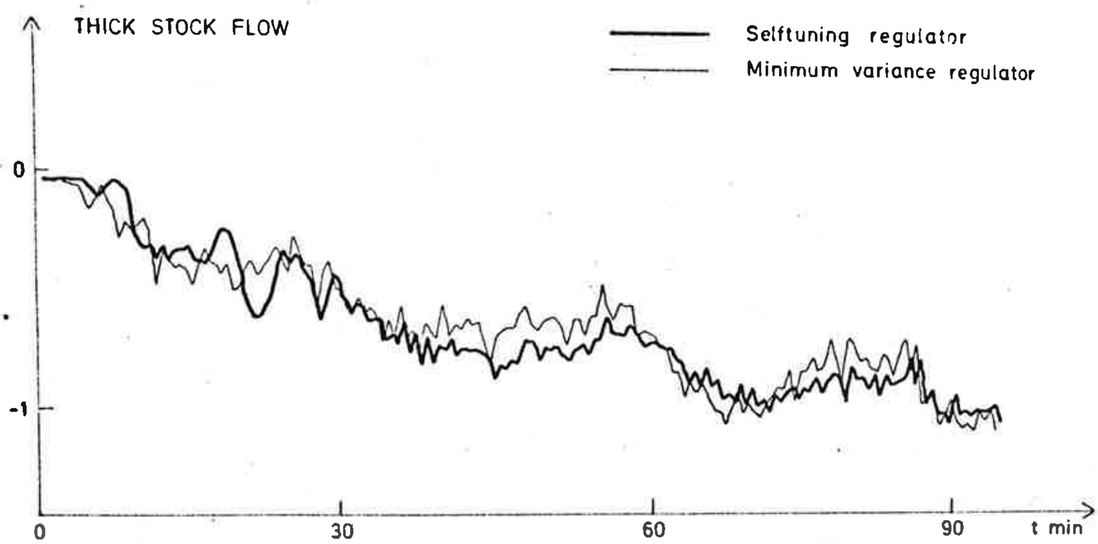


Fig 7.8 Control signal in normalized units for Example 7.4

*Automatica*, Vol. 9, pp. 185–199. Pergamon Press, 1973. Printed in Great Britain.

# On Self Tuning Regulators*

## Sur les Régulateurs Auto-Syntonisants

## Über selbsteinstellende Regler

## О самонастраивающихся регуляторах

K. J. ÅSTRÖM and B. WITTENMARK

*Control laws obtained by combining a least squares parameter estimator and a minimum variance strategy based on the estimated parameters have asymptotically optimal performance.*

**Summary**—The problem of controlling a system with constant but unknown parameters is considered. The analysis is restricted to discrete time single-input single-output systems. An algorithm obtained by combining a least squares estimator with a minimum variance regulator computed from the estimated model is analysed. The main results are two theorems which characterize the closed loop system obtained under the assumption that the parameter estimates converge. The first theorem states that certain covariances of the output and certain cross-covariances of the control variable and the output will vanish under weak assumptions on the system to be controlled. In the second theorem it is assumed that the system to be controlled is a general linear stochastic $n$th order system. It is shown that if the parameter estimates converge the control law obtained is in fact the minimum variance control law that could be computed if the parameters of the system were known. This is somewhat surprising since the least squares estimate is biased. Some practical implications of the results are discussed. In particular it is shown that the algorithm can be feasibly implemented on a small process computer.

### 1. INTRODUCTION

IT HAS been shown in several cases that linear stochastic control theory can be used successfully to design regulators for the steady state control of industrial processes. See Ref. [1]. To use this theory it is necessary to have mathematical models of the system dynamics and of the disturbances. In practice it is thus necessary to go through the steps of plant experiments, parameter estimation, computation of control strategies and implementation. This procedure can be quite time consuming in particular if the computations are made off-line. It might also be necessary to repeat the procedure if the system dynamics or the characteristics of the disturbances are changing as is often the case for industrial processes.

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From a practical point of view it is thus meaningful to consider the control of systems with constant but unknown parameters. Optimal control problems for such systems can be formulated and solved using non-linear stochastic control theory. The solutions obtained are extremely impractical since even very simple problems will require computations far exceeding the capabilities of todays computers. For systems with constant but unknown parameters it thus seems reasonable to look for strategies that will converge to the optimal strategies that could be derived if the system characteristics were known. Such algorithms will be called *self-tuning* or *self-adjusting* strategies. The word adaptive is not used since adaptive, although never rigorously defined, usually implies that the characteristics of the process are changing. The problem to be discussed is thus simpler than the adaptive problem in the sense that the system to be controlled is assumed to have constant parameters.

The purpose of the paper is to analyse one class of self-adjusting regulators. The analysis is restricted to single-input single-output systems. It is assumed that the disturbances can be characterized as filtered white noise. The criterion considered is the minimization of the variance of the output. The algorithms analysed are those obtained on the basis of a separation of identification and control. To obtain a simple algorithm the identification is simply a least squares parameter estimator.

The main result is a characterization of the closed loop systems obtained when the algorithm is applied to a general class of systems. It is shown in Theorem 5.1 that if the parameter estimates converge the closed loop loop system obtained will be such that certain covariances of the inputs and the outputs of the closed loop system are zero. This is shown under weak assumptions on the system to be controlled. Moreover if it is assumed that the system to be controlled is a sampled finite dimensional linear stochastic system with a time delay in the control signal it is demonstrated in Theorem 5.2 that, if the parameter estimates converge, the corresponding regulator will actually converge to the minimum variance regulator. This is true, in spite of the fact that the least squares estimate is biased.

The major assumptions are that the system is minimum phase, that the time delay is known and that a bound can be given to the order of the system. The first two assumptions can be removed at the price of a more complicated algorithm.

The paper is organized as follows: sections 2 and 3 provide background and a motivation. The algorithm is given in section 4. Control strategies for systems with known parameters are given in section 2. Least squares parameter estimation is briefly reviewed in section 3. Some aspects on the notion of identifiability is also given in section 3. The algorithm presented in section 4 is obtained simply by fitting the parameters of a least squares structure as was described in section 3 and computing the corresponding minimum variance control strategy as was described in section 2. The possible difficulty with non-identifiability due to the feedback is avoided by fixing one parameter.

The main result is given as two theorems in section 5. We have not yet been able to prove that the algorithm converges in general. In section 6 it is, however, shown that a modified version of the algorithm converges for a first order system. The convergence properties of the algorithm are further illustrated by the examples in section 7. Some practical aspects of the algorithm as well as some problems which remain to be solved are given in section 8. In particular it is shown that the algorithm is easily implemented on a minicomputer.

## 2. MINIMUM VARIANCE CONTROL

This section gives the minimum variance strategy for a system with known parameters. Consider a system described by

$$y(t)+a_1y(t-1)+\ldots+a_ny(t-n)=b_1u(t-k-1)$$
$$+\ldots+b_nu(t-k-n)+\lambda[e(t)+c_1e(t-1)$$
$$+\ldots+c_ne(t-n)],$$
$$t=0,\ \pm 1,\ \pm 2,\ldots \qquad (2.1)$$

where $u$ is the control variable, $y$ is the output and $\{e(t),\ t=0, \pm 1, \pm 2, \ldots\}$ is a sequence of independent normal (0, 1) random variables. If the forward shift operator $q$, defined by

$$qy(t)=y(t+1)$$

and the polynomials

$$A(z)=z^n+a_1z^{n-1}+\ldots+a_n$$

$$B(z)=b_1z^{n-1}+\ldots+b_n,\ b_1 \neq 0$$

$$C(z)=z^n+c_1z^{n-1}+\ldots+c_n$$

are introduced, the equation (2.1) describing the system can be written in the following compact form:

$$A(q)y(t)=B(q)u(t-k)+\lambda C(q)e(t). \qquad (2.2)$$

It is well known that (2.1) or (2.2) is a canonical representation of a sampled finite dimensional single-input single-output dynamical system with time delays in the output and disturbances that are gaussian random processes with rational spectral densities.

The model (2.1) also admits a time delay $\tau$ in the system input which need not be a multiple of the sampling interval. The number $k$ corresponds to the integral part of $\tau/h$, where $h$ is the sampling interval.

Let the criterion be

$$V_1=Ey^2(t) \qquad (2.3)$$

or

$$V_2=E\frac{1}{N}\sum_1^N y^2(t). \qquad (2.4)$$

The optimal strategy is then

$$u(t)=-\frac{q^kG(q)}{B(q)F(q)}y(t) \qquad (2.5)$$

where $F$ and $G$ are polynomials

$$F(z)=z^k+f_1z^{k-1}+\ldots+f_k \qquad (2.6)$$

$$G(z)=g_0z^{n-1}+g_1z^{n-2}+\ldots+g_{n-1} \qquad (2.7)$$

determined from the identity

$$q^kC(q)=A(q)F(q)+G(q). \qquad (2.8)$$

Proofs of these statements are given in [2]. The following conditions are necessary:

- —The polynomial $B$ has all zeroes inside the unit circle. Thus the system (2.1) is minimum phase.
- —The polynomial $C$ has all zeroes inside the unit circle.

These conditions are discussed at length in [1]. Let it suffice to mention here that if the system (2.1) is non-minimum phase the control strategy (2.5) will still be a minimum variance strategy. This strategy will, however, be so sensitive that the slightest variation in the parameters will create an unstable closed loop system. Suboptimal strategies which are less sensitive to parameter variations are also well known. This paper will, however, be limited to minimum phase systems.

## 3. PARAMETER ESTIMATION

For a system described by (2.1) it is thus straight forward to obtain the minimum variance regulator, if the parameters of the model are known. If the parameters are not known it might be a possibility to try to determine the parameters of (2.1) using some identification scheme and then use the control law (2.5) with the true parameters substituted by their estimates. A suitable identification algorithm is the maximum likelihood method which will give unbiased estimates of the coefficients of the $A$, $B$ and $C$ polynomials. The maximum likelihood estimates of the parameters of (2.1) are, however, strongly non-linear functions of the inputs and the outputs. Since finite dimensional sufficient statistics are not known it is not possible to compute the maximum likelihood estimate of the parameters of (2.1) recursively as the process develops. Simpler identification schemes are therefore considered.

*The least squares structure*

The problem of determining the parameters of the model (2.1) is significantly simplified if it is assumed that $c_i=0$ for $i=1, 2, \ldots, n$. The model is then given by

$$A(q)y(t)=B(q)u(t-k)+\lambda e(t+n). \tag{3.1}$$

The parameters of this model can be determined simply by the least squares method [3]. The model (3.1) is therefore referred to as a *least squares model.*

The least squares estimate has several attractive properties. It can easily be evaluated recursively. The estimator can be modified to take different model structures, e.g. known parameters, into account. The least squares estimates will converge to the true parameters e.g. under the following conditions.

- —The output $\{y(t)\}$ is actually generated from a model (3.1).
- —The residuals $\{e(t)\}$ are independent.
- —The input is persistently exciting, see Ref. [3].
- —The input sequence $\{u(t)\}$ is independent of the disturbance sequence $\{e(t)\}$.

These conditions are important. If the residuals are correlated the least squares estimate will be biased. If the input sequence $\{u(t)\}$ depends on $\{e(t)\}$ it may not be possible to determine all parameters.

When the inputs are generated by a feedback they are correlated with the disturbances and it is not obvious that all the parameters of the model can be determined. Neither is it obvious that the input generated in this way is persistently exciting of sufficiently high order. A simple example illustrates the point.

*Example* 3.1

Consider the first order model

$$y(t)+ay(t-1)=bu(t-1)+e(t). \tag{3.2}$$

Assume that a linear regulator with constant gain

$$u(t)=\alpha y(t) \tag{3.3}$$

is used. If the parameters $a$ and $b$ are known the gain $\alpha=a/b$ would obviously correspond to a minimum variance regulator. If the parameters are not known the gain $\alpha=\hat{a}/\hat{b}$ where $\hat{a}$ and $\hat{b}$ are the least squares estimates of $a$ and $b$ could be attempted. The least squares parameter estimates are determined in such a way that the loss function

$$V(a, b)=\sum_{1}^{N}[y(t+1)+ay(t)-bu(t)]^2 \tag{3.4}$$

is minimal with respect to $a$ and $b$. If the feedback control (3.3) is used the inputs and outputs are linearly related through

$$u(t)-\alpha y(t)=0. \tag{3.5}$$

Multiply (3.5) by $-\gamma$ and add to the expression within brackets in (3.4).

Hence

$$V(a, b)=\sum_{1}^{N}[y(t+1)+(a+\alpha\gamma)y(t)-(b+\gamma)u(t)]^2$$

$$=V(a+\alpha\gamma,\ b+\gamma).$$

The loss function will thus have the same value for all estimates $a$ and $b$ on a linear manifold. Thus the two parameters $a$ and $b$ of the model (3.2) are not identifiable when the feedback (3.3) is used.

The simple example shows that it is in general not possible to estimate all the parameters of the model (3.1) when the input is generated by a feedback. Notice, however, that all parameters can be estimated if the control law is changed. In the particular example it is possible to estimate both parameters of the model, if the control law (3.3) is replaced by

$$u(t)=\alpha y(t-1)$$

or

$$u(t)=\alpha_1 y(t)+\alpha_2 y(t-1)$$

or if a time varying gain is used.

## 4. THE ALGORITHM

In order to control a system with constant but unknown parameters the following procedure could be attempted. At each step of time determine the parameters of the system (3.1) using least squares estimation based on all previously observed inputs and outputs as was described in section 3. Then determine a control law by calculating the minimum variance strategy for the model obtained. To compute the control law the identity (2.8) must be resolved in each step. The problem of computing the minimum variance regulator is simplified if it is observed that by using the identity (2.8) the system (3.1) can be written as

$$\begin{aligned} y(t+k+1)+\alpha_1 y(t)+ \dots +\alpha_m(t-m+1) \\ =\beta_0[u(t)+\beta_1 u(t-1)+ \dots +\beta_l u(t-l)] \\ +\varepsilon(t+k+1) \end{aligned} \tag{4.1}$$

where $m=n$ and $l=n+k-1$. The coefficients $\alpha_i$ and $\beta_i$ are computed from the parameters $a_i$ and $b_i$ in (3.1) using the identity (2.8). The disturbance $\varepsilon(t)$ is a moving average of order $k$ of the driving noise $e(t)$.

The minimum variance strategy is then simply

$$\begin{aligned} u(t)=\frac{1}{\beta_0}[\alpha_1 y(t)+ \dots +\alpha_m y(t-m+1)] \\ -\beta_1 u(t-1)- \dots -\beta_l u(t-l). \end{aligned} \tag{4.2}$$

In order to obtain simple computation of the control strategy it could thus be attempted to use the model structure (4.1) which also admits least squares estimation. The trade-off for the simple calculation of the control law is that $k$ more parameters have to be estimated.

As was shown in Example 3.1 all parameters of the model (4.1) can not necessarily be determined from input–output observations if the input is generated by a feedback (4.2) with constant parameters. In order to avoid a possible difficulty it is therefore assumed that the parameter $\beta_0$ is given. It will be shown in section 6 that the choice of $\beta_0$ is not crucial.

Summing up, the algorithm can be described as follows.

*Step* 1 *parameter estimation*

At the sampling interval $t$ determine the parameters $\alpha_1, \dots \alpha_m, \beta_1, \dots, \beta_l$ of the model

$$\begin{aligned} y(t)+\alpha_1 y(t-k-1)+ \dots +\alpha_m y(t-k-m) \\ =\beta_0[u(t-k-1)+\beta_1 u(t-k-2) \\ + \dots +\beta_l u(t-k-l-1)]+\varepsilon(t) \end{aligned} \tag{4.1}$$

using least squares estimation based on all data available at time $t$, i.e.

$$\sum_{k=0}^{t} \varepsilon^2(k)$$

minimum. The parameter $\beta_0$ is assumed known.

*Step* 2 *control*

At each sampling interval determine the control variable from

$$\begin{aligned} u(t)=\frac{1}{\beta_0}[\alpha_1 y(t)+ \dots +\alpha_m y(t-m+1)] \\ -\beta_1 u(t-1)- \dots -\beta_l u(t-l) \end{aligned} \tag{4.2}$$

where the parameters $\alpha_i$ and $\beta_i$ are those obtained in Step 1.

The control law (4.2) corresponds to

$$\begin{aligned} u(t)&=\frac{\alpha_1+\alpha_2 q^{-1}+ \dots +\alpha_m q^{-m+1}}{\beta_0[1+\beta_1 q^{-1}+ \dots +\beta_l q^{-l}]}y(t) \\ &=\frac{q^{l-m+1}\mathscr{A}(q)}{\mathscr{B}(q)}y(t). \end{aligned} \tag{4.3}$$

Since the least squares estimate can be computed recursively the algorithm requires only moderate computations.

It should be emphasized that the algorithm is not optimal in the sense that it minimizes the criterion (2.3), or the criterion (2.4). It fails to minimize (2.3) because it is not taken into account that the parameter estimates are inaccurate and it fails to minimize (2.4) because it is not dual in FELDBAUM's sense [4]. These matters are discussed in [2]. It will however, be shown in section 5 that the algorithm has nice asymptotic properties.

The idea of obtaining algorithms by a combination of least squares identification and control is old. An early reference is KALMAN [5]. The particular algorithm used here is in essence the same as the one presented by PETERKA [6]. A similar algorithm where the uncertainties of the parameters are also considered is given in WIESLANDER–WITTENMARK [7].

## 5. MAIN RESULTS

The properties of the algorithm given in the previous section will now be analysed. We have

*Theorem* 5.1

Assume that the parameter estimates $\alpha_i(t)$, $i=1,\ldots,m$, $\beta_i(t)$, $i=1,\ldots l$ converge as $t\to\infty$ and that the closed loop system is such that the output is *ergodic* (in the second moments). Then the closed loop system has the properties

$$Ey(t+\tau)y(t)=r_y(\tau)=0 \qquad \tau=k+1,\ldots k+m \tag{5.1}$$

$$Ey(t+\tau)u(t)=r_{yu}(\tau)=0 \qquad \tau=k+1,\ldots,k+l+1. \tag{5.2}$$

*Proof*

The least squares estimates of the parameters $\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_l$ is given by the equations

$$\left[\begin{array}{cccc|ccc}
\Sigma y(t)^2 & \Sigma y(t)y(t-1) & \ldots & \Sigma y(t)y(t-m+1) & -\beta_0\Sigma y(t)u & \ldots & -\beta_0\Sigma y(t)u(t-l)\\
\Sigma y(t)y(t-1) & & & \Sigma y(t-1)y(t-m+1) & \cdot & & \cdot\\
\cdot & & & \cdot & \cdot & & \cdot\\
\cdot & & & \cdot & \cdot & & \cdot\\
\cdot & & & \cdot & & & \\
\Sigma y(t)y(t-m+1) & & & \Sigma y^2(t-m+1) & -\beta_0\Sigma y(t-m+1)u(t-1) & \ldots & -\beta_0\Sigma y(t-m+1)u(t-l)\\
\hline
-\beta_0\Sigma y(t)u(t-1) & & & & \beta_0^2\Sigma u^2(t-1) & \ldots & \beta_0^2\Sigma u(t-1)u(t-l)\\
\cdot & & & & & & \cdot\\
\cdot & & & & & & \cdot\\
\cdot & & & & & & \cdot\\
-\beta_0\Sigma y(t)u(t-l) & \ldots & & & \ldots & & \beta_0^2\Sigma u^2(t-l)
\end{array}\right]\cdot$$

$$\begin{bmatrix}\alpha_1\\ \alpha_2\\ \cdot\\ \cdot\\ \cdot\\ \alpha_m\\ \beta_1\\ \cdot\\ \cdot\\ \cdot\\ \beta_l\end{bmatrix}=\begin{bmatrix}-\Sigma y(t+k+1)y(t)+\beta_0\Sigma u(t)y(t)\\ \Sigma y(t+k+1)y(t-1)+\beta_0\Sigma u(t)y(t-1)\\ \cdot\\ \cdot\\ \cdot\\ -\Sigma y(t+k+1)y(t-m+1)+\beta_0\Sigma u(t)y(t-m+1)\\ -\beta_0\Sigma y(t+k+1)u(t-1)-\beta_0^2\Sigma u(t)u(t-1)\\ \cdot\\ \cdot\\ \cdot\\ \beta_0\Sigma y(t+k+1)u(t-l)-\beta_0^2\Sigma u(t)u(t-l)\end{bmatrix} \tag{5.3}$$

where the sums are taken over $N_0$ values. See Ref. [3].

Assume that the parameters converge. For sufficiently large $N_0$ the coefficients of control law (4.2) will then converge to constant values. Introduction of (4.2) into (5.3) gives

$$\begin{aligned}
&\Sigma y(t+k+1)y(t)=0\\
&\Sigma y(t+k+1)y(t-1)=0\\
&\cdot\\
&\cdot\\
&\cdot\\
&\Sigma y(t+k+1)y(t-m+1)=0\\
&\Sigma y(t+k+1)u(t-1)=0\\
&\cdot\\
&\cdot\\
&\cdot\\
&\Sigma y(t+k+1)u(t-l)=0.
\end{aligned}$$

Using the control law (4.2) it also follows that

$$\Sigma y(t+k+1)u(t)=0.$$

Under the ergodicity assumption the sums can furthermore be replaced by mathematical expectations and the theorem is proven.

*Remark* 1

Notice that the assumptions on the system to be controlled are very weak. In particular it is not necessary to assume that the system is governed by an equation like (2.1) or (3.1).

*Remark* 2

It is sufficient for ergodicity that the system to be controlled is governed by a difference equation of finite order, e.g. like (2.1), and that the closed loop system obtained by introducing the feedback law into (2.1) gives a stable closed loop system.

*Remark* 3

The self tuning algorithm can be compared with a PI-regulator. If the state variables of a deterministic system with a PI-regulator converge to steady state values, the control error must be zero irrespective of the properties of the system. Analogously theorem 5.1 implies that if the parameter estimates of the self tuning algorithm converge the covariances (5.1) and (5.2) are zero.

*Remark* 4

Theorem 5.1 holds even if the algorithm is modified in such a way that the parameter estimation (Step 1) is not done in every step.

If it is assumed that the system to be controlled is governed by an equation like (2.1) it is possible to show that the conditions (5.1) and (5.2) in essence imply that the self tuning regulator will converge to a minimum variance regulator. We have

*Theorem* 5.2

Let the system to be controlled be governed by the equation (2.1). Assume that the self tuning algorithm is used with $m=n$ and $l=n+k-1$. If the parameter estimates converge to values such that the corresponding polynomials $\mathscr{A}$ and $\mathscr{B}$ have no common factors, then the corresponding regulator (4.2) will converge to a minimum variance regulator.

*Proof*

Assume that the least squares parameter estimates converge. The regulator is then given by (4.3) i.e.

$$u(t)=\frac{q^{l-m+1}\mathscr{A}(q)}{\mathscr{B}(q)}y(t)=\frac{q^{k}\mathscr{A}(q)}{\mathscr{B}(q)}$$

where the coefficients of $\mathscr{A}$ and $\mathscr{B}$ are constant. Since the system to be controlled is governed by (2.1) the closed loop system becomes

$$[A(q)\mathscr{B}(q)-B(q)\mathscr{A}(q)]y(t)=\lambda C(q)\mathscr{B}(q)e(t). \tag{5.4}$$

The closed loop system is of order $r=n+l$. Introduce the stochastic process $\{v(t)\}$ defined by

$$v(t)=\lambda\frac{q^{l}C(q)}{A(q)\mathscr{B}(q)-B(q)\mathscr{A}(q)}e(t). \tag{5.5}$$

Then

$$y(t)=q^{-l}\mathscr{B}(q)v(t) \tag{5.6}$$

and

$$u(t)=q^{-m+1}\mathscr{A}(q)v(t). \tag{5.7}$$

Multiplying (5.6) and (5.7) by $y(t+\tau)$ and taking mathematical expectations gives

$$r_{y}(\tau)=r_{yv}(\tau)+\beta_1 r_{yv}(\tau+1)\ldots+\beta_l r_{yv}(\tau+l) \tag{5.8}$$

$$r_{yu}(\tau)=\alpha_1 r_{yv}(\tau)+\alpha_2 r_{yv}(\tau+1)+\ldots+\alpha_m r_{yv}(\tau+m-1). \tag{5.9}$$

Furthermore it follows from Theorem 5.1, equations (5.1) and (5.2), that the left member of (5.8) vanishes for $\tau=k+1,\ldots,k+m$ and that the left member of (5.9) vanishes for $\tau=k+1,\ldots,k+l+1$. We thus obtain the following equation for $r_{yv}(\tau)$.

$$\begin{bmatrix} 1 & \beta_1 & \ldots & & \beta_l & 0 & \ldots & 0 \\ 0 & 1 & \beta_1 & & & \cdot & & \cdot \\ \cdot & & \cdot & & & \cdot & & \cdot \\ \cdot & & & \cdot & & & \cdot & \cdot \\ \cdot & & & & \cdot & & & 0 \\ 0 & \ldots & 0 & 1 & \beta_1 & & & \beta_l \\ 0 & \alpha_1 & \ldots & & \alpha_m & 0 & \ldots & 0 \\ \cdot & & \cdot & & & \cdot & & \cdot \\ \cdot & & & \cdot & & & \cdot & \cdot \\ \cdot & & & & \cdot & & & \cdot \\ & & & & & & & 0 \\ 0 & \ldots & & & \alpha_1 & \ldots & & \alpha_m \end{bmatrix} \begin{bmatrix} r_{yv}(k+1) \\ r_{yv}(k+2) \\ \cdot \\ \cdot \\ \cdot \\ \\ \\ \\ \\ \\ \\ r_{yv}(k+l+m) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \\ \\ \\ \\ \\ \\ 0 \end{bmatrix}. \tag{5.10}$$

Since the polynomials $\mathscr{A}$ and $\mathscr{B}$ have no common factor it follows from an elementary result in the theory of equations [8, p. 145] that the $(l+m) \times (l+m)$-matrix of the left member of (5.10) is nonsingular.

Hence

$$r_{yv}(\tau)=0 \qquad \tau=k+1, \ldots, k+l+m. \qquad (5.11)$$

Since $v$ is the output of a dynamical system of order $r=n+l=m+l$ driven by white noise it follows from (5.11) and the Yule–Walker equation that

$$r_{yv}(\tau)=0 \qquad \tau \geqslant k+1. \qquad (5.12)$$

The equation (5.8) then implies

$$r_{y}(\tau)=0 \qquad \tau \geqslant k+1. \qquad (5.13)$$

The output of the closed loop system is thus a moving average of white noise of order $k$. Denote this moving average by

$$y(t)=q^{-k}F(q)e(t) \qquad (5.14)$$

where $F$ is a polynomial of degree $k$. It follows from (5.4) and (5.14) that

$$q^k C = FA - \frac{BF\mathscr{A}}{\mathscr{B}}.$$

Since $q^k C$ and $FA$ are polynomials $BF\mathscr{A}/\mathscr{B}$ must also be a polynomial.

Hence

$$q^k C = FA + G \qquad (5.15)$$

where

$$G = -\frac{BF\mathscr{A}}{\mathscr{B}} \qquad (5.16)$$

is of degree $n-1$ and $F$ of degree $k$. A comparison with (2.8) shows, however, that (5.15) is the identity which is used to derive the minimum variance strategy. It thus follows from (2.5) that the minimum variance strategy is given by

$$u(t) = -\frac{q^k G}{BF} y(t).$$

The equation (5.16) then implies that

$$-\frac{q^k G}{BF} = \frac{q^k \mathscr{A}}{\mathscr{B}} \qquad (5.17)$$

and it has thus been shown that (4.2) is a minimum variance strategy.

*Remark*

The conditions $m=n$ and $l=n+k-1$ mean that there are precisely the number of parameters in the model that are required in order to obtain the correct regulator for the process (2.1). Theorem 5.2 still holds if there are more parameters in the regulator in the following cases.

(i) Theorem 5.2 still holds if $m=n$ and $l \geqslant n+k-1$. In this case the order of the system is $r=n+l$ and since $m=n$ the equation (5.10) implies (5.11)–(5.13) and the equation (5.16) is changed to

$$G = -\frac{q^{l-k-m+1} BF\mathscr{A}}{\mathscr{B}}. \qquad (5.16')$$

The rest of the proof remains unchanged.

(ii) If $m \geqslant n$ and $l=n+k-1$ the theorem will also hold. The closed loop system is of order $r \leq m+l$ but the equation (5.10) will still imply (5.11) and (5.16) is changed to (5.16′). The rest of the proof remains unchanged.

(iii) If $m>n$ and $l>n+k-1$ the theorem does not hold because $\mathscr{A}$ and $\mathscr{B}$ must have a common factor if the parameter estimates converge. It can, however, be shown that if the algorithm is modified in such a way that common factors of $\mathscr{A}$ and $\mathscr{B}$ are eliminated before the control signal is computed, Theorem 5.2 will still hold for the modified algorithm.

## 6. CONVERGENCE OF THE ALGORITHM

It would be highly desirable to have general results giving conditions for convergence of the parameter estimates. Since the system (2.1) with the regulator (4.2) and the least squares estimator is described by a set of nonlinear time dependent stochastic difference equations the problem of a general convergence proof is difficult. So far we have not been able to obtain a general result. It has, however, been verified by extensive numerical simulations that the algorithm does in fact converge in many cases. The numerical simulations as well as analysis of simple examples have given insight into some of the conditions that must be imposed in order to ensure that the algorithm will converge.

A significant simplification of the analysis is obtained if the algorithm is modified in such a way that the parameter estimates are kept constant over long periods of time. To be specific a simple example is considered.

*Example* 6.1

Let the system be described by

$$y(t)+ay(t-1)=bu(t-1)+e(t)+ce(t-1) \qquad |c|<1. \qquad (6.1)$$

Assume that the control law

$$u(t)=\alpha_n y(t) \tag{6.2}$$

is used in the time interval $t_n<t<t_{n+1}$ where the parameter $\alpha_n$ is determined by fitting the least squares model

$$y(t+1)+\alpha y(t)=u(t)+\varepsilon(t+1) \tag{6.3}$$

to the data $\{u(t), y(t), t=t_{n-1}, \ldots, t_n-1)\}$.

The least squares estimate is given by

$$\alpha_n=-\frac{\sum_{t=t_{n-1}}^{t_n-2} y(t)[y(t+1)-u(t)]}{\sum_{t=t_{n-1}}^{t_n-2} y^2(t)}$$

$$=\alpha_{n-1}-\frac{\sum_{t=t_{n-1}}^{t_n-2} y(t+1)y(t)}{\sum_{t=t_{n-1}}^{t_n-2} y^2(t)} \tag{6.4}$$

where the last equality follows from (6.2). Assume that $t_n-t_{n-1}\to\infty$ and that

$$|a-b\alpha_{n-1}|<1 \tag{6.5}$$

which means that the closed loop system used during the time interval $t_{n-1}<t<t_n$ is stable then

$$\alpha_n=\alpha_{n-1}-\frac{r_y(1)}{r_y(0)} \tag{6.6}$$

where $r_y(\tau)$ is the covariance function of the stochastic process $\{y(t)\}$ defined by

$$y(t)+(a-b\alpha_{n-1})y(t-1)=e(t)+ce(t-1). \tag{6.7}$$

Straightforward algebraic manipulations now give

$$\alpha_n=\alpha_{n-1}-\frac{(c-a+b\alpha_{n-1})(1-ac+bc\alpha_{n-1})}{1+c^2-2ac+2bc\alpha_{n-1}}. \tag{6.8}$$

The problem is thus reduced to the analysis of the nonlinear differerence equation given by (6.8).

Introduce

$$x_n=\alpha_n-\frac{a-c}{b} \tag{6.9}$$

the equation (6.8) then becomes

$$x_{n+1}=g(x_n)=(1-b)x_n+\frac{b^2cx_n^2}{1-c^2+2bcx_n}. \tag{6.10}$$

The point $x=0$ is a fixed point of the mapping $g$ which corresponds to the optimal value of gain of the feedback loop, i.e. $\alpha=(a-c)/b$. The problem is thus to determine if this fixed point is stable. Since the closed loop system is assumed to be stable it is sufficient to consider

$$\frac{c-1}{b}<x<\frac{c+1}{b}. \tag{6.11}$$

Three cases have to be investigated

1. $c=0$
2. $c>0$ or $c<0$, $0<b\leqslant 1$
3. $c>0$ or $c<0$, $1<b<2$.

For all cases $g(x)\approx(1-b)x$ if $x$ is small. This implies that solutions close to 0 converge to $x=0$ if $0<b<2$.

*Case* 1

The equation (6.10) reduces to

$$x_{n+1}=(1-b)x_n$$

and the fixed point $x=0$ is stable if $|1-b|<1$ i.e. $0<b<2$.

*Case* 2

The pincipal behavior of $g(x)$ when $c>0$ and $0<b\leqslant 1$ is shown in Fig. 1. It is straightforward to verify that all initial values in the stability region $(c-1)/b<x<(c+1)/b$ will give solutions which converge to zero.

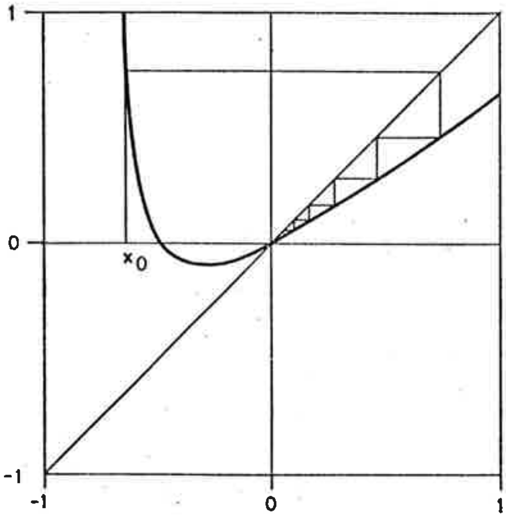


FIG. 1. Graph of the function $g$ when $0<b\leqslant 1$. The figure is drawn with the parameter values $a=-0{\cdot}5$, $b=0{\cdot}5$ and $c=0{\cdot}7$.

*Case* 3

The function $g(x)$ for this case is shown in Fig. 2. It is not obvious that $x$ will converge to zero, because there might exist a "limit cycle".

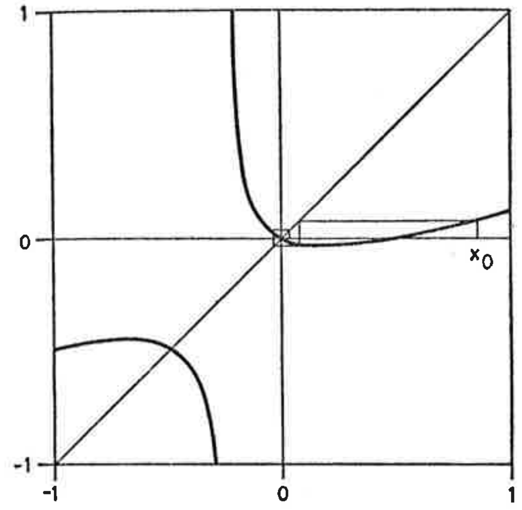


FIG. 2. Graph of the function $g$ when $1<b<2$. The figure is drawn with the parameter values $a=-0{\cdot}5$, $b=1{\cdot}5$ and $c=0{\cdot}7$.

If $c>0$ and starting with $x_o>0$ then if $g(x_o)<0$ it can be shown that after two iterations the new value $x_2$ will satisfy $0<x_2<x_o$ i.e.

$$0<g(g(x))<x \qquad \text{if } x>0,\, g(x)<0.$$

If $(c-1)/b<x<0$ then $g(x)$ will be positive and can be taken as a new initial value for which the condition above will be satisfied. If $c<0$ then it can be shown that $g(g(x))>x$ if $x<0$ and $g(x)>0$.

*Summary*

From the analysis above we can conclude that $x=0$ is a stable fixed point if

$$-1<c<1$$

and

$$0<b<2. \tag{6.12}$$

The example shows that under the condition (6.12) the version of the self-tuning algorithm where the parameters of the control law are kept constant over long intervals will in fact converge. In the analysis above $\beta_o=1$ was chosen. If $\beta_o\neq 1$ then the condition (6.12) is replaced by

$$0<b/\beta_o<2 \tag{6.12'}$$

or

$$0.5b<\beta_o<\infty. \tag{6.13}$$

The condition (6.13) implies that it is necessary to pick the parameter $\beta_o$ in a correct manner. The algorithm will always converge if $\beta_o$ is greater than $b$. Under-estimation may be serious and the value $\beta_o<0{\cdot}5b$ gives an unstable algorithm.

The analysis presented in the simple example can be extended to give stability conditions for the modified algorithm in more complex cases. The analysis required is tedious.

## 7. SIMULATIONS

The results in section 5 are given under the assumption that the least squares estimator really converges, but yet we have not been able to give general conditions for convergence. But simulation of numerous examples have shown that the algorithm has nice convergence properties.

This section presents a number of simulated examples which illustrate the properties of the self-tuning algorithm.

*Example* 7.1

Let the system be

$$y(t)+ay(t-1)=bu(t-1)+e(t)+ce(t-1) \tag{7.1}$$

with $a=-0{\cdot}5$, $b=3$ and $c=0{\cdot}7$. The minimum variance regulator for the system is

$$u(t)=\frac{a-c}{b}y(t)=-0{\cdot}4y(t). \tag{7.2}$$

A regulator with this structure can be obtained by using the self-tuning algorithm based on the model

$$y(t+1)+\alpha y(t)=\beta_o u(t)+\varepsilon(t+1). \tag{7.3}$$

Figure 3 shows for the case $\beta_o=1$ how the parameter estimate converges to the value $\alpha=-0{\cdot}4$ which corresponds to the minimum variance strategy (7.2).

In Figure 4 is shown the expected variance of the output if the current value of $\alpha$ should be used for all future steps of time. Notice that the algorithm has practically adjusted over 50 steps.

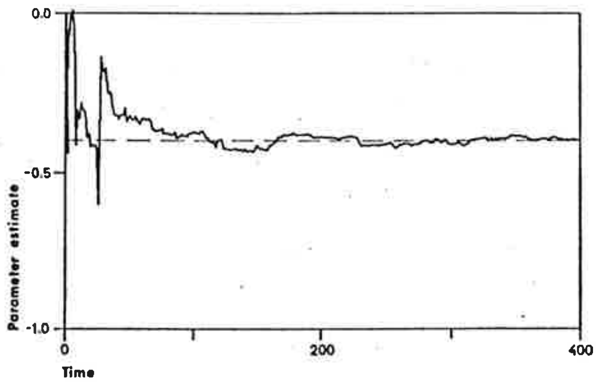


FIG. 3. Parameter estimate $a(t)$ obtained when the self tuning algorithm based on the model (7.3) is applied to the system given by (7.1). The minimum variance regulator corresponds to $a=-0{\cdot}4$ and is indicated by the dashed line.

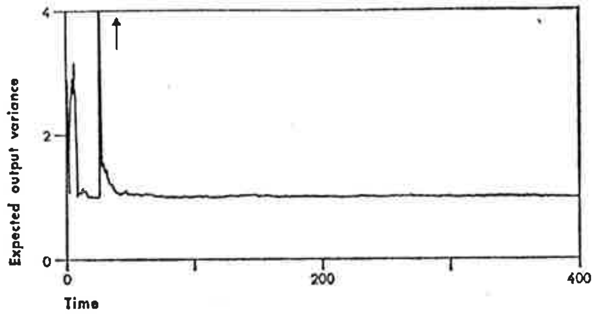


FIG. 4. Expected variance of the output of Example 7.1 if the control law obtained at time $t$ is kept constant for all future times. Notice that the estimate at time $t=26$ would give an unstable system.

The analysis of Example 6.1 shows that, since $b>2$, and $\beta_o=1$ the modified self-tuning algorithm obtained when the parameters of the controller are kept constant over long intervals is unstable. The simulation in Example 7.1 shows that at least in the special case a conservative estimate of the convergence region is obtained by analysing the modified algorithm. If the value of $b$ is increased further it has been shown that the algorithm is unstable. Unstable realizations have been found for $b=5$. In such cases it is of course easy to obtain a stable algorithm by increasing $\beta_o$. This requires, however, a knowledge of the magnitude of $b$.

The system of Example 7.1 is very simple. For instance, if no control is used the variance will still be reasonably small. The next example is more realistic in this aspect.

*Example* 7.2

Consider the system

$$y(t)-1.9y(t-1)+0{\cdot}9y(t-2)=u(t-2)$$
$$+u(t-3)+e(t)-0{\cdot}5e(t-1). \quad (7.4)$$

If no control is used the variance of the output is infinite. Also notice that $B(z)=z-1$. The assumption that $B$ has all zeroes inside the unit circle is thus violated. The minimum variance strategy for the system is

$$u(t)=-1{\cdot}76y(t)+1{\cdot}26y(t-1)-0{\cdot}4u(t-1)$$
$$+1{\cdot}4u(t-2). \quad (7.5)$$

A regulator with this structure is obtained by using the self-tuning algorithm with the model

$$y(t+2)+\alpha_1 y(t)+\alpha_2 y(t-1)$$
$$=u(t)+\beta_1 u(t-1)+\beta_2 u(t-2)+\varepsilon(t+2). \quad (7.6)$$

The convergence of the parameters is shown in Fig. 5. Figure 6 shows the accumulated losses when

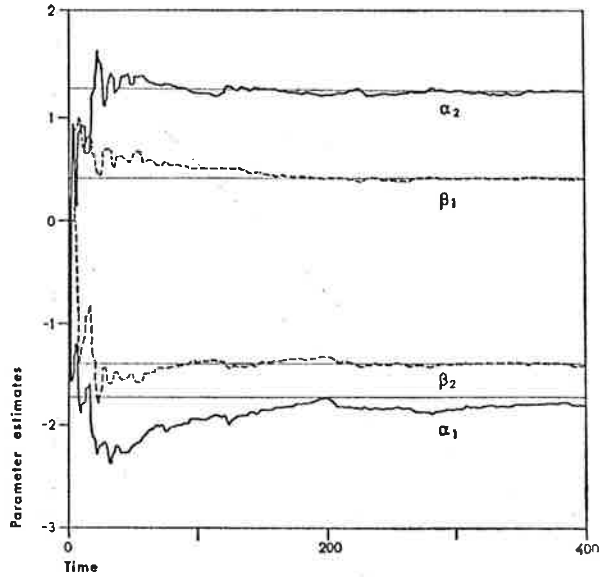


FIG. 5. Parameter estimates $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ obtained when the self tuning algorithm based on (7.6) is applied to the system given by (7.4). The thin lines indicate the parameter values of the minimum variance strategy.

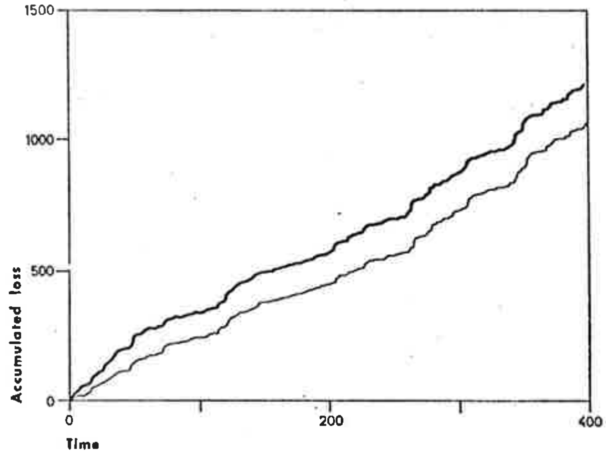


FIG. 6. Accumulated loss

$$\sum_{s=1}^{t} y^2(s)$$

for a simulation of the system (7.4) when using the self tuning algorithm (thick line) and when using the optimal minimum variance regulator (7.5) (thin line).

using the self-tuning algorithm and when using the optimal minimum variance regulator (7.5).

In both examples above, the models in the self-tuning algorithm have had enough parameters so it could converge to the optimal minimum variance regulator. The next example shows what happens when the regulator has not enough parameters.

*Example* 7.3

Consider the system

$$y(t)-1{\cdot}60y(t-1)+1{\cdot}61y(t-2)-0{\cdot}776y(t-3)$$
$$=1{\cdot}2u(t-1)-0{\cdot}95u(t-2)+0{\cdot}2u(t-3)+e(t)$$
$$+0{\cdot}1e(t-1)+0{\cdot}25e(t-2)+0{\cdot}87e(t-3). \quad (7.7)$$

The polynomial $A(z)$ has two complex zeroes near the unit circle ($+0{\cdot}4\pm 0{\cdot}9i$) and one real zero equal to 0·8.

If a self-tuning regulator is determined based on a model with $m=3$ and $l=2$ it will converge to the minimum variance regulator as expected. Figure 7(a) shows a small sample of the output together with the sample covariance of the output, $\hat{r}_y(\tau)$.

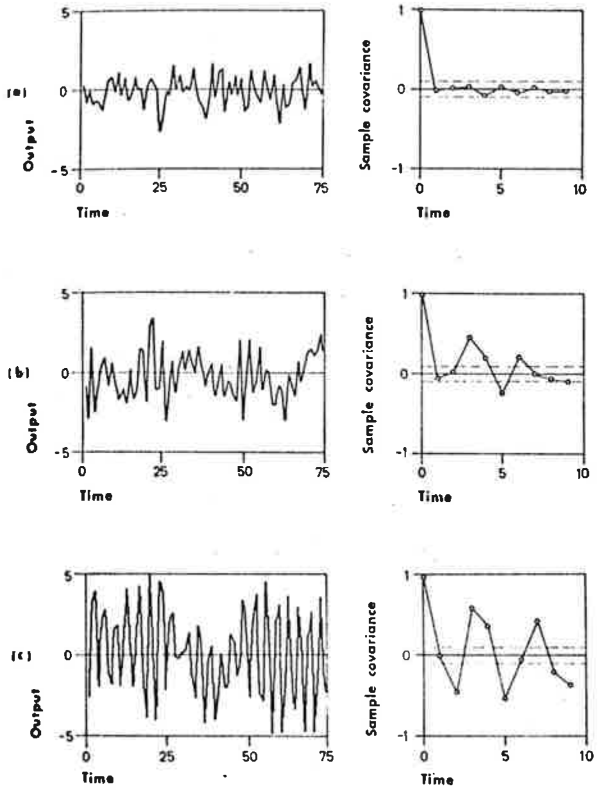


FIG. 7. Output of the system (7.7) and sample covariance of the output $\hat{r}_y(\tau)$ when controlling with self tuning regulator is having different number of parameters

(a) $m=3$, $l=2$ (b) $m=2$, $l=1$ (c) $m=1$, $l=0$.

The dashed lines show the 5 per cent confidence intervals for $\tau\neq 0$.

If the self-tuning algorithm instead is based on a model with $m=2$ and $l=1$ it is no longer possible to obtain the minimum variance regulator for the system since there are not parameters enough in the self-tuning regulator. Theorem 5.1 indicates, however, that if the self-tuning regulator converges, its parameters will be such that the covariances $r_y(1)$, $r_y(2)$, $r_{yu}(1)$ and $r_{yu}(2)$ are all zero. The simulation shows that the algorithm does in fact converge with $\beta_o=1{\cdot}0$. The covariance function of the output is shown in Fig. 7(b). It is seen that the sample covariances $\hat{r}_y(1)$ and $\hat{r}_y(2)$ are within the 5 per cent confidence interval while $\hat{r}_y(3)$ is not as would be expected from Theorem 5.1.

If a self-tuning algorithm is designed based on a model with $m=1$, $l=0$ then Theorem 5.1 indicates that $r_y(1)$ should vanish. Again the simulation shows that the algorithm does in fact converge and that the sample covariance $\hat{r}_y(1)$ does not differ significantly from zero. See Fig. 7(c).

When using regulators of lower order than the optimal minimum variance regulator, the parameters in the controller will not converge to values which for the given structure gives minimum variance of the output. In Table 1 is shown the variance of the output for the system above when using different regulators.

The loss when using the self-adjusting regulator is obtained through simulations. The optimal regulator is found by minimizing $r_y(0)$ with respect to the parameters in the controller.

TABLE 1

| | | Loss $\frac{1}{N}\sum_{t=1}^{N} y^2(t)$ | |
|---|---|---|---|
| $m$ | $l$ | Self-adjusting | Optimal |
| 3 | 2 | 1·0 | 1·0 |
| 2 | 1 | 2·5 | 1·9 |
| 1 | 0 | 4·8 | 3·4 |

The previous examples are all designed to illustrate various properties of the algorithm. The following example is a summary of a feasibility study which indicates the practicality of the algorithm for application to basis weight control of a paper machine.

*Example* 7.4

The applicability of minimum variance strategies to basis weight control on a paper machine was demonstrated in [9]. In this application the control loop is a feedback from a wet basis weight signal to thick stock flow. The models used in [9] were obtained by estimating the parameters of (2.1) using the maximum likelihood method. In one particular case the following model was obtained.

$$y(t)=\frac{b_1q^{-1}+b_2q^{-2}}{1+a_1q^{-1}+a_2q^{-2}}u(t-2)+v(t) \quad (7.8)$$

where the output $y$ is basis weight in g/m$^2$ and the control variable is thick stock flow in g/m$^2$. The disturbance $\{v(t)\}$ was a drifting stochastic process which could be modelled as

$$v(t)=\lambda\frac{1+c_1q^{-1}+c_2q^{-2}}{(1+a_1q^{-1}+a_2q^{-2})(1-q^{-1})}e(t) \quad (7.9)$$

where $\{e(t)\}$ is white noise. The sampling interval was 36 sec and the numerical values of the parameters obtained through identification were as follows

$$a_1=-1{\cdot}283$$

$$a_2=0{\cdot}495$$

$$b_1=2{\cdot}307$$

$$b_2=-2{\cdot}025$$

$$c_1=-1{\cdot}438$$

$$c_2=0{\cdot}550$$

$$\lambda=0{\cdot}382.$$

To investigate the feasability of the self-tuning algorithm for basis weight control, the algorithm was simulated using the model (7.8) where the disturbance $v$ was the actual realization obtained from measurements on the paper machine. The parameters of the regulator were chosen as $k=1$, $l=3$, $m=4$ and $\beta_o=2{\cdot}5$ and the initial estimates were set to zero. The algorithm is thus tuning 7 parameters.

The results of the simulation are shown in Figs. 8–10. Figure 8 compares the output obtained when using the self-tuning algorithm with the result obtained when using the minimum variance regulator computed from the process model (7.8) with the disturbance given by (7.9). The reference value was 70 g/m$^2$. In the worst case the self-tuning regulator gives a control error which is about 1 g/m$^2$ greater than the minimum variance regulator. This happens only at two sampling intervals.

After about 75 sampling intervals (45 min) the output of the system is very close to the output obtained with the minimum variance regulator.

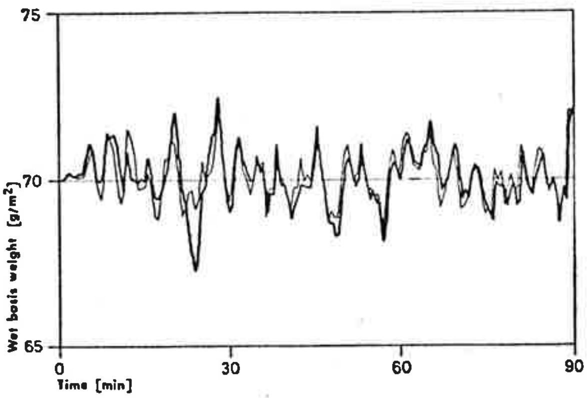


FIG. 8. Wet basis weight when using the self tuning regulator (thick line) and when using the minimum variance regulator based on maximum likelihood identification (thin line). The reference value for the controller was 70g/m².

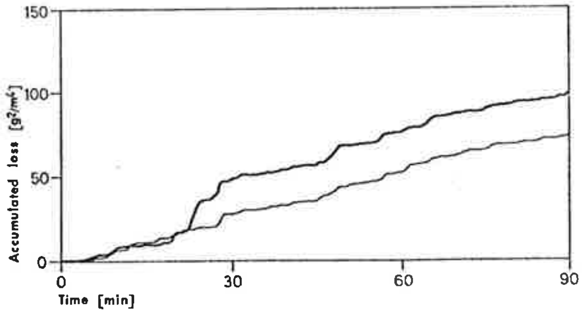


FIG. 9. Accumulated loss for Example 7.4 when using the self tuning regulator (thick line) and when using the minimum variance regulator (thin line).

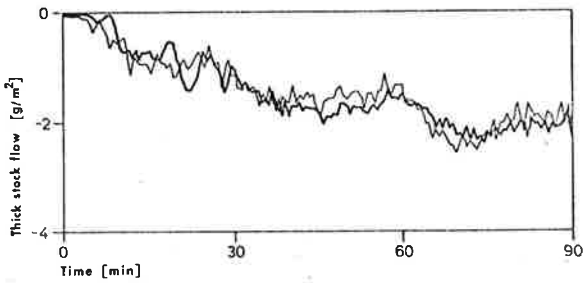


FIG. 10. The control signal in g/m² for Example 7.4 when using the self tuning regulator (thick line) and when using the minimum variance regulator (thin line).

Figure 9 compares the accumulated losses

$$V(t)=\sum_{n=0}^{t} y^2(n)$$

obtained with the minimum variance regulator and the self-tuning regulator. Notice that in the time interval (21, 24) minutes there is a rapid increase in the accumulated loss of the self-tuning regulator of about 17 units. The largest control error during this interval is 2·7 g/m$^2$ while the largest error of the minimum variance regulator is 1 g/m$^2$. The accumulated losses over the last hour is 60 units for the self-tuning regulator and 59 units for the minimum variance regulator.

The control signal generated by the self-tuning algorithm is compared with that of the minimum variance regulator in Fig. 10. There are differences in the generated control signals. The minimum variance regulator generates an output which has more rapid variations than the output of the self-tuning regulator.

The parameter estimates obtained have not converged in 100 sampling intervals. In spite of this the regulator obtained will have good performance as has just been illustrated. The example thus indicates that the self-tuning algorithm could be feasible as a basis weight regulator.

## 8. PRACTICAL ASPECTS

A few practical aspects on the algorithm given in section 4 are presented in this section which also covers some possible extensions of the results

*A priori knowledge*

The only parameters that must be known à *priori* are $k$, $l$, $m$ and $\beta_o$. If the algorithm converges it is easy to find out if the à *priori* guesses of the parameters are correct simply by analyzing the sample covariance of the output. Compare Example 7.3. The parameter $\beta_o$ should be an estimate of the corresponding parameter of the system to be controlled. The choice of $\beta_o$ is not critical as was shown in the Examples 6.1 and 7.1. In the special cases studied in the examples an under-estimate led to a diverging algorithm while an over-estimate was safe.

*Implementation on process computers*

It is our belief that the self-tuning algorithm can be conveniently used in process control applications. There are many possibilities. The algorithm can be used as a tool to tune regulators when they are installed. It can be installed among the systems programs and cycled through different control loops repetitively to ensure that the regulators are always properly tuned. For critical loops where the parameters are changing it is also possible to use a dedicated version which allows slowly drifting parameters.

A general self-tuning algorithm requires about 40 FORTRAN statements. When compiled using the standard PDP 15/30 FORTRAN compiler the code consists of 450 memory locations. The number of memory locations required to store the data is $(l-1+m)^2+3(l-1+m)+2k+4$. Execution times on a typical process computer (PDP 15) without floating point hardware are given in the table below. The major part of the computing is to update the least squares estimate.

| Number of parameters $l+m$ | Execution time ms |
|---|---|
| 1 | 5 |
| 3 | 16 |
| 5 | 34 |
| 8 | 69 |

*Improved convergence rates*

The results of this paper only shows that if the parameters converge the regulator obtained will tend to a minimum variance regulator. Nothing is said about convergence rates, which of course is of great interest from practical as well as theoretical points of view. There are in fact many algorithms that have the correct asymptotic properties. Apart from the algorithm given in section 4 we have the algorithm which minimizes (2.4). But that algorithm is impossible to use due to the computational requirements. It is of interest to investigate if other possible algorithms have better convergence rates than the algorithm of section 4. No complete answer to this problem is yet known. A few possibilities will be indicated. It could be attempted to take into account that the parameter estimates are uncertain. See Refs. [2, 7 and 10]. The least squares identifier can be improved upon by introducing exponential weighting of past data. This has in some cases shown to be advantageous in simulations. Algorithms of this type have in simulations been shown to handle slowly drifting parameters.

Another possibility is to assume that the parameters are Wiener processes, which also can be incorporated fairly easily [2, 7]. It has been verified by simulation that the region of convergence can be improved by introducing a bound on the control signal.

*Feed forward*

In many industrial applications the control can be improved considerable if feed forward is used. The self tuning regulators in this paper can include feed forward control by changing the process model (4.1) to

$$\begin{aligned} y(t+k+1)+\alpha_1 y(t)+ \dots +\alpha_m y(t-m+1) \\ =\beta_0[u(t)+\beta_1 u(t-1)+ \dots +\beta_l u(t-l)]+\gamma_1 s(t) \\ + \dots +\gamma_p s(t-p+1)+\varepsilon(t+k+1) \end{aligned} \tag{8.1}$$

where $s(t)$ is a known disturbance.

The parameters $\alpha_i$, $\beta_i$ and $\gamma_i$ can be identified as before and the control law (4.2) will be changed to

$$\begin{aligned} u(t)=\frac{1}{\beta_0}[\alpha_1 y(t)+ \dots +\alpha_m y(t-m+1)-\gamma_1 s(t) \\ - \dots -\gamma_p s(t-p+1)]-\beta_1 u(t-1) \\ - \dots -\beta_l u(t-l). \end{aligned} \tag{8.2}$$

*Nonminium phase systems*

Difficulties have been found by a straightforward application of the algorithm to nonminimum phase systems, i.e. systems where the polynomial $B$ has zeroes outside the unit circle.

Several ways to get around the difficulty have been found. By using a model with $B(z)=\beta_o$ it has in many cases been possible to obtain stable algorithms at the sacrifice of variance.

It is well-known that the minimum variance regulators are extremely sensitive to parameter variations for nonminimum phase systems [1]. This is usually overcome by using suboptimal strategies which are less sensitive [1]. The same idea can be used for the self-tuning algorithms as well. The drawback is that the computations increase because the polynomials $F$ and $G$ of an identity similar to (2.8) must be determined at each step of the iteration. An alternative is to solve a Riccati-equation at each step.

*Multivariable and nonlinear systems*

It is possible to construct algorithms that are similar to the one described in section 4 for multivariable and nonlinear systems as long as a model structure which is linear in the parameters [3, p. 131] is chosen. For multivariable systems the structure given in equation (3.2) of Ref. [3] can thus be attempted. Analyses of the properties of the algorithm obtained when applied to a multivariable or a nonlinear system are not yet done.

## 9. CONCLUSIONS

The paper has been concerned with control of systems with constant but unknown parameters. The analysis has been limited to single-input single-output systems with disturbances in terms of filtered white noise. A control algorithm based on least squares parameter estimation and a minimum variance regulator computed from the estimated parameters has been analysed. Assuming that the parameter estimates converge the closed loop system has been analysed. A characterization of the closed loop system has been given under weak assumption on the system to be controlled. Under stronger assumptions on the system to be controlled it has been shown that the regulator obtained will actually converge to the minimum variance regulator if the estimates converge.

Since the closed loop system is characterized as a nonlinear stochastic system it is very difficult to give general conditions that guarantee that the estimates converge. The convergence has only been treated for simple examples and under further assumptions as in section 6. But simulations of numerous examples indicate that the algorithm has nice convergence properties.

The simplicity of the algrorithm in combination with its asymptotic properties indicate that it can be useful for industrial process control. The feasibility has also been demonstrated by experiments on real processes in the paper and mining industries.

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**Résumé**—On considère le problème du contrôle d'un système avec paramètres constants mais inconnus. L'analyse se limite aux systèmes discrets à entrée unique et sortie unique. Un algorithme obtenu en combinant un estimateur à carrés minimum avec un régulateur à variance minimum calculée du modèle estimé, est analysé. Les résultats principaux sont deux théorêmes qui caractérisent le système à boucle fermée obtenu en supposant que les estimations des paramètres convergent. Le premier théorême dit que certaines covariances de la sortie et certaines covariances transverses de la variable de contrôle et de la sortie disparaitront avec des suppositions faibles du système contrôlé. Dans le second théorême il est supposé que le système à contrôler est un système général linéaire stochastique du nème ordre. Il est montré que si les estimations des paramètres convergent, la loi de contrôle obtenue est en fait la loi de contrôle de variance minimale qui pourrait être calculée si les paramètres du système étaient connus. Ceci est quelque peu surprenant car l'estimation des carrés minimum est partiale. On discute certaines des implications pratiques des résultats. Il est montré en particulier qu'il est possible d'appliquer l'algorithme à un petit ordinateur.

**Zusammenfassung**—Betrachtet wird das Problem der Steuerung eines Systems mit konstanten, aber unbekannten Parametern. Die Analyse wird auf Systeme mit Diskretzeit und einem Eingang bzw. einem Ausgang beschränkt. Ein durch Kombination einer Schätzeinrichtung und der Methode der kleinsten Quadrate mit einer aus dem Schätzmodell berechneten Regeleinrichtung erhaltener Algorithmus wird analysiert. Die Hauptergebnisse sind zwei Theoreme, die unter der Annahme, daß die Parameterschätzungen konvergieren, den erhaltenen geschlossenen Regelkreis charakterisieren. Das erste Theorem konstatiert, daß bestimmte Kovarianzen des Ausganges und bestimmte Kreuz-Kovarianzen der Steuervariablen und des Ausgangs unter schwachen Annahmen über das zu regelnde System verschwinden. Im zweiten Theorem wird angenommen, daß das zu regelnde System ein allgemeines lineares stochastisches System n-ter Ordnung ist. Gezeigt wird, daß bei Konvergenz der Parameterschätzung des erhaltenen Steuergesetzes in der Tat das Steuergesetz bei minimaler Varianz ist, das berechnet werden kann, wenn die Parameter des Systems bekannt waren. Das ist etwa überraschend, weil die Schätzung nach den kleinsten Quadraten angesteuert wird. Einige praktische Folgerungen aus den Ergebnissen werden diskutiert. Speziell wird gezeigt, daß der Algorithmus auf einem kleinen Prozeßrechner leicht verwirklicht werden kann.

**Резюме**—Рассматривается проблема регулирования систем при помощи постоянных но неизвестных параметров. Анализ ограничивается системами дискретного времени с одним вводом и с одним выводом. Анализируется алгоритм, полученный путем комбинирования оценок наименьших квадратов с регулировкой минимальных колебаний, вычисленных из расчетной модели. Основными результатами являются две теоремы, характеризующие систему замкнутого контура, полученного в предположении, что оценки параметров сходятся. Первая теорема утверждает, что некоторая ковариантность вывода и некоторая пересекающая ковариантность контрольных переменных вывода исчезнут при небольших допущениях в регулируемой системе. Во второй теореме предполагается, что контролируемая система является общей линейной стохастической системой порядка n:th. Показано, что, если оценки параметров сходятся, то полученный закон регулировкя ивляется фактически законом регулировки минимального колебания, которое могло бы быть вычислено при известных параметрах системы. Это до некоторой степени неожиданно, так как оценка наименьших квадратов явияется смещенной. Обсуждаются некоторые практические выводы из результатов. В частности, показано, что этот алгоритм может быть применен на небольшой счетно-вычислительной машине для продессов.