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On Duality in Robustness Analysis

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Abstract

Frequency domain conditions involving multipliers is a powerful tool for robustness analysis. The resulting analysis problem is generally infinite dimensional and numerical solutions restricted to finite dimensional subspaces need to be considered. The finite dimensional problem can be transformed to a linear matrix inequality, which can be solved with efficient algorithms. This paper presents a format for the dual of the infinite dimensional problem. The dual can be used to investigate if the primal robustness problem is feasible. It can also be used to estimate the conservatism of a particular finite dimensional subspace of the primal

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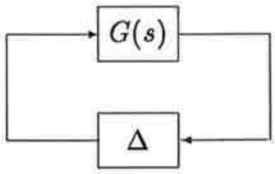

Figure 1. LTI system with perturbation

1. Introduction

Many practical systems can be modeled as a feedback interconnection of a linear time-invariant (LTI) plant G and a perturbation Δ, as in Figure 1. The perturbation contains everything in the system that cannot be modeled as an LTI plant. For example, it can contain nonlinear elements, time-varying elements, and uncertain elements with various assumptions on the uncertainty.

Several classical results from 1960–1975 give sufficient conditions for stability in terms of the Nyquist curve in the case when G is a single-input single-output (SISO) plant for various nonlinear and/or time-varying perturbations, see for example [4]. From the early 1980s much progress has been made on computational methods for robustness analysis in the case of MIMO plants. For example, Doyle introduced μ-analysis, which can be used for robustness test of a large class of systems with structured LTI perturbations by solving an optimization problem at a preselected grid of frequencies, see [5], [15]. However, in the case of nonlinear and/or time-varying perturbations there exists coupling between frequencies. It is then not possible to do frequency by frequency optimization.

Recently, Megretski and Rantzer introduced a unified approach for analysis of systems on the form in Figure 1, [12], [17]. The main idea is to find a description of the perturbation in terms of integral quadratic constraints. If Π is a measurable Hermitean matrix function, which is bounded on the imaginary axis and satisfies $\Pi(j\omega) = \overline{\Pi(-j\omega)}$, then a bounded perturbation Δ is said to satisfy the IQC defined by Π if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{\Delta(u)}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{u}(j\omega) \\ \widehat{\Delta(u)}(j\omega) \end{bmatrix} d\omega \geq 0$$

for all $u \in \mathbf{L}_2[0,\infty)$, where $\widehat{u}$ and $\widehat{\Delta(u)}$ denotes the Fourier transforms of u and $\Delta(u)$. It is often possible to find a convex set Π_Δ such that Δ satisfies the IQC defined by any $\Pi \in \Pi_\Delta$. It was then proved in [17], that the system is robustly stable if under some weak conditions there exists $\Pi \in \Pi_\Delta$ and $\varepsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < -\varepsilon I, \quad \forall \omega \geq 0 \tag{1}$$

The problem of finding $\Pi \in \Pi_\Delta$ such that (1) is satisfied is a convex but generally infinite dimensional problem. An approach for solving this problem is to introduce a finite dimensional basis for the multipliers in Π_Δ. The frequency domain inequality in (1) can then be transformed by use of the positive real lemma to an equivalent Linear Matrix Inequality (LMI), which can be solved by efficient numerical algorithms such as LMI-lab [6]. More detailed description of this approach were given in [8] and [9]. Similar ideas in a more restricted setting can be found in [11] and [2]. The effectiveness of the approach described above is generally dependent on the choice of basis multipliers.

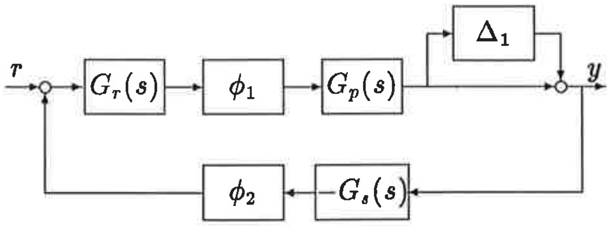

Figure 2. Feedback system

We will in this paper study the dual to the problem of finding a suitable $\Pi \in \Pi_\Delta$ such that (1) is satisfied. The dual problem will give

- Indication on existence of a feasible solution.
- A bound for the optimization problem, which gives an indication of the quality of the basis multipliers.
- Insight into what is important for the choice of suitable basis multipliers.

For the examples in this paper it will turn out that a solution to the dual problem can be found by solving an LMI optimization problem for a preselected set of frequencies.

The organization of the paper is as follows. The next section gives a motivating example where the basic ideas of the paper are introduced. In particular the primal optimization problem is formulated, which characterise the ability of a certain convex set Π_Δ to prove robustness. The next two sections give the mathematical preliminaries on duality theory and functional analysis needed in this paper. In Section 5 a format for Π_Δ is introduced, which can be used to describe a large class of perturbations appearing in practical problems. The dual optimization problem is then derived for the corresponding class of robustness problems. Section 6 gives a refinement of the format for Π_Δ such that we can derive the dual for a problem with complicated and highly structured perturbation Δ by putting together dual problems corresponding to the components of Δ. We derive the dual for common perturbations in Section 7. Section 8 discusses how we can achieve upper bounds for the dual optimization problem using LMI computations and Section 9 gives some numerical examples.

2. A Motivating Example

We will in this section give a simple example that illustrates the main points of this paper. Consider the feedback system in Figure 2. The system is a single loop control system with the following specifications. The plant is modeled as an LTI system G_p with multiplicative output uncertainty $\Delta_1(j\omega)$, where $\|\Delta_1\| \le \varepsilon$ ($\|\cdot\|$ denotes the $\mathbf{H}_\infty$-norm). G_r and G_s represents the controller and sensor dynamics, respectively. ϕ_1 and ϕ_2 are actuator and sensor nonlinearities. We assume that $k_{\min}x^2 \le \phi_i(x)x \le k_{\max}x^2$, $i = 1, 2$, which covers the case with saturations in the actuator and the sensor.

For a study of input/output stability we represent the system in Figure 1, with

$$G(s) = \begin{bmatrix} 0 & G_p & 0 \\ 0 & 0 & G_r \\ -\varepsilon G_s & -G_s G_p & 0 \end{bmatrix}$$

and $\Delta = \operatorname{diag}[\widetilde{\Delta}(j\omega), \phi_1, \phi_2]$, where $\widetilde{\Delta} = \Delta_1/\varepsilon$. We notice that Δ satisfies the

IQC's defined by a matrix function of the form

$$\Pi(j\omega) = \left[\begin{array}{ccc|ccc} x_1(j\omega) & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha x_2 & 0 & 0 & \beta x_2 & 0 \\ 0 & 0 & \alpha x_3 & 0 & 0 & \beta x_3 \\ \hline 0 & 0 & 0 & -x_1(j\omega) & 0 & 0 \\ 0 & \beta x_2 & 0 & 0 & -2x_2 & 0 \\ 0 & 0 & \beta x_3 & 0 & 0 & -2x_3 \end{array}\right] \qquad (2)$$

where x_1 is such that $x(j\omega) \geq 0$ for all $\omega \in \mathbf{R}$, $x_2, x_3 \geq 0$, $\alpha = -2k_{\min}k_{\max}$ and $\beta = k_{\max} + k_{\min}$. We can choose a basis for x_1 in terms of functions in $\mathbf{RH}_\infty$ and then transform the stability condition in (1) to an LMI optimization of the coordinates for the basis by using the positive real lemma. Two questions arise. The first question is if there exists any suitable basis at all, i.e. if there is any a Π matrix with structure as in (2) such that (1) it satisfied. The second question regards the quality of a particular basis. We will be able to answer both questions by solving the dual to the following optimization problem.

PRIMAL OPTIMIZATION PROBLEM 1

$$\mu_p = \inf -\gamma$$

$$\text{subj to} \begin{cases} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\gamma I \\ \Pi \in \Pi_\Delta, \quad \gamma \in \mathbf{R} \end{cases} \qquad (3)$$

for all $\omega \geq 0$, where Π_Δ is defined as the closed, bounded and convex set consisting of the Π-matrices in (2), when the multipliers are restricted such that $0 \leq x_1(j\omega) \leq I$, $\forall \omega \in \mathbf{R}$ and $0 \leq x_i \leq 1$, $i = 1, 2$.

We say that the robustness test is feasible if $\mu_p < 0$ and if this is the case then $|\mu_p|$ is a measure of the robustness margin. The boundedness of Π_Δ implies that $\mu_p > -\infty$ and the maximal value of (3) is $\mu_p = 0$, since $0 \in \Pi_\Delta$. We say that the robustness test is unfeasible if $\mu_p = 0$.

3. A Result from Duality Theory

We will in this section present a special case of the Fenchel duality theorem, see for example [10]. This conic formulation of the duality theorem has been used in the finite dimensional case in for example [14] and [18].

Mathematical Preliminaries

Let X be a normed vector space. The dual of X is the normed space consisting of all bounded linear functionals on X and it is denoted by X^*. If $x \in X$ and $x^* \in X^*$, then $\langle x, x^* \rangle$ denotes the value of the linear functional x^* at x.

DEFINITION 1
A set C in a vector space is said to be a *cone* if $x \in C$ implies that $\alpha x \in C$ for all $\alpha \geq 0$.

Definition 2
We define a *positive cone* P in X as any convex cone in X defining the following relation. For any $x, y \in X$, the notation $x \geq y$, means that $x - y \in P$.

Definition 3
Given a positive cone $P \subset X$, then we define $P^{\oplus}$ as

$$P^{\oplus} = \{x^* \in X^* : \langle x, x^* \rangle \geq 0, \quad \forall x \in P\}$$

Definition 4
Given a subspace $\mathcal{L} \subset X$, then its orthogonal complement $\mathcal{L}^{\perp}$ is defined as

$$\mathcal{L}^{\perp} = \{x^* \in X^* : \langle x, x^* \rangle = 0, \quad \forall x \in \mathcal{L}\}$$

Definition 5
A translated subspace, $\mathcal{L} + c$, where $c \in X$, is said to be a *linear variety*.

Definition 6
If $S \subset X$ is a nonempty set then the closed linear variety generated by S, $v(S)$, is the intersection of all closed linear varieties in X that contain S.

Definition 7
The *relative interior* of a nonempty set S is the collection of points in S, which are interior points of S relative to $v(S)$. This means that for every x_0 in the relative interior of S, there exists $\varepsilon > 0$ such that all $x \in v(S)$ satisfying $\|x - x_0\|$ are also members of the relative interior of S. Hence, the relative interior of S is an open subset of $v(S)$.

The Duality Result

If we define the primal problem as the following minimization problem

$$\mu_p = \inf_{x \in P \cap (\mathcal{L}+D)} \langle x, C \rangle \tag{4}$$

where $\mathcal{L}$ is a subspace of X, $C \in X^*$, and $D \in X$. The corresponding dual problem is defined as the following minimization problem

$$\mu_d = \min_{x^* \in P^{\oplus} \cap (\mathcal{L}^{\perp}+C)} \langle D, x^* \rangle \tag{5}$$

The primal and dual problems are related as follows

Proposition 1
If $P \cap (\mathcal{L} + D)$ contains points in the relative interior of P and $\mathcal{L} + D$, and if $\inf_{x \in P \cap (\mathcal{L}+D)} \langle x, C \rangle$ is finite, then

$$\inf_{x \in P \cap (\mathcal{L}+D)} \langle x, C \rangle + \min_{x^* \in P^{\oplus} \cap (\mathcal{L}^{\perp}+C)} \langle D, x^* \rangle = \langle D, C \rangle \tag{6}$$

□

Remark 1. Our main usage of this proposition will be to investigate the following inequality. Given $x \in P \cap (\mathcal{L} + D)$ and $x^* \in P^{\oplus} \cap (\mathcal{L}^{\perp} + C)$, then

$$\langle x, C \rangle \geq \inf_{x \in P \cap (\mathcal{L}+D)} \langle x, C \rangle \geq \langle D, C \rangle - \langle D, x^* \rangle \tag{7}$$

This shows that the primal problem is lower bounded by $\langle D, C\rangle - \langle D, x^*\rangle$ for any dual feasible $x^* \in P^{\oplus} \cap (\mathcal{L}^{\perp} + C)$. In our applications $\langle D, C\rangle$ will be zero.

Remark 2. The assumption that $P \cap (\mathcal{L} + D)$ contains points in the relative interior of P and $\mathcal{L} + D$ means that $\mathcal{P} \cap (\mathcal{L} + D)$ contains interior points of P and $\mathcal{L} + D$ relative to linear varieties generated by P and $\mathcal{L} + D$, respectively.

We show in Appendix A that Proposition 1 follows from the Fenchel duality theorem.

4. More Mathematical Preliminaries

We will in this section give some further mathematical preliminaries that will be used in the sequel. We will also introduce the notation and the vector spaces that will be used in our applications. The following definitions are standard and can be found in [10].

DEFINITION 8
The (Cartesian) product of two vector spaces X_1 and X_2, which are defined over the same field of scalars, is denoted $X_1 \times X_2$ and it consists of all ordered pairs $x = (x_1, x_2)$, with $x_1 \in X_1$ and $x_2 \in X_2$. x_1 and x_2 are said to be the coordinates of $X_1 \times X_2$. Addition and scalar multiplication is defined as $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ and $\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$.

DEFINITION 9
The dual of $X_1 \times X_2$ is given as $X_1^* \times X_2^*$, where X_1^* and X_2^* are the duals of X_1 and X_2 respectively. Given $x = (x_1, x_2) \in X_1 \times X_2$ and $x^* = (x_1^*, x_2^*) \in X_1^* \times X_2^*$, then $\langle x, x^*\rangle = \langle x_1, x_1^*\rangle + \langle x_2, x_2^*\rangle$.

Extension of the above definitions to the product of n vector spaces $X_1, \ldots, X_n$ is obvious. We use the following notation

DEFINITION 10
The product of $X_1, \ldots, X_N$ will be denoted $\prod_{k=1}^{N} X_k$ and the elements of this product space are denoted $\{x_k\}_{k=1}^{N} \stackrel{def}{=} (x_1, \ldots, x_N)$. If $X_i = X$ for $i = 1, \ldots n$, then we use the notation X^n for the product.

If every X_k is a product space, i.e. $X_k = \prod_{l=1}^{M_k} X_{kl}$, then we use the notation $\prod_{k=1}^{N} \prod_{l=1}^{M_k} X_{kl}$ for the product space and the elements are denoted $\{\{x_{kl}\}_{l=1}^{M_k}\}_{k=1}^{N}$.

DEFINITION 11
$B(X, Y)$ denotes the normed space of all bounded linear operators from X into Y, where X, Y are normed vector spaces.

DEFINITION 12
Let $H \in B(X, Y)$, then the adjoint operator $H^{\times} : Y^* \mapsto X^*$ is defined by

$$\langle Hx, y^*\rangle = \langle x, H^{\times} y^*\rangle$$

for all $x \in X$ and $y^* \in Y^*$. We will use the following properties of the adjoint

- If $H_1, H_2 \in B(X, Y)$ then $(H_1 + H_2)^{\times} = H_1^{\times} + H_2^{\times}$.
- If $H_1 \in B(X, Y)$ and $H_2 \in B(Y, Z)$ then $(H_2 H_1)^{\times} = H_1^{\times} H_2^{\times}$.

Here is a list of notation and vector spaces that will be used from now on.

- $*$ denotes Hermitian conjugation of a complex matrix.
- We will from now on denote the elements from any dual space X^* with with lower case letter z in order not to have ambiguity in the interpretation of the notation x^*.
- $\mathbf{L}_\infty^{n\times n}$ is the Banach space of measurable $n \times n$ complex valued functions that are essentially bounded on the imaginary axis and satisfy $F(-i\omega) = \overline{F(j\omega)}$ for almost all $\omega \in \mathbf{R}$. $\mathbf{RL}_\infty^{n\times n} \subset \mathbf{L}_\infty^{n\times n}$ is the subspace consisting of proper real rational functions with no poles on the imaginary axis.
- $\mathbf{RH}_\infty^{m\times m} \subset \mathbf{RL}_\infty^{m\times m}$ is the space consisting proper real rational matrix functions with no poles in the closed right half plane. Note that G^* generally means the Hilbert adjoint of $G(s)$, defined as $G^T(-s)$. The Hilbert adjoint reduces to the Hermitean conjugate of G when $s = i\omega$.
- $\mathcal{S}_\mathrm{R}^{m\times m} \subset \mathbf{R}^{m\times m}$ denotes the space of symmetric $m \times m$ matrices. The dual space can be identified with $\mathcal{S}_\mathrm{R}^{m\times m}$ itself and the linear functionals are defined as $\langle x, z\rangle = \mathrm{tr}(xz)$, where $x, z \in \mathcal{S}_\mathrm{R}^{m\times m}$.
- $\mathcal{A}_\mathrm{R}^{m\times m}$ denotes the space of skew-symmetric $m \times m$ matrices. The dual space can be identified with $\mathcal{A}_\mathrm{R}^{m\times m}$ itself and the linear functionals are defined as $\langle x, z\rangle = \mathrm{tr}(x^T z)$, where $x, z \in \mathcal{A}_\mathrm{R}^{m\times m}$.
- $\mathcal{S}_\mathrm{C} \subset \mathbf{C}^{m\times m}$ denotes the space of Hermitean $m \times m$ matrices. The dual space can be identified with $\mathcal{S}_\mathrm{C}^{m\times m}$ itself and the linear functionals are defined as $\langle x, z\rangle = \mathrm{tr}\{\mathrm{Re} xz\} = \mathrm{tr}\{xz\}$, where $x, z \in \mathcal{S}_\mathrm{C}^{m\times m}$.
- $\mathbf{B}^{m\times m}$ is the Banach space of $n \times n$ complex valued functions that are bounded on the imaginary axis and satisfy $x(-i\omega) = \overline{x(j\omega)}$ for all $\omega \in \mathbf{R}$. We define the norm on $\mathbf{B}^{m\times m}$ as $\|x\|_\mathrm{B} = \sup_{\omega\in\mathrm{R}} |x(j\omega)|_\infty$. The matrix norm $|\cdot|_\infty$ is defined as $|x|_\infty = \max_{ij} |x_{ij}|$ when $x \in \mathbf{C}^{m\times m}$.
- $\mathcal{S}_\mathrm{B}^{m\times m} \subset \mathbf{B}^{m\times m}$ denotes the subspace consisting of functions in $\mathbf{B}^{m\times m}$, which satisfy $x(j\omega) = x^*(j\omega)$ for all $\omega \in \mathbf{R}$.
- $\mathbf{AM}^{m\times m}$ is the Banach space consisting of the distributions of the form

$$z(\omega) = \sum_{k=1}^{\infty} z_k \delta(\omega - \omega_k) + \overline{z_k}\delta(\omega + \omega_k)$$

  where $z_k \in \mathbf{C}^{m\times m}$ and $\sum_{k=1}^{\infty} |z_k|_1 < \infty$. The norm on $\mathbf{AM}$ is defined as $\|z\|_\mathrm{AM} = \sum_{k=1}^{\infty} |z_k|_1 < \infty$. The matrix norm $|\cdot|_1$ is defined as $|z|_1 = \sum_{i,j} |z_{ij}|$ when $z \in \mathbf{C}^{m\times m}$.
- $\mathcal{S}_\mathrm{AM} \subset \mathbf{AM}$ denotes the subspace consisting of functions in $\mathbf{AM}^{m\times m}$, which satisfy $z(\omega) = z^*(\omega)$ for all $\omega \in \mathbf{R}$. This implies that the coefficients of the distributions δ satisfy $z_k = z_k^*$ for all k.
- $\mathcal{P}_\mathrm{R}^{m\times m} \subset \mathcal{S}_\mathrm{R}^{m\times m}$ is the positive cone of positive semidefinite symmetric matrices.
- $\mathcal{P}_\mathrm{C}^{m\times m} \subset \mathcal{S}_\mathrm{C}^{m\times m}$ is the positive cone of positive semidefinite Hermitean matrices.
- $\mathcal{P}_\mathrm{B}^{m\times m} \subset \mathcal{S}_\mathrm{B}^{m\times m}$ is the positive cone of functions $x \in \mathcal{S}_\mathrm{B}^{m\times m}$ satisfying $x(j\omega) \geq 0$ for all $\omega \in \mathbf{R}$.
- $\mathcal{P}_\mathrm{AM}^{m\times m} \subset \mathcal{S}_\mathrm{AM}^{m\times m}$ is the positive cone of functions $z \in \mathcal{S}_\mathrm{AM}^{m\times m}$ having coefficients satisfying $z_k \geq 0$ for all k.

Remark. Note that the function spaces $\mathcal{S}_\mathrm{R}$, $\mathcal{S}_\mathrm{C}$ $\mathcal{S}_\mathrm{B}$ and $\mathcal{S}_\mathrm{AM}$ are defined over the scalar field $\mathbf{R}$. Also note that $\mathcal{S}_\mathrm{R}$ can be viewed as a subspace of $\mathcal{S}_\mathrm{C}$ and that $\mathcal{S}_\mathrm{C}$ can be viewed as a subspace of $\mathcal{S}_\mathrm{B}$, i.e. we have $\mathcal{S}_\mathrm{R} \subset \mathcal{S}_\mathrm{C} \subset \mathcal{S}_\mathrm{B}$.

Remark. We search for a $\Pi \in \Pi_\Delta \subset \mathbf{L}_\infty^{2m\times 2m}$ such that (1) is satisfied. It is, however, convenient do the search over functions in $\mathcal{S}_\mathrm{B}^{2m\times 2m}$ when deriving the dual. The reason is that it turns out that it is non-conservative to specify the dual in terms of functions in $\mathcal{S}_\mathrm{AM}$. Note that $\mathcal{S}_\mathrm{B}^{2m\times 2m}$ contains nonmeasurable functions. The next theorem shows that nothing is gained in searching over $\mathcal{S}_\mathrm{B}^{2m\times 2m}$, which justifies our choice of function spaces.

THEOREM 1
The Primal Optimization Problem 1 gives the same objective value regardless if we consider $\Pi_\Delta \subset \mathbf{L}_\infty^{2m\times 2m}$ or $\Pi_\Delta \subset \mathcal{S}_\mathrm{B}^{2m\times 2m}$.

Proof: The proof is given in Appendix B

□

Remark. The dual $(\mathcal{S}_\mathrm{B}^{m\times m})^*$ of $\mathcal{S}_\mathrm{B}^{m\times m}$ is not easily characterized. It is clear that $\mathcal{S}_\mathrm{AM}^{m\times m} \subset (\mathcal{S}_\mathrm{B}^{m\times m})^*$ and we will prove that in the applications we consider there is nothing lost in restricting attention to the functions in $\mathcal{S}_\mathrm{AM}^{m\times m}$.

Remark. If $G \in \mathbf{RH}_\infty^{m\times m}$ and $z \in \mathbf{AM}^{m\times m}$ then $Gz \in \mathbf{AM}^{m\times m}$ and

$$\begin{aligned}(Gz)(\omega) &= \sum_{k=1}^{\infty} G(j\omega_k)z_k\delta(\omega-\omega_k) + \overline{G(j\omega_k)z_k}\delta(\omega+\omega_k)\\ &=: \int_{-\infty}^{\infty} G(j\omega)z(\omega)d\omega\end{aligned}$$

LEMMA 1
The dual of $\mathcal{S}_\mathrm{AM}^{m\times m}$ can be identified with $\mathcal{S}_\mathrm{B}^{m\times m}$ and if $z \in \mathcal{S}_\mathrm{AM}^{m\times m}$ and $x \in \mathcal{S}_\mathrm{B}^{m\times m}$, then

$$\langle z, x\rangle = \sum_{k=1}^{\infty} 2\mathrm{tr}\{\mathrm{Re} z_k x(j\omega_k)\} = \sum_{k=1}^{\infty} 2\mathrm{tr}\{z_k x(j\omega_k)\}$$

and

$$|\langle z, x\rangle| \le \|z\|_\mathrm{AM}\|x\|_\mathrm{B}$$

Proof: The proof of the lemma follows from the proof of Theorem 3 in [3].
□

5. Dual Formulation of a Class of Robustness Problems

The general robustness test of finding $\Pi \in \Pi_\Delta$ such that (1) is satisfied can be formulated as Primal Optimization Problem in 1. Π_Δ is the convex set of structured Π-matrices corresponding to the robustness problem under consideration.

We will consider problems where the perturbation can be described by IQC's defined by a convex set Π_Δ, with elements defined in terms of a set of multipliers. More precisely we consider bounded and convex sets Π_Δ with $0 \in \Pi_\Delta$ defined as follows

$$\Pi_\Delta = \{\Pi = \Pi_\mathrm{M} x_\mathrm{M} : x_\mathrm{M} \in X_\mathrm{M} \text{ and } L_\mathrm{M} x_\mathrm{M} + D_\mathrm{M} \in P_\mathrm{M}\} \tag{8}$$

Here x_M is an element of a suitably defined normed vector space X_M. The range of x_M is restricted by the constraint $L_M x_M + D_M \in P_M$, where $L_M : X_M \mapsto Y_M$ is a bounded linear operator from X_M into the normed vector space Y_M, P_M is a positive cone in Y_M, and D_M is an element in Y_M. The constraint $L_M x_M + D_M \in P_M$ is defined such that the coordinates in x_M are upper and lower bounded in such a way that $x_M = 0$ satisfies the constraint, which implies that $0 \in \Pi_\Delta$. Further, $\Pi_M : X_M \mapsto \mathcal{S}_B^{2m\times 2m}$ is a bounded linear operator. We assume that X_M and Y_M are products of vector spaces from the listing in Section 4.

The optimization problem in (3) can now be stated as

PRIMAL OPTIMIZATION PROBLEM 2

$$
\begin{aligned}
\mu_p = & \inf -\gamma \\
& \text{subj to} \begin{cases} -M_G \Pi_M x_M - \gamma I \in P_B^{m\times m} \\ L_M x_M + D_M \in P_M \\ x_M \in X_M, \quad \gamma \in \mathbf{R} \end{cases}
\end{aligned} \tag{9}
$$

where $M_G : \mathcal{S}_B^{2m\times 2m} \mapsto \mathcal{S}_B^{m\times m}$ is the bounded linear operator defined as

$$
M_G \Pi = \begin{bmatrix} G \\ I \end{bmatrix}^* \Pi \begin{bmatrix} G \\ I \end{bmatrix}
$$

for any $\Pi \in \mathcal{S}_B^{2m\times 2m}$.

□

This optimization problem can now be stated in the formalism for the primal problem in (4). This follows if we let

$$
\begin{aligned}
X &= \mathcal{S}_B^{m\times m} \times Y_M \times \mathbf{R} \\
P &= P_B^{m\times m} \times P_M \times \mathbf{R} \\
D &= (0, D_M, 0) \in X \\
C &= (0, 0, -1) \in X^* \\
\mathcal{L} &= \{(-M_G \Pi_M x_M - \gamma I, L_M x_M, \gamma) : x \in X_M, \gamma \in \mathbf{R}\}
\end{aligned} \tag{10}
$$

Remark. We will in our applications always choose Π_Δ such that the constraint set $P \cap (\mathcal{L} + D)$ contains points in the relative interior of P and $\mathcal{L}$. Note that it is important to allow γ to be arbitrary in $\mathbf{R}$ in order to ensure that this is the the case for the first coordinate.

In order to obtain the corresponding dual optimization problem formulated as in (5) we need to derive $P^\oplus$ and $\mathcal{L}^\perp$. It is straightforward to see that we have

$$
\begin{aligned}
X^* &= (\mathcal{S}_B^{m\times m})^* \times Y_M^* \times \mathbf{R} \\
P^\oplus &= (P_B^{m\times m})^\oplus \times P_M^\oplus \times \{0\}
\end{aligned}
$$

where, Y_M^* is the dual of Y_M and $P_M^\oplus$ is the positive convex cone in the dual space, which corresponds to the positive cone P_M in Y_M. In order to derive $\mathcal{L}^\perp$ we need to find the set of $z \in X^*$ satisfying $\langle x, z\rangle = 0$ for all $x \in \mathcal{L}$. Let $z = (z_0, z_M, z_\gamma) \in X^*$, then

$$
\begin{aligned}
\langle x, z\rangle =& \langle -M_G \Pi_M x_M - \gamma I, z_0\rangle + \langle L_M x_M, z_M\rangle + \gamma z_\gamma \\
=& \gamma(z_\gamma - \langle I, z_0\rangle) + \langle x_M, L_M^\times z_M - \Pi_M^\times M_G^\times z_0\rangle = 0
\end{aligned}
$$

should hold for arbitrary $\gamma \in \mathbf{R}$ and arbitrary $x_{\mathrm{M}} \in X_{\mathrm{M}}$. Hence it follows that $z_\gamma = \langle I, z_0 \rangle$ and $L_{\mathrm{M}}^{\times} z_{\mathrm{M}} = \Pi_{\mathrm{M}}^{\times} M_G^{\times} z_0$. We get

$$\mathcal{L}^{\perp} = \{(z_0, z_{\mathrm{M}}, \langle I, z_0 \rangle) \in X^* : L_{\mathrm{M}}^{\times} z_{\mathrm{M}} = \Pi_{\mathrm{M}}^{\times} M_G^{\times} z_0\} \tag{11}$$

The dual optimization problem is defined in terms of functions in $P^{\oplus} \cap (\mathcal{L}^{\perp} + C)$, which contains coordinates from $\mathcal{S}_{\mathrm{B}}^*$. In order to use the dual in practice we need to restrict the coordinate z_0 to be in $P_{\mathrm{AM}}^{m\times m}$. A similar restriction must hold for the coordinates in z_{M} that are in $\mathcal{S}_{\mathrm{B}}^*$. We use the notation $(P_{\mathrm{M}}^{\oplus})_{\mathrm{AM}}$ for this restriction of $P_{\mathrm{M}}^{\oplus}$. Similarly, $(Y_{\mathrm{M}}^*)_{\mathrm{AM}}$ denotes the subspace of Y_{M}^* obtained by restricting the coordinates in $\mathcal{S}_{\mathrm{B}}^*$ to be in $\mathcal{S}_{\mathrm{AM}}$. The dual optimization problem can now be stated as

Dual Optimization Problem 1

$$\mu_d = \inf \langle D_{\mathrm{M}}, z_{\mathrm{M}} \rangle$$
$$\text{subj to} \begin{cases} z_0 \in P_{\mathrm{AM}}^{m\times m}, z_{\mathrm{M}} \in (P_{\mathrm{M}})_{\mathrm{AM}} \\ \langle I, z_0 \rangle = 1 \\ L_{\mathrm{M}}^{\times} z_{\mathrm{M}} = \Pi_{\mathrm{M}}^{\times} M_G^{\times} z_0 \end{cases} \tag{12}$$

where restricted adjoint $M_G^{\times} : \mathcal{S}_{\mathrm{AM}}^{m\times m} \mapsto \mathcal{S}_{\mathrm{AM}}^{2m\times 2m}$ is defined as

$$M_G^{+} Z = \begin{bmatrix} G \\ I \end{bmatrix} Z \begin{bmatrix} G \\ I \end{bmatrix}^*$$

for any $Z \in \mathcal{S}_{\mathrm{AM}}^{m\times m}$.

□

Remark. It would be more appropriate to us the notation $M_{G|\mathrm{AM}}^{\times}$ for the adjoint restricted to AM. However, for convenience of notation, we will not distinguish the adjoint and the adjoint restricted to AM. This remark will hold throughout the paper.

The resulting optimization problem can be viewed as an LMI involving complex matrices, which easily is transformed to an LMI involving real-valued matrices. We can use the dual in (12) to estimate the quality of a particular finite dimensional restriction of the primal in (2) as follows. Given the optimal dual objective μ_d and the optimal objective $\widehat{\mu}_p$ corresponding to the restricted primal, then the relative duality gap $(\mu_d + \widehat{\mu}_p)/|\widehat{\mu}_p|$ is a measure of the quality of the basis for the multipliers in the primal. In general, we only have an approximate solution to the dual $\widehat{\mu}_d$, and the duality gap $(\widehat{\mu}_d + \widehat{\mu}_p)/|\widehat{\mu}_p|$ gives a measure of both the dual and primal approximation. It is shown in Section 9 how the example in Section 2 can be treated in this way.

We say that robustness condition corresponding to Primal Optimization Problem 2 is unfeasible if $\mu_p = 0$. The associated unfeasibility test can be formulated in the following way.

Unfeasibility Test 1

The robustness condition corresponding to Primal Optimization Problem 2 is unfeasible if there exists $z_0 \in P_{\mathrm{AM}}^{m\times m}$ and $z_{\mathrm{M}} \in (P_{\mathrm{M}}^{\oplus})_{\mathrm{AM}}$ such that

$$L_{\mathrm{M}}^{\times} z_{\mathrm{M}} = \Pi_{\mathrm{M}}^{\times} M_G^{\times} z_0$$
$$\langle D_{\mathrm{M}}, z_{\mathrm{M}} \rangle = 0$$

□

We will show in the next section that the unfeasibility test corresponding to the example in Section 2 can be formulated as an attractive LMI problem.

The next theorem gives conditions which ensure that the dual optimization problem in (12) is non-conservative.

THEOREM 2
Assume P_{M} is closed and that the restriction of $\Pi_{\mathrm{M}}^{\times} M_G^{\times}$ to $S_{\mathrm{AM}}^{m\times m}$ and the restriction of $L_{\mathrm{M}}^{\times}$ to $(Y_{\mathrm{M}}^{*})_{\mathrm{AM}}$ have closed range spaces in $(X_M^{*})_{\mathrm{AM}}$. Further assume that the equation $L_{\mathrm{M}}^{\times} z_{\mathrm{M}} = \Pi_{\mathrm{M}}^{\times} M_G^{\times} z_0$ has a solution $(z_0, z_{\mathrm{M}}) \in P_{\mathrm{AM}}^{m\times m} \times (P_{\mathrm{M}}^{\oplus})_{\mathrm{AM}}$ in the relative interior of $P_{\mathrm{AM}}^{m\times m} \times (P_{\mathrm{M}}^{\oplus})$. Then the dual optimization problem in (12) is non-conservative, i.e. $\mu_d = \mu_p$.

Proof: The proof is given in Appendix C.

□

In Section 7 we show that all the conditions of the theorem are satisfied in the applications we will consider. Note that the positive cones $P_{\mathrm{B}}, P_{\mathrm{C}}$ and P_{R} are closed.

6. Refinement of the Multiplier specification

The definition of Π_Δ in (8) is compact but it does not reveal how the different multipliers come into play. We will here give a refined definition of Π_Δ from which it is possible to derive a dual robustness test where the contribution of every multiplier is explicitly revealed.

For this we will use two properties of IQC's, which are useful when deriving IQC:s for complex perturbations with diagonal structure.

PROPOSITION 2
Assume Δ satisfies the IQC's defined by $\Pi_1, \ldots, \Pi_n$, then Δ also satisfies the IQC defined by $\sum_{i=1}^{n} \alpha_i \Pi_i$, for any $\alpha_i \geq 0$, $i \in [1, \ldots, n]$

□

PROPOSITION 3
Assume Δ has the block-diagonal structure $\Delta = \mathrm{diag}[\Delta_1, \ldots, \Delta_n]$, and that Δ_i satisfies the IQC defined by Π_i. Then Δ satisfies the IQC defined by $\Pi = \mathrm{daug}[\Pi_1, \ldots, \Pi_n]$, where the operation daug is defined as follows. If

$$\Pi_i = \begin{bmatrix} \Pi_{i1} & \Pi_{i2} \\ \Pi_{i2}^{*} & \Pi_{i3} \end{bmatrix}, \quad i = 1, 2$$

where the block structures are consistent with the size of Δ_1 and Δ_2, respectively, then

$$\mathrm{daug}(\Pi_1, \Pi_2) = \left[\begin{array}{cc|cc} \Pi_{11} & 0 & \Pi_{12} & 0 \\ 0 & \Pi_{21} & 0 & \Pi_{22} \\ \hline \Pi_{12}^{*} & 0 & \Pi_{13} & 0 \\ 0 & \Pi_{22}^{*} & 0 & \Pi_{23} \end{array}\right]$$

□

The propositions above suggests the following refined definition of a set Π_Δ describing a perturbation $\Delta = \mathrm{diag}[\Delta_1, \ldots, \Delta_N]$, where the size of Δ_k is $m_k \times m_k$.

$$\Pi_\Delta = \{\Pi = \mathrm{daug}(\Pi_1, \ldots, \Pi_N) : \Pi_k \in \Pi_{\Delta_k}\}$$

where

$$\Pi_{\Delta_k} = \{\Pi_k = \sum_{l=1}^{M_k} \Pi_{kl} x_{kl} : x_{kl} \in X_{kl}, L_{kl} x_{kl} + D_{kl} \in P_{kl}\}$$

Here x_{kl} is a multiplier defined on a suitably defined normed function space X_{kl}. The range of x_{kl} is restricted by the constraint $L_{kl}x_{kl} + D_{kl} \in P_{kl}$, where $L_{kl} : X_{kl} \mapsto Y_{kl}$ is a bounded linear operator from X_{kl} into the normed function space Y_{kl}, P_{kl} is a positive cone in Y_{kl}, and D_{kl} is an element in Y_{kl}. Further, $\Pi_{kl} : X_{kl} \mapsto \mathcal{S}_{\mathrm{B}}^{2m_k \times 2m_k}$ is a bounded linear operator. Let us illustrate this by continuing the example from Section 2.

EXAMPLE 1—Example from Section 2 continued
The convex set Π_Δ can be described in the format given in (8) as follows.

$$\Pi_\Delta = \{\mathrm{daug}(\Pi_1, \Pi_2, \Pi_3) | \Pi_i \in \Pi_{\Delta i}, i = 1, 2, 3\}$$

where

$$\Pi_{\Delta 1} = \{\Pi_1 x_1 = \begin{bmatrix} x_1 & 0 \\ 0 & -x_1 \end{bmatrix} | 0 \leq x_1(j\omega) = L_1 x_1 + D_1 \in P_1\}$$

$$\Pi_{\Delta 2} = \{\Pi_2 x_2 = x_2 \begin{bmatrix} -2k_{\min}k_{\max} & k_{\min} + k_{\max} \\ k_{\min} + k_{\max} & -2 \end{bmatrix} | L_2 x_2 + D_2 \in P_2\}$$

$$\Pi_{\Delta 3} = \{\Pi_3 x_3 = x_3 \begin{bmatrix} -2k_{\min}k_{\max} & k_{\min} + k_{\max} \\ k_{\min} + k_{\max} & -2 \end{bmatrix} | L_2 x_3 + D_2 \in P_2\}$$

where

$$\begin{aligned} &L_1 : \mathcal{S}_{\mathrm{B}} \mapsto \mathcal{S}_{\mathrm{B}} \times \mathcal{S}_{\mathrm{B}}, \quad \text{is defined as } L_1 x = (x, -x) \\ &D_1 = (0, I) \\ &P_1 = P_{\mathrm{B}} \times P_{\mathrm{B}} \\ &L_2 : \mathcal{S}_{\mathrm{R}} \mapsto \mathcal{S}_{\mathrm{R}} \times \mathcal{S}_{\mathrm{R}}, \quad \text{is defined as } L_2 x = (x, -x) \\ &D_2 = (0, I) \\ &P_2 = P_{\mathrm{R}} \times P_{\mathrm{R}} \end{aligned}$$

Let $X_k = \prod_{l=1}^{M_k} X_{kl}$ for $k = 1, \ldots, N$. Then in the notation above we can describe the primal optimization problem in terms of

$$\begin{aligned} X_{\mathrm{M}} &= \prod_{k=1}^{N} X_k = \prod_{k=1}^{N} \prod_{l=1}^{M_k} X_{kl} \\ Y_{\mathrm{M}} &= \prod_{k=1}^{N} \prod_{l=1}^{M_k} Y_{kl} \\ P_{\mathrm{M}} &= \prod_{k=1}^{N} \prod_{l=1}^{M_k} P_{kl} \in Y_{\mathrm{M}} \\ D_{\mathrm{M}} &= \{\{D_{kl}\}_{l=1}^{M_k}\}_{k=1}^{N} \in Y_{\mathrm{M}} \end{aligned}$$

The operators $L_{\mathrm{M}} : X_{\mathrm{M}} \mapsto Y_{\mathrm{M}}$, defined as $L_{\mathrm{M}} x = \{\{L_{kl} x_{kl}\}_{l=1}^{M_k}\}_{k=1}^{N}$, for any $x \in X_{\mathrm{M}}$, and the operator $\Pi_{\mathrm{M}} : X_{\mathrm{M}} \mapsto \mathcal{S}_{\infty}^{2m \times 2m}$ is defined as

$$\begin{aligned} \Pi_{\mathrm{M}} x &= \mathrm{daug}(\Pi_1 x_1, \ldots, \Pi_N x_N) \\ &= \mathrm{daug}(\sum_{l=1}^{M_1} \Pi_{1l} x_{1l}, \ldots, \sum_{l=1}^{M_N} \Pi_{Nl} x_{Nl}) \end{aligned}$$

for any $x = \{x_k\}_{k=1}^{N} = \{\{x_{kl}\}_{l=1}^{M_k}\}_{k=1}^{N}$.

The dual optimization is defined in terms of

$$X_{\mathrm{M}}^* = \prod_{k=1}^{N}\prod_{l=1}^{M_k} X_{kl}^*$$

$$Y_{\mathrm{M}}^* = \prod_{k=1}^{N}\prod_{l=1}^{M_k} Y_{kl}^*$$

$$P_{\mathrm{M}}^{\oplus} = \prod_{k=1}^{N}\prod_{l=1}^{M_k} P_{kl}^{\oplus} \subset Y_{\mathrm{M}}^*$$

where X_{kl}^* and Y_{kl}^* is the dual of X_{kl} and Y_{kl}, respectively, and $P_{kl}^{\oplus}$ is the positive cone in the dual space, which corresponds to the positive cone P_{kl} in Y_{kl}. It remains to determine the adjoints $L_{\mathrm{M}}^{\times}$ and $\Pi_{\mathrm{M}}^{\times}$. We have that $L_{\mathrm{M}}^{\times} : Y_{\mathrm{M}}^* \mapsto X_{\mathrm{M}}^*$ is defined as

$$L_{\mathrm{M}}^{\times} z = \{\{L_{kl}^{\times} z_{kl}\}_{l=1}^{M_k}\}_{k=1}^{N}$$

for any $z \in Y_{\mathrm{M}}^*$. To derive $\Pi_{\mathrm{M}}^{\times} : (\mathcal{S}_{\mathrm{B}}^{2m\times 2m})^* \mapsto X_{\mathrm{M}}^*$, let $x \in X_{\mathrm{M}}$ and $z \in (\mathcal{S}_{\mathrm{B}}^{2m\times 2m})^*$. Then

$$\begin{aligned}
\langle \Pi_{\mathrm{M}} x, z\rangle &= \langle \mathrm{daug}(\Pi_1 x_1, \ldots, \Pi_N x_N), z\rangle \\
&= \sum_{k=1}^{N} \langle \Pi_k x_k, \mathcal{P}_k z\rangle \\
&= \sum_{k=1}^{N}\sum_{l=1}^{M_k} \langle x_k, \Pi_{kl}^{+} \mathcal{P}_k z\rangle = \langle x, \Pi_{\mathrm{M}}^{\times} z\rangle
\end{aligned}$$

where $\mathcal{P}_k : (\mathcal{S}_{\mathrm{B}}^{2m\times 2m})^* \mapsto (\mathcal{S}_{\mathrm{B}}^{2m_k\times 2m_k})^*$ is defined as follows. Introduce $E_k^T = \mathrm{diag}(0, \ldots, 0, I_{m_k}, 0, \ldots, 0)$, then

$$\mathcal{P} z = [\, E_k \quad E_{k+N}\,]^T z\, [\, E_k \quad E_{k+N}\,]$$

Hence,

$$\Pi_{\mathrm{M}}^{\times} = \{\{\Pi_{kl}^{\times} \mathcal{P}_k z\}_{l=1}^{M_k}\}_{k=1}^{N}$$

The dual optimization problem becomes (restricted to **AM**) becomes.

Dual Optimization Problem 2

$$\mu_d = \min \sum_{k=1}^{N}\sum_{l=1}^{M_k} \langle D_{kl}, z_{kl}\rangle$$

$$\text{subj to} \begin{cases} z_0 \in P_{\mathrm{AM}}^{m\times m},\ z_{kl} \in (P_{kl}^{\oplus})_{\mathrm{AM}} \\ \langle I, z_0\rangle = 1 \\ L_{kl}^{\times} z_{kl} = \Pi_{kl}^{+} \mathcal{P}_k M_G^{\times} z_0 \end{cases} \tag{13}$$

□

The corresponding unfeasibility test is

unfeasibility test 2

The robustness condition corresponding to Primal Optimization Problem 2 is unfeasible if there exists $z_0 \in P_{\mathrm{AM}}^{m\times m}$ and $z_{kl} \in (\mathcal{P}_{kl}^{\oplus})_{\mathrm{AM}}$ such that

$$\mathcal{L}_{kl}^{\times} z_{\mathrm{M}} = \Pi_{kl}^{\times} \mathcal{P}_k M_G^{\times} z_0$$

$$\sum_{k=1}^{N}\sum_{l=1}^{M_k} \langle D_{kl}, z_{kl}\rangle = 0$$

It is clear from (13) and (2) that we can derive the dual optimization problem for a system with block-diagonal perturbation Δ from the dual of its diagonal parts. We will in the next section derive the dual of some perturbations that are common in practice. The conditions of Theorem 5 will be satisfied for all of them.

7. Examples

In this section the dual is derived for four perturbations that are common in applications. We will only consider one block perturbations. More complicated perturbation structures built by diagonalization of these four perturbations are treated using the ideas in Section 6. We will restrict dual spaces and the domain of the adjoints to **AM**. It is trivial to check that the conditions of Theorem (5) are satisfied for all examples in this section.

The Real Structured Singular Value

We consider the case with one repeated bounded real parameter, i.e. the case when $\Delta = \delta I$, where $\delta \in [-1, 1]$. Then Π_Δ consists of the matrix functions of the form

$$\Pi = \begin{bmatrix} X & (iY) \\ (iY)^* & -X \end{bmatrix}$$

where $X, Y \in \mathcal{S}_{\mathrm{B}}^{m\times m}$ satisfies $0 \leq X(j\omega) = X^*(j\omega) \leq I$ and $-I \leq Y(j\omega) = Y^*(j\omega) \leq I$ for all ω. The multiplier constraints can be written as $LX + D_x \in P_x$ and $LY + D_y \in P_y$, where

$$\begin{aligned} &P_x = P_y = \mathcal{P}_B^{m\times m} \times \mathcal{P}_B^{m\times m} \\ &D_x = (0, I) \quad \text{and} \quad D_y = (I, I) \\ &L : \mathcal{S}_{\mathrm{B}}^{m\times m} \mapsto \mathcal{S}_{\mathrm{B}}^{m\times m} \times \mathcal{S}_{\mathrm{B}}^{m\times m}, \quad \text{defined as} \quad LX = (X, -X) \end{aligned}$$

The adjoint $L^\times : \mathcal{S}_{\mathrm{AM}}^{m\times m} \times \mathcal{S}_{\mathrm{AM}}^{m\times m} \mapsto \mathcal{S}_{\mathrm{AM}}^{m\times m}$ is defined as $L^\times z = z_1 - z_2$ for any $z = (z_1, z_2) \in \mathcal{S}_{\mathrm{AM}}^{m\times m} \times \mathcal{S}_{\mathrm{AM}}^{m\times m}$.

We have $\Pi = \Pi_x X + \Pi_y Y$, where

$$\Pi_x X = \begin{bmatrix} X & 0 \\ 0 & -X \end{bmatrix}, \quad \Pi_y Y = \begin{bmatrix} 0 & (iY) \\ (iY)^* & 0 \end{bmatrix}$$

and the corresponding adjoints $\Pi_x^\times, \Pi_y^\times : \mathcal{S}_{\mathrm{AM}}^{2m\times 2m} \mapsto \mathcal{S}_{\mathrm{AM}}^{m\times m}$ are given as

$$\Pi_x^\times Z = Z_{11} - Z_{22}, \quad \Pi_y^\times Z = iZ_{12} + (iZ_{12})^*$$

for any

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{bmatrix} \in \mathcal{S}_{\mathrm{AM}}^{2m\times 2m}$$

The dual optimization problem is according to (13)

$$\mu_d = \min \langle I, z_{x2} \rangle + \langle I, z_{y1} + z_{y2} \rangle$$

$$\text{subj to} \begin{cases} z_0, z_{x1}, z_{x2}, z_{y1}, z_{y2} \in P_{\mathrm{AM}}^{m\times m} \\ \langle I, z_0 \rangle = 1 \\ z_{x1} = z_{x2} + G z_0 G^* - z_0 \\ z_{y1} = z_{y2} + iGz_0 + (iGz_0)^* \end{cases}$$

It is possible to simplify this optimization problem by noticing

1. We can eliminate z_{x1} from the optimization problem. This makes the constraint involving z_{x1} and z_{x2} into a positivity constraint. For notation we use $z_x = z_{x2}$.
2. We can eliminate z_{y1} from the optimization problem. This makes the constraint involving z_{y1} and z_{y2} into a positivity constraint. For notation we use $z_y = z_{y2}$.

The resulting dual becomes

DUAL OPTIMIZATION PROBLEM 3—The Real Structured Singular Value

$$\mu_d = \min\langle I, z_x\rangle + \langle I, z_y + iGz_0 + (iGz_0)^*\rangle$$
$$\text{subj to} \begin{cases} z_0, z_x, z_y \in P_{\text{AM}}^{m\times m} \\ \langle I, z_0\rangle = 1 \\ z_x + Gz_0G^* - z_0 \in P_{\text{AM}}^{m\times m} \\ z_y + iGz_0 + (iGz_0)^* \in P_{\text{AM}}^{m\times m} \end{cases} \tag{14}$$

□

This dual optimization problem can also be formulated as

$$\min_{\begin{cases} z_0 \in P_{\text{AM}}^{m\times m} \\ \langle I, z_0\rangle = 1 \end{cases}} \int_{-\infty}^{\infty} \operatorname{tr}(\{z_0 - Gz_0G^*\}^+)d\omega + \int_{-\infty}^{\infty} \operatorname{tr}|iGz_0 + (iGz_0)^*|d\omega$$

where $|T|$ denotes the positive square root of $T \in \mathcal{S}_{\text{B}}^{m\times m}$ and $\{T\}^+ = \frac{1}{2}(|T|+T)$ denotes the positive part of T.

It is clear that for the case when we only consider a perturbation $\Delta = \delta I_m$ with delta as above then this becomes a one frequency optimization problem. This means that we should let z_0, z_x and z_y have only one dirac at some frequency ω, and the problem becomes an LMI optimization problem involving complex matrices. We can solve for the optimum by sweeping over ω. We also notice that if $\Delta = \Delta(j\omega)I$, with $\Delta(j\omega)$ the transfer function of a SISO LTI system then the primal and dual are as above with the exception that only the variables X and z_x appear in the optimization problems.

UNFEASIBILITY TEST 3
The robustness test for the real structured singular value is unfeasible iff there exists a frequency $\omega_0 \geq 0$ such that $G(j\omega_0)$ has a real valued eigenvalue with magnitude greater that one.

Proof: This is a well known result, but we give the proof in any case. We first conclude from (14) that the dual corresponding to real-μ is unfeasible iff there exists $\omega_0 \geq 0$ and $z_0 \in P_{\text{C}}$ such that

$$\begin{aligned} G(j\omega_0)z_0G^*(j\omega_0) - z_0 &\in P_{\text{C}} \\ G(j\omega_0)z_0 &= z_0G^*(j\omega_0) \end{aligned} \tag{15}$$

Now assume $G(j\omega_0)$ has a real valued eigenvalue with corresponding eigenvector v. Let $z_0 = vv^*$. With this z_0 we have

$$\begin{aligned} G(j\omega_0)z_0G^*(j\omega_0) - z_0 &= z_0(\lambda^2 - 1) \in P_{\text{C}} \\ G(j\omega_0)z_0 &= z_0G^*(j\omega_0) = 0 \end{aligned}$$

For the converse we need the following lemma, which follows from a result in [16]

LEMMA 2
If $R \in \mathbf{C}^{m\times m}$ satisfies $RS = SR^*$ for some $S \in P_{\mathrm{C}}^{m\times m}$, then $S = \sum_{k=1}^{m} s_k s_k^*$, where $R s_k s_k^* = s_k s_k^* R^*$.

The second condition from (15) and the lemma implies that $z_0 = \sum_{k=1}^{m} z_k z_k^*$, where $G(j\omega_0) z_k z_k^* = z_k z_k^* G(j\omega_0)^*$ for $k = 1, \ldots, m$. Assume $z_k \neq 0$, then multiplication with z_k from the right in the last identity gives

$$G(j\omega_0) z_k = \lambda_k z_k, \quad \lambda_k = z_k^* G(j\omega_0) z_k / (z_k^* z_k)$$

This implies

$$0 = G(j\omega_0) z_k z_k^* - z_k z_k^* G^*(j\omega_0) = z_k z_k^* (\lambda_k - \overline{\lambda}_k)$$

from which we conclude that λ_k is real-valued for all $k = 1, \ldots, m$ with nonzero z_k. Hence, with $z_0 = \sum_{k=1}^{m} z_k z_k^*$, the first condition in (15) gives

$$0 \leq \sum_{k=1}^{m} G(j\omega_0) z_k z_k^* G^*(j\omega_0) - z_k z_k^* = \sum_{k=1}^{m} (\lambda_k^2 - 1) z_k z_k^*$$

which implies that $|\lambda_k| \geq 1$ for at least one $k \in \{1, \ldots, m\}$.

$\square$

Time-varying Parameter

Consider the case with a repeated time-varying parameter, i.e. the case when $\Delta = \delta(t) I$, where $\delta(t) \in [-1, 1]$. We assume that the rate of time-variation is arbitrary. One possible choice of Π_Δ is the set of matrix functions

$$\Pi = \begin{bmatrix} X & Y \\ Y^T & -X \end{bmatrix}$$

where $0 \leq X = X^T \leq I$ and where $Y = -Y^T$ is bounded as $-I \leq iY \leq I$. The multiplier constraint on X can be written as $L_x X + D_x \in P_x$ where

$$\begin{aligned} P_x &= P_{\mathrm{R}}^{m\times m} \times P_{\mathrm{R}}^{m\times m} \\ D_x &= (I_m, I_m) \\ L_x &: \mathcal{S}_{\mathrm{R}}^{m\times m} \mapsto \mathcal{S}_{\mathrm{R}}^{m\times m} \times \mathcal{S}_{\mathrm{R}}^{m\times m}, \quad \text{defined as} \quad L_x X = (X, -X) \end{aligned}$$

The adjoint of L_x is defined as

$$L_x^\times : \mathcal{S}_{\mathrm{R}}^{m\times m} \times \mathcal{S}_{\mathrm{R}}^{m\times} \mapsto \mathcal{S}_{\mathrm{R}}^{m\times m}, \quad \text{defined as} \quad L_x^\times z_x = z_{x1} - z_{x2}$$

for any $z_x = (z_{x1}, z_{x2}) \in \mathcal{S}_{\mathrm{R}}^{m\times m} \times \mathcal{S}_{\mathrm{R}}^{m\times m}$. The constraint on Y can be written as $L_y Y + D_y \in P_y$, where

$$\begin{aligned} P_y &= P_{\mathrm{C}}^{2m\times 2m} \times P_{\mathrm{C}}^{2m\times 2m} \\ D_y &= (I_m, I_m) \\ L_y &: \mathcal{A}_{\mathrm{R}}^{m\times m} \mapsto \mathcal{S}_{\mathrm{C}}^{m\times m} \times \mathcal{S}_{\mathrm{C}}^{m\times m}, \quad \text{defined as} \quad L_y Y = (iY, -iY) \end{aligned}$$

The adjoint of L_y is defined as

$$L_y^\times : \mathcal{S}_{\mathrm{C}}^{m\times m} \times \mathcal{S}_{\mathrm{C}}^{m\times m} \mapsto \mathcal{A}_{\mathrm{R}}^{m\times m}, \quad \text{defined as} \quad L_y^\times z_y = \mathrm{Im} z_{y2} - \mathrm{Im} z_{y1}$$

for any $z_y = (z_{y1}, z_{y2}) \in \mathcal{S}_{\mathrm{C}}^{m\times m} \times \mathcal{S}_{\mathrm{C}}^{m\times m}$.

We have $\Pi = \Pi_x X + \Pi_y Y$, where

$$\Pi_x : \mathcal{S}_{\mathrm{R}}^{m\times m} \mapsto \mathcal{S}_{\mathrm{B}}^{2m\times 2m}, \quad \text{defined as} \quad \Pi_x X = \begin{bmatrix} X & 0 \\ 0 & -X \end{bmatrix}$$

$$\Pi_y : \mathcal{A}_{\mathrm{R}}^{m\times m} \mapsto \mathcal{S}_{\mathrm{B}}^{2m\times 2m}, \quad \text{defined as} \quad \Pi_y Y = \begin{bmatrix} 0 & Y \\ Y^T & 0 \end{bmatrix}$$

The corresponding adjoints are given as

$$\Pi_x^\times : \mathcal{S}_{\mathrm{AM}}^{2m\times 2m} \mapsto \mathcal{S}_{\mathrm{R}}^{m\times m}, \quad \text{defined as} \quad \Pi_x^\times Z = \int_{-\infty}^{\infty} (z_{11} - z_{22}) d\omega$$

$$\Pi_y^\times : \mathcal{S}_{\mathrm{AM}}^{2m\times 2m} \mapsto \mathcal{A}_{\mathrm{R}}^{m\times m}, \quad \text{defined as} \quad \Pi_x^\times z = \int_{-\infty}^{\infty} (z_{12}^* - z_{12}) d\omega$$

for any

$$z = \begin{bmatrix} z_{11} & z_{12} \\ z_{12}^* & z_{22} \end{bmatrix} \in \mathcal{S}_{\mathrm{AM}}^{2m\times 2m}$$

The dual optimization becomes

$$\mu_d = \min \mathrm{tr}\{z_{x2}\} + \mathrm{tr}\{z_{y1} + z_{y2}\} = \min \mathrm{tr}\{z_{x2}\} + \mathrm{tr}\{\mathrm{Re} z_{y1} + \mathrm{Re} z_{y2}\}$$

$$\text{subj to} \begin{cases} z_0 \in P_{\mathrm{AM}}^{m\times m}, z_{y1}, z_{y2} \in P_{\mathrm{C}}^{m\times m}, z_{x1}, z_{x2} \in P_{\mathrm{R}}^{m\times m} \\ \langle I, z_0 \rangle = 1 \\ z_{x1} = z_{x2} + \int_{-\infty}^{\infty} (G z_0 G^* - z_0) d\omega \\ \mathrm{Im} z_{y2} = \mathrm{Im} z_{y1} + \int_{-\infty}^{\infty} (z_o G^* - G z_0) d\omega \end{cases} \tag{16}$$

We can simplify this optimization problem by noticing the following

1. We can eliminate z_{x1} from the optimization problem. This makes the constraint involving z_x into a positivity constraint. For notation we use $z_x = z_{x2}$.
2. If we invoke Lemma 3 in the next Section, then the constraints involving z_y becomes

$$\begin{aligned} &\begin{bmatrix} \mathrm{Re} z_{y1} & \mathrm{Im} z_{y1} \\ -\mathrm{Im} z_{y1} & \mathrm{Re} z_{y1} \end{bmatrix} \geq 0 \\ &\begin{bmatrix} \mathrm{Re} z_{y2} & \mathrm{Im} z_{y1} + \int_{-\infty}^{\infty} \mathrm{Hm}\{G z_0\} d\omega \\ -\mathrm{Im} z_{y1} - \int_{-\infty}^{\infty} \mathrm{Hm}\{G z_0\} d\omega & \mathrm{Re} z_{y2} \end{bmatrix} \geq 0 \end{aligned} \tag{17}$$

where $\mathrm{Hm}\{G z_0\} = z_0 G^* - G z_0$. It is possible reduce these two constraints into

$$\begin{bmatrix} z_y & \int_{-\infty}^{\infty} (z_o G^* - G z_0) d\omega \\ -\int_{-\infty}^{\infty} (z_o G^* - G z_0) d\omega & z_y \end{bmatrix} \geq 0 \tag{18}$$

where $z_y \in P_{\mathrm{R}}^{m\times m}$. This follows from the following argument. First it is clear that (18) follows from (17) when $\mathrm{Re} z_{y1} = \mathrm{Im} z_{y1}$ and if we let $z_y = \mathrm{Re} z_{y2}$. Next we need to show that the reduced constraint in (18) does not impose conservativity on the objective value in (16). First notice that

$$\begin{bmatrix} \mathrm{Re}z_{y1} & \mathrm{Im}z_{y1} \\ -\mathrm{Im}z_{y1} & \mathrm{Re}z_{y1} \end{bmatrix} \geq 0 \quad \Longleftrightarrow \quad \begin{bmatrix} \mathrm{Re}z_{y1} & -\mathrm{Im}z_{y1} \\ \mathrm{Im}z_{y1} & \mathrm{Re}z_{y1} \end{bmatrix} \geq 0 \qquad (19)$$

which follows since the matrices are related by a similarity transform. Next assume that $\mathrm{Re}z_{y1}, \mathrm{Re}z_{y2}$ and $\mathrm{Im}z_{y1}$ are such that the constraints in (17) are satisfied. Then using the equivalence in (19) it follows that the sum of the constraints gives

$$\begin{bmatrix} \mathrm{Re}z_{y1} + \mathrm{Re}z_{y2} & \int_{-\infty}^{\infty}(z_0G^* - Gz_0)d\omega \\ -\int_{-\infty}^{\infty}(z_0G^* - Gz_0)d\omega & \mathrm{Re}z_{y1} + \mathrm{Re}z_{y2} \end{bmatrix} \geq 0$$

from which our statement follows. The resulting dual optimization problem becomes

DUAL OPTIMIZATION PROBLEM 4—Time-Varying Parameter

$$\mu_d = \min \mathrm{tr}\{z_x\} + \mathrm{tr}\{z_y\}$$
$$\text{subj to}$$
$$\begin{cases} z_0 \in P_{\mathrm{AM}}^{m\times m}, z_y, z_x \in P_{\mathrm{R}}^{m\times m} \\ \langle I, z_0\rangle = 1 \\ \begin{bmatrix} z_y & \int_{-\infty}^{\infty}(z_oG^* - Gz_0)d\omega \\ -\int_{-\infty}^{\infty}(z_oG^* - Gz_0)d\omega & z_y \end{bmatrix} \geq 0 \\ z_x + \displaystyle\int_{-\infty}^{\infty}(Gz_0G^* - z_o)d\omega \geq 0 \end{cases} \qquad (20)$$

□

This dual can equivalently be formulated as

$$\min_{\begin{cases} z_0 \in P_{\mathrm{AM}}^{m\times m} \\ \langle I, z_0\rangle = 1 \end{cases}} \mathrm{tr}\{\int_{-\infty}^{\infty}(z_0 - Gz_0G^*)d\omega\}^+ + \mathrm{tr}|\int_{-\infty}^{\infty}(z_0G^* - Gz_0)d\omega|$$

where $|T|$ denotes the positive square root of $T \in \mathcal{S}_{\mathrm{R}}^{m\times m}$ and $\{T\}^+ = \frac{1}{2}(|T| + T)$ denotes the positive part of T.
The corresponding unfeasibility test is

UNFEASIBILITY TEST 4
The robustness test for a time-varying real parameter is unfeasible if there exists a set of frequencies $\omega_k \geq 0$ and corresponding $z_{0k} \in P_{\mathrm{C}}^{m\times m}$ such that

$$\sum_{k=1}^{\infty} \mathrm{Re}\{z_{0k}G^*(i\omega_k) - G(i\omega_k)z_{0k}\} = 0$$
$$\sum_{k=1}^{\infty} \mathrm{Re}\{G(i\omega_k)z_{0k}G^*(i\omega_k) - z_{0k}\} \geq 0$$

□

Sector Bounded Nonlinearity

Here we consider the case when the perturbation is a sector bounded static nonlinear gain, possibly time-varying, defined by the relation $K_{\min}x(t) \leq \phi(t, x(t)) \leq K_{\max}x(t)$ for all $(t, x(t)) \in \mathbf{R} \times \mathbf{R}$. ϕ satisfies the set of IQC:s defined by the following matrices

$$\Pi = x \begin{bmatrix} -2K_{\min}K_{\max} & K_{\min} + K_{\max} \\ K_{\min} + K_{\max} & -2 \end{bmatrix}$$

where $0 \leq x \leq 1$. The multiplier constraint is treated exactly as in the last section. We can write $\Pi = \Pi_x x$, where $\Pi : \mathbf{R} \mapsto \mathcal{S}_{\mathrm{B}}^{2\times 2}$ is defined in an obvious way and the adjoint $\Pi_x^{\times} : \mathcal{S}_{\mathrm{AM}}^{2\times 2} \mapsto \mathbf{R}$ is defined as

$$\Pi_x^{\times} z = \int_{-\infty}^{\infty} [(K_{\min} + K_{\max})(z_{12} + z_{12}^*) - 2K_{\min}K_{\max}z_{11} - 2z_{22}]d\omega$$

The dual becomes

DUAL OPTIMIZATION PROBLEM 5—Sector Nonlinearity

$$\mu_d = \min z_x$$
$$\text{subj to} \begin{cases} z_0 \in P_{\mathrm{AM}}^{2\times 2}, z_x \in P_{\mathrm{R}} \\ z_x + \displaystyle\int_{-\infty}^{\infty} [(K_{\min} + K_{\max})(z_0 G^* + G z_0) \\ \qquad - 2K_{\min}K_{\max} G z_0 G^* - 2z_0]d\omega \geq 0 \end{cases} \tag{21}$$

□

This dual can also be formulated as

$$\min_{\begin{cases} z_0 \in P_{\mathrm{AM}}^{2\times 2} \\ \langle I, z_0 \rangle = 1 \end{cases}} \max\left(0, -\int_{-\infty}^{\infty} \Pi_x^{\times} M_G^{\times} z_0 d\omega\right)$$

where

$$\Pi_x^{\times} M_G^{\times} z_0 = (K_{\min} + K_{\max})(z_0 G^* + G z_0) - 2K_{\min}K_{\max} G z_0 G^* - 2z_0$$

Multiplication with Harmonic Parameter

Consider the case when $(\Delta u)(t) = u(t)\cos(\omega_0 t)$, then Δ satisfies the IQC defined by any matrix function with structure, [13].

$$\begin{bmatrix} 0.5(X(j(\omega + \omega_0)) + X(j(\omega - \omega_0))) & 0 \\ 0 & -X(j\omega) \end{bmatrix} \tag{22}$$

where $0 \leq X(j\omega) = X^*(j\omega)$, $\forall \omega \in \mathbf{R}$. The multiplier constraint can be written as $L_x X + D_x \in P_x$, where

$$P_x = P_{\mathrm{B}}^{m\times m} \times P_{\mathrm{B}}^{m\times m}$$
$$D_x = (I_m, I_m)$$
$$L_x : \mathcal{S}_{\mathrm{B}}^{m\times m} \mapsto \mathcal{S}_{\mathrm{B}}^{m\times m} \times \mathcal{S}_{\mathrm{B}}^{m\times m}, \quad \text{defined as} \quad L_x X = (X, -X)$$

The adjoint $L_x^\times : \mathcal{S}_{\mathrm{AM}}^{m\times m} \times \mathcal{S}_{\mathrm{AM}}^{m\times m} \mapsto \mathcal{S}_{\mathrm{AM}}^{m\times m}$ is defined as $L^\times z = z_1 - z_2$ for any $z = (z_1, z_2) \in \mathcal{S}_{\mathrm{AM}}^{m\times m}$. We have $\Pi = \Pi_x X$, where $\Pi_x : \mathcal{S}_{\mathrm{B}}^{m\times m} \mapsto \mathcal{S}_{\mathrm{B}}^{2m\times 2m}$ is defined as

$$\Pi_x X = \begin{bmatrix} 0.5(X(j(\omega+\omega_0)) + X(j(\omega-\omega_0))) & 0 \\ 0 & -X(j\omega) \end{bmatrix}$$

The adjoint $\Pi_x^\times : \mathcal{S}_{\mathrm{AM}}^{2m\times 2m} \mapsto \mathcal{S}_{\mathrm{AM}}^{m\times m}$ is defined as

$$\Pi_x^\times z = 0.5(z_{11}(\omega+\omega_0) + z_{11}(\omega-\omega_0)) - z_{22}(\omega)$$

for any

$$z = \begin{bmatrix} z_{11} & z_{12} \\ z_{12}^* & z_{22} \end{bmatrix} \in \mathcal{S}_{\mathrm{AM}}^{2m\times 2m}$$

As in the derivation of the dual corresponding to real-μ we arrive at the following dual optimization problem

Dual Optimization Problem 6—Harmonic Oscillation

$$\begin{aligned} &\mu_d = \min\langle I, z_x\rangle \\ &\quad \text{subj to} \\ &\qquad \begin{cases} z_0, z_x \in P_{\mathrm{AM}}^{m\times m} \\ \langle I, z_0\rangle = 1 \\ z_x + 0.5G(j(\omega+\omega_0))z_0(\omega+\omega_0)G^*(j(\omega+\omega_0)) + \\ 0.5G(j(\omega-\omega_0))z_0(\omega-\omega_0)G^*(j(\omega-\omega_0)) - z_0(\omega) \in P_{\mathrm{AM}}^{m\times m} \end{cases} \end{aligned} \tag{23}$$

□

This can equivalently be formulated as

$$\begin{cases} \min\limits_{\substack{z_0 \in P_{\mathrm{AM}}^{m\times m} \\ \langle I, z_0\rangle = 1}} \int_{-\infty}^{\infty} \{-\Pi_x^\times M_G^\times z_0\}^+ d\omega \end{cases}$$

where

$$\begin{aligned} \Pi_x^\times M_G^\times z_0(\omega) =& 0.5G(j(\omega+\omega_0))z_0(\omega+\omega_0)G^*(j(\omega+\omega_0)) \\ &+ 0.5G(j(\omega-\omega_0))z_0(\omega-\omega_0)G^*(j(\omega-\omega_0)) - z_0(\omega) \end{aligned}$$

We can get an approximate solution to the dual optimization problem in (23) by restricting attention to a finite set of frequencies $\Omega = \{\omega_1, \ldots, \omega_N\}$. The structure of the last constraint in (23) suggests that we choose the set Ω such that

$$\begin{aligned} &\omega_j = \omega_{M+j} - \omega_0 \\ &\omega_{(k-1)M+j} + \omega_0 = \omega_{kM+j} = \omega_{(k+1)M+j} - \omega_0 \\ &\omega_{(L-1)M+j} + \omega_0 = \omega_{LM+j} \end{aligned}$$

for $k = 1, \ldots, L-1$ and $j = 1, \ldots, M$. This gives M diracs within a frequency spann of ω_0. The corresponding dual optimization problem becomes

Dual Optimization Problem 7—Harmonic Oscillation

$$\mu_d = \inf_{\Omega} \sum_{k=0}^{L} \sum_{j=1}^{M} \mathrm{tr}\{z_{x(kM+j)}\}$$

subj to

$$\begin{cases} z_{0(kM+j)}, z_{x(kM+j)} \in P_{\mathbf{C}}^{m\times m} \\ \sum_{k=0}^{L}\sum_{j=1}^{M} \mathrm{tr}\{z_{0(kM+j)}\} = 1 \\ z_{xj} + 0.5 G_{M+j} z_{0(M+j)} G_{M+j}^T - z_{0j} \geq 0 \\ z_{x[kM+j]} + 0.5 G_{(k+1)M+j} z_{0[(k+1)M+j]} G_{(k+1)M+j}^T + \\ 0.5 G_{(k-1)M+j} z_{0[(k-1)M+j]} G_{(k-1)M+j}^T - z_{0[kM+j]} \geq 0 \\ z_{x[LM+j]} + 0.5 G_{(L-1)M+j} z_{0[(L-1)M+j]} G_{(L-1)M+j}^T - z_{0[LM+j]} \geq 0 \end{cases} \tag{24}$$

for $k = 1, \ldots, L-1$ and $j = 1, \ldots, M$,

□

This is an LMI problem involving complex matrices. We will in the next section see how such problems can be transformed into LMI problems involving only real-valued matrices.

8. Computational Issues

In problems involving block-diagonal perturbations with elements from the last section gives a dual optimization problem as in (13) and an unfeasibility test as in (2) that are infinite-dimensional LMI problems involving complex matrices. We can obtain upper bounds of the dual by restricting the optimization to a finite set of matrices $\Omega = \{\omega_0, \ldots, \Omega_N\}$. The corresponding optimization problem is a finite dimensional LMI problem which can be transformed to an LMI problem involving only real-valued matrices by invoking the following lemma.

Lemma 3
Let $z = z_r + iz_i \in \mathbf{C}^{m\times m}$ be a complex valued matrix with $z_r, z_i \in \mathbf{R}^{m\times m}$. We can represent z as a matrix in $\mathbf{R}^{2m\times 2m}$ as

$$Z = \begin{bmatrix} z_r & z_i \\ -z_i & z_r \end{bmatrix}$$

We then have the following properties.

1. The conditions for z to be Hermitian can be stated as $z = z^* \Leftrightarrow Z = Z^T$, which implies that $z_r = z_r^T$ and $z_i = -z_i^T$.
2. If z is Hermitian, then $z \geq 0 \Leftrightarrow Z \geq 0$.
3. Multiplication and addition of complex matrices corresponds to multiplication and addition of the corresponding real valued matrices. Hence, we have $z_1 + z_2 \Leftrightarrow Z_1 + Z_2$ and $z_1 z_2 \Leftrightarrow Z_1 Z_2$.
4. The identity $I_m \in \mathbf{C}^{m\times m}$ corresponds to $I_{2m} \in \mathbf{R}^{2m\times 2m}$.

5. $jI_m \in \mathbf{C}^{m\times m}$ corresponds to

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \in \mathbf{R}^{2m\times 2m}$$

Proof: This is a trivial fact from algebra.

□

9. Numerical Examples

In this section we apply the ideas of this paper in two examples.

EXAMPLE 2—Example from Section 2 continued
We obtain the dual optimization problem by putting together the duals corresponding to the three components as in (13). The component duals are obtained as described in Section 7. We use the following notation, where e_i is the i^{th} identity vector in $\mathbf{R}^6$.

$$\begin{aligned}
\Pi_1^\times \mathcal{P}_1 z &= \widetilde{\mathcal{P}}_1 z \stackrel{def}{=} e_1^T z e_1 - e_4^T z e_4 \\
\Pi_2^\times \mathcal{P}_2 z &= \int_{-\infty}^{\infty} \widetilde{\mathcal{P}}_2 z(\omega) d\omega = \sum_{k=1}^{\infty} 2\mathrm{Re}\widetilde{\mathcal{P}}_2 z_k \\
\Pi_3^\times \mathcal{P}_3 z &= \int_{-\infty}^{\infty} \widetilde{\mathcal{P}}_3 z(\omega) d\omega = \sum_{k=1}^{\infty} 2\mathrm{Re}\widetilde{\mathcal{P}}_3 z_k
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{\mathcal{P}}_2 z &\stackrel{def}{=} (k_{\min} + k_{\max})(e_2^T z e_5 + e_5^T z e_2) - 2k_{\min}k_{\max} e_2^T z e_2 - 2e_5^T z e_5 \\
\widetilde{\mathcal{P}}_3 z &\stackrel{def}{=} (k_{\min} + k_{\max})(e_3^T z e_6 + e_6^T z e_3) - 2k_{\min}k_{\max} e_3^T z e_3 - 2e_6^T z e_6
\end{aligned}$$

for any $z \in \mathcal{S}_{\mathrm{C}}^{6\times 6}$.

Let z_1, z_2 and z_3 correspond to x_1, x_2 and x_3 respectively. Then the dual optimization problem can be stated as

$$\begin{aligned}
\mu_d = \inf \sum_{k=1}^{\infty} & 2z_{1k} + z_2 + z_3 \\
\text{subj to} & \\
& \begin{cases}
z_{0k} \in \mathcal{S}_{\mathrm{C}}^{3\times 3}, z_{1k} \geq 0, z_2 \geq 0, z_3 \geq 0 \\
2\sum_{k=1}^{\infty} \mathrm{tr}\{z_{0k}\} = 1 \\
z_{1k} + \widetilde{\mathcal{P}}_1 M_G^+(i\omega_k) z_{0k} \in P_{\mathrm{C}} \\
z_2 + \sum_{k=1}^{\infty} 2\mathrm{Re}\widetilde{\mathcal{P}}_2 M_G^+(i\omega_k) z_{0k} \geq 0 \\
z_3 + \sum_{k=1}^{\infty} 2\mathrm{Re}\widetilde{\mathcal{P}}_3 M_G^+(i\omega_k) z_{0k} \geq 0
\end{cases}
\end{aligned} \tag{25}$$

If we restrict attention to a finite set of frequencies $\Omega = \{\omega_1, \ldots, \omega_N\}$, then (25) becomes an LMI involving complex-valued matrices, which can be transformed to an LMI involving only real-valued matrices by invoking Lemma 3.

Let

$$Z_{0k} = \begin{bmatrix} \mathrm{Re} z_{0k} & \mathrm{Im} z_{0k} \\ -\mathrm{Im} z_{0k} & \mathrm{Re} z_{0k} \end{bmatrix}$$

and

$$Z_{1k} = \begin{bmatrix} z_{1k} & 0 \\ 0 & z_{1k} \end{bmatrix}$$

Further let $\widetilde{\mathcal{P}}_1$ be defined in terms of

$$E_i = \begin{bmatrix} e_i & 0 \\ 0 & e_i \end{bmatrix}$$

and let $\widetilde{\mathcal{P}}_i$ $i = 2, 3$ be defined in terms of $(e_i^T, 0)$, for $i = 1, \ldots, 6$. Finally define the map $M_G^+(\omega) : \mathbf{R}^{6\times 6} \mapsto \mathbf{R}^{12\times 12}$ as

$$M_G^+(\omega)Z = \begin{bmatrix} \mathrm{Re}G_\omega & \mathrm{Im}G_\omega \\ I & 0 \\ -\mathrm{Im}G_\omega & \mathrm{Re}G_\omega \\ 0 & I \end{bmatrix} Z \begin{bmatrix} \mathrm{Re}G_\omega & \mathrm{Im}G_\omega \\ I & 0 \\ -\mathrm{Im}G_\omega & \mathrm{Re}G_\omega \\ 0 & I \end{bmatrix}^T$$

where $G_\omega = G(i\omega)$. The dual becomes

$$\mu_d = \inf_{\Omega} \sum_{k=1}^{N} \mathrm{tr}\{Z_{1k}\} + z_2 + z_3$$
$$\text{subj to}$$
$$\begin{cases} Z_{0k}, Z_{1k}, z_2, z_3 \geq 0, \quad k = 1, \ldots, N \\ \sum_{k=1}^{N} \mathrm{tr}\{Z_{0k}\} = 1 \\ Z_{1k} + \widetilde{\mathcal{P}}_1 M_G^+(\omega_k) Z_{0k} \geq 0, \quad k = 1, \ldots, N \\ z_2 + 2\sum_{k=1}^{N} \widetilde{\mathcal{P}}_2 M_G^+(\omega_k) Z_{0k} \geq 0 \\ z_3 + 2\sum_{k=1}^{N} \widetilde{\mathcal{P}}_3 M_G^+(\omega_k) Z_{0k} \geq 0 \end{cases} \tag{26}$$

The corresponding unfeasibility test becomes

UNFEASIBILITY TEST 5

The robustness test in (3) is unfeasible if there exists a set of frequencies $\Omega = \{\omega_1, \ldots, \omega_N\}$ and corresponding matrices $Z_{0k} \geq 0$ such that

$$\widetilde{\mathcal{P}}_1 M_G^+(\omega_k) Z_{0k} \geq 0, \quad k = 1, \ldots, N$$
$$\sum_{k=1}^{N} \widetilde{\mathcal{P}}_2 M_G^+(\omega_k) Z_{0k} \geq 0$$
$$\sum_{k=1}^{N} \widetilde{\mathcal{P}}_3 M_G^+(\omega_k) Z_{0k} \geq 0$$

□

Ω	$\widehat{\mu}_d$
$\{0\}$	0.8725
$\{1\}$	0.0188
$\{0,1\}$	0.0101

Table 1. Dual for $k = 0.05$

$R(s)$	$\widehat{\mu}_p$
1	-0.0077
$Ritz(1,1)$	-0.0093
$Ritz(1,2)$	-0.010
$Ritz(1,3)$	-0.010

Table 2. Primal for $k = 0.05$

Numerical Results

Assume that the plant G_p is a resonant system and that the controller G_r is a lag compensator and finally that the sensor dynamics is of first order. Let the transfer functions be

$$\begin{aligned} G_p(s) &= \frac{1}{s^2 + 0.2s + 1} \\ G_r(s) &= k\frac{s+1}{s+0.1} \\ G_s(s) &= \frac{10}{s+10} \end{aligned}$$

Further, let $k_{\max} = 1$, $k_{\min} = 0$, and $\varepsilon = 0.3$.

If the controller gain $k = 0.05$ then LMI optimization as in (26) with LMI-lab gave numerical results as in Table 1. No other combination of frequencies appear to give lower dual objective. We solved the primal problem in (3) with $x_1 = R^*UR$, where $R \in \mathbf{RH}_\infty^{M\times 1}$ is the basis multiplier and $U = U^T$ the coordinates that should be optimized. We achieved numerical results as Table 2. where

$$Ritz(p,n) = \begin{bmatrix} 1 & \frac{s-p}{s+p} & \cdots & \frac{(s-p)^n}{(s+p)^n} \end{bmatrix}^T$$

We see that even for simple choices of basis functions there is a relative duality gap $(\widehat{\mu_p} + \widehat{\mu_d})/|\widehat{\mu_p}| = 0.5\%$, which is very small.

The control system we have studied is not of much practical intersest, since it has a small loop gain, $(G_r(0)G_p(0)G_s(0) = 0.5)$. The problem here is that we describe the nonlinearities with IQC:s that are very general in the sense that they allow a large class of nonlinear functions. This is restrictive if ϕ_1 and ϕ_2 are for example saturations. If we use a more precise IQC description of the saturations then we expect that we can allow for a larger loop gain. However, the purpose of the example is to study how a primal-dual analysis can indicate the quality of certain finite dimensional restictions of the primal problem. The example is very succesful in this sense.

EXAMPLE 3—Ship Steering

We will in this example consider ship steering dynamics as in Example 9.6 in [1]. The dynamics for the ship can, with notation as in Figure 3, be

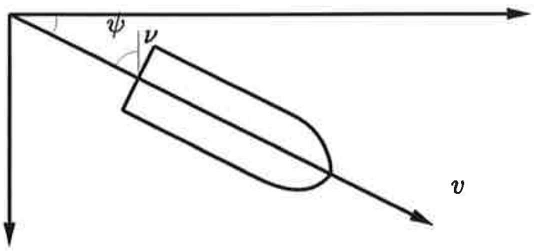

Figure 3. Notation used to describe the motion of ships.

approximated by the Nomoto model

$$\begin{aligned}\dot{x}(t) &= v(t)(-ax(t) + bv(t)\nu(t))\\ \dot{\psi}(t) &= x(t)\end{aligned}$$

where ψ denotes the heading of the ship, ν denotes the rudder angle and v is the speed of the ship. It is assumed that $v(t) \geq 0$. We will as in [1] study stability of the ship dynamics for an unstable tanker, which is controlled by a PD regulator

$$\begin{aligned}\nu &= -K\psi\\ K(s) &= k(1 + sT_d)\end{aligned}$$

where $k = 2.5$ and $T_d = 0.86$. It is also assumed that $a = -0.3$ and $b = 0.8$. The closed loop system can be illustrated as in Figure 1, with $\delta(t) = v(t)I_2$ and $G(s)$ replaced by

$$G_0(s) = \begin{pmatrix} -a/s & b \\ -K(s)/s^2 & 0 \end{pmatrix}$$

We will here study the particular case when $v(t) = v_0 + \tilde{v}(t)$, and $\tilde{v}(t) = A\cos(\omega_0 t)$. It is easy to see that the closed loop system is stable for a constant speed of $v_{const} > b_{nom}/(a_{nom}kT_d) = 0.1744$. This implies that $v_0 > 0.1744 + A$ is necessary for stability. We transform the system as in Figure 4 which gives an equivalent system with $\delta(t) = \cos(\omega_0 t)$ and the transformed system

$$G(s) = A(I - v_0 G_0(s))^{-1} G_0(s) \in \mathbf{RH}_\infty^{2\times 2}$$

The primal optimization problem becomes

$$\begin{aligned}&\mu_p = \inf -\gamma\\ &\quad \text{subj to}\\ &\quad \begin{cases} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\gamma I \\ \Pi \in \Pi_{\Delta harmonic}, \quad \gamma \in \mathbf{R} \end{cases}\end{aligned}$$

for all $\omega \geq 0$, where $\Pi_{\Delta harmonic}$ is the set of matrix functions defined as in (22). We solve the primal using LMI-computations as described in [7] with $X = R^* U R$, where $R \in \mathbf{RH}_\infty^{N\times 2}$ is a basis for X and $U = U^T \geq$ is the

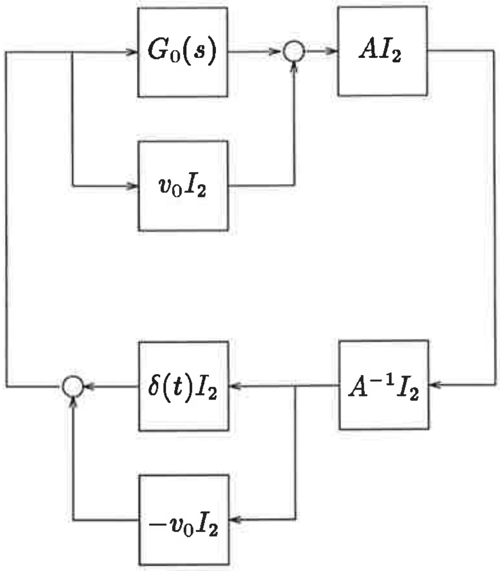

Figure 4. Transformed system

corresponding coordinates. For the dual optimization problem we transform (24) to an LMI involving real valued matrices. Let

$$Z_{ik} = \begin{bmatrix} Z_{ikR} & Z_{ikI} \\ -Z_{ikI} & Z_{ikR} \end{bmatrix}$$

for $i = 0, x$ and $k = 1, \ldots, N$, where $N = (L+1)M$. Here $Z_{ikR} = Z_{ikR}^T \in \mathbf{R}^{2\times 2}$ and $Z_{ikI} = -Z_{ikI}^T \in \mathbf{R}^{2\times 2}$. Further let

$$G_k = \begin{bmatrix} \mathrm{Re}G(j\omega_k) & \mathrm{Im}G(j\omega_k) \\ -\mathrm{Im}G(j\omega_k) & \mathrm{Re}G(j\omega_k) \end{bmatrix}$$

Then we arrive at the following LMI optimization problem

$$\begin{aligned}
\mu_d = \inf_{\Omega} & \sum_{k=0}^{L}\sum_{j=1}^{M} \mathrm{tr}\{Z_{x(kM+j)}\} \\
& \text{subj to} \\
& \begin{cases} Z_{0(kM+j)}, Z_{x(kM+j)} \geq 0 \\ \sum_{k=0}^{L}\sum_{j=1}^{M} \mathrm{tr}\{Z_{0(kM+j)}\} \\ Z_{xj} + 0.5 G_{M+j} Z_{0(M+j)} G_{M+j}^T - Z_{0j} \geq 0 \\ Z_{x[kM+j]} + 0.5 G_{(k+1)M+j} Z_{0[(k+1)M+j]} G_{(k+1)M+j}^T + \\ 0.5 G_{(k-1)M+j} Z_{0[(k-1)M+j]} G_{(k-1)M+j}^T - Z_{0[kM+j]} \geq 0 \\ Z_{x[LM+j]} + 0.5 G_{(L-1)M+j} Z_{0[(L-1)M+j]} G_{(L-1)M+j}^T - Z_{0[LM+j]} \geq 0 \end{cases}
\end{aligned} \tag{27}$$

for $k = 1, \ldots, L-1$ and $j = 1, \ldots, M$.

Numerical Results

We have have made computations for two cases.

$R(s)$	μ_p
I_2	-0.6543
$R(1,2,1)$	-0.6643
$R(1,2,3)$	-0.6793
$R(1,2,4)$	-0.6830
$R(1,2,5)$	-0.6882

L	M	ω_1	μ_d
2	1	0.28	0.7847
4	1	0.28	0.6906
6	1	0.28	0.6900
8	1	0.28	0.6899
8	2	0.28	0.6899

Table 3. Numerical results for the case $v_0 = 1$, $A = 0.2$ and $\omega_0 = 0.5$.

$R(s)$	μ_p
I_2	unfeasible
$R(1,2,1)$	unfeasible
$R(1,2,2)$	-0.0651
$R(1,2,3)$	-0.1718
$R(1,2,4)$	-0.1844
$R(1,2,5)$	-0.2173
$R(1,2,6)$	-0.2182

L	M	ω_1	μ_d
2	1	0.185	0.2305
4	1	0.185	0.2264
6	1	0.185	0.2264
2	2	0.185	0.2304
4	2	0.185	0.2264

Table 4. Numerical results for the case $v_0 = 0.5$, $A = 0.1$ and $\omega_0 = 0.5$.

- $v_0 = 1$, $A = 0.2$ and $\omega_0 = 0.5$. The numerical results are given in Table 3, where
$$R(p,m,n) = \begin{bmatrix} I_m & \frac{s}{s+p} & \cdots & \frac{s^n}{(s+p)^n I_m} \end{bmatrix}^T$$
The relative duality gap is $(\mu_d + \mu_p)/|\mu_p| = 0.0025$.
- $v_0 = 0.5$, $A = 0.1$ and $\omega_0 = 0.5$. The numerical results are given in Table 4. The relative duality gap is $(\mu_d + \mu_p)/|\mu_p| = 0.0375$.

We see that the relative duality gap is quite small in both cases.

10. Conclusions

We have derived a format for deriving the dual of a large class of robustness problems involving multipliers. Solutions to the dual optimization problem can be achieved by LMI optimization. The results of the paper have been applied to two examples.

11. Appendix A: Proof of the Duality Result

We will here put Proposition 1 into perspective of the Fenchel duality theorem as stated in [10]. If $\mathcal{C}$ and $\mathcal{D}$ are convex sets and f and g is a convex and concave functional respectively, then we define the convex sets $[f, \mathcal{C}]$ and $[g, \mathcal{D}]$ as

$$[f, \mathcal{C}] = \{(r, x) \in \mathbf{R} \times X : x \in \mathcal{C}, f(x) \leq r\}$$
$$[g, \mathcal{D}] = \{(r, x) \in \mathbf{R} \times X : x \in \mathcal{D}, r \leq g(x)\}$$

THEOREM 3—Fenchel Duality Theorem
Assume that f and g are, respectively, convex and concave functionals on the convex sets $\mathcal{C}$ and $\mathcal{D}$ in a normed space X. Assume that $\mathcal{C} \cap \mathcal{D}$ contains points

in the relative interior of $\mathcal{C}$ and $\mathcal{D}$ and that either $[f, \mathcal{C}]$ or $[g, \mathcal{D}]$ has nonempty interior. Suppose further that $\inf_{x \in \mathcal{C} \cap \mathcal{D}}\{f(x) - g(x)\}$ is finite. Then

$$\inf_{x \in \mathcal{C} \cap \mathcal{D}}\{f(x) - g(x)\} = \max_{x^* \in \mathcal{C}^* \cap \mathcal{D}^*}\{g^*(x^*) - f^*(x^*)\} \tag{28}$$

where the maximum on the right is achieved by some $x_0{}^* \in \mathcal{C}^* \cap \mathcal{D}^*$.

Let $\mathcal{C} = \mathcal{L} + D$, $\mathcal{D} = P$, $f(x) = \langle x, C\rangle$ and $g(x) = 0$. We can then state the primal optimization problem in (4) as

$$\mu_p = \inf_{x \in \mathcal{C} \cap \mathcal{D}}\{f(x) - g(x)\}$$

It is clear that $\mathcal{C}$ and $\mathcal{D}$ are convex sets and by the assumption in Proposition 1 we have that $\mathcal{C} \cap \mathcal{D}$ contains points in the relative interior of $\mathcal{C}$ and $\mathcal{D}$, and that μ_p is finite. Further f is a continuous convex functional by the definition of X^* and g is clearly continuous and concave. The continuity of f and g implies that the convex sets $[f, \mathcal{C}]$ and $[g, \mathcal{D}]$ have interior points. This follows from Proposition 1 in Section 7.9 of [10]. The conjugate sets $\mathcal{C}^*$ and $\mathcal{D}^*$ are defined as

$$\begin{aligned}
\mathcal{C}^* &= \{x^* \in X^* : \sup_{x \in \mathcal{C}}[\langle x, x^*\rangle - f(x)] < \infty\} \\
&= \{x^* \in X^* : \sup_{x \in \mathcal{L}+D} \langle x, x^* - C\rangle < \infty\} = \mathcal{L}^\perp + C \\
\mathcal{D}^* &= \{x^* \in X^* : \inf_{x \in \mathcal{D}}[\langle x, x^*\rangle - g(x)] > -\infty\} \\
&= \{x^* \in X^* : \inf_{x \in P}\langle x, x^*\rangle > -\infty\} = P^\oplus
\end{aligned}$$

The conjugate functionals f^* and g^* defined on $\mathcal{C}^*$ and $\mathcal{D}^*$, respectively are defined as

$$\begin{aligned}
f^*(x^*) &= \sup_{x \in \mathcal{C}}[\langle x, x^*\rangle - f(x)] \\
&= \sup_{x \in P}\langle x, x^* - C\rangle = \langle D, x^*\rangle - \langle D, C\rangle \\
g^*(x^*) &= \inf_{x \in \mathcal{D}}[\langle x, x^*\rangle - g(x)] \\
&= \inf_{x \in P}\langle x, x^*\rangle = 0
\end{aligned}$$

From Theorem 3 we have

$$\begin{aligned}
\mu_p &= \inf_{x \in \mathcal{C} \cap \mathcal{D}}\{f(x) - g(x)\} \\
&= \max_{x^* \in \mathcal{C}^* \cap \mathcal{D}^*}\{g^*(x^*) - f^*(x^*)\} \\
&= \langle D, C\rangle - \min_{x^* \in P^\oplus \cap (\mathcal{L}^\perp + C)}\langle D, x^*\rangle
\end{aligned}$$

from which Proposition 1 follows.

12. Appendix B: The proof of Theorem 1

Assume that we consider optimization over a set $\Pi_\Delta \subset \mathcal{S}_{\mathrm{B}}^{2m \times 2m}$ and that μ_p is the resulting objective value. Let $\varepsilon > 0$. We will show that there exists a piece-wise constant function $\Pi_{pc} \in \Pi_\Delta$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi_{pc}(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < (\mu_p + \varepsilon)I, \quad \forall \omega \geq 0 \tag{29}$$

Then the theorem follows since ε is arbitrary. We will actually prove more than that since we will construct a Π_{pc}, with only a finite number of discontinuities.

By assumption there exists $\Pi_0 \in \Pi_\Delta \subset \mathcal{S}_{\mathrm{B}}^{2m\times 2m}$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi_0(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < (\mu_p + \varepsilon/2)I, \quad \forall \omega \geq 0$$

Let $M_G : \mathcal{S}_{\mathrm{C}}^{2m\times 2m} \times \mathbf{R} \mapsto \mathcal{S}_{\mathrm{C}}^{m\times m}$ be the operator defined as

$$M_G(\Pi, \omega) = \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}$$

By the properties of $G \in \mathbf{RH}_\infty^{m\times m}$, the exists $\bar{\omega}$ such that

$$\max_{\omega \geq \omega} \rho(M_G(\Pi_0(j\omega), \omega) - M_G(\Pi_0(\bar{\omega}), \bar{\omega})) < \varepsilon/2$$

where $\rho(H) = \lambda_{\max}(H)$, for $H \in \mathcal{S}_{\mathrm{C}}^{m\times m}$. Further by the uniform continuity of $G \in \mathbf{RH}_\infty^{m\times m}$ on $[0, \bar{\omega}]$, there exist $\delta > 0$ and N such that

$$\cup_{k=1}^N [(2k-2)\delta, 2k\delta] = [0, \bar{\omega}]$$

and

$$\max_{\omega \in [(2k-2)\delta, 2k\delta]} \rho(M_G(\Pi_0(j\omega, \omega) - M_G(\Pi_0(j(2k-1)\delta), (2k-1)\delta)) < \varepsilon/2$$

Then the choice

$$\Pi_{pc}(\omega) = \begin{cases} \Pi_0(j(2k-1)\delta), & \omega \in [(2k-2)\delta, 2k\delta), k = 1, \ldots, N \\ \Pi_0(\bar{\omega}), & \omega \geq \bar{\omega} \end{cases}$$

satisfies (29).

13. Appendix C: The proof of Theorem 5

The main idea for our proof is to use the symmetry in Proposition 1. We will show the dual optimization problem in (12) satisfies the conditions of the the *primal* in Proposition 1 and that the corresponding *dual* in Proposition 1 is the optimization problem in (2).

We will first derive a compact description of $\mathcal{L}$ in (10) and $\mathcal{L}^\perp$ in (11). Let $L : X_{\mathrm{M}} \times \mathbf{R} \mapsto X$ be the operator defined as

$$Lx = (-M_G \Pi_{\mathrm{M}} x_{\mathrm{M}} - \gamma I, L_{\mathrm{M}} x_{\mathrm{M}}, \gamma)$$

for any $x = (x_{\mathrm{M}}, \gamma) \in X_{\mathrm{M}} \times \mathbf{R}$. The corresponding adjoint $L^\times : X^* \mapsto X_{\mathrm{M}}^* \times \mathbf{R}$ is

$$L^\times z = (L_{\mathrm{M}}^\times z_{\mathrm{M}} - \Pi_{\mathrm{M}}^\times M_G^\times z_0, z_\gamma - \langle I, z_0 \rangle)$$

for any $z = (z_0, z_{\mathrm{M}}, z_\gamma) \in X^*$. We then have

$$\mathcal{L} = \{Lx : x \in X_{\mathrm{M}} \times \mathbf{R}\} = \mathcal{R}(L)$$
$$\mathcal{L}^\perp = \{z \in X^* : L^\times z = 0\} = \mathcal{N}(L^\times)$$

where $\mathcal{R}(L)$ denotes the *range space* of L and $\mathcal{N}(L^\times)$ denotes the *nullspace* of $L^\times$.

Next let us define Z and P_Z as

$$Z = (X^*)_{\mathrm{AM}} = \mathcal{S}_{\mathrm{AM}}^{m\times m} \times (Y_{\mathrm{M}}^*)_{\mathrm{AM}} \times \mathbf{R}$$
$$P_Z = (P^\oplus)_{\mathrm{AM}} = P_{\mathrm{AM}}^{m\times m} \times (P_{\mathrm{M}}^\oplus)_{\mathrm{AM}} \times \{0\}$$

Further, let L_Z be the restriction of $L^\times$ to Z. By the assumptions of the theorem we have that L_Z has closed range in $(X_{\mathrm{M}}^*)_{\mathrm{AM}} \times \mathbf{R}$. Finally let $\mathcal{L}_Z = \mathcal{N}(L_Z)$. The dual problem in (12) can now be stated as

$$\inf_{z\in P_Z\cap(\mathcal{L}_Z+C)} \langle z, D\rangle \tag{30}$$

We will next show that (30) qualifies as the primal in Proposition 1 by proving that $P_Z \cap (\mathcal{L}_Z + C)$ contains points in the relative interior of P_Z and $\mathcal{L}_Z + C$. The second statement follows trivially since $\mathcal{L}_Z$ is a closed subspace of Z. The second statement follows from the assumption that there are solutions (z_0, z_{M}) of $L_{\mathrm{M}}^\times z_{\mathrm{M}} = \Pi_{\mathrm{M}}^\times M_G^\times z_0$ in the relative interior of $P_{\mathrm{AM}}^{m\times m} \times (P_{\mathrm{M}}^\oplus)_{\mathrm{AM}}$.

It now remains to show that the dual of (30) is the optimization problem in (3). In order to do this we need to show

1. $P_Z^\oplus = P$, which follows since $P_{\mathrm{B}}^{m\times m}$ and P_{M} are closed.
2. $\mathcal{L}_Z^\perp = \mathcal{L}$, which follows since $L_Z^\times = L$ and that $\mathcal{N}(L_Z)^\perp = \mathcal{R}(L_Z^\times) = \mathcal{R}(L)$. The last statement follows from Theorem 6.2 in [10] since L_Z has closed range.

To see that $L_Z^\times = L$, we first note that $L_Z^\times : X_{\mathrm{M}} \times \mathbf{R} \mapsto X$, which follows from Lemma 1 and the fact that the coordinates in X_{M} are taken from $\mathcal{S}_{\mathrm{R}}$, $\mathcal{S}_{\mathrm{C}}$ and $\mathcal{S}_{\mathrm{B}}$. We have by the definition of L_Z

$$\langle L_Z z, x\rangle = \langle z, Lx\rangle$$

for any $x \in X_{\mathrm{M}} \times \mathbf{R}$ and $z \in (X^*)_{\mathrm{AM}}$. Hence $\langle z, (L_Z^\times - L)x\rangle = 0$ for any for any $x \in X_{\mathrm{M}} \times \mathbf{R}$ and $z \in (X^*)_{\mathrm{AM}}$, which implies that $L_Z^\times = L$, by the properties of the vector spaces involved.

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