Stability Theory Using Lipschitz and Dahlquist Functionals

Jönsson, Ulf; Olsson, Henrik

1993

Document Version:
Publisher's PDF, also known as Version of record

Link to publication

Citation for published version (APA):
Jönsson, U., & Olsson, H. (1993). *Stability Theory Using Lipschitz and Dahlquist Functionals.* (Technical Reports TFRT-7502). Department of Automatic Control, Lund Institute of Technology (LTH).

Total number of authors:
2

LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

ISSN 0280–5316
ISRN LUTFD2/TFRT--7502--SE

Stability Theory Using Lipschitz and Dahlquist functionals

Ulf Jönsson
Henrik Olsson

Department of Automatic Control
Lund Institute of Technology
April 1993

Department of Automatic Control
Lund Institute of Technology
P.O. Box 118
S-221 00 Lund Sweden

Document name
INTERNAL REPORT

Date of issue
April 1993

Document Number
ISRN LUTFD2/TFRT--7502--SE

Supervisor

Sponsoring organisation

Author(s)
Ulf Jönsson and Henrik Olsson

Title and subtitle
Stability Theory Using Lipschitz and Dahlquist Functionals

Abstract

In this report we will introduce the Lipschitz and Dahlquist functionals, and derive them for linear time invariant systems and for simple static diagonal nonlinearities. Algorithms for the computation of the functionals are given in the linear case. A stability theory for feedback systems is developed. With stability we mean that the signals in the feedback loop are bounded and depend Lipschitz continuously on the input signals. The stability theorems developed state conditions for stability in terms of the Lipschitz and Dahlquist functionals. The theorems are similar to those found in the classical control systems litterature, but stated in terms of the introduced functionals.

Key words
stability theory, nonlinear functional analysis, Lipschitz functionals, Dahlquist functionals, small gain theorem.

Classification system and/or index terms (if any)

Supplementary bibliographical information

ISSN and key title
0280–5316

ISBN

Language
English

Number of pages
23

Recipient's notes

Security classification

The report may be ordered from the Department of Automatic Control or borrowed through the University Library 2, Box 1010, S-221 03 Lund, Sweden, Fax +46 46 110019, Telex: 33248 lubbis lund.

1. Introduction

In this report we will introduce the Lipschitz and Dahlquist functionals, [Söderlind, 1992], and derive them for linear time invariant systems and for simple static nonlinearities. Algorithms for the computation of the functionals will be given in the linear case.

A stability theory for feedback systems will be developed. With stability we will mean that the signals in the feedback loop are bounded and depend Lipschitz continuously on the input signals. The stability theorems developed will state conditions for stability in terms of the Lipschitz and the Dahlquist functionals. The theorems are similar to those found in, for example, [Desoer and Vidyasagar, 1975], but stated in terms of the introduced functionals.

The report puts, into a feedback system stability context, results from nonlinear functional analysis that have been used in numerical analysis [Söderlind, 1984], [Söderlind, 1986], and [Söderlind, 1992]. The main stability theorem in this report appeared in [Söderlind, 1992].

2. The Lipschitz and the Dahlquist functionals

In this section the Lipschitz and the Dahlquist functionals presented in [Söderlind, 1984], [Söderlind, 1986], and [Söderlind, 1992] will be introduced. We will be concerned with operators H mapping a subset of a Banach space X into the same Banach space, i.e. $H : \mathrm{Dom}(H) \mapsto \mathrm{Im}(H)$, where $\mathrm{Dom}(H), \mathrm{Im}(H) \in X$ denote the domain and range of H respectively.

Lipschitz functionals

For $u, v \in \mathrm{Dom}(H)$ the upper and lower Lipschitz functionals are defined as follows

$$L[H] = \sup_{u \neq v} \frac{\|H(u) - H(v)\|}{\|u - v\|}, \qquad l[H] = \inf_{u \neq v} \frac{\|H(u) - H(v)\|}{\|u - v\|} \tag{1}$$

We will call operators with $L[H] < \infty$ Lipschitz continuous. The following properties of the Lipschitz functionals can easily be proven [Söderlind, 1986].

Theorem 1

Let H, H_1 and H_2 be Lipschitz continuous with $\mathrm{Dom}(H_1) = \mathrm{Dom}(H_2)$ and $\mathrm{Im}(H_1) \subset \mathrm{Dom}(H_2)$. Then

1. $L[H] \geq 0; L(H) = 0 \Leftrightarrow H = const.$
2. $L[\alpha H] = |\alpha| L[H]$
3. $L[H_1] - L[H_2] \leq L[H_1 + H_2] \leq L[H_1] + L[H_2]$
4. $l[H_2]L[H_1] \leq L[H_2 H_1] \leq L[H_2]L[H_1]$ □

It follows from the theorem above that $L[\cdot]$ is a seminorm on the space of Lipschitz continuous operators. It is possible to construct equivalence classes of Lipschitz continuous operators differing only by a constant term. Then we get a linear quotient space called $\mathcal{L}[X]$ on which $L[\cdot]$ reduces to an operator norm, see [Söderlind, 1992]. It was shown in [Söderlind, 1992] that $\mathcal{L}(X)$ is complete and open. Let $\bar{\mathcal{L}}(X)$ be the closure of $\mathcal{L}(X)$, then it follows that the boundary $\partial\mathcal{L}(X) = \bar{\mathcal{L}}(X) \setminus \mathcal{L}(X)$ consists of unbounded operators.

The functional $l[\cdot]$ has the following properties, [Söderlind, 1992]

THEOREM 2
Let $H, H_1, H_2 \in \mathcal{L}(X)$, with $\mathrm{Dom}(H_1) = \mathrm{Dom}(H_2)$ and $\mathrm{Im}(H_1) \subset \mathrm{Dom}(H_2)$. Then

1. $0 \leq l[H] \leq L[H]$
2. $l[\alpha H] = |\alpha| l[H]$
3. $l[H_1] - L[H_2] \leq l[H_1 + H_2] \leq l[H_1] + L[H_2]$
4. $l[H_2]l[H_1] \leq l[H_2 H_1] \leq L[H_2]l[H_1]$ □

Remark. We may define $l[\cdot]$ even for unbounded operators. An example of this will be given in Example 3.

We will, as discussed in the next section, have reason to restrict ourselves to operators satisfying $H(0) = 0$. In what follows we will only consider this class of operators. We define the linear space of operators $\mathcal{L}_0[X] = \{H : L[H] < \infty \text{ on } \mathrm{Dom}(H) \in X, \text{with } H(0) = 0\}$. The theorems and the comments above hold also for $\mathcal{L}_o(X)$.

The following theorem is useful when studying invertibility properties of operators.

THEOREM 3
If $l[H] > 0$ then $H^{-1} \in \mathcal{L}_0(X)$; $L[H^{-1}] = l[H]^{-1}$ □

Remark. In the case when $\mathrm{Dom}(H) = X$, the condition $l[H] > 0$ only implies that H^{-1} is defined on $\mathrm{Im}(H)$ and therefore in general not on all of X.

We will conclude the presentation of the Lipschitz functionals with an important lemma that will be used in a proof later on.

LEMMA 1
If $L[H] < 1$, then $(I + H)^{-1} \in \mathcal{L}_0(X)$ and $L[(I + H)^{-1}] \leq \frac{1}{1-L[H]}$. Further, if $\mathrm{Dom}(H) = X$ then $\mathrm{Dom}((I + H)^{-1}) = X$.

Proof: Since $l[I+H] \geq l[I] - L[H] = 1 - L[H] > 0$, $I+H$ is injective and thus invertible on $\mathrm{Im}(I + H)$. The bound on the Lipshitz functional of $(I + H)^{-1}$ follows easily from Theorem 3. It remains to prove that if $\mathrm{Dom}(H) = X$ then $\mathrm{Dom}((I + H)^{-1}) = X$. This follows if we can show that the equation $(I + H)x = y$ has a unique solution for any $y \in X$. The equation can be rewritten as $x = y - Hx =: f(y, x)$, where $f(y, x)$ is a contraction in X for any $y \in X$. This follows since for arbitrary $x, x' \in X$ we have $\|f(y,x) - f(y,x')\| = \|Hx - Hx'\| \leq L(H)\|x - x'\|$, and from the fact that $L(H) < 1$. Hence, it follows from Banach's fixed point theorem that $x = f(y, x)$ has a unique solution for any $y \in X$ and the lemma is proven. □

Dahlquist functionals

The Dahlquist functionals $M[\cdot]$ and $m[\cdot]$ are defined for $H \in \mathcal{L}_0(X)$ as, see [Söderlind, 1992],

$$M[H] = \lim_{\varepsilon \to 0+} \frac{L[I + \varepsilon H] - 1}{\varepsilon}; \qquad m[H] = \lim_{\varepsilon \to 0-} \frac{L[I + \varepsilon H] - 1}{\varepsilon} \tag{2}$$

It follows from the definition that $m[H] = -M[-H]$. Further it should be noted that the Dahlquist functionals are only defined for bounded operators, i.e. operators in $\mathcal{L}_0(X)$. However if $X = \mathcal{H}$, a Hilbert space, then the definition of the Dahlquist functionals reduce to

$$M[H] = \sup_{u \neq v} \frac{Re\langle u - v, H(u) - H(v)\rangle}{||u - v||^2} \quad m[H] = \inf_{u \neq v} \frac{Re\langle u - v, H(u) - H(v)\rangle}{||u - v||^2} \tag{3}$$

and if we have a real valued Hilbert space then this reduce to

$$M[H] = \sup_{u \neq v} \frac{\langle u - v, H(u) - H(v)\rangle}{||u - v||^2} \quad m[H] = \inf_{u \neq v} \frac{\langle u - v, H(u) - H(v)\rangle}{||u - v||^2} \tag{4}$$

These definitions also hold for operators that are not in $\mathcal{L}_0(X)$, i.e. unbounded operators.
The functionals $M[\cdot]$ and $m[\cdot]$ satisfy the following properties [Söderlind, 1986].

THEOREM 4
Let H, H_1 and H_2 be Lipschitz continuous, with $\text{Dom}(H_1) = \text{Dom}(H_2)$. Then

1. $-l[H] \leq M[H] \leq L[H]$
2. $M[H + zI] = M[H] + \text{Re}z$
3. $M[\alpha H] = \alpha M[H], \quad \alpha \geq 0$
4. $m[H_1] + M[H_2] \leq M[H_1 + H_2] \leq M[H_1] + M[H_2]$
5. $-L[H] \leq m[H] \leq l[H]$
6. $m[H + zI] = m[H] + \text{Re}z$
7. $m[\alpha H] = \alpha m[H], \quad \alpha \geq 0$
8. $m[H_1] + m[H_2] \leq m[H_1 + H_2] \leq M[H_1] + m[H_2]$ □

In what follows we will mainly be concerned with the functional $m[\cdot]$. The reason for this is that concepts in systems theory such as passivity of an operator H can be stated as $m[H] \geq 0$. However, also the operator $M[\cdot]$ has well known applications. For example, when A is a matrix we have that $M[A] = \mu[A]$, the logarithmic norm of the matrix.

The next theorem will be of tremendous importance in this report. It states a condition in terms of $m[\cdot]$ for the operator $H \in \mathcal{L}_0(X)$ to be invertible with the inverse defined on all of X.

THEOREM 5
If $H \in \mathcal{L}_0(X)$ and $m[H] > 0$, then $H^{-1} \in \mathcal{L}_0(X)$, with $L[H^{-1}] \leq \frac{1}{m[H]}$. Further, if $\text{Dom}(H) = X$, then $\text{Dom}(H^{-1}) = X$.

Proof: $l[H] \geq m[H] > 0 \Rightarrow H$ is injective, and thus invertible on $\text{Im}(H)$. It follows from Theorem 3 that $L[H^{-1}] = \frac{1}{l[H]} \leq \frac{1}{m[H]}$. It remains to prove that $\text{Im}(H) = X$ when $\text{Dom}(H) = X$. It is no restriction to redefine the problem as follows. Let $H = I + (H - I) = I + H'$. We need to prove that if $\text{Dom}(H') = X$ and if $m[H'] > -1$ then $(I + H')^{-1}$ is defined on all of X. Introduce the operator $F(\alpha) = I + \alpha H'$, for $\alpha \in [0, 1]$. It is clearly true that $F(0) = I$ is invertible with the inverse defined on all of X and we want to show that $F(1) = I + H'$ has the same property. Assume that $F(\alpha)$ is invertible, then the following identity holds $F(\alpha') = (I + (\alpha' - \alpha)H'(I + \alpha H')^{-1})F(\alpha)$. Now if $|\alpha' - \alpha| < \frac{1}{\beta}$, where $\beta = \max(\frac{L[H']}{1+m[H']}, L[H']) < \infty$, i.e. $\frac{1}{\beta} > \varepsilon$ for some $\varepsilon > 0$, then $L[(\alpha' - \alpha)H'(I + \alpha H')^{-1}] < 1, \forall \alpha \in [0, 1]$, and it follows from Lemma 2 that the first factor on the right hand side of the identity above is invertible on all of X. Now divide $[0, 1]$ into N intervals $[\alpha_i, \alpha_{i+1}]$ each of length smaller than $1/\beta$, (that is $\lceil\beta\rceil \geq N < \infty$, where $\lceil\cdot\rceil$ denote the smallest integer larger than the argument). Then invertibility of $F(\alpha_i)$ on all of X implies invertibility of $F(\alpha_{i+1})$ on all of X, and since $F(0)$ is invertible on all of X the theorem follows from an induction type argument. $\square$

From the proof of the previous theorem we have the following important corollary

COROLLARY 1
If $H \in \mathcal{L}_0(X)$ has $\mathrm{Dom}(H) = X$ and if $m[H] > -1$, then $(I + H)^{-1} \in \mathcal{L}_0(X)$ and $\mathrm{Dom}((I + H)^{-1}) = X$. $\square$

In the proof of Theorem 5 we made crucial use of the assumption that $H \in \mathcal{L}_0(X)$, i.e. that H is Lipschitz continuous. If X is a Hilbert space $\mathcal{H}$, then it is possible to weaken the requirement of Lipschitz continuity and only require continuity. This follows from the classical uniform monotonicity theorem, see [Browder, 1963] or [Minty, 1962]. This theorem goes as follows.

THEOREM 6—Uniform momotonicty theorem
If $H : \mathcal{H} \mapsto \mathcal{H}$ is a continuous mapping on a Hilbert space $\mathcal{H}$ and if there exists a real constant $c > 0$ for which

$$\mathrm{Re}\langle H(x) - H(x'), x - x'\rangle \geq c\|x - x'\|, \quad \forall x, x' \in \mathcal{H}$$

Then H is one-to-one and onto $\mathcal{H}$ and has a continuous inverse. $\square$

Remark. Note that the conditions of the theorem could be stated as $H : \mathcal{H} \mapsto \mathcal{H}$ should be a continuous mapping on a Hilbert space $\mathcal{H}$, with $m[H] > 0$.

3. The Functionals L, l, M and m for Linear Time–Invariant Dynamical Systems

Introductory example
We will start this section with an example to see what the introduced functionals are in a simple matrix case.

EXAMPLE 1
A linear operator from $\mathbb{R}^n$ to $\mathbb{R}^n$ is defined by an $n \times n$ matrix A. If the 2-norm is used on $\mathbb{R}^n$, we can derive the following results that are well-known in matrix theory.

$$\begin{array}{ll} L_2[A] = \max_i \sqrt{\lambda_i(A^TA)} & l_2[A] = \min_i \sqrt{\lambda_i(A^TA)} \\ M_2[A] = \max_i \lambda_i(\frac{1}{2}(A + A^T)) & m_2[A] = \min_i \lambda_i(\frac{1}{2}(A + A^T)) \end{array} \tag{5}$$

We recognize $L_2[A]$ as what normally is denoted by$\|A\|_2$, the induced 2-norm of the matrix. We conclude that $L_2[A]$ is the largest singular value of the matrix A, and $l_2[A]$ is the smallest singular value of A. $M_2[A]$ is sometimes called the matrix measure or the logarithmic norm, see [Desoer and Vidyasagar, 1975].

If we let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

we get $L_2[A] = \sqrt{2}$, $l_2[A] = 0$, $M_2[A] = (1 + \sqrt{2})/2$ and $m_2[A] = (1 - \sqrt{2})/2$. We note that all the inequalities from the previous section are satisfied. The matrix is not invertible which is consistent with $l_2[A] = 0$. $\square$

After this introductory example we will look at L, l, M and m for linear time invariant dynamical systems.

L, l, M, m for the signal space $L_2^p[0,\infty)$

In this subsection we will show how the functionals $L_2[G]$, $l_2[G]$, $M_2[G]$ and $m_2[G]$ are computed when G is a linear time–invariant dynamical system with a stable, proper and rational transfer function matrix and when the signal space is $X = L_2^p[0,\infty)$. The subscripts on the functionals indicate the signal space used.

Let the linear time–invariant dynamical system G be given by

$$\dot{x} = Ax + Bu \tag{6}$$

$$y = Cx + Du \tag{7}$$

with the transfer function defined by

$$G(s) = C(sI - A)^{-1}B + D$$

We will sometimes use the notation

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

to mean either the state space realization or the transfer function.

Instead of looking at G as a function from $L_2^p[0,\infty)$ to $L_2^p[0,\infty)$ we can define it by means of $G(s)$ as a mapping between the Laplace transforms of the signals in $L_2^p[0,\infty)$, see [Francis, 1987], i.e. $G(s) : U(s) \mapsto Y(s)$. The operator is then a square matrix in the complex variable s. The different functionals can now be obtained from simple matrix theory as in the introductory example. The only difference is that here we have a matrix in a complex variable and we therefore have to find the infimum or the supremum in the open right half plane over this complex variable.

$L_2[G]$ and $l_2[G]$

We can derive, see [Francis, 1987]

$$\begin{aligned} L(G) &= \sup_{u \neq v} \frac{\|Gu - Gv\|_2}{\|u - v\|_2} = \sup_{u \neq 0} \frac{\|Gu\|_2}{\|u\|_2} \\ &= \|G(i\omega)\|_\infty = \sup_{\omega} \sigma_{\max}(G(i\omega)) \end{aligned}$$

For $l(G)$ we get

$$l(G) = \inf_{u \neq v} \frac{\|Gu - Gv\|_2}{\|u - v\|_2} = \inf_{u \neq 0} \frac{\|Gu\|}{\|u\|} = \inf_{\omega} \sigma_{\min}(G(i\omega))$$

We can of course also use the relation

$$l[G] = L[G^{-1}]^{-1}$$

if the inverse of G exists and is stable and proper.

Several comments regarding $l[G]$ are in place. As pointed out in previous sections $l[G] > 0$ implies that G is invertible on $\mathrm{Im}(G)$ but not necessarily on the whole Banach space, which is $L_2^p[0,\infty)$ in our case. On $\mathrm{Im}(G)$ we also have that $L(G^{-1}) = \frac{1}{l(G)}$. This is illustrated by an example.

EXAMPLE 2
The system $G(s) = \frac{s-1}{s+1}$ has a zero in the right half plane. It is therefore not considered as an invertible system in ordinary control context. However $l_2[G] = 1$ so the system should be invertible. This is true but $\text{Im}(G)$ is just a subset of $L_2^p[0, \infty)$ and it is only on this subset that G is invertible. The subset is the set of signals

$$\{u(t) = \dot{x}(t) - x(t) | x(t) \text{ is a differentiable signal in } L_2[0, \infty)\}$$

This shows that it is extremely important to be aware of what spaces the operators are defined on when using the theory in this report, and most importantly one must be very careful when drawing conclusions about existence of inverses from inequalities such as $l[G] > 0$. If we instead use the Dahlquist functional we find that $m[G] = -1$ and therefore it does not conclude anything about invertibility on all of the Banach space as discussed previously. □

We will now give a necessary condition for G to be invertible with G^{-1} stable and proper. If $G(s)$ has the realization

$$G(s) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = C(sI - A)^{-1}B + D$$

then G is invertible, but not necessarily stable, if D is invertible, i.e. if $\sigma_{\min}(D) > 0$. This follows since $l(G) \geq \sigma_{\min}(D)$. The inverse has the realization

$$G(s) = \left(\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right) = -D^{-1}C(sI - A + BD^{-1}C)^{-1}BD^{-1} + D^{-1}$$

The condition $\sigma_{\min}(D) > 0$ is, however, not sufficient for the inverse to be stable since it only ascertains that the inverse of $G(s)$ is proper. We need one more condition, namely that $G(s)$ has no right half plane zeros. In conclusion we have that the condition $G(s)$ has no right half plane zeros together with the condition $\sigma_{\min}(D) > 0$ are necessary and sufficient for $G(s)$ to be invertible on the whole Banach space with the inverse being stable.

Another example will be used to show that the definition of $l[\cdot]$ may hold even for unstable systems.

EXAMPLE 3
Consider the system $G(s) = \frac{s+1}{s-1}$ which is unstable. As before all initial conditions are zero. It has the realization

$$\dot{x} = x + 2u$$
$$y = x + u$$

and the output is for an arbitrary input $u \in L_2[0, \infty)$ given as

$$y(t) = 2e^t \int_0^t e^{-\tau} u(\tau) d\tau + u(t)$$

If we take the input signal $u(t) = e^{-t} - 2te^{-t}$ which is in $L_2[0, \infty)$ then we get after simple calculations $y(t) = e^{-t}$, which means that the definition for $l[G]$ still makes sense and $l[G] < \infty$, since the infimum in the definition will be bounded. □

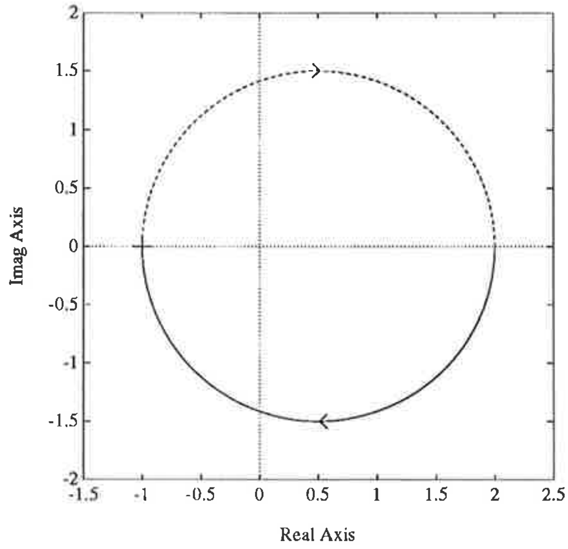

Figure 1. Nyquist curve for the transfer function in Example 4.

$M_2[G]$ and $m_2[G]$
For $M_2[G]$ we get

$$
\begin{aligned}
M_2[G] &= \lim_{\varepsilon \to 0+} \frac{L_2[I + \varepsilon G] - 1}{\varepsilon} = \lim_{\varepsilon \to 0+} \frac{\sup_\omega \sigma_{max}(I + \varepsilon G(i\omega)) - 1}{\varepsilon} \\
&= \lim_{\varepsilon \to 0+} \frac{\sup_\omega \sqrt{\lambda_{max}((I + \varepsilon G(i\omega))^*(I + \varepsilon G(i\omega)))} - 1}{\varepsilon} \\
&= \sup_\omega \frac{\lambda_{\max}(G^*(i\omega) + G(i\omega))}{2}
\end{aligned}
$$

and $m_2[G]$ is given in the same way by

$$
m_2[G] = -M_2[-G] = \inf_\omega \frac{\lambda_{min}(G^*(i\omega) + G(i\omega))}{2}
$$

In the SISO case this reduces to

$$
\begin{aligned}
M_2[G] &= \sup_\omega \operatorname{Re} G(i\omega) \\
m_2[G] &= \inf_\omega \operatorname{Re} G(i\omega)
\end{aligned}
$$

EXAMPLE 4
Let us look at the single-input single-output first order transfer function

$$
G(s) = \frac{3}{s+1} - 1
$$

The Nyquist curve of this transfer function is shown in Figure 1. For that system we find that $L_2[G] = 2$, $l_2[G] = 1$, $M_2[G] = 2$ and $m_2[G] = -1$. Which easily can be found either from the formulas or directly from the Nyquist plot.
□

To summarize in words, what we have for a SISO system is that $L_2[G]$ is the largest distance from the Nyquist curve to the origin, $l_2[G]$ is the smallest

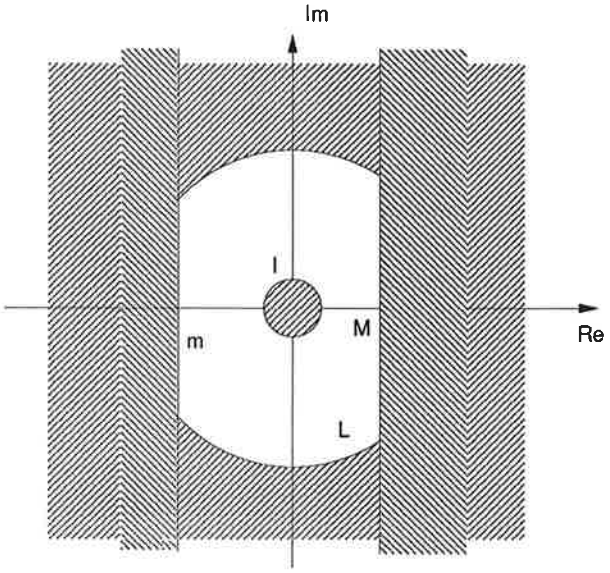

Figure 2. The bounds on the Nyquist curve imposed by the Lipschitz and Dahlquist functionals.

distance from the Nyquist curve to the origin, $M_2[G]$ is the smallest value such that the Nyquist curve is to the left of the line $\mathrm{Re}(s) = M_2[G]$ and $m_2[G]$ is the largest value such that the Nyquist curve is to the right of the line $\mathrm{Re}(s) = m_2[G]$. Figure 2 shows these bounds on the Nyquist curve. Similarily we have in the MIMO case that $m[G(i\omega)] > 0$ implies that all the characteristic loci of $G(i\omega)$ are in the right half plane. Of course one can relate this to classical control theory concepts such as passivity and positive realness.

Relation to classical control concepts

What has been stated in the previous subsection is almost the same as saying that a system is positive real except for a shift along the real axis. Let us look at the problem of determining if $m[G] \geq \delta$. This problem is transformed to the ordinary positive realness problem by including the term $-\delta I$ in the D-matrix of the system. From the theory of positive real (PR) transfer functions we know that the Kalman-Yakubovitch Lemma determines when a system in state-space form is PR. For our MIMO-system the lemma reads as follows.

LEMMA 2

Let the stable transfer function matrix $G(s)$ in state space form have the matrices A,B,C and D as a minimal realization . The poles of $G(s)$ lie in the open left half plane except for simple ones on the imaginary axis and with the associated residue matrix non-negative definite hermitian. Then $m[G] \geq \delta$ if and only if a positive definite matrix P and matrices Q and S exist such that

$$PA + A^T P = -QQ^T \tag{8}$$

$$C^T - PB = QS \tag{9}$$

$$S^T S = D + D^T - 2\delta I \tag{10}$$

Proof: The proof can be found in [Andersson and Vongpanitlerd, 1973] or [Narendra and Taylor, 1973] □

From the lemma we can easily deduce some variations, which we state in a few remarks.

Remark 1. The equations in the lemma above can be reformulated as a Linear Matrix Inequality (LMI). Rewriting them in a slightly different way as a single matrix equation yields.

$$\begin{pmatrix} -PA - A^T P & C^T - PB \\ C - B^T P & D + D^T - 2\delta I \end{pmatrix} = \begin{pmatrix} Q \\ S^T \end{pmatrix} \begin{pmatrix} Q^T & S \end{pmatrix}$$

The matrix on the right hand side is clearly positive semidefinite and therefore the problem is to find a positive definite matrix P such that

$$\begin{pmatrix} -PA - A^T P & C^T - PB \\ C - B^T P & D + D^T - 2\delta I \end{pmatrix} \geq 0$$

Thus the system is positive real if there exists such a P satisfying the LMI, see [Willems, 1971].

Remark 2. Assuming that 2δ is not an eigenvalue of $D + D^T$, then $D + D^T - 2\delta I$ is invertible and the equations in the Kalman-Yakubovitch lemma above reduce to a Riccati equation. We can solve for S in the third equation of the lemma and substitute into the second, solve for Q and then substitute into the first to get a Riccati equation. A positive definite solution to this equation will then guarantee the positive realness of $G(s) - \delta I$ or in other terms, $m[G] \geq \delta$. The Riccati equation looks as follows.

$$P(A - BR^{-1}C) + (A^T - C^T R^{-1} B^T)P + C^T R^{-1} C + PBR^{-1}B^T P = 0 \quad (11)$$

where

$$R = D + D^T - 2\delta I \quad (12)$$

Remark 3. The Riccati equation in the previous remark has a corresponding Hamiltonian matrix, see [Bittanti et al., 1991], which looks as follows

$$H = \begin{pmatrix} A - BR^{-1}C & BR^{-1}B^T \\ -C^T R^{-1} C & -A^T + C^T R^{-1} B^T \end{pmatrix}$$

Remark 4. A singular $D + D^T - 2\delta I$ can be handled if the Kalman-Yakubovitch formalism is used. The Riccati equation does not work nor does a test on the Hamiltonian matrix.

Computation of the Dahlquist and Lipschitz functionals for the signal space $L_2^p[0, \infty)$

We need a way to compute $L_2[G] = \sigma_{\max}(G(i\omega))$ of a transfer function matrix. This is the same problem as finding the so called H_∞-norm of a transfer function. A bisection method for doing this was presented in [Boyd et al., 1989]. The method is based on the following theorem.

THEOREM 7

Let the transfer matrix $G(s)$ have a state space realization with the matrices A, B, C and D which are real and of dimension $n \times n$, $n \times p$, $p \times n$ and $p \times p$, respectively. Then if A is a strictly stable matrix, i.e. all eigenvalues in the open left half plane, and $\gamma > \sigma_{\max}(D)$. Then $\|G(i\omega)\|_\infty \geq \gamma$ if and only if M_γ has imaginary eigenvalues. Where M_γ is the Hamiltonian matrix

$$M_\gamma = \begin{pmatrix} A - B(D^T D - \gamma^2 I)^{-1} D^T C & -\gamma B(D^T D - \gamma^2 I)^{-1} B^T \\ \gamma C^T (DD^T - \gamma^2 I)^{-1} C & -A^T + C^T D(D^T D - \gamma^2 I)^{-1} B^T \end{pmatrix}$$

□

Remark. There are no assumptions on controllability or observability of the realization of the system.

The bisection algorithm for computing $\|G(i\omega)\|_\infty$ is.

$\gamma_L := \gamma_{lb}$;
$\gamma_U := \gamma_{ub}$;
repeat{
  $\gamma := (\gamma_L + \gamma_U)/2$;
  Form M_γ;
  if M_γ *has no imaginary eigenvalues then*
    $\gamma_H := \gamma$
  else
    $\gamma_L := \gamma$
until $\gamma_H - \gamma_L \le 2\varepsilon\gamma_L$

In [Boyd et al., 1989] it was also noted that a similar bisection method could be used for computing what is equivalent to $M_2(G)$ and $m_2(G)$. We will now derive that method. Similarly to Theorem 7 we have

THEOREM 8
Let the transfer function matrix $G(s)$ be realized with the matrices A, B, C and D which are real and of dimension $n\times n$, $n\times p$, $p\times n$ and $p\times p$, respectively. Then, if A is a strictly stable matrix and $\delta > \lambda_{max}(\frac{D^T+D}{2})$, $M_2[G] \ge \delta$ if and only if N_δ has imaginary eigenvalues, where N_δ is the Hamiltonian matrix

$$N_\delta = \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix} - \begin{pmatrix} B \\ C^T \end{pmatrix} (D^T + D - 2\delta I)^{-1} \begin{pmatrix} C & -B^T \end{pmatrix} \tag{13}$$

Proof: Since $\lim_{\omega\to\infty} G(i\omega) = D$ it is clear that $M_2[G] \ge \lambda_{\max}(\frac{D^T+D}{2})$, so when $\delta > \lambda_{\max}(\frac{D^T+D}{2})$ the following statements are equivalent

1. $M_2[G] = \sup_\omega \lambda_{\max}(\frac{1}{2}(G^*(i\omega) + G(i\omega)) \ge \delta$
2. $\exists\omega_0$ and $\exists$ eigenvector x such that $\frac{1}{2}(G^*(i\omega_0) + G(i\omega_0))x = \delta x$
3. The matrix $\tilde{G}(s) = G^*(s) + G(s) - 2\delta I$ looses rank at $s = i\omega_0$
4. The matrix $\tilde{G}(s)$ has at least one zero at $s = i\omega_0$

where the equivalence between 1 and 2 follows since the maximal eigenvalue depends continuously on ω. The other equivalences follow trivially. A realization of $\tilde{G}(s)$ is given by

$$\tilde{G}(s) = \left(\begin{array}{cc|c} A & 0 & B \\ 0 & -A^T & C^T \\ \hline C & -B^T & D^T + D - 2\delta I \end{array}\right) \tag{14}$$

$\tilde{G}$ is clearly invertible since $D^T + D - 2\delta I$ is invertible. A realization of the inverse is given as

$$\tilde{G}(s)^{-1} = \left(\begin{array}{c|c} \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix} - \begin{pmatrix} B \\ C^T \end{pmatrix} R^{-1} \begin{pmatrix} C & -B^T \end{pmatrix} & \begin{pmatrix} B \\ C^T \end{pmatrix} R^{-1} \\ \hline \begin{pmatrix} C & -B^T \end{pmatrix} R^{-1} & R^{-1} \end{array}\right) \tag{15}$$

where $R = D^T + D - 2\delta I$. The necessity follows since $\tilde{G}(s)^{-1}$ has by assumption at least one pole at $s = i\omega_0$ and from (15) we see that this implies that the Hamiltonian matrix N_δ has at least one eigenvalue equal to $i\omega_0$. The sufficiency part follows if we can prove that the realization of $\tilde{G}^{-1}$ in (15) cannot have any pole zero cancellations on the imaginary axis. However, since A by assumption does not have any imaginary eigenvalues, relation (16) below together with the PBH test shows that there are no uncontrollable modes on the imaginary axis. Similarly, it can be shown that there are no unobservable modes on the imaginary axis, and therefore no pole zero cancellation can occur on the imaginary axis.

$$\begin{aligned} &\left[\begin{pmatrix} sI - A + BR^{-1}C & -BR^{-1}B^T \\ C^T R^{-1} C & sI + A^T - C^T R^{-1} B^T \end{pmatrix} \quad \begin{pmatrix} BR^{-1} \\ C^T R^{-1} \end{pmatrix} \right] \\ &= \left[\begin{pmatrix} sI - A & 0 \\ 0 & sI + A^T \end{pmatrix} \quad \begin{pmatrix} B \\ C^T \end{pmatrix} \right] \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -C & B^T & R \end{pmatrix}^{-1} \end{aligned} \tag{16}$$

□

Remark 1. There are no assumptions on controllability or observability of the system.

Remark 2. The Hamiltonian N_δ is the same as in Remark 3 of Lemma 2.

The bisection algorithm presented earlier can be modified for computing $M_2[G]$.

To compute $m_2[G]$ we need the following theorem, which is proven in the same way as Theorem 8.

THEOREM 9

Under the same assumptions as in Theorem 8, we have that when $\delta < \lambda_{\min}(\frac{D^T+D}{2})$ then $m_2[G] \leq \delta$ if and only if N_δ has imaginary eigenvalues, where N_δ was defined in (13). □

The bisection algorithm can also easily modified for computations of $m_2[G]$.

L, l, M and m for the signal space $L_\infty^n[0, \infty)$.

In this subsection we give formulas for computing $L_\infty[G], l_\infty[G], M_\infty[G]$ and $m_\infty[G]$ when G is a linear time invariant and stable dynamical system for the case when the signals are in $L_\infty^n[0, \infty)$.

$L_\infty[G]$ **and** $l_\infty[G]$
We start with the SISO case, when the realization of the transfer function is

$$G(s) = \left(\begin{array}{c|c} A & b \\ \hline c & d \end{array}\right) = c(sI - A)^{-1}b + d$$

The impulse response of the LTI operator is $g(t) = ce^{At}b\theta(t) + d\delta(t)$, where $\theta(t)$ is the unit step function and $\delta(t)$ is the dirac function. We have, see [Desoer and Vidyasagar, 1975] or [Dahleh and Pearson, 1987]

$$\begin{aligned} L_\infty[G] &= \sup_{u \neq v} \frac{\|Gu - Gv\|_\infty}{\|u - v\|_\infty} = \sup_{u \neq 0} \frac{\|Gu\|_\infty}{\|u\|_\infty} \\ &= \int_0^\infty |ce^{At}b|dt + |d| \stackrel{def}{=} |g|_A \end{aligned}$$

This follows since

$$\begin{aligned} |y(t)| &= |\int_{-\infty}^{\infty} g(\tau)u(t-\tau)d\tau| \\ &\leq \int_{-\infty}^{\infty} |ce^{A\tau}b\theta(\tau)u(t-\tau)|d\tau + |d \cdot u(t)| \\ &\leq (\int_0^\infty |ce^{At}b|dt + |d|)\|u\|_\infty \end{aligned}$$

which imply that $L_\infty[G] \leq \int_0^\infty |ce^{At}b|dt + |d|$. For the reversed inequality fix t and let

$$u(t-\tau) = \begin{cases} \operatorname{sign}(ce^{A(\tau)}b), & 0 < \tau < t \\ \operatorname{sign}(d), & \tau = 0 \\ 0, & \text{otherwise} \end{cases} \tag{17}$$

Then we have $\|u\|_\infty = 1$ and we get

$$y(t) = \int_{-\infty}^{\infty} g(\tau)u(t-\tau)d\tau = \int_0^t |ce^{At}b|dt + |d|$$

which imply that $y(t) \to \int_0^\infty |ce^{At}b|dt + |d|$ as $t \to \infty$. Hence we have $L_\infty[G(i\omega)] \geq \int_0^\infty |ce^{At}b|dt + |d|$.

For the MIMO case we assume that G has the realization

$$G(s) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = C(sI - A)^{-1}B + D$$

which have the impulse response matrix $g(t) = Ce^{At}B\theta(t) + D\delta(t) = \{g_{i,j}\}$. We have

$$\begin{aligned} L_\infty[G] &= \sup_{u \neq v} \frac{\|Gu - Gv\|_\infty}{\|u - v\|_\infty} = \sup_{u \neq 0} \frac{\|Gu\|_\infty}{\|u\|_\infty} \\ &= \max_i \sum_{j=1}^n \|g_{i,j}\|_A \end{aligned} \tag{18}$$

where the norm $\|\cdot\|_\infty$ is defined as $\|u\|_\infty = \max_i \|u_i\|_\infty$ for any vector $u \in L_\infty^n[0, \infty)$. The result follows since

$$\begin{aligned} \|Gu\|_\infty &= \max_i \|\sum_{j=1}^n g_{i,j} * u_j\|_\infty \leq \max_i \sum \|g_{i,j}\|_A \|u_j\|_\infty \\ &\leq (\max_i \sum_{j=1}^n \|g_{i,j}\|_A) \max_i \|u_i\|_\infty \end{aligned}$$

That is

$$L[G] \leq \max_i \sum_{j=1}^{n} \|g_{i,j}\|_A$$

and we can achieve the reversed inequality by taking u as $u(t) = [u_1^*(t)...u_n^*(t)]^T$, where the components of u are chosen as

$$u_j^*(t-\tau) = \begin{cases} \text{sign}(\{Ce^{A(\tau)}B\}_{i^*,j}), & 0 < \tau < t \\ \text{sign}(\{D\}_{i^*,j}), & \tau = 0 \\ 0, & \text{otherwise} \end{cases}$$

where i^* is the i which gives the maxima in (18). Hence $\|u\|_\infty = 1$ and we get the i^*th component of y to be

$$y_{i^*}(t) = \sum_{j=0}^{n}(\int_0^t |\{C^{At}B\}_{i^*,j}|dt + |\{D\}_{i^*,j}|)$$

By letting $t \to \infty$ we see that $L_\infty[G] \geq \max_i \sum_{j=1}^{n} \|g_{i,j}\|_A$.

For $l_\infty[G]$ we cannot derive a nice looking expression but $l_\infty[G]$ can be computed from

$$\frac{1}{L_\infty[G^{-1}]}$$

when G is invertible. The necessary and sufficient conditions for G to be invertible with an inverse mapping from $L_\infty[0,\infty)$ into itself are the same as previously.

$M_\infty[G]$ and $m_\infty[G]$

From the definition of $M[G]$ we derive

$$\begin{aligned} M_\infty[G] &= \lim_{\varepsilon \to 0+} \frac{L_\infty[I+\varepsilon G]-1}{\varepsilon} \\ &= \lim_{\varepsilon \to 0+} \frac{\max_i(\sum_{j=1}^{n}\int_0^\infty \varepsilon|\{Ce^{At}B\}_{i,j}(t)|dt + |\delta_{i,j}+\varepsilon d_{i,j}|) - 1}{\varepsilon} \\ &= \max_i(\sum_{j=1}^{n}\int_0^\infty |\{Ce^{At}B\}_{i,j}(t)|dt + d_{i,i} + \sum_{j \neq i}|d_{i,j}|) \end{aligned}$$

which in the SISO case reduces to

$$M[G] = \int_0^\infty |ce^{At}b|dt + d$$

For $m_\infty[G]$ we get

$$m_\infty[G] = -M_\infty[-G] = -\max_i(\sum_{j=1}^{n}\int_0^\infty |\{Ce^{At}B\}_{i,j}(t)|dt - d_{i,i} + \sum_{j\neq i}|d_{i,j}|)$$

which reduces to

$$m_\infty[G] = -\int_0^\infty |ce^{At}b|dt + d$$

in the SISO case.

4. L, l, M and m for static diagonal nonlinearities

In this section we will determine the Lipschitz and Dahlquist constants for static diagonal nonlinearities. To define what we mean by a static diagonal nonlinearity, let the input to the linearity be $u \in L_p^n[0,\infty)$ and the output be $y \in L_p^n[0,\infty)$. Our static diagonal nonlinearity is an operator from the input to the output, i.e. $K : u \mapsto y$. It is defined by a function K' from $\mathbb{R}^n$ to $\mathbb{R}^n$ by $y(t) = (y_1(t), y_2(t), \cdots, y_n(t)) = K'(u(t)) = (K_1'(u_1(t)), K_2'(u_2(t)), \cdots, K_n'(u_n(t)))$. This means that for each time instant the output of the nonlinearity does only depend on the input at that time instant, not on the history of the input, i.e. it is memoryless. Furthermore we restrict us to look at time invariant nonlinearities. We assume that all K_i' are Lipschitz continuous functions from $\mathbb{R}$ to $\mathbb{R}$, i.e.

$$L[K_i'] = \sup_{u \neq v} \frac{|K_i'(u) - K_i'(v)|}{|u - v|} < \infty$$

which means that we can define $K_{i,\min}$ and $K_{i,\max}$ as follows

$$K_{i,\min} \stackrel{def}{=} \inf_{u \neq v} \frac{K_i'(u) - K_i'(v)}{u - v} > -\infty$$

$$K_{i,\max} \stackrel{def}{=} \sup_{u \neq v} \frac{K_i'(u) - K_i'(v)}{u - v} < \infty$$

clearly

$$L[K_i'] = \max(|K_{i,\max}|, |K_{i,\min}|)$$

$K_{i,\max}(K_{i,\min})$ is the supremum(infimum) of the slope of K_i', where the slope is considered with its sign. Note that K_i' need not be differentiable, but if so the slope is equal to the derivative of K_i'. Now we look at the Lipschitz constant for K when the input and output are signals in $L_p^n[0,\infty)$ and when we use the induced norm. We get

$$\begin{aligned} L_p[K] &= \sup_{u \neq v} \frac{(\int_0^\infty \|K'(u(s)) - K'(v(s))\|_p^p ds)^{1/p}}{(\int_0^\infty \|u(s) - v(s)\|_p^p ds)^{1/p}} \\ &\leq \frac{(\int_0^\infty (L_p[K'])^p \|u(s) - v(s)\|_p^p ds)^{1/p}}{(\int_0^\infty \|u(s) - v(s)\|_p^p ds)^{1/p}} = L_p[K'] \end{aligned} \tag{19}$$

where

$$\begin{aligned} L_p[K'] &= \sup_{u \neq v} \frac{\|K'(u) - K'(v)\|_p}{\|u - v\|_p} \\ &= \sup_{u \neq v} \frac{\|(K_1'(u_1) - K_1'(v_1), K_2'(u_2) - K_2'(v_2), \cdots, K_n'(u_n) - K_n'(v_n))\|_p}{\|(u_1 - v_1, u_2 - v_2, \cdots, u_n - v_n)\|_p} \\ &\leq \max_i L[K_i'] = L[K_{i^*}'] \end{aligned} \tag{20}$$

We can show that we, in fact, can achieve equalities above. Let the i^*th component of $u(t)$ and $v(t)$ be constant during a bounded time interval T and zero thereafter and let the other components be zero all the time, i.e.

$$u_i(t) = \begin{cases} c_1 & 0 \leq t \leq T, i = i^* \\ 0 & \text{otherwise} \end{cases} \qquad v_i(t) = \begin{cases} c_2 & 0 \leq t \leq T, i = i^* \\ 0 & \text{otherwise} \end{cases}$$

In the definition of $L_p[K]$ we take the supremum over all possible $u(t)$ and $v(t)$. This clearly includes taking the supremum over c_1 and c_2 for $u_{i*}(t)$ and $v_{i*}(t)$ defined as above. We have

$$\begin{aligned} L_p[K] &= \sup_{u\neq v} \frac{(\int_0^\infty \|K'(u(s)) - K'(v(s))\|_p^p ds)^{1/p}}{(\int_0^\infty \|u(s) - v(s)\|_p^p ds)^{1/p}} \\ &\geq \sup_{c_1\neq c_2} \frac{(\int_0^T \|(0,\cdots,0,K'_{i*}(c_1) - K'_{i*}(c_2),0,\cdots,0)\|_p^p ds)^{1/p}}{(\int_0^T \|(0,\cdots,0,c_1 - c_2,0,\cdots,0)\|_p^p ds)^{1/p}} \\ &= \sup_{c_1\neq c_2} \frac{|K'_{i*}(c_1) - K'_{i*}(c_2)|}{|c_1 - c_2|} = L[K'_{i*}] \end{aligned}$$

So we have shown that in fact

$$L_p[K] = L[K'_{i*}] \tag{21}$$

which means that the Lipschitz constant is the same irrespective of the norm used on the time functions. It is equal to the supremum of the absolute value of the slope of any K'_i where K'_i is a function from $\mathbb{R}$ to $\mathbb{R}$. In the same way the lower Lipschitz constant $l[K]$ can be seen to equal the infimum of the absolute value of the slope of any K'_i.

Let us now look at the Dahlqvist constants for the same type of nonlinearity. We want to determine

$$M_p[K] = \lim_{\varepsilon\to 0+} \frac{L_p[I + \varepsilon K] - 1}{\varepsilon}$$

Since $I + \varepsilon K$ is just a static diagonal nonlinearity as discussed above, we know that

$$M_p[K] = \lim_{\varepsilon\to 0+} \frac{L_p[I + \varepsilon K'] - 1}{\varepsilon}$$

and since we are looking at the limit when ε is small and positive

$$L_p[1 + \varepsilon K'] = 1 + \varepsilon \max_i K_{i,\max}$$

and hence

$$M_p[K] = \max_i K_{i,\max}$$

By the same reasoning we find that

$$m_p[K] = \min K_{i,\min}$$

Put in words we can conclude that $M_p[K]$ equals the supremum of the slope of any K'_i and $m_p[K]$ equals the infimum of the slope of any K'_i, where the slope is considered with its sign.

5. Stability theorems

In this section we will derive stability theorems for feedback interconnections using the operators $L[\cdot]$ and $m[\cdot]$. The first theorem is the classic incremental small gain theorem and the other three are new theorems that involve the use of the operator $m[\cdot]$. An example will also be given that show cases where the

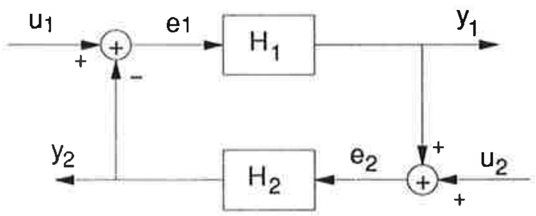

Figure 3. Feedback system under consideration

new theorems give less conservative conditions for stability. With stability we will mean existence of a unique solution that depends Lipschitz continuously on the input signals.

We will state and prove the theorems in a general Banach space. There are, however, several drawbacks with working in a Banach space when deriving stabilty theorems for feedback systems. For example, all signals must be contained in the Banach space, which sometimes deny interesting and in practice often appearing signals from consideration. One way to get around this is to work in extended spaces. We will in this report avoid the technicalities involved in defining Lipschitz and Dahlquist functionals in extended space. For details on extended spaces and the derivation of extended space versions of the theorems in this section, see [Jönsson, 1993].

Before stating any stability theorems we need to define the Banach space X and the type of operators and the feedback system that will be considered in this section. The Banach space X is specified to be a signal space according to the following definition.

DEFINITION 1
The space X is a linear space of functions x of the type

$$x : \mathbf{T} \mapsto V$$

where $\mathbf{T}$ is a half infinite subset of the real numbers $\mathbb{R}$, i.e.$\mathbf{T}=[T_0, \infty) \subset \mathbb{R}$, or the integers Z, i.e. $\mathbf{T}=[T_0, \infty) \subset Z$. It is assumed that X is a Banach space with norm bounded elements, i.e. $x \in X \iff \|x\| < \infty$. □

Examples of signal spaces X are the $L_p^n[T_0, \infty)$and $l_p^n[T_0, \infty)$ spaces, where $p \in [1, \infty]$.

We will study the properties of the feedback system in Figure 3 with respect to boundedness, continuity, existence and uniqueness of solutions. The equations of the feedback system are

$$\begin{aligned} e_1 &= u_1 - y_2 \\ e_2 &= u_2 + y_1 \\ y_1 &= H_1 e_1 \\ y_2 &= H_2 e_2 \end{aligned} \tag{22}$$

The operators H_1 and H_2 are assumed to be nonlinear operators in the class $\mathbf{H}$ defined as follows.

DEFINITION 2
An operator H in $\mathbf{H}$ maps X into itself and has the property $H(0) = 0$. □

Remark 1. This essentially means that we do not consider constant signals in the feedback system. The reason is that constant signals are not bounded in important signal spaces such as $L_2^n[0,\infty)$. We can always redefine an operator that does not satisfy the condition $H(0) = 0$ of the definition above. This means that we disregard the constant signals in the feedback system when analyzing stability.

Remark 2. It is in practice only sensible to consider causal operators, that is, operators whose output does not depend on future inputs. We will make the implicit assumption that the operators are causal. However no attempt to define causality mathematically will be done since this requires the introduction of extended spaces. We refer to [Willems, 1971], [Zames, 1966] or [Desoer and Vidyasagar, 1975] for more details on extended spaces.

The operator $H \in \mathbf{H}$ is Lipschitz continuous if $H \in \mathcal{L}_0(X)$, i.e. when

$$L[H] = \sup_{u \neq v} \frac{\|Hu - Hv\|}{\|u - v\|} < \infty$$

It follows that a Lipschitz continuous operator is also bounded. Since $H(0) = 0$, we have

$$\gamma(H) = \sup_{u \neq 0} \frac{\|Hu\|}{\|u\|} \leq \sup_{u \neq v} \frac{\|Hu - Hv\|}{\|u - v\|} = L[H] < \infty$$

that is, the gain of H is less than or equal to the Lipschitz functional of H.

We will only regard operators in $\mathbf{H}$ and therefore when stated $H \in \mathcal{L}_0(X)$, we mean a Lipschitz continuous operator with $H(0) = 0$. We will also assume that the initial conditions of the feedback system in Figure 3 are zero. If this is not the case then it is still possible to use the theorems of this section if we can replace the effect of the initial conditions with corresponding input signals. This is for example the case when the operators H_1 and H_2 are linear. We will also assume that the operators have stable internal states (uncontrollable and unobservable modes), since otherwise we could have an operator that "explodes" even though our analysis indicates that the system is stable. Note that the operators $L[\cdot]$ and $m[\cdot]$ only consider the input-output map of an operator and does not take internal modes into consideration.

Our first stability theorem is the classical incremental small gain theorem, see [Desoer and Vidyasagar, 1975]

THEOREM 10—Incremental small gain theorem
If the operators H_1 and H_2 in Figure 3 are in $\mathcal{L}_0(X)$ with Lipschitz constants satisfying the condition $L[H_1]L[H_2] < 1$ then

a. $\forall u_1, u_2 \in X$, there exists a unique solution e_1, e_2, y_1 and $y_2 \in X$.
b. e_1, e_2, y_1 and y_2 depend Lipschitz continuously on u_1 and u_2.

Proof: The proof is similar to a more general proof (valid in extended spaces) in [Desoer and Vidyasagar, 1975].

a. If there exists a solution to the feedback system in X, then from the feedback equations (22) we have

$$e_1 = u_1 - H_2(u_2 + H_1 e_1) \tag{23}$$

However, this equation makes no sense until we have proven that there really exists a solution $e_1 \in X$ to it. We can regard (23) as a map of the form $e_1 = f(u_1, u_2, e_1)$. We will show that the map f is a contraction. Then it follows from Banach's fixed point theorem that there is a unique solution $e_1 \in X$ to (23). Let u_1 and u_2 be given signals in X and let e_1 and e_1' be arbitrary in X, then

$$\begin{aligned}\|e_1 - e_1'\| &= \|u_1 - H_2(u_2 + H_1 e_1) - (u_1 - H_2(u_2 + H_1 e_1'))\| \\ &\leq L[H_2]\|H_1 e_1 - H_1 e_1'\| \leq L[H_2]L[H_1]\|e_1 - e_1'\|\end{aligned}$$

From the assumption it follows that f is a contraction. It follows that $y_1 = H_1 e_1$, $e_2 = y_1 + u_2$ and $y_2 = H_2 e_2$ also exist and are unique.

b. It follows from a that for arbitrary $u_1, u_2 \in X$ and $u_1', u_2' \in X$ there exist unique solutions $e_1, e_2, y_1, y_2 \in X$ and $e_1', e_2', y_1', y_2' \in X$. We get

$$\begin{aligned}\|e_1 - e_1'\| &= \|u_1 - H_2(u_2 + H_1 e_1) - (u_1' - H_2(u_2' + H_1 e_1'))\| \\ &\leq \|u_1 - u_1'\| + L[H_2]\|u_2 - u_2'\| + L[H_2]L[H_1]\|e_1 - e_1'\|\end{aligned}$$

From the assumption $L[H_2]L[H_1] < 1$ it follows that

$$\|e_1 - e_1'\| \leq \frac{1}{1 - L[H_2]L[H_1]}(\|u_1 - u_1'\| + L(H_2)\|u_2 - u_2'\|)$$

It is proven in a similar way that

$$\|e_2 - e_2'\| \leq \frac{1}{1 - L[H_1]L[H_2]}(\|u_2 - u_2'\| + L(H_1)\|u_1 - u_1'\|)$$

and since $\|y_1 - y_1'\| \leq L[H_1]\|e_1 - e_1'\|$ and $\|y_2 - y_2'\| \leq L[H_2]\|e_2 - e_2'\|$ **b** is proven. □

Remark 1. In the case that $H_2 = G_2$ is linear the conditions of the theorem reduce to $G_2, H_1 \in \mathcal{L}_0(X)$ with $L[G_2 H_1] < 1$. Similarly if $H_1 = G_1$ is linear then the conditions of the theorem reduce to $G_1, H_2 \in \mathcal{L}_0(X)$ with $L[G_1 H_2] < 1$.

Remark 2. It can be shown that if the operators H_1 and H_2 are causal then the closed loop system is also causal. The idea of the proof is that the solution to the feedback system can be obtained by a fixed point iteration involving causal operators, hence the signals e_1, e_2, y_1 and y_2 will depend causally on u_1 and u_2.

In the case that one of the operators H_1 or H_2 are linear we can achieve less conservative theorems for stability. Before we state any theorems it is in place to introduce the notion of well posedness. We will denote the feedback system in Figure 3 well posed if, from both a mathematical and a practical point of view, there exists a solution $e_1, e_2, y_1, y_2 \in X$ to every input pair $u_1, u_2 \in X$. There are cases when a solution to the feedback system exists from a mathematical point of view even though the system makes no sense from a practical point of view. Such feedback systems will not be considered as well posed. The following example shows such a feedback system.

EXAMPLE 5—[Willems, 1971]
Consider the feedback system in Figure 4, where $K > 0$. If a mathematical point of view is taken when analyzing the response of the system, then $y(t) = \frac{K}{1+K}u_1(t)$ when $K \neq -1$. However, in practice we will always have some slight delay in the system. This follows since the transmission speed of the signals in a physical system is finite. A simple calculation shows that the response at

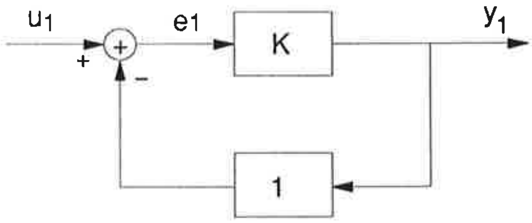

Figure 4. The feedback system of Example 5

$t = 1$ to a unit pulse (i.e. the response to $u_1(t) = \theta(t) - \theta(t-1)$ at $t = 1$) when a delay e^{-sT_ε} is inserted in the feedback system is

$$y(1) = K\sum_{n=0}^{n_\varepsilon}(-K)^n, \quad \text{where} \quad n_\varepsilon = \lfloor\frac{1}{T_\varepsilon}\rfloor$$

where $\lfloor\cdot\rfloor$ denotes the truncation operator, which gives the integer part of its argument. It follows that $y(1) \to \infty$ as $T_\varepsilon \to 0$ when $K > 1$, i.e. the output $y(1)$ can be extremely large when the transmission time is small. Our conclusion is that from a practical point of view, a solution to the feedback system in Figure 4 exists only if $K < 1$. This feedback system is in other words only well posed when $K < 1$ □

We have the following two theorems

THEOREM 11
If $H_2 = G_2$ is linear and if the feedback system in Figure 3 is well posed, then a sufficient condition for the solution $e_1, e_2, y_1, y_2 \in X$ to depend Lipschitz continuously on $u_1, u_2 \in X$ is that $H_1, G_2 \in \mathcal{L}_0(X)$ and $m[G_2H_1] > -1$. The solution to the feedback system will also be unique.

Proof: From the feedback equations (22) we have

$$e_1 = u_1 - G_2(u_2 + H_1e_1) \tag{24}$$

by adding $G_2H_1e_1$ on both sides of this equation we get

$$(I + G_2H_1)e_1 = u_1 + G_2H_1e_1 - G_2(u_2 + H_1e_1) = u_1 - G_2u_2 \tag{25}$$

From Corollary 1 we know that $m[G_2H_1] > -1$ is a sufficient condition for the operator on the left hand side to be invertible and the inverse is bounded by $L[(I + G_2H_1)^{-1}] \leq \frac{1}{1+m[G_2H_1]}$. Hence, we get

$$e_1 = (I + G_2H_1)^{-1}(u_1 - G_2u_2)$$

Which also shows that e_1 is unique. We also have

$$\|e_1 - e_1'\| \leq \frac{1}{1 + m[G_2H_1]}(\|u_1 - u_1'\| + L(G_2)\|u_2 - u_2'\|$$

where e_1 and e_1' are the solution to (25) when the inputs are $u_1, u_2 \in X$ and $u_1', u_2' \in X$ respectively. Since $y_1 = H_1e_1$, $e_2 = u_2 + y_1$ and $y_2 = G_2e_2$, it follows that y_1, e_2, y_2 also depend Lipschitz continuously on u_1 and u_2. □

THEOREM 12
If $H_1 = G_1$ is linear and if the feedback system in Figure 3 is well posed, then a sufficient condition for a solution $e_1, e_2, y_1, y_2 \in X$ to depend Lipschitz continuously on $u_1, u_2 \in X$ is that $H_2, G_1 \in \mathcal{L}_0(X)$ and $m[G_1 H_2] > -1$. The solution to the feedback system will also be unique.

Proof: This theorem is proven in the same way as the last theorem □

If $u_2 \equiv 0$ then we have a theorem similar to those above where both operators H_1 and H_2 are allowed to be nonlinear [Söderlind, 1992].

THEOREM 13
If $u_2 \equiv 0$ and if the feedback system in Figure 3 is well posed, then a sufficient condition for a solution $e_1, e_2, y_1, y_2 \in X$ to depend Lipschitz continuously on $u_1, u_2 \in X$ is that $H_1, H_2 \in \mathcal{L}_0(X)$ and $m[H_2 H_1] > -1$. The solution to the feedback system will also be unique.

Proof: From the feedback equations in (22) we get

$$e_1 = u_1 - H_2 H_1 e_1$$

Since $m[H_2 H_1] > -1$, we can solve for e_1 and get $e_1 = (I + H_2 H_1)^{-1} u_1$, and unicity of the solution follows. From this equation it is easy to derive the following inequalities

$$\begin{aligned} \|e_1 - e_1'\| &\leq \frac{1}{1 + m[H_2 H_1]} \|u_1 - u_1'\| \\ \|y_1 - y_1'\| = \|e_2 - e_2'\| &\leq \frac{L[H_1]}{1 + m[H_2 H_1]} \|u_1 - u_1'\| \\ \|y_2 - y_2'\| &\leq \frac{L[H_2] L[H_1]}{1 + m[H_2 H_1]} \|u_1 - u_1'\| \end{aligned}$$

so the system is Lipschitz continuous. □

Remark. In Theorem 11, the assumption on well posedness guarantees that equation (24) makes sense. However, if we take a strictly mathematical attitude to the problem of existence of a solution, we see that the condition $m[G_2 H_1] > -1$ is sufficient to guarantee that a solution exists to equation (25) and therefore to equation (24). The reason we have the assumption on well posedness of the feedback system is that there are examples of systems that do not work in practice even though the condition $M[G_2 H_1] > -1$ is satisfied. This is shown in the example below. This remark also holds for Theorem 12 and Theorem 13

We will now show that the system in Example 5 is a system where the conditions on the Lipschitz and the Dahlquist functionals in the theorems above are satisfied even though the system is not well posed.

EXAMPLE 6—Continuation of Example 5
Both H_1 and H_2 in the example are linear and in $\mathcal{L}_0(X)$. Further, we have that $m[H_1 H_2] = m[H_2 H_1] = m[K] = K > -1$. Therefore the conditions for continuity of Theorem 11 – 13 are fulfilled even though the system is not well posed. □

From the example we see that systems with direct feedthrough greater than or equal to one might not be well posed. It will be shown in [Jönsson,

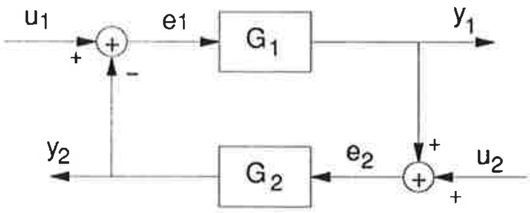

Figure 5. Feedback system in Example 7

1993] that the condition of well posedness is satisfied for systems with direct feedthrough less than one and for systems containing a delay.

In practice all well designed feedback systems satisfy the condition on direct feedthrough less than one and there will be no problem using the theorems involving the Dahlquist functionals when the model of the physical system is good.

We will now give an example that shows that there are cases when the theorems involving $m[\cdot]$ give less conservative conditions for stability than the incremental small gain theorem.

EXAMPLE 7

In this example it is assumed that the feedback interconnection consists of two linear time invariant SISO systems as in Figure 5. We also assume that the signal space under consideration is $L_2[0,\infty)$. We know from Section 3 that the Lipschitz functional is $L_2[G] = \sup_\omega |G(i\omega)|$ and the Dahlquist functional is $m_2[G] = \inf_\omega \text{Re } G(i\omega)$. The conditions for stability in the incremental small gain theorem reduces in this case to the condition that both G_1 and G_2 are stable and that the open loop gain $G_o = G_1G_2$ has a Nyquist curve strictly inside a disc with radius 1. This should be compared to the conditions for stability when using the theorems involving $m[\cdot]$. There we need G_1 and G_2 to be stable and that the real part of the nyquist curve of the open loop gain G_o is strictly in the half plane Re $s > -1$. The condition of well posedness must of course also be satisfied.

It is easy to find examples where the conditions for stability in the theorem involving $m[\cdot]$ are satisfied even though the conditions of the small gain theorem are violated. For example, let $G_1(s) = \frac{1}{s+1}$ and $G_2 = 4\frac{s+1}{s+2}$ then $G_o = G_1G_2 = \frac{4}{s+2}$ and it is seen from the Nyquist curve in Figure 6 that the theorems involving $m[\cdot]$ guarantee stability while the small gain theorem does not. The closed loop system is $G_{cl} = \frac{s+2}{s^2+7s+6}$, which of course is stable.

□

The passivity theorem and M and m for interconnected systems

The application of the classical passivity theorem to two LTI systems in a simple feedback loop says that if either one of $G_1(i\omega)$ or $G_2(i\omega)$ lies in the open right half plane and the other in the closed right half plane then the closed loop is stable. The conditions on $G_1(i\omega)$ and $G_2(i\omega)$ implies that the Nyquist curve of the product of the two systems must lie in the complex plane cut open along the negative real axis, i.e. the phase of the loop gain lies in the open interval$(-\pi,\pi)$. If we try to state this theorem in terms of $m_2[\cdot]$ we find that we can easily formulate the two restrictions on $G_1(s)$ and $G_2(s)$ respectively in terms of $m_2[\cdot]$ but that the resulting system $G_2(s)G_1(s)$ does not necessarily satisfy any bound on $m_2[\cdot]$. Only if both $G_1(s)$ and $G_2(s)$ have finite gains can we guarantee that $m_2[G_2G_1] > R$ for some R. The problem is of course that the Dahlquist functionals do not have any submultiplicative

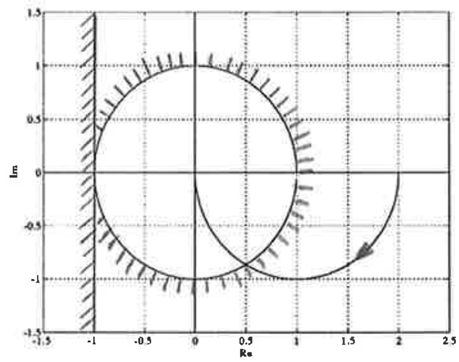

Figure 6. Nyquist curve for the open loop gain of the feedback system in example 7

property irrespectively of the norm used, which means that we cannot deduce anything about $m[G_2G_1]$ from $m[G_1]$ and $m[G_2]$ only. If we for instance have two LTI systems in a simple feedback loop as above with realizations

$$G_1 = \left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right)$$

and

$$G_2 = \left(\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right)$$

and we want to decide if $m[G_2(s)G_1(s)] > \delta$, then we cannot look at $m[G_1]$ and $m[G_2]$ but have to study the interconnection of the two system matrices, i.e. the realization of G_2G_1 namely

$$G_2G_1 = \left(\begin{array}{cc|c} A_2 & B_2C_1 & B_2D_1 \\ 0 & A_1 & B_1 \\ \hline C_2 & D_2C_1 & D_2D_1 \end{array}\right)$$

Now the results above can be applied to this system matrix.

6. Conclusions

We have in this report applied Lipschitz and Dahlquist functionals for stability analysis of feedback systems. Closed expressions for the functionals have been given in the Banach spaces $L_2^p[0,\infty)$ and $L_\infty^p[0,\infty)$ for linear time invariant dynamical systems and for static diagonal nonlinearities. In the case of $L_2^p[0,\infty)$ we get familiar results from system theory. We have also given a simple example where a stability theorem using Dahlquist functionals give less restrictive conditions than the small gain theorem. The problem with using Dahlquist functionals for stability analysis is that they do not posses a submultiplicative property. We have also seen that it is restrictive to derive stability theorems in Banach space since we cannot always treat signals, such as sinusoidals, commonly appearing in practice.

7. References

[Andersson and Vongpanitlerd, 1973] Andersson, B. and Vongpanitlerd, S. (1973). *Network Analysis and Sythesis: A Modern Systems Theory Approach.* Prentice Hall.

[Bittanti et al., 1991] Bittanti, S., Laub, A., and Willems, J., editors (1991). *The Riccati Equation.* Springer–Verlag.

[Boyd et al., 1989] Boyd, S., Balakrishnan, V., and Kabamba, P. (1989). A Bisection Method for Computing the H_∞ Norm of a Transfer Matrix and Related Problems. *Mathematics of Control, Signals, and Systems*, (2):207–219.

[Browder, 1963] Browder, F. (1963). The Solvability of Non-Linear Functional Equations. *Duke Math. J.*, (30):557–560.

[Dahleh and Pearson, 1987] Dahleh, M. and Pearson, B. (1987). L^1–Optimal Compensators for Continuous–Time Systems. *IEEE Trans. Automat. Contr.*, AC–32(10):889–895.

[Desoer and Vidyasagar, 1975] Desoer, C. and Vidyasagar, M. (1975). *Feedback Systems: Input-Output Properties.* Academic Press.

[Francis, 1987] Francis, B. A. (1987). *A Course in H_∞ Control Theory.* Springer-Verlag.

[Jönsson, 1993] Jönsson, U. (1993). Stability Theory Using Dahlquist and Lipschitz Functionals, Part II: Extended Spaces and Causality. Technical report, Dept of Automatic Control, Lund Institute of Technology.

[Minty, 1962] Minty, G. (1962). Monotone (nonlinear) Operators in Hilbert Space. *Duke Math. J.*, (29):341–346.

[Narendra and Taylor, 1973] Narendra, K. and Taylor, J. (1973). *Frequency Domain Criteria for Absolute Stability.* Academic Press.

[Söderlind, 1984] Söderlind, G. (1984). On Nonlinear Difference and Differential Equations. *BIT*, (24):667–680.

[Söderlind, 1986] Söderlind, G. (1986). Bounds on Nonlinear Operators in Finite-dimensional Banach Spaces. *Num. Math*, (50):27–44.

[Söderlind, 1992] Söderlind, G. (1992). The Lipschitz Algebra and its Extensions. Technical Report LU-CS-TR:92-100, Dept of Computer Science, Lund Institute of Technology.

[Willems, 1971] Willems, J. (1971). *The Analysis of Feedback Systems.* The M.I.T. Press.

[Zames, 1966] Zames, G. (1966). On the Input-Output Stability of Time Varying Nonlinear Feedback Systems, Part I: Conditions Derived Using Concepts of Loop Gain, and Positivity. *IEEE Trans. Automat. Contr.*, AC-11(2):228–238.