Completeness in modal logic

Sebastian Enqvist (821023-4111)
Course: FTE603:2
Supervisor: Bengt Hansson
University of Lund 2005

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1 Introduction

This paper is mainly inspired by formal research results and philosophical considerations in "A guide to intensional semantics", written in 1973 by Bengt Hansson and Peter Gärdenfors. That essay was focused on the philosophy of semantics for modal logics, with special attention to completeness results. The purpose of the essay was to exhibit a negative trend in modal logic regarded as a philosophical discipline; the enterprise of completeness proving has become largely "*l'art pour l'art*", as the authors express it, and semantics often seem to be viewed by logicians as mere "collectors of completeness results" rather than tools for actually interpreting modal logics. To illustrate this point, they constructed a semantics that makes every classical modal logic come out complete. Completeness as such is a trivial property for all even moderately interesting modal logics.

The first sections of this essay will be devoted to the guide to intensional semantics. In the following sections, I turn to another yet, I shall argue, related essay written by David Lewis in 1974 called "Intensional logics without iterative axioms". In that essay, Lewis proves a theorem stating that all non-iterative logics are Scott-Montague complete. Some implications of this will be discussed, as well as the relation to the guide to intensional semantics.

I address not one but several problems in this essay. Some of these will be formal, mathematical problems concerning the properties of logical systems and their semantics. Others will be of a more philosophical nature. However, all parts of the essay will be tied together in that they revolve around perhaps the most central concept in metalogic: the notion of completeness. The importance of Hanssons and Gärdenfors' essay lies in that it makes the idea of completeness problematic, and so raises several difficult questions: how are different types of completeness related to each other? What type of completeness holds for what systems? And most pressing: *what does a completeness result mean?*

2 What is modal logic?

Informally put, modal logic is the logic of necessity and possibility. It is a class of systems devised to give formal accounts of the idea that propositions may not merely be true or false, but necessarily true/false, or false yet possibly true etc. Essential to modal logics is the addition of the operators "□" (for necessity) and "◊" (for possibility) to the language.

The *language* (L) of the basic non-modal *propositional calculus*, or PC, can be defined as follows. The *alphabet* of L consists of infinitely many *propositional variables*, usually written p, q, r, and so on, and the symbols $\vee$ and ~ for disjunction and negation respectively ("or" and "not"). To form the set of all sentences in the language, or all *well formed formulae* (usually abbreviated "wff"), we have the following recursive formation rules[1]:

[1] any propositional variable is a wff

[2] if Φ and Ψ are wff, so is $\Phi \vee \Psi$

[3] if Φ is a wff, so is $\sim\Phi$

[4] nothing else than the formulae specified by [1], [2] and [3] is a wff

Formulae consisting of only one propositional variable are said to be *atomic*, whilst all other formulae are *molecular*. To give the language of standard modal logics, we need only add □ to the alphabet together with the following formation rule (with an obvious modification to [4]):

[5] if Φ is a wff, so is $\Box\Phi$

We can now *define* some more connectives using the ones we already have. I give a list of definitions for connectives $\wedge$, $\supset$, $\equiv$ and $\Diamond$ (meaning "and", "if, then", "if and only if" and "it is possible that" respectively):

Definition 2.1 ($\wedge$): $\Phi \wedge \Psi =_{df} \sim(\sim\Phi \vee \sim\Psi)$

Definition 2.2 ($\supset$): $\Phi \supset \Psi =_{df} \sim\Phi \vee \Psi$

Definition 2.3 ($\equiv$): $\Phi \equiv \Psi =_{df} (\Phi \supset \Psi) \wedge (\Psi \supset \Phi)$

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[1] Symbols like Φ, Ψ, θ etc. are *metalinguistic* symbols that can stand for whole complex formulae as well as atomic ones. They allow formulae to be schematic: $\Phi \supset (\Psi \supset \Phi)$ for example can stand for $p \supset (q \supset p)$, q ⊃ (p

Definition 2.4 (◊): $\Diamond\Phi =_{df} \sim\Box\sim\Phi$

All these definitions are quite standard, and we shall get some motivation for them when we move to look at the semantics for PC and modal logic, as they all rely on semantic equivalencies.

As we have defined the language of modal logic (we may call it $L_\Box$) as a set of formulae closed under the formation-rules, we may define a *modal logic* or *system* S as a *subset* of $L_\Box$ containing every PC-valid formula (see the next section) and such that it is closed under the following rules:

US (uniform substitution): if Φ is in S, and Ψ is the result of uniformly substituting formulae for propositional variables in Φ, then Ψ is in S.

MP (modus ponens): if $\Phi \supset \Psi$ is in S, and Φ is in S, then Ψ is in S.

We say that a modal logic is *consistent* iff it is a *proper subset* of $L_\Box$, i. e. not every wff is in S[2]. The formulae of which a logic consists are called its *theorems*. To determine which ones those formulae are, we simply stipulate that some designated formulae in $L_\Box$ are to be theorems of S. The class of all theorems is then determined by those formulae and the set of rules of the system, minimally containing US and MP. The class of stipulated theorems is called the *axiom-set* of the logic, and its members are called *axioms*.

Every modal logic needs to contain all PC-valid formulae, and an easy way to take care of this is to state that every PC-valid formula is to count as an axiom. PC can of course be treated axiomatically as well, but modalities are what interests us here, so this method is a good way to focus our concentration on the specifically modal parts of the logics. PC-semantics will thus determine the basic axiom-set for modal logics. I will not explicitly state that this set is in the axiom-set for any logic up for study, but merely give the modal axioms. Before moving on, I give a few definitions that will be important as we move on, and list a few well known modal logics together with their axioms.

$\supset q), (p \wedge \sim p) \supset (p \supset (p \wedge \sim p))$ and so forth.

[2] Some readers may be more acquainted with an alternative definition of consistency, where consistency is defined as never having both Φ and ~Φ in S for any Φ. That definition however is equivalent to the present one, as may be checked.

Definition 2.5: a logic is *classical* if whenever it contains $\Phi \equiv \Psi$, it also contains $\Box\Phi \equiv \Box\Psi$.

Definition 2.6: a logic is *normal* if it contains **K**: $\Box(p \supset q) \supset (\Box p \supset \Box q)$, and additionally contains $\Box\Phi$ whenever it contains Φ (we call the latter principle the *rule of necessitation*).

A few logics:

K	**K** + all PC-valid formulae
D	K + $\Box p \supset \Diamond p$
T	K + $\Box p \supset p$
S4	T + $\Box p \supset \Box\Box p$
S5	T + $\Diamond p \supset \Box\Diamond p$

3 Semantics for modal logic

I first briefly present truth-functional semantics for PC. The idea is that each proposition always takes one of two values: 1, for “true”, and 0 for “false”. For every complex formula, the value of that formula is to be determined completely by the values of its atomic sub-formulae. To achieve this effect, we determine *truth-functions* for all sentence-forming connectives. The basic truth tables for those connectives should be familiar, but I give matrixes for the primitive connectives ~ and ∨ here:

∨	1	0
1	1	1
0	1	0

~	
1	0
0	1

You may now check that in the definitions for ∧, ⊃ and ≡, the defining and the defined terms are semantically equivalent, they always take the same value. The most commonly addressed semantics for modal logic is the Kripke-type semantics, employing the concept of a “possible world” as a different state of affairs than the one actually obtaining. Necessity is thought of as truth in *all* possible worlds, possibility as truth in *some* possible world. To model this concept formally, we use something we call *frame semantics*. These semantics are just like ordinary bivalent truth-functional semantics for PC, except that we evaluate the atomic formulas not just once, but at many places simultaneously, at various *points*. PC-formulae are evaluated as usual at every point. Given this structure, we can add *intensional operators* to the language, that are distinguished from the PC-operators (the *extensional* operators) in that their values at a point are calculated not only from the values of their arguments[3] at that point, but at other points as well. But which points are the ones that matter?

To decide this, we introduce a relation between points, called the *accessibility relation.* For that reason, this type of semantics is often called *relational semantics*. The points taken into consideration when obtaining values at some point are just the points which are available by that relation from the point in question. We say that the point can *see* these points. In modal logic, the intensional operators are the modal operators introduced in the last section, $\Box$ for necessity and $\Diamond$ for possibility. $\Box\Phi$ is true at a point p iff Φ is true at every point p can see. $\Diamond\Phi$ is true at p iff Φ is true at some such point. The points may be called *possible worlds* or just *worlds,* since that is the most illustrative way to think of them in modal applications. We can now define the concepts of a *Kripke frame,* a *Kripke model, truth* and *validity.*

A Kripke frame is an ordered pair < W, R >, where W is a non-empty set of worlds, and R is a set of ordered pairs < w, w’ > of worlds, defining the extension of the accessibility-relation over the set W. We call W the *universe* of the frame. If < w, w’ > is in R, we say that w *sees* w’, or that w’ is possible relative to w. We denote this by wRw’. A Kripke model is an ordered triple < W, R, V > where W and R are as before, and V is an assignment of the truth-values 1 and 0 to each atomic formula at every world in W. A formula Φ is *true in a model* iff it has value 1 in every world in W for some valuation V. Φ is *valid in a frame* iff it is true in every model based on that frame, i.e. remains true in the model whatever values are assigned to the atomic formulas. Finally, a formula is *valid with respect to a class C of frames* iff it is valid in every frame in C. Such classes are usually defined by posing restraints on the relation R. Thus, the formula

T: $\Box p \supset p$

is valid in the class of *reflexive* frames, the class of all frames where R is reflexive (i. e. wRw for all w in W), as you can check. We say that a system S is *sound* with respect to a class of frames if all theorems (provable formulae) in the system are valid in that class. The *range* of a system is the class of all frames where it is sound. We say that a system is *complete* with respect to a class of frames if its theorems are *exactly* the formulae valid in the class. The

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[3] The *arguments* of a connective are the formulae it binds. For example, in $p \supset (q \wedge r)$, the arguments of ”$\supset$” are

basic system K is complete with respect to the class of all Kripke frames. We say that this class *characterizes* K. Adding **T** to K gives us T, the system characterized by the class of all reflexive frames. K contains the following formula (usually taken as its modal axiom):

K: $\Box(p \supset q) \supset (\Box p \supset \Box q)$

We can’t have systems characterized by Kripke-frames, that do not contain K. K is as low as Kripke semantics will go (in the terminology of Hansson and Gärdenfors, to be introduced later, K determines the *width* of Kripke-semantics.) So what do we do if we don’t want **K** to be a theorem in our system? We do *Scott-Montague semantics*.

Scott-Montague semantics, also commonly called *neighborhood* semantics, is a kind of generalization from Kripke semantics. The alteration consists in how we determine which worlds shall be taken into consideration when evaluating formulae at some world. Instead of the relation R, we have a function N assigning a class of sets of worlds to every w, such that every set in the class is a subset of W. $\Box\Phi$ is true at w if the set of all and only the worlds at which Φ is true is in N(w). We say that $\Box\Phi$ is true at w if *the truth-set of Φ is a neighborhood of w*. Truth in a model and validity are as before (except in the equivalent formulation of the semantics presented by Hansson and Gärdenfors. We postpone the presentation of this variant until we need it – this section is merely intended to introduce the basic concepts of modal semantics). This is as far as we go at present; more concepts will be introduced later. We conclude this section however by proving that Scott-Montague semantics do just what we want them to do: they allow systems weaker than K.

Theorem 3.1: $\Box(p \supset q) \supset (\Box p \supset \Box q)$ is not valid in the class of all Scott-Montague frames.

Proof: we construct a falsifying model. Let this model be < W, N, V >, and let W be $\{w_1, w_2, w_3\}$. We give the following values to p and q at the different worlds: p and q are both false at w_1, q alone is true at w_2, and both are true at w_3. Let $N(w_1)$ be $\{\{w_1, w_2, w_3\}, \{w_3\}\}$. $\{w_1, w_2, w_3\}$ is the truth-set of $p \supset q$, so $\Box(p \supset q)$ is true at w_1. $\{w_3\}$ is the truth-set of p, so $\Box p$ is also true at w_1. But the truth set of q is $\{w_2, w_3\}$ and this is not in $N(w_1)$. So $\Box q$ fails at w_1, and thus so does $\Box p \supset \Box q$ and therefore the entire formula $\Box(p \supset q) \supset (\Box p \supset \Box q)$ fails at w_1, since the antecedent is true there.

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p and $q \wedge r$.

QED

4 Hansson and Gärdenfors on intensional semantics

I shall try to summarize the “guide to intensional semantics” as briefly as possible. The article starts by introducing two new concepts: *width* and *depth*. Width and depth are measures of how many systems some type of semantics, e. g. relational semantics, makes complete. The width of some semantics is determined by the weakest system complete in that semantics, the smallest system characterized by some class of frames in it. Any system stronger than that weakest system is in the width of the semantics. The width of relational semantics is thus set by K, and that class is called the class of *quasi-normal* systems. A quasi-normal system is any system which contains K. There are quasi-normal systems that are not normal, although I shall not prove it. The width of neighborhood semantics is set by a system called “E”, and logics in this class are called *quasi-classical*. The depth of a semantics relative to some class of systems is a measure of how many of the systems in that class are complete in the semantics. The depth is said to be *maximal* relative to that class if all systems in the class are complete. Of very high interest, of course, is the depth of a semantics relative to its width.

Scott-Montague semantics are then put in a different yet equivalent formulation, where the “N”-function from worlds to classes of subsets of W is replaced by the function “*f*”, simply from subsets of W to subsets of W. The idea is that when the evaluation on a Scott-Montague frame is set, all we need to know in order to know where $\Box\Phi$ is true for some Φ is where Φ is true. So *f* can be regarded as a function from the truth-set of Φ to the truth-set of $\Box\Phi$. If $/\Phi/$ is the truth-set of Φ, then the truth-set of $\Box\Phi$ is to be $f\,(/\Phi/)$ (in case we have an equivalent Scott-Montague model, f (A) would correspond to the set of worlds that have A as a neighborhood). A formula is true in a model if its truth-set is W. So, for example, $\Box\Phi$ is true in a model if $f\,(/\Phi/) = W$, and valid if this holds for every evaluation on the frame.

This formulation allows for a simple generalization, to what we call *Boolean semantics*. A boolean frame is an ordered pair $< B, f >$. B is a Boolean algebra, i. e. a distributive lattice with an operation “–” such that $A = - -A$ for all elements A in the algebra, and with a top element 1 and a bottom element 0. Most Boolean algebras can be thought of as the full subset-algebra of some set, where $-$ is the complement operation, intersections form lower and unions form upper bounds, and 1 is the whole set, 0 is $\varnothing$. Distributivity on such a lattice means that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for all A, B, C, and the lattice will be ordered by the subset-relation. I regret that there isn't enough space to introduce lattice theory and Boolean algebras more thoroughly, but the relevant facts about Boolean algebras for my present purposes can be understood without knowing exactly what they look like. Not all Boolean algebras are of the above kind, and that makes this semantics non-equivalent to Scott-Montague semantics.

The function *f* goes from elements of B to other elements of B. Truth for some valuation V holds if V(Φ) = 1. So truth in a model for $\Box\Phi$ holds if $V(\Box\Phi) = f(V(\Phi)) = 1$. Validity is just as before. A *very* important fact to note about Boolean semantics is that it completely dispenses with the whole idea of possible worlds. Individual worlds started to get redundant in the alternative formulation of Scott-Montague semantics, as *sets* of worlds came into the foreground. Here, possible worlds are gone altogether. Possible worlds, however we interpret them, are arguably the best tools we have for understanding modality, and Boolean semantics misses them completely.

The width of Boolean semantics is the same as for Scott-Montague semantics, the class of all quasi-classical systems. The two most important results in the article (from the present point of view) can now be stated:

Theorem 4.1: the depth of Boolean semantics with respect to classical systems is maximal.

Theorem 4.2: the depth of Scott-Montague semantics with respect to classical systems is not maximal.

(Theorem 4.1 is in fact made even stronger, by showing that the depth of Boolean semantics can be made maximal with respect to all quasi-classical logics, by the addition of so called filters to the frames). We have presented three distinct types of semantics so far: Kripke semantics, Scott-Montague semantics, and Boolean semantics. The Boolean semantics seems not to explain anything of how modality works. Scott-Montague semantics incorporates possible worlds and is thus better off; but Kripke semantics clearly provides the best models for modality. Possible worlds are related to each other by the relation of worlds "being possible" to one another, and altering that relation alters the logic. Kripke semantics explicates "what's going on" with striking clarity. How are these semantics related to each other? Well, we've seen that Boolean semantics and Scott-Montague semantics share the same width but differ in depth when regarded relative to classical logics. Boolean semantics includes almost every conceivable modal logic in its depth. Scott-Montague semantics and

Kripke semantics differ in width, and so of course also in depth relative to classical logics. But what about the comparative depth in the class of *normal* systems?

5 The normal depth of Scott-Montague semantics

The biggest mathematical contribution in the guide to intensional semantics, apart from the maximal depth theorems, I believe is the invention of the concepts of width and depth. These measures unify the project of completeness proving in a neat and illuminating way, and provide a framework for the discussion. One question raised within this framework in the guide to intensional semantics is whether or not the depth in normal systems is greater for Scott-Montague semantics than it is for Kripke semantics. The question shall remain open, and I will prove neither the affirmative nor the negative here. I shall however show that if it is so, then all of the normal systems that are Kripke incomplete but neighborhood complete lack the so-called finite model property[4].

Lemma 5.1: in any Scott-Montague frame such that for some w, N(w) is not closed under intersections, soundness fails for any normal system.

Proof: The following formula is a theorem in K[5]

$$\Rightarrow_K \Box(p \wedge q) \equiv (\Box p \wedge \Box q)$$

Now let B and C be two sets in N(w) such that their intersection is not in N(w). Set the valuation on the frame so that the truth set of p is exactly B, and the truth-set of q is exactly C. Then obviously the truth-set of $p \wedge q$ is exactly the intersection of B and C. So $\Box p$ holds at w since B is in N(w), and $\Box q$ holds at w since C is in N(w), so $\Box p \wedge \Box q$ holds at w. But the intersection of B and C is not in N(w), and so $\Box(p \wedge q)$ fails at w contradicting the K-theorem. Since K is the weakest normal system, this shows that any system characterized by the frame must be less than normal.

QED

[4] My original intention in this chapter was to give a conclusive proof that the normal depth of Scott-Montague semantics equals that of Kripke semantics. I owe thanks to Bengt Hansson for discovering a flaw in that proof and explaining it to me, thus helping me rewrite the chapter to its current form.

5 I allow myself to refer to Hughes & Cresswell 1996 for the proof.

Lemma 5.2: the rule of necessitation holds in a Scott-Montague frame iff W is in N(w) for all w.

Proof: if W is in N(w) for all w, then necessitation holds since when some formula Φ is valid, it is true in all w, so its truth set will be in N(w) for all w and thus $\Box\Phi$ is true for all w. If there is some w in W such that W is not in N(w), then for any valid formula Φ, $\Box\Phi$ fails at that world. This proves the lemma.

QED

Lemma 5.3: **K** fails in any Scott-Montague frame where we have W in N(w) for all w, yet we have some w such that there are some subsets A, B of W for which $A \in N(w)$, $A \subseteq B$ but $B \notin N(w)$.

Proof: take any such frame, and set the valuation so that the truth-set of p is exactly A, the truth-set of q is exactly B. The truth-set of $p \supset q$ is then W, since all worlds were p is true are in A, and so since $A \subseteq B$ and B is the truth-set of q, q holds at all worlds in W where p holds. Since W is in N(w), this means that $\Box(p \supset q)$ holds at w. $\Box p$ also holds at w, since the truth-set of p is A which is in N(w). $\Box q$ however fails at w, since the truth set of q is B which is not in N(w). So $\Box(p \supset q) \supset (\Box p \supset \Box q)$ fails at w and so is not valid on the frame.

QED

Theorem 5.4: any finite Scott-Montague frame that validates all theorems of a normal system is equivalent to some relational frame.

Proof: we first show that in any such frame, for all w, N(w) contains some set A such that $A \subseteq B$ for all sets B in N(w). By lemma 5.1 N(w) is closed under intersections, so that $B \cap C$ is in N(w) whenever both of B and C are. So clearly if $N(w) = \{D_1,\ldots,D_n\}$, then $D_1 \cap\ldots\cap D_n \in N(w)$, and that set is contained in any set in N(w) since it is the set of all members common to all sets in N(w)[6]. Let us call A the *base of N(w).*

We may construct the equivalent Kripke-frame in the following manner. W in the Kripke frame is the same as in the original neighborhood frame. For all w in W, we let wRw' iff w' is in the base of N(w) in the corresponding Scott-Montague frame. Any formula Φ is valid on this frame iff it is valid on the corresponding Scott-Montague frame, since for the same valuation V to all propositional variables, for any w in W and any formula Φ, V gives Φ the same value at w on the Kripke frame as on the Scott-Montague frame. To show this, all we are really interested in is whether the $\Box$-operators behave in the same way. So let $\Box\Phi$ be true at some w in the Kripke frame for some valuation V. Then Φ is true at all w' such that wRw'. This means that Φ is true at all worlds in A, the base of N(w) in the corresponding Scott-Montague model, so the truth set of Φ must contain A. By lemma 5.2 and 5.3 together this means that the truth-set of Φ must be in N(w), so $\Box\Phi$ holds at w. Now let $\Box\Phi$ hold at w in the Scott-Montague model. Then the truth-set of Φ is in some set in N(w). By the above reasoning A is contained in that set, so Φ is true at all worlds in A making it true at all w' such that wRw' in the Kripke model. So $\Box\Phi$ holds at w in the Kripke model. So for any w in W, $\Box\Phi$ is true at w in for some valuation on the Kripke frame iff it is true at w for the same valuation on the Scott-Montague frame.

QED

Definition 5.6: A system has the *finite model property* iff it is characterized by a class of finite frames[7].

Corollary 5.7: If S is a normal system, and S is Kripke-incomplete yet Scott-Montague complete, S lacks the finite model property in Scott-Montague semantics.

Proof: let F be some class of Scott-Montague frames, and let F characterize some Kripke-incomplete normal system S. If every frame in F where to be finite, by theorem 5.4 every frame in F would be equivalent to some Kripke frame. For every Scott-Montague frame in F, take one equivalent Kripke frame to form a class F' of Kripke frames. Clearly F' would characterize S, contradicting our assumption. So at least one member of F has to be infinite.

QED

[6] This reasoning is not guaranteed to hold in infinite cases, hence the restriction that the frame be finite.
[7] This definition is taken from Hughes & Cresswell 1996.

This is the main result in this chapter, but we have two additional results that may be of some interest.

Theorem 5.5: any class F of Scott-Montague frames that characterizes a Kripke-incomplete normal system must have some member satisfying the following condition:

"Neighborhood denseness": for at least some w, for every member A of N(w), there is always some member C such that C is strictly between A and $\varnothing$ (i.e. $\varnothing \subset C \subset A$).

Proof: by the assumption that F characterizes a normal system, lemmas 5.2 and 5.3 hold for every frame in F. We have seen from theorem 5.4 that if additionally there is some A in N(w) for every w in every frame in F, such that $A \subseteq B$ for all B in N(w), then the system characterized by F must be Kripke complete. So there must be some frame in F such that there is at least some w such that for every A in N(w), there is some B in N(w) such that not $A \subseteq B$. This cannot hold for $\varnothing$ (since $\varnothing \subseteq B$ trivially for any set B), so $\varnothing$ is not in N(w). Let A be any set in N(w). There is some set B in N(w) such that not $A \subseteq B$, i.e. there are members in A not in B. This means that $A \cap B \subset A$, and by lemma 5.1 and by the hypothesis that the frame characterizes a normal system, $A \cap B \in N(w)$. Furthermore, since $\varnothing$ is not in N(w), $A \cap B \neq \varnothing$, so clearly $\varnothing \subset A \cap B$, so we have $\varnothing \subset A \cap B \subset A$ as desired.

QED

Definition 5.8: A *filter* is a set D of sets such that D is closed under intersections and furthermore contains B whenever it contains some A such that $A \subseteq B$.

Theorem 5.9: if F is a class of Scott-Montague frames characterizing some normal system, then for every member *F* in F, for every w in *F*, N(w) is a filter.

Proof: this follows immediately from lemmas 5.1, 5.2 and 5.3 together.

These last two results, I think, are interesting as mathematical facts for the curious, but whether they have any more substantial implications I shall not go into. In this chapter, we have studied purely formal affairs, regarding only the mathematical properties of logics and semantics. In the next chapter, we move to say something about the philosophy of these structures.

6 The philosophical implications of the maximal depth theorems

What makes completeness results interesting? What information can we gain by a completeness result? Formally, completeness is just a convergence between two types of validity: deductive validity (for systems) and semantic validity (for frames). A completeness result simply tells us that for some system and some class of semantic structures, the two notions of validity coincide. What does that really tell us? One immediate response might be that if we can show that a certain logic is complete, then it should be reasonable to expect that result to speak in *favor* of the logic in some sense. Semantics as such deals with *meaning*, with *content*, so since a completeness result shows that there is a semantic structure that exactly fits some logic, completeness proving should ideally help us separate *meaningful* logics from empty sets of formulae closed under purely syntactical rules. But Boolean semantics shouldn't be fit to do that, since they don't seem to give any account at all of what modality is and how it works – it does not incorporate the idea of possible worlds. The maximal depth theorem *confirms* this: completeness in Boolean semantics is trivial for all even moderately interesting modal logics, so a Boolean completeness result as such is devoid of informational content. Although of course it might be informative to learn that some specific Boolean frame characterizes a system, merely establishing the property of "completeness" for a system in Boolean semantics is a trivial result. When we move up to Scott-Montague semantics, possible worlds come into play and so we start to move closer to a proper account of the modal notions. So fewer systems can be expected to pass as complete, and indeed we know that to be the case.

At this point we may want to argue that there are two distinct roles for completeness results to play: one is to *explain* systems, to give them interpretations that explicate the implicit content in the system, the other is to *evaluate* systems, completeness results may be used to give a sense of "correctness" to a system. If a system is granted a complete and strong semantic interpretation, then this may count as a virtue of that system. It is clear then that Boolean semantics may play the former role but not the latter. We might perhaps still find philosophically interesting Boolean interpretations of modal logics, but Boolean semantics couldn't possibly provide a filter separating better logics from worse through completeness.

The positive thing about the results in the guide to intensional semantics on the present view is that they show a *correlation* between these two roles. Scott-Montague semantics are more explanatory in nature than Boolean semantics are, and Kripke semantics are better off than both. Corresponding to this, Boolean semantics allow all classical logics to pass the "completeness-test", Scott-Montague semantics allow less than all classical systems and Kripke semantics allow only quasi-normal ones. So the better a semantics plays the explanatory role, the better it plays also the role as a test for deciding the viability of a system, since discrimination increases with explanatory content.

Hansson has pointed out to me that an implicit premise in the foregoing argument is that we construct logics in something like the following manner: we start with some intuitions as to what follows from what, at the level of language. We then formalize the language and provide axioms and rules matching those intuitions, and so obtain a system of valid sentences. Then we go on to find a matching semantic interpretation of the system, that explains and motivates the system. We might however turn this picture around, and put intuitions on semantics first, and so reverse the order of the construction process. On this view, we start with an intuition on, so to speak, what goes on in the world, and then try to construct a language with the means to reason about the imagined structure. Lastly we construct a system by giving axioms and inference rules, and if we cannot provide a complete system, then the problem probably lies in some insufficiency in the language. (This view is reminiscent of the view proposed by what Greg Restall calls the "Amsterdam school", according to which formal logics are essentially tools for reasoning about semantic structures). If we look at the situation with predicate logic, this view seems to accord very well with the facts, since the extension of the language by second order quantification is motivated largely by Gödels incompleteness theorem for first order arithmetic.

I agree with Hansson that this is indeed a possible alternative (perhaps even a better one), but I restrict my discussion within the framework of the former model. I should however say something, in the light of the above remark, to motivate that the model I imagine really does resemble actual procedure in philosophical logic at least in some cases. To give a historical example, one of the main motivations for modal logic was the possibility of having a strict implication, usually defined in the following manner:

Definition 6.1 ($\rightarrow$): $\Phi \rightarrow \Psi =_{df} \Box(\Phi \supset \Psi)$

Lewis constructed his earliest systems largely in order to use strict implication to strengthen the notion of entailment as to avoid the so-called “paradoxes of material implication”, counter-intuitive PC theorems such as (p ⊃ q) ∨ (q ⊃ p) or p ⊃ (q ⊃ p), formulas that are not valid if we substitute → for ⊃. The first considerations that motivated systems like S1 and S2 where thus found at the language level rather than in semantics, and probably this order is as common in actual practice as its converse, or at least common.

7 David Lewis’ general completeness result

In 1974 David Lewis proved an important theorem that bears some relevance to the present discussion. This result states a very simple fact, but since the proof is quite large and complicated, I will merely state the theorem here:

Definition 7.1: a logic is iterative iff every axiom-set for that logic contains axioms containing at least one instance of an intensional operator being within the scope of another intensional operator.

Theorem 7.2: every non-iterative logic is Scott-Montague complete.

Modal operators are of course intensional, PC operators aren’t. So if a modal logic can be axiomatized without ever putting a modal operator in the scope of another, that logic is characterized by some class of neighborhood frames. How shall we interpret this result? Well, we know that Scott-Montague semantics are weaker in depth than Boolean semantics (due to Hansson and Gärdenfors), but we don’t know how much weaker. Lewis result sets a lower limit: the depth of Scott-Montague semantics must include at least all non-iterative modal logics. An interesting relationship holds between this theorem and the maximal depth theorem in the guide to intensional semantics. They are both the same sorts of theorems, they are completeness results with a very high degree of generality, but they come from different angles: one states a completeness fact about a class of semantics, the other states a fact about a class of systems. Furthermore, the likeness between the results goes deeper, as I shall try to show that from the point of view of philosophy they point in the same direction

To show this, we turn our attention to systems rather than semantics. Consider for example one particularly interesting class of systems, the class of multi-modal logics. Multi-modal logics are modal logics that employ more than one type of necessity. Hansson and Gärdenfors explicitly point out the expressive richness these logics provide, and in fact the system used to provide the incompleteness result giving theorem 4.2 is multiply modal. Now, according to Lewis' theorem, a logic can be Scott-Montague incomplete only if every axiomatic basis for that logic contains iterative axioms. As soon as iterative axioms come into play, the system falls outside the scope of Lewis' result, and completeness becomes an exclusive thing. In multiply modal systems, this applies in particular to axioms where a modal operator is in the scope of one of another type. The possibility of having such axioms is, I feel, what makes multiply modal logics interesting. Without them the relationship between the different operators becomes fairly weak. We can have some amount of interaction between different operators without requiring iterative axioms of course, for instance we can have $\Box_1$ be stronger than $\Box_2$ by having the formula $\Box_1 p \supset \Box_2 p$ as an axiom. However, this appears *prima facie* to me as an interaction of a very low degree compared to what we can achieve in iterative systems. Compare it for example with the very strong interaction between operators in Tense Logic, for which the consequence of the axioms in Kripke semantics is that the corresponding accessibility relations are conversely related to each other, so that wR_1w' iff $w'R_2w$ for any w, w'. We shall study Tense Logic closer in chapter 9, where we shall see that the system is in fact iterative.

This is one case where iterativeness corresponds to greater strength of a system, but although this example is particularly illustrative, I believe that the correlation is much more general. To give an example of how iterativeness generally has this property, it is interesting to view the correspondence iterativeness for normal logics has to the ordering of the accessibility relation on Kripke frames. It can be seen that if a logic is non-iterative and Kripke-complete, then the accessibility relations on the characterizing frames are restricted by constraining conditions of no more than a certain complexity. This is because for any non-iterative formula, its value at some world depends only on values of atomic formulae at that world and all worlds related to it. The value assignments at additional worlds related to those worlds, i.e. worlds that are so to speak two steps away, strictly do not matter. I shall not prove formally in this section that it is so, but this fact is the key to the iterativeness theorems I prove later on. So in non-iterative logics, we cannot "force" the accessibility relations to take on properties like transitivity, for instance, where worlds that are "two steps away" are required to state the condition (transitivity = if wRw' and w'Rw'', then wRw''). This seems to reveal a great limitation in how strong non-iterative logics can be. (This reasoning gives an informal iterativeness proof for systems with only transitive Kripke frames, such as for example the well-known logic S5).

Lewis' general completeness theorem holds only for non-iterative logics, there are Scott-Montague incomplete logics amongst the iterative systems. This means that, if we take the step up to iterative systems, some of those will be denied an interpretation in the stronger semantics (neighborhood semantics and relational semantics – where by the weaker semantics I of course refer to Boolean frames). So as the maximal depth theorem of Hansson and Gärdenfors implies that semantics with more explanatory force also discriminate systems harder, Lewis' result shows that particularly strong systems are discriminated harder than weaker ones. Both theorems then show a correspondence between having a greater meaning-content and having greater effectiveness in semantics sorting the better systems from the less viable ones. For anyone who wants to put mathematical logic in a philosophical context, such a correspondence must be regarded as a positive thing.

The following sections are devoted mostly to providing the iterativeness result for Tense Logic promised above, in order to strengthen the argumentation in this chapter. However, the project of constructing iterativeness proofs is of a more general interest. In metalogic, certain properties that systems may have are of special importance. These are properties such as completeness, decidability and so forth. Most metalogical theorems aim to show that some system or class of systems has some of those properties. In the light of Lewis' theorem, one might argue that iterativeness too is an important metalogical property, due to its close correspondence to neighborhood completeness. There is a line dividing logics into iterative ones and non-iterative ones that are all Scott-Montague complete, and it should be interesting in several cases to know more about where the line goes.

Proving that a system is non-iterative is quite straightforward; one merely needs to provide an axiomatic basis that doesn't contain any iterated formulae. Showing that a system is iterative is trickier, since one needs to show that every axiomatic basis contains iterated axioms. In section 8, I prove iterativeness for a logic constructed mostly for this purpose, thus drawing the outlines of a general basis for iterativeness proving. In section 8, I apply parts of the apparatus developed in that proof in order to prove iterativeness for Tense Logic.

8 An iterativeness theorem

In this section, I shall attempt to show iterativeness for a simple multi-modal logic (for simplicity I choose to name it "MM"), obtained by the addition of the following formula:

MM: $\Box_1 p \supset \Box_2\Box_1 p$

to K (where by K I mean the bimodal equivalent to K, where we have two operators satisfying the usual axioms and rules). MM is normal and closed under necessitation. The range[8] of MM is the class of frames satisfying the following condition:

C: If wR_2w', and $w'R_1w''$, then also wR_1w''.

It is provable by quite standard methods[9] that this class of frames actually characterizes MM. To prove that MM is iterative, I shall require a few lemmas. I shall also require the following method of decomposing formulae: for some formula Φ, we define a set *S*(Φ) of subsets $\{\Psi_1 \ldots \Psi_n\}$ of Φ accordingly:

[1] let all atomic formulae in Φ occurring at least once outside the scope of any modal operator be members of $\{\Psi_1 \ldots \Psi_n\}$.

[2] also, let all subsets of Φ of the form □θ, where θ is some subset of Φ be members of $\{\Psi_1 \ldots \Psi_n\}$.

Lastly, to ensure that *S* is always uniquely defined, we add:

[3] nothing else is a member of $\{\Psi_1 \ldots \Psi_n\}$.

Lemma 8.1: If Φ is non-iterative, then for every member of *S*(Φ) of the form □θ, θ is strictly PC.

Proof: this follows immediately from the definition of non-iterativeness.

QED

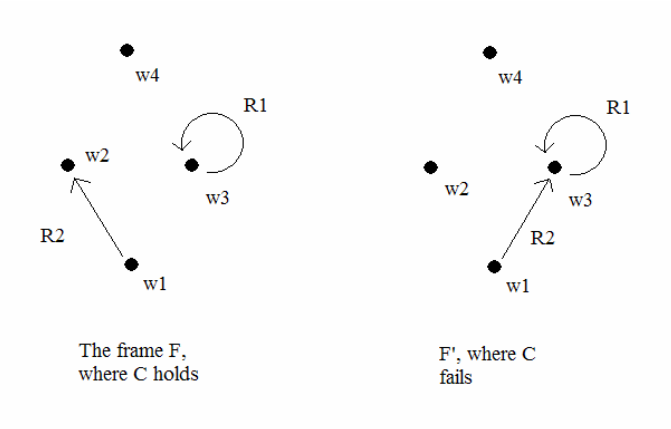
Lemma 8.2: every formula Φ is a truth-functional compound of members from *S*(Φ)

[8] In the class of Kripke-frames, Scott-Montague frames are put out of consideration for the moment.
[9] I used a canonical model type proof to check it.

Proof: this is easily established by a straightforward induction on the length of Φ. First, the lemma holds trivially for all atomic Φ (any formula can of course be regarded as a truth-functional "compound" of itself, and any atomic Φ is itself a member of *S*(Φ) since certainly it occurs at least once outside the scope of a modal operator). We also have that if the lemma holds for two formulae Φ, Ψ, it holds for Φ ∨ Ψ. Clearly *S*(Φ ∨ Ψ) = *S*(Φ) ∪ *S*(Ψ), i.e. the members of *S*(Φ ∨ Ψ) are simply the members of *S*(Φ) together with the members of *S*(Ψ). So we have *S*(Φ) ⊆ *S*(Φ ∨ Ψ) and we also have *S*(Ψ) ⊆ *S*(Φ ∨ Ψ), which means of course that both Φ and Ψ are truth-functional compounds of members from *S*(Φ ∨ Ψ) since they are truth-functional compounds of members from *S*(Φ) and *S*(Ψ) respectively. So since Φ ∨ Ψ is a truth-functional compound of Φ and Ψ, it too is a truth-functional compound of members from *S*(Φ ∨ Ψ). If the lemma holds for Φ, it holds for ~Φ. *S*(~Φ) = *S*(Φ) since no atomic formulae or formulae of the form □θ have been added or subtracted in ~Φ. So since ~Φ is simply a truth-functional operation on Φ, clearly the lemma holds for ~Φ. Finally, we have that if the lemma holds for Φ, it holds for □Φ. This holds trivially, since every formula of the form □Φ is itself a member of *S*(□Φ). So the case is exactly analogous to the case of atomic formulae.

QED

We shall soon see that these very simple lemmas provide a good ground for iterativeness proofs. We now define two frames, *F* and *F'*, sharing the same universe but with different extensions of the accessibility relation. Let the universe of *F* consist of $\{w_1, w_2, w_3, w_4\}$ and let R_1 be $\{<w_3, w_3>\}$, R_2 be $\{<w_1, w_2>\}$. The world w_2 is thus a dead end, i.e. it cannot see any w. Let *F'* be *F* with the modification that R_2 is now $\{<w_1, w_3>\}$ instead (so w_1 does not see_2 w_2 anymore). Note that C holds on *F* but not on *F'*. It holds on *F* since the antecedent of the condition is never true for any w, w', w''. It fails on F' since we have $w_1R_2w_3$, $w_3R_1w_3$ but not $w_1R_1w_3$.



Lemma 8.3: for any formula Φ, if Φ has a falsifying model m' on *F*', it has a falsifying model m* on *F*' such that $V^*(p, w_3) = V^*(p, w_2)$ for all atomic formulae.

Proof: obviously, no assignment of values to atomic formulae at w_2 will disturb any valuations of Φ on the other worlds since it cannot be seen by any other world. So if the falsifying world for Φ in m' is w_1, w_3 or w_4, then the lemma holds, since w_2 is free to take on the values of w_3. If the falsifying world is w_2 itself, then since w_2 and w_4 are completely similar in *F*', clearly we can let w_4 take over the role as the falsifying world in m* by providing it with the proper values, and w_2 is again free to take on the values of our choice. (This latter fact is especially crucial in the cases where we actually need a dead end to falsify Φ, for example if Φ were $\Box(p \wedge \sim p) \supset q$, which holds at all worlds that are not dead ends.)

QED

Lemma 8.4: if Φ is non-iterative and has a falsifying model m' on F', then it also has a falsifying model m on F.

Proof: By lemma 8.3, if Φ has a falsifying model m' on *F'*, it also has a model m* such that $V^*(p, w_3) = V^*(p, w_2)$ for all atomic formulae. For any m', let m be defined such that $V(p, w) = V^*(p, w)$ for all w and all atomic formulae, where V* is the valuation on some m* corresponding to m'. By lemma 8.2, Φ is a truth-functional compound of members from *S*(Φ). So it suffices to show that at the world w at which Φ fails in m*, $V(\Psi_k, w) = V^*(\Psi_k, w)$ for all members of *S*(Φ). Let Ψ_k be an atomic formula. The result follows immediately from the definition of m. Let Ψ_k be $\Box_1\theta$. $V(\theta, w) = V^*(\theta, w)$ for all w since, by lemma 8.1, θ is fully truth-functional, and so since R_1 remains unchanged, $V(\Box_1\theta, w) = V^*(\Box_1\theta, w)$ for any w.

Finally, let Ψ_k be $\Box_2\theta$. θ again retains its value at all worlds. The only place where the value of $\Box_2\theta$ could possibly change is at w_1, since only that world is related differently to other worlds by R_2 than before. But w_1 now $sees_2$ w_2, and since by lemma 8.1 again θ is truth-functional and $V^*(p, w_2) = V^*(p, w_3)$ for all atomic formulas, and consequently $V(p, w_2) = V(p, w_3)$ by definition of m, $V(\theta,w_2) = V(\theta,w_3)$. $V^*(\theta,w_2) = V^*(\theta,w_3)$ also, and so we have $V(\theta,w_2) = V^*(\theta,w_3)$, which by a simple reasoning on the change of R_2 gives $V(\Box_2\theta, w_1) = V^*(\Box_2\theta, w_1)$.

QED

Theorem 8.5: MM is iterative

Proof: first note that since C holds on *F*, *F* is a frame for MM. So every theorem of MM is valid in *F*. We want to ask: can any set of non-iterative theorems in MM provide an axiom-set for MM? By lemma 8.4, any non-iterative formula that has a falsifying model on *F'* also has one on *F*. This is equivalent to saying that every non-iterative formula which is valid on *F* is still valid on *F'*. So every non-iterative MM-theorem is valid on *F'*. But C fails on *F'*, and so since C determines the range of MM, no set of non-iterative theorems in MM is sufficient to axiomatize MM.

QED

The lemmas on the decomposition of Φ can be used to prove iterativeness for a well-known and oft-studied logic, the logic of tensed sentences.

9 Tense Logic

Tense Logic serves as a very nice example of how strong Kripke semantics can be as an analytic tool. The points in the frames are now no longer interpreted quite as "possible worlds" as such, but rather as *time-points* (illustrating that the possible-worlds interpretation is just one of many ways to use frames with points). We have two distinct necessity-operators evaluated just as before, where $\Box_1 p$ is best read "it always has been the case that p", and $\Box_2 p$ should be read as "it always will be the case that p". The basic axioms for tense logic are the following:

TL1 $\sim p \supset \Box_1 \sim \Box_2 p$

TL2 $\sim p \supset \Box_2 \sim \Box_1 p$

I call the normal system resulting from adding these axioms "TL". This system is Kripke-complete although I only refer to the proof[10]. It is characterized by the class of frames where the relations R_1 and R_2 corresponding to each operator are conversely related to each other, i.e. wR_1w' iff $w'R_2w$. Note how nicely this class of frames matches the basic intuitive idea of a time-line – the time-series is made up of individual time-points at which certain facts are the case and others are not. These time-points are related to each other such that some time-points lie in the future relative to some time-point, and others are in the past. And if from the perspective of one time-point some other time-point is a future one, then from the latter's perspective the first one is in the past. Thus the foundation of a genuinely metaphysical model of the time-line is laid, and we can continue altering it to get closer to the real thing. For example, it is a straightforward thing to impose transitivity on R_1 and R_2, which seems like a reasonable thing to do. This puts the philosophical discussion on the nature of time right into formal logic. Standpoints on certain discussions in the field can be formalized as semantic frames, and so we can see directly what happens in the logic depending on how we alter our view on time. The most prominent problem within this research field, addressed among others by Arthur Prior[11], is whether time will come to an end or not. Is the future as such endless, or will time itself at some point cease? That question goes far outside the scope of this essay of course. But what is relevant is that we have a metaphysical question that, via relational semantics, translates directly into the question of what we shall take as the correct version of Tense Logic. Tense Logic changes depending on whether or not we allow last time-points in the frames.

The basic system of Tense Logic is iterative, as I will show.

Theorem 9.1: the Kripke-range of TL is the class of frames satisfying C:

C: $\forall w \forall w' (wR_1w' \equiv w'R_2w)$

[10] The proof is found at page 218 in Hughes & Creswell
[11] See for example "Time and Tense"

Proof: Informally put, C means that R_1 and R_2 are each other's converses. I prove that this is really the range of TL before proceeding. First to show that any such frame validates the axioms. Let ~p be true at some w. To show that $\Box_1{\sim}\Box_2 p$ is true, let w' be any world such that wR_1w'. $w'R_2w$ by C, so $\Box_2 p$ fails at w' and thus $\sim\Box_2 p$ holds. So $\Box_1{\sim}\Box_2 p$ holds at w. The same reasoning goes for **TL2**. To show that only such frames validate the axioms, take any frame where C fails. Then either **TL1** or **TL2** fails. I show how this happens in the case of **TL1**. There are two ways for C to fail for two w, w' – we may have the one see_1 the other but the latter not see_2 the former, or vice versa. So let wR_1w' but not $w'R_2w$. Then we can construct a falsifying model for **TL1** by letting p be false at w but true everywhere else. ~p is true at w, but $\Box_1{\sim}\Box_2 p$ fails since $\Box_2 p$ is true at w'.

QED

Theorem 9.2: Tense Logic is iterative

Proof: consider the following two frames *F*, *F'* sharing the universe $\{w_1, w_2, w_3\}$. In *F*, R_1 is $\{<w_1, w_2>\}$ and R_2 is $\{<w_2, w_1>\}$. To give *F'* we change R_1 so that w_1 $sees_1$ w_3 instead of w_2. C holds in *F* but fails on *F'*, so showing that all non-iterative formulas valid on *F* are valid on *F'* as well gives us the result in the same manner as before.

Let then Φ be a non-iterated formula true in every model m on *F*. Then V(Φ, w) = 1 for all w and V. Let V' denote some valuation on *F'*. For all V', V'(p, w) = V(p, w) for all w, for some V. This means that $V'(\Phi, w_3) = 1$ for all V', since that world is still a dead end and thus is not affected by the switch of R_1. w_2 still $sees_2$ w_1, and the fact that the accessibility relations from w_1 have changed clearly does not affect the value of Φ at w_2 since Φ is non-iterative, and thus the value of all members of the form $\Box\theta$ from *S*(Φ) remain unchanged since by lemma 8.1 θ is strictly PC. So $V'(\Phi, w_2) = 1$ for all V' also.

As for w_1, Φ is true at w_1 in *F* for every V. For any valuation to the Ψ_k in *S*(Φ) occurring at w_1 in all valuations V on *F*, Φ comes out true at w_1. Furthermore, by lemma 8.2, Φ depends for its value on w_1 only on the values of the Ψ_k in *S*(Φ) at w_1. So if $V'(\Psi_k, w_1) = V(\Psi_k, w_1)$ for some V, for all Ψ_k and all V', clearly $V'(\Phi, w_1) = 1$ for all valuations V' on *F'*. I show that this is the case.

Let V' be some valuation on *F'*. Let V be a valuation on *F* such that $V(p, w_2) = V'(p, w_3)$ for all atomic formulae, and also $V(p, w_1) = V'(p, w_1)$. $V'(\Psi_k, w_1) = V(\Psi_k, w_1)$ immediately if the Ψ_k is either atomic or is $\Box_2\theta$ for some θ. If Ψ_k is $\Box_1\theta$, then by lemma 8.1, θ is strictly PC and so V'(θ, w_3) = V(θ, w_2), so since R_1 has simply switched over from w_2 to w_3 in *F'*, V'($\Box_1\theta$, w_1) = V($\Box_1\theta$, w_1).

We have thus shown that if Φ is non-iterative and is true at all w in *F* for all valuations V, it is true at all w in *F'* for all V' also.

QED

10 Summary

In this essay I have attempted to continue the discussion started in the "guide to intensional semantics". After an introduction to the results presented there, I proved that the depth of Scott-Montague semantics in the class of normal systems is greater compared to that of Kripke semantics only if the added systems lack the finite model property in neighborhood semantics. I also proved that in Scott-Montague frames for normal systems all neighborhoods are filter type structures, and that any class of Scott-Montague frames that characterizes a Kripke incomplete normal system has some member satisfying what I called the "neighborhood denseness" condition, these last two results being "bonus" results that followed easily from the main arguments. After that, I discussed the philosophical implications of Hanssons and Gärdenfors' results, the maximal depth theorems of Boolean semantics as well as the neighborhood incompleteness result for a classical system. In the following chapter I discussed Lewis' general completeness theorem, and argued that it has philosophical repercussions similar to those of Hanssons and Gärdenfors' results. The implications of both essays were said to consist in that there is a correlation between the amount of meaning content in the formal structures employed in logic and the discriminating role of completeness proving, but that Hanssons and Gärdenfors' showed this from the point of view of semantics, whereas the completeness theorem for non-iterative logics showed it from the angle of systems. Lastly I proceeded to prove iterativeness for the system "MM" as well as ordinary Tense Logic.

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