

Dynamic and Static Hedging of Barrier options

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Abstract

In this paper the performance of a static hedging strategy of European barrier options are evaluated, first introduced by Carr, Ellis and Gupta in 1998. Two dynamic hedging strategies are used as benchmarks; the delta hedging and delta-gamma hedging strategy, respectively. To increase realism to the system only discrete rebalancing of the replicating portfolios are possible. In addition, transaction costs are assumed and included. The analysis is limited to the down-and-out and up-and-out call European options. Hedging of European put barrier options could easily be constructed from the put-call-parity and European in-barrier options from the fact that a European option is equal to the sum of the in- and out-barrier option with the same barrier. The static hedge outperforms the two dynamic hedging strategies in particular when transaction costs are included. The static hedge is more successful in reducing the risk and delivers a higher average return.

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1 Introduction

Barrier options constitute a substantial part of the Global Over The Counter equity derivatives market, which has an estimated value of \$ 8364 billion¹. The popularity of barrier options is much due to the fact that they are cheaper than the corresponding standard options. The sheer size of the market is an incentive for considering the possibilities of hedging the Barrier options, i.e. the issuers of the contracts needs to protect their position against excessive risk. Despite the size of the market for Barrier options, they will strictly be treated as Over The Counter contracts (OTC) in this thesis. That is we assume that the contracts are tailor made for the customer and in general not traded on a market.

A Barrier option is a regular option with one (consider this case in the thesis) or several additional restraints. A barrier options either seizes or starts to exist when an upper or lower level is reached by the underlying asset (which henceforth will be labelled as a Stock, to make the notation less cumbersome) during the lifetime of the option. The Barrier option will always be cheaper than the corresponding regular option, since it is equal to a regular option conditioned on that some other event occurs. In this thesis, the authors will assume the position of an issuer of a Barrier option, i.e. the position of a bank. The bank is assumed to have the objective to reduce its risk as much as possible, that is, the bank is assumed to make money primarily on the price the bank charge for the transaction, and not on the return of the position it takes on the market when dealing with its clients. Thus the bank wants to hedge the short position in the Barrier contract. To this end, three different hedging strategies will be considered. Two out of these will be Dynamic hedging strategies, that is, strategies where the hedging portfolio is frequently rebalanced during the lifetime of the Barrier contract. The third hedging strategy is a Static hedge that was first introduced by Carr, Ellis and Gupta [4] in 1998. The static hedge is not re-balanced during the lifetime of the Barrier contract. As for the Dynamic hedging strategies, one Delta-hedging strategy and one Delta-Gamma hedging strategy will be considered.

 $^{^{1}2006, [8]}$

When hedging the Barrier option, we assume that the bank has a liquid market which is free of arbitrage, where the Stock, a risk free asset and the Vanilla options (with the addition of Binary Call options) are traded, at its disposal. The bank may thus use any of these assets to hedge the Barrier option.

In the thesis, the authors especially aim to answer the question; Which one out of the Static-hedging strategy, the Delta-hedging strategy and the Delta-Gamma hedging strategy is most efficient in reducing the risk of a Barrier option?

To answer this question we have chosen to simulate the market using Monte Carlo simulation. A large number of simulations are carried out. For each realization, the performance of each strategy is calculated, and statistical analysis of the total number of simulations is used to evaluate the performance of each hedging strategy.

The analysis will be restricted to Down-and-out and Up-and-out Call options, which are At-the-money at the time when the Barrier contract is issued. All contracts considered in the thesis are of European style, that is, they can only be exercised at the day of expiration. No explicit currency is assumed, and thus all of the results below are quoted without a specific unit.

The theoretical framework used in the thesis is presented in section 2. Section 3 describes how the data is produced using Monte Carlo simulation. The method of how the income for each strategy and for each simulation is calculated, is presented in section 4, together with the statistical tools that are used to evaluate the performance of each strategy. The results are given in section 5 and finally, the conclusions are given in section 6, together with suggestions of further extensions of the subject.

2 Theoretical framework

In the introduction, a clear distinction was made between those contracts that are traded on the market, and those contracts that are not. However, in order for the bank to be able to hedge the Barrier option dynamically, it needs to calculate the movements of the price of the Barrier option throughout its lifetime. The price of the Barrier option will be calculated using risk-neutral valuation, which will also be used to calculate the prices of the options that are traded in the market. Thus the price of all options will be calculated using risk-neutral valuation. However we emphasize the fact that although we calculate the prices using the same method, it is only the Vanilla options that can be traded on the market at the prices given by risk-neutral valuation.

Throughout the text one or several of the parameters that the price of a given asset depends on will be suppressed in the notation. However, the value of the parameters should be clear from the context.

In subsection 2.1 the Vanilla options are introduced. Subsection 2.2 states how the assets that are traded in the market are priced. The Barrier contracts are introduced in subsection 2.3, which also includes the pricing formulas of the Barrier options. The Greeks are introduce in subsection 2.4. Finally, subsection 2.5 and subsection 2.6 gives an introduction of Dynamic and Static hedging respectively.

2.1 Vanilla options

The Vanilla options are simple contingent claims, That is, the contract function Φ only depends on the price of the Stock at the day of expiration that is, $\Phi = \Phi(S(T))$. Below the Binary Call option is included amongst the Vanilla options to simplify matters, which may be considered as being out of convention.

2.1.1 Call option

The holder of a Call option written on the Stock S(t), with the strike price K and the expiration date T, has an option (but not an obligation) to buy the Stock on the day of expiration to the fixed price K. The contract function $\Phi(S(T))$ of a Call option is equal to max[S(T) - K, 0]. The terminal payoff of a Call option with the strike price K = 100 can be found in figure 1 a.

2.1.2 Put option

The holder of a Put option written on the Stock S(t), with the strike price K and the expiration date T, has an option (but not an obligation) to sell the Stock on



Figure 1: The terminal payoff of an European Call option (a) and a European Put option (b), both with the strike price K = 100

the day of expiration to the fixed price K. The contract function $\Phi(S(T))$ of a Put option is equal to max[K - S(T), 0]. In figure 1 b, the terminal payoff of a Put option (b) with the strike price K = 100, is on display.

2.1.3 The Binary Call option

The holder of a European Binary Call option receives 1 unit of currency if the price of the Stock S(T) is above the strike price K at the time of expiration, and receives 0 otherwise.

2.2 The prices of the contracts that are traded in the market

The prices for the assets that are traded in the marked are given in this subsection.

2.2.1 Risk free asset

The price process of the risk free asset B(t) is given by

$$dB(t) = rB(t) dt \tag{1}$$

which is shorthand for:

$$B(t) = \int_{0}^{t} rB(s) \, ds \tag{2}$$

solving this equation gives:

$$B(t) = B(0) \exp(rt) \tag{3}$$

2.2.2 Stock

The price process of the Stock S(t) is assumed to follow the *Geometric Brownian* Motion:

$$dS(t) = rS(t) dt + \sigma S(t) dW(t), \qquad (4)$$

where the *drift* r and the *volatility* σ are constants, and where W(t) is a Wiener process, which has the following properties:

- W(0) = 0.
- The process W(t) has independent increments, i.e. if $r < s \leq t < u$ then W(u) W(t) and W(s) W(r) are independent stochastic variables.
- For s < t the stochastic variable W(t) W(s) has the Gaussian distribution $N\left[0, \sqrt{t-s}\right]$.
- W has continuous trajectories.

Without loss of generality, the drift of the Stock has been chosen to be equal to the risk free interest rate r. Solving the stochastic differential eq. 4 using Itô's Lemma² gives:

$$S(t) = S(0) \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right\}$$
(5)

 $^{^{2}}$ see e.g. pp. 65-66 in [7]

2.2.3 Pricing the derivatives

The prices of the options are calculated using risk-neutral valuation. That is, the price of the Derivative expiring at T, $\Pi(t, S(t))$, with the contract function $\Phi(S(T))$ is equal to:

$$\Pi(t,s) = \exp\left(-r\left(T-t\right)\right) E_{t,s}^{Q}\left[\Phi\left(S\left(T\right)\right)\right]$$
(6)

where the Q implies that the expectation value has to be taken over the price process of the Stock that is given in eq. 4. Eq. 6 can be derived using Itô's Lemma, Black-Scholes differential equation and Feynman-Kac stochastic representation formula.³

2.2.4 Call option

The price of the European Call option C(t, s) is calculated using *Black-Scholes for*mula, which can be derived from eq. 6^4 :

$$C(t,s) = sN[d_1(t,s)] - \exp(-r(T-t))KN[d_2(t,s)],$$
(7)

where N is the cumulative distribution function for the Gaussian distribution with zero mean and unit variance, and

$$d_1(t,s) = \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln \frac{s}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right\}$$
(8)

$$d_{2}(t,s) = d_{1}(t,s) - \sigma\sqrt{T-t}.$$
(9)

In figure 2, the price of a Call option is plotted as a function of the Stock price and the time to maturity, with the strike price K = 100 and constant rate and volatility. The smooth property of the surface makes the option straightforward to hedge, if the assumption of a constant rate and constant volatility is accurate.

 $^{^{3}}$ A complete derivation of eq. 6 can be found in e.g. ch. 3, 4 and 6 in [5]

⁴For a complete derivation of Black-Scholes formula, see e.g. ch. 6 in [5]



Figure 2: The price of a Call option as a function of the Stock price and the time to maturity, with the strike price K = 100 and constant interest rate and volatility.

2.2.5 Put option

The price of the European Put option P(t,s) is calculated using *Black-Scholes for*mula:

$$P(t,s) = -sN[-d_1(t,s)] + \exp(-r(T-t))KN[-d_2(t,s)], \quad (10)$$

where $d_1(t,s)$ and $d_2(t,s)$ is the same as above.

2.2.6 Binary Call option

The price of the European Binary Call option BC(t, s) is calculated as⁵:

$$BC(t,s) = \exp(-r(T-t))N[d_2(t,s)]$$
(11)

where $d_{2}(t,s)$ is the same as above.

2.3 Barrier options

Barrier options are, on the contrary to Vanilla options, path dependent. That is the terminal payoff does not only depend on the price of the Stock at T, but it also depends on the price of the Stock for all times between the time of issue, to the time of expiration. There are two classes of Barrier contracts; out-contracts and in-contracts. An out-contract becomes worthless if the price of the Stock hits the barrier before the time of expiration. An in-contract on the other hand will expire worthless if the barrier has not been hit before the time of expiration.

The barrier can either be a down-barrier or an up-barrier. A down-barrier is hit if the price of the Stock becomes less or equal to the value of the barrier, and an up-barrier is hit if the price of the Stock becomes greater than or equal to the value of the barrier. A Barrier option can have more than one barrier, however in this thesis we will only consider single Barrier options.

⁵The formula is taken from p. 88 [3]

Barrier option	Category	Description					
Down-and-out Call	Regular	The option dies out-of-the-money					
Down-and-in Call	Regular	The option is born out-of-the-money					
Up-and-out Call	Reverse	The option dies in-the-money					
Up-and-in Call	Reverse	The option is born in-the-money					
Down-and-out Put	Reverse	The option dies in-the-money					
Down-and-in Put	Reverse	The option is born in-the-money					
Up-and-out Put	Regular	The option dies out-of-the-money					
Up-and-in Put	Regular	The option is born out-of-the-money					

Table 1: The eight different flavours of Single-Barrier options.

If one considers both Call and Put Barrier options, then there are eight different flavours of Barrier options all together. These are listed in table 1. The Barrier options that are categorized as Regular dies/are born (in most cases) out-of-themoney, while the options that are categorized as Reverse dies/are born in-the-money. In general, the Reverse options are much more difficult to hedge.

In this thesis we will focus on Call options. It can easily be realized that a position that holds one Down-and-out Call and one Down-and-in Call (or one Up-and-out Call and one Up-and-in Call), with the same barrier H, the same strike price K and the same time to maturity T - t, is identical to the position holding one standard Call option with the same strike price K and time to maturity T - t. That is:

$$OC(K,H) + IC(K,H) = C(K)$$
(12)

where the time to maturity dependence has been suppressed. Since an In-contract can be replicated by the corresponding Out-contract and a Vanilla Call option, we have chosen to limit our investigation to Out-Call options.

2.3.1 Down-and-out Call option

For any Down-and-out contract, The spot price $S(t_0)$ when the contract is issued must be greater than the barrier H, otherwise the contract is dead from the begin-



Figure 3: The terminal payoff of two Down-and-out Call option with the strike price K = 100. In the left panel the barrier H = 90, and in the right panel the barrier H = 120.

ning.

If the barrier has not been hit before expiration, the terminal payoff of a Downand-out Call option will have different features depending on if the barrier H is below or above the strike price K. This is illustrated in figure 3, where the strike price K = 100 in both panels, with the barrier H = 90 in the left panel, and with the barrier H = 120 in the right panel.

In this thesis, the investigation is limited to the case where the barrier is set to be below the strike price.

2.3.2 Up-and-out Call option

The spot price at t_0 for an Up-and-out contract must be below the barrier H, otherwise the contract is dead from the beginning. If the barrier H is below the strike price K, then the contract can never have a non-zero terminal payoff, since the contract will die before the Stock price moves above K. Therefore we are only interested in the case where H > K.

The terminal payoff for the Up-and-out Call option, where the barrier has not



Figure 4: The terminal payoff of an Up-and-out Call option with the strike price K = 100, and the barrier H = 140.

been hit before expiration, is on display in figure 4.

2.3.3 The Prices of the Down-and-out and the Up-and-out Call options

The expressions for calculating the prices of the Barrier options are taken from $Haug^6$. Since the expressions are of some length, they have been placed in appendix A.1.

2.4 The Greeks

The Greeks is the common name of the set of the derivatives of the price of the instrument, with respect to the Stock and the model parameters. The knowledge of the Greeks provides information about how sensitive the price is to changes in the Stock and the model parameters.

⁶pp. 70-71 in [3]

2.4.1 Delta

Provides information of how sensitive the price of the derivative is to a change in the Stock price

$$\Delta_{\Pi} = \frac{\partial \Pi}{\partial S} \tag{13}$$

2.4.2 Gamma

Provides information of how sensitive the delta is to a change in the Stock price (the second derivative of the price of the derivative is equal to the first derivative of the delta)

$$\Gamma_{\Pi} = \frac{\partial^2 \Pi}{\partial S^2} \tag{14}$$

2.4.3 Rho, Theta and Vega

Rho provides information of how sensitive the price of the derivative is to a change in the interest rate.

$$\rho_{\Pi} = \frac{\partial \Pi}{\partial r} \tag{15}$$

Theta provides information of how sensitive the price of the derivative is to changes in t.

$$\Theta_{\Pi} = \frac{\partial \Pi}{\partial t} \tag{16}$$

Vega provides information of how sensitive the price of the derivative is to a change in the volatility.

$$v_{\Pi} = \frac{\partial \Pi}{\partial \sigma} \tag{17}$$

In the rest of the thesis we will mainly focus our attention to the Delta and the Gamma.

2.5 Dynamic Hedging

The general idea of Dynamic hedging is to construct a portfolio consisting of the instrument that is to be hedged, and the hedge, that is locally neutral with respect to one or several of the *Greeks*. E.g., if a portfolio is delta-neutral, then the value of the portfolio remain constant when the price of the Stock changes. However, the value of the delta changes over time and as the price of the Stock changes. Thus the portfolio is only *locally* delta neutral. To ensure that the portfolio remains delta neutral over time, it needs to be re-balanced frequently over the lifetime of the instrument, i.e. hedged *dynamically*. The more often the portfolio is re-balanced, the less the value of the portfolio is continuously re-balanced, the value of the portfolio is continuously re-balanced, the value of the portfolio is continuously re-balanced. However, in real life there are costs associated with re-balancing the portfolio. There is thus a trade-off between risk-reduction (re-balancing often) and return (re-balance less often).

2.5.1 Delta hedging

The objective is to reduce the delta to *zero*, ideally continuously but in practice discreetly, by adding some instrument to the derivative. In this thesis we will use the Stock to construct a portfolio that is delta-neutral on a daily basis. The Delta of the Stock S is trivially equal to 1:

$$\Delta_S = \frac{\partial S}{\partial S} = 1 \tag{18}$$

Conciser the portfolio V consisting of one option and the weight w_s of the Stock S, where w_s is the number of the units of the Stock (w_s is permitted to be smaller than one, which is not a drastic assumption since in real life a portfolio generally consists of hundreds or thousands of assets), i.e:

$$V = \Pi + w_s S \tag{19}$$

Then the delta of the portfolio V is equal to:

$$\Delta_V = \frac{\partial V}{\partial S} = \frac{\partial \Pi}{\partial S} + w_s \frac{\partial S}{\partial S} = \Delta_\Pi + w_s \tag{20}$$

Thus it can readily be seen that to obtain a delta that is zero for the portfolio V, we take a short position of w_s Stocks, where w_s is equal to the delta of the derivative, i.e.

$$\Delta_V = 0 \Rightarrow \tag{21}$$

$$w_s = -\Delta_{\Pi} \tag{22}$$

By creating a portfolio consisting of one long option and Δ_{Π} short Stocks, we have locally obtained a portfolio that for which the value of the portfolio does not change when the price of the Stock changes.

However, as can be seen in figure 5, where the delta of a Call option is on display for different Stock prices and different times to maturity, the delta changes over time and when the price of the Stock changes. Thus the portfolio that was constructed above is only locally delta-neutral. The portfolio therefore needs to be re-balanced as delta changes. When the price of the Stock is close to the strike price, the delta is increasingly more sensitive to changes in the Stock price as the time to maturity approaches (notice how the slope of the curve increases around K = 100).

2.5.2 Delta Gamma hedging

In the previous section the fact that the delta depends on the price of the Stock was mentioned. In particular, if the Delta is very sensitive to changes in the price of the Stock, then the Delta-neutral Strategy will not be very effective. An idea is then to create a portfolio that is both Delta and Gamma neutral. That is, if the Gamma of the portfolio is zero, then the Delta will not change when the price of the underlying change, and this portfolio will be less sensitive to price changes. In this thesis we will use the following strategy to create a Delta and Gamma neutral portfolio V:

In addition to the derivative Π that we wish to hedge, we will add w_F units of



Figure 5: The delta of a Call option as a function of the Stock price and the time to maturity, with the strike price K = 100 and constant interest rate and volatility.

some asset that has both a non-zero Delta and Gamma; Π_F , and w_S units of the Stock (note that this w_S is not the same as in the Delta hedge above). Since the Delta of the Stock is 1, the Gamma of the Stock is 0. If w_F is chosen such that the Gamma of the portfolio is equal to zero, i.e.

$$\Gamma_{\Pi} + w_F \Gamma_F = 0 \tag{23}$$

and w_S is chosen such that the Delta of the portfolio is zero, i.e.

$$\Delta_{\Pi} + w_F \Delta_F + w_S \Delta_S = 0 \tag{24}$$

Then the portfolio;

$$V = \Pi + w_F \Pi_F + w_S S \tag{25}$$

will be both Delta and Gamma neutral (notice that since the Gamma of the Stock is zero, the Gamma of V will still be zero when we add the w_s units of the Stock). Since we have a system of two equations and two unknowns (w_f and w_s), the system can be easily solved. Doing so yields:

$$w_F = -\frac{\Gamma_{\Pi}}{\Gamma_F} \tag{26}$$

$$w_S = \frac{\Gamma_\Pi \Delta_F}{\Gamma_F} - \Delta_\Pi \tag{27}$$

In figure 6, the Gamma of a Call option is on display as a function of the Stock price and the time to maturity. As expected when considering the Delta in figure 5, the Gamma, when the price of the Stock is close to the strike price K = 100, is increasingly more sensitive to changes in the price of the Stock as the time to maturity approaches.

2.6 Static Hedging

A Static hedge is a position that is taken when the contract is purchased, and which is maintained throughout the lifetime of the contract (buy and hold). In



Figure 6: The Gamma of a Call option as a function of the Stock price and the time to maturity, with the strike price K = 100 and constant interest rate and volatility.

general, the Static hedge does not provide a satisfying reduction of the risk of the contract. It has previously been mentioned that the Delta and the Gamma changes considerably over time and over different values of the Stock price. Therefore, if a position is taken to hedge the risk of a contract at the time of purchasing, by creating a portfolio that is Delta and Gamma neutral as above, this position will not be Delta and Gamma neutral throughout the life-time of the contract. However, when considering Barrier options, the Delta and the Gamma does not display the smooth behaviour that was the case for the Call option. This is apparent in figure 7 and in figure 8, where the Delta and the Gamma are on display for a Down-and-out Call option with the strike price K = 100 and the barrier H = 90. It is apparent that both the Delta and the Gamma are very sensitive to changes in the price of the Stock when the spot price is in the vicinity of the barrier. This makes the Dynamichedging strategy described above complicated, and it may be of interest to consider Static hedging.

The general idea of the Static-hedging strategy considered in this thesis (Carr, Ellis, Gupta [4]) is to construct a Static hedge that both match the price of the contract along the barrier and the terminal payoff of the contract. If we can do so successfully, then the two possible outcome of the (European) Out-contract are covered, namely either the barrier is hit before expiration, or it delivers the payoff at the time of expiration. The task is then to find a combination of the contracts that are traded in the market that full-fill the desired properties.

2.6.1 Down-and-out Call option, H < K

From section 2.3.1, we recall that if the barrier is hit before expiration, the downand-out Call option is worthless. If the barrier has not been hit, the terminal payoff is equal to the terminal payoff of a Vanilla Call option with the same strike price. Therefore we start by matching the terminal payoff by purchasing a Call option with the strike price K. The Call option has non-zero value along the barrier for t < T. This is illustrated for the case when the barrier H = 95 in figure 9, where the price of a Call option is plotted as a function of the Stock price and the time to maturity. In the inset of figure 9, the Call price is plotted as a function of time, when the Stock



Figure 7: The Delta of a Down-and-out Call option as a function of the Stock price and the time to maturity, with the strike price K = 100, the barrier H = 90 and constant interest rate and volatility.



Figure 8: The Gamma of a Down-and-out Call option as a function of the Stock price and the time to maturity, with the strike price K = 100, the barrier H = 90 and constant interest rate and volatility.



Figure 9: The price of a Call option as a function of the Stock price and time to maturity, with K = 100. In the inset, the Call price is plotted as a function of time when the Stock price $S(t) = 95 \forall t$.

price $S(t) = 95 \forall t$. The task is now to add a contract to the Call option that creates a portfolio that has zero value along the barrier, without affecting the terminal payoff. To do so, The Put-Call symmetry relation⁷ is used. Put-Call symmetry states that if the drift of the price process in eq. 4 is *zero* (which it is not, however the drift is assumed to be *zero* for the time being), then the following relation holds:

$$C(K_C) K_C^{-\frac{1}{2}} = P(K_P) K_P^{-\frac{1}{2}}$$
(28)

where the geometric mean of the Call strike K_C and the Put strike K_P is the forward price F, i.e.

$$(K_C K_P)^{\frac{1}{2}} = F \tag{29}$$

If the forward price is equal to the barrier, i.e. let F = H, then:

$$K_P = \frac{F^2}{K_C} = \frac{H^2}{K_C} \tag{30}$$

Substituting eq. 30 into eq. 28 and rearranging the terms yields:

$$C(K_C) = \frac{K_C}{H} P\left(\frac{H^2}{K_C}\right) \tag{31}$$

Next the relation in eq. 31, with $K_C = K$ is considered in the present case of the Down-and-out Call option where the barrier H is below the strike price K. We construct a replicating portfolio, Π_{CEG} , consisting of a long position in the Call option with the strike K and a short position in $\frac{K}{H}$ units of the Put option with the strike $\frac{H^2}{K}$:

$$\Pi_{CEG} = C\left(K\right) - \frac{K}{H}P\left(\frac{H^2}{K}\right)$$
(32)

Both the Call and the Put option have the same time to maturity as the Downand-out Call option. Since H < K, $\frac{H}{K} < 1$ and $\frac{H^2}{K} < H$. If the barrier has not been hit before expiration, S(T) > H. This means that the Put option will expire

 $^{^7 {\}rm pp.}$ 1166-1169 [4]



Figure 10: The price of a Call option with the strike price K = 100 (top left), the price of $\frac{K}{H}$ units of the Put option with the strike price $K = \frac{H^2}{K}$ (top right) and the price of a portfolio consisting of long the Call option and short $\frac{K}{H}$ units of the Put options (bottom left). All prices are plotted as function of the Stock price and the time to maturity T - t.

worthless (see figure 1 b), and the payoff of the Call option will match the payoff of the Down-and-out Call option. It then remains to make certain that the price of the replicating portfolio is zero along the barrier throughout the lifetime of the contracts.

That this is indeed found to be the case, can readily be observed in figure 10 and in figure 11. Thus to hedge the Down-and-out Call option, the portfolio consisting of short the Down-and-out Call option, long the Call option and short the $\frac{K}{H}$ units of the Put option will be used in the sequel.

If a non-zero drift is introduced, it will cause the price of the Call option to rise, and the price of the Put option to fall, which will introduce an error in the hedging portfolio. The size of the error will be commented in the results below.



Figure 11: The price of a Call option with the strike price K = 100 and the Stock price s = 95 (solid line, top figure), the price of $\frac{K}{H}$ units of the Put option with the strike price $K = \frac{H^2}{K}$ and the Stock price s = 95 (stars, top figure) and the price of a portfolio consisting of long the Call option and short the $\frac{K}{H}$ units of the Put option, with the same Stock price. Plotted as functions of time in units of years, and where the initial time to maturity is equal to 1 year.



Figure 12: The price of an Up-and-out Call option as a function of the Stock price and time to maturity, with the strike price K = 100 and the barrier H = 120.

2.6.2 Up-and-out Call option

The price of an Up-and-out Call option with the strike price K = 100 and the barrier H = 120, is plotted as a function of the Stock price and time to maturity in figure 12.

In section 2.3.2, the terminal payoff below the barrier, if the barrier has not been hit before the time of expiration, of an Up-and-out Call was shown to be equal to the terminal payoff of a Call option with the same strike, or zero otherwise. Thus we again start constructing the hedge with the Call option. Along the barrier, the Call option is trivially non-zero (see figure 2), so the hedge must be complemented. According to Carr, Ellis, Gupta⁸, the following hedge, denoted Π_{CEG} , can be used to hedge the Up-and-out Call option:

$$\Pi_{CEG} = C\left(K\right) + \Pi_{CEG(2nd)} \tag{33}$$

where

$$\Pi_{CEG(2nd)} = -\left(\frac{K}{H}\right)C\left(\frac{H^2}{K}\right) - (H-K)\left(2BC\left(H\right) + \frac{1}{H}C\left(H\right)\right)$$
(34)

The terminal payoff of $\Pi_{CEG(2nd)}$, if the barrier has not been hit before expiration (S(T) < H), is zero. This is trivial for the Binary Call option, BC(H) and the

⁸p. 1172 [4]



Figure 13: The price of a Call option with the strike price K = 100 (top left), the price of the second part of the static hedge $\Pi_{CEG(2nd)}$ in eq. 34 (top right), the price of the hedge Π_{CEG} in eq 33 (bottom left) and the price of the hedge Π_{CEG} along the barrier H = 120 (bottom right). All prices in the three dimensional figures are plotted as function of the Stock price and the time to maturity, and the price of the two dimensional figure is plotted as a function of time.

Call option, C(H). Since H > K, $\left(\frac{H}{K} > 1\right)$ and $\left(\frac{H^2}{K}\right) > H$, and it is thus clear that the terminal payoff of $C\left(\frac{H^2}{K}\right)$ must also be zero when S(T) < H. It then remains to make certain that the hedge Π_{CEG} in eq 33 is zero-valued along the barrier throughout the life-time of the contracts. That this is indeed found to be the case can be observed in figure 13.

We end this section by commenting on the use of the Binary Call. The Binary Call can be well approximated using a vertical spread of Vanilla Call options⁹. Thus the assumption that the Binary Call is traded on the market may be relaxed.

⁹p. 1173 [4]

3 Data

As stated previously, no real market will be used to produce the results. In stead the market data will be simulated using Monte Carlo simulation. Or more precisely, the price process of the Stock is simulated. With the knowledge of the Stock price, the Vanilla, the Binary and the Barrier options can be calculated using the expressions in the previous section. With the knowledge of the complete price process of the Stock, the terminal payoff of the Barrier options can also be calculated. Finally, from the knowledge of the price process of each asset, the Delta and the Gamma for each of the assets for all t may also be calculated. The complete price process of the Stock will be simulated 1000 times.

An account for how the market data for each asset and the corresponding Greeks are calculated, is presented below.

3.1 Simulating the Stock price

From eq. 4, the price process of the Stock is given as:

$$dS = rSdt + \sigma SdW \tag{35}$$

where r and σ are known constant. The price process of the Stock is calculated discretely. The discrete version of eq. 35 is given by:

$$\Delta S(t) = rS(t)\,\Delta t + \sigma S(t)\,\Delta W \tag{36}$$

Given the properties of the Wiener process, eq. 36 can be rewritten as:

$$\Delta S(t) = rS(t) \,\Delta t + \sigma S(t) \,\sqrt{\Delta t} Z, \tag{37}$$

where Z is drawn form a Gaussian distribution with zero mean and unit variance.

Let t_0 denote the time at t = 0. The price of the Stock at t_0 is *chosen* as $S(t_0)$. The price of the Stock at the next time-step t_1 is then calculated as:

$$S(t_1) = S(t_0) + rS(t_0)(t_1 - t_0) + \sigma S(t_0)\sqrt{(t_1 - t_0)}Z_1$$
(38)

In the same way, the price of the Stock at t_i is calculated as:

$$S(t_i) = S(t_{i-1}) + rS(t_{i-1})(t_i - t_{i-1}) + \sigma S(t_{i-1})\sqrt{(t_i - t_{i-1})}Z_i,$$
(39)

where each Z_i is drawn independently. The elapsed time between t_i and t_{i-1} is set to be equal to $\Delta t \forall i$, and eq. 39 is rewritten as:

$$S(t_i) = S(t_{i-1}) + rS(t_{i-1})\Delta t + \sigma S(t_{i-1})\sqrt{\Delta t}Z_i$$

$$\tag{40}$$

Since the price of the Stock is known at t_0 , the price at all later times t_n , $n = 1, 2, ..., \frac{T}{\Delta t}$, can be calculated using the iterative process defined by eq. 40. In the thesis Δt is chosen to be equal to one day.

3.2 Calculating the price of the options

With the knowledge of $S(t_i)$, the price of any Call option $C(t_i, S(t_i))$ at t_i , any Put option $P(t_i, S(t_i))$ at t_i and any Binary Call option $BC(t_i, S(t_i))$ at t_i may be calculated as in sub-subsection 2.2.4 - 2.2.6, since all the model parameters are known.

Similarly, the price of any Barrier option may be calculated as in sub-subsection 2.3.3.

3.3 Calculating the Greeks discreetly

In this thesis, the Greeks will be calculated discreetly, as described below.

3.3.1 Delta

From eq. 13 we have:

$$\Delta_{\Pi}(S) = \frac{\partial \Pi(S)}{\partial S} = \lim_{dS \to 0} \frac{\Pi(S + dS) - \Pi(S)}{dS}$$
(41)

The delta for the option is thus calculated using infinitesimal changes in the Stock price. However, the market does not move in infinitesimal intervals, and it is therefore more practical to consider a discrete delta:

$$\Delta_{\Pi}(S) = \frac{\Pi(S + \Delta S) - \Pi(S)}{\Delta S}$$
(42)

where ΔS will be calculated as $0.01 \cdot S(t_0)$. Note that the discrete move in the Stock price ΔS , should not be confused with the delta of the Stock price; Δ_S . Eq. 42 only considers the upward move in the Stock price, which might differ considerably from the downward move, and we will therefore calculate the delta in the following way in the sequel¹⁰:

$$\Delta_{\Pi}(S) = \frac{1}{2} \frac{\Pi(S + \Delta S) - \Pi(S)}{\Delta S} + \frac{1}{2} \frac{\Pi(S) - \Pi(S - \Delta S)}{\Delta S}$$
(43)

$$\Delta_{\Pi}(S) = \frac{\Pi(S + \Delta S) - \Pi(S - \Delta S)}{2\Delta S}$$
(44)

3.3.2 Gamma

The Gamma will be calculated discreetly in the same manor as the Delta is calculated above. From section 2.4.2, we remember that the Gamma is equal to the Delta of the Delta. From eq. 44, we have:

$$\Gamma_{\Pi}(S) = \frac{\Delta_{\Pi}(S + \Delta S) - \Delta_{\Pi}(S - \Delta S)}{2\Delta S}$$
(45)

where

$$\Delta_{\Pi} \left(S + \Delta S \right) = \frac{\Pi \left(S + 2\Delta S \right) - \Pi \left(S \right)}{2\Delta S} \tag{46}$$

and

 $^{^{10}}$ As suggested in p. 118 of [2]

$$\Delta_{\Pi} \left(S - \Delta S \right) = \frac{\Pi \left(S \right) - \Pi \left(S - 2\Delta S \right)}{2\Delta S} \tag{47}$$

Inserting eq. 46 and eq. 47 in eq. 45 yields:

$$\Gamma_{\Pi}(S) = \frac{\Pi\left(S + 2\Delta S\right) - \Pi\left(S\right)}{4\left(\Delta S\right)^2} - \frac{\Pi\left(S\right) - \Pi\left(S - 2\Delta S\right)}{4\left(\Delta S\right)^2} \tag{48}$$

$$\Gamma_{\Pi}(S) = \frac{\Pi\left(S + 2\Delta S\right) - 2\Pi\left(S\right) + \Pi\left(S - 2\Delta S\right)}{4\left(\Delta S\right)^2} \tag{49}$$

4 Method

For each of the 1000 simulations of the Stock price (with accompanying option prices), the *income* is calculated for the No-hedging strategy and for each of the three hedging strategies. Subsection 4.1 describes how the *income* for each strategy is calculated. Subsection 4.2 describes how the performance of each strategy is evaluated using statistical analysis of the 1000 realizations of income for each of the four strategies. As a measure of the performance of each strategy, the mean value, the standard deviation, the Value at Risk and the Conditional Value at Risk will be calculated.

4.1 Calculating the income for each realization

The *income* for a given strategy is defined as the total value of all transactions together with the total amount of interest during the life-time of the portfolio, after all assets have been sold off at the time of expiration.

4.1.1 Transaction costs

The transaction cost for taking the short position in the Barrier contract is set to equal to zero. This is natural since we, as the bank issue the contract. All other transactions, i.e. the hedging position the bank take to reduce the risk, will be accompanied with a cost for buying or selling the contracts in the market. These transaction costs are set to equal 0.5% of the absolute value of the amount of each transaction.

4.1.2 Notation

Below the following notation will be used:

- t_h = Time when the barrier is hit (if the barrier is hit)
- I = The total income for a given strategy
- $Tc(t_j)$ = The total value of the transaction costs derived from trading in the j^{th} day.
- $\Pi_{sh}(t_j, S(t_j))$ = The price of the Static hedge at t_j
- $\Pi_{dh}(t_j, S(t_j)) =$ The price of the Dynamic hedge at t_j
- $\Pi_{reb}(t_j, S(t_j)) =$ The cost of re-balancing the Dynamic hedge in the j^{th} day, excluding transaction costs.
- $\Phi_b(S(T)) =$ The terminal payoff of the Barrier option.
- $\Phi_{dh}(S(T)) =$ The terminal value of the Dynamic hedge at the time of expiration.
- $\Phi_{sh}(S(T)) =$ The terminal value of the Static hedge at the time of expiration.

4.1.3 No hedge

For the No-hedging strategy, the income I for a given realization is equal to the price of the Barrier option at t_0 , the interest gained from investing the money obtained from selling the Barrier option in the risk-free asset subtracted by the terminal payoff of the Barrier option, i.e.:

$$I = \Pi_b (t_0, S(t_0)) \exp (rT) - \Phi_b (S(T))$$
(50)

4.1.4 Static hedge

The Static-hedging strategy is executed as described below.

At t_0 , a short position in the barrier contract $\Pi_b(t_0, S(t_0))$ is taken, and a long position in the static hedge $\Pi_{sh}(t_0, S(t_0))$. If the barrier H is hit before expiration at time t_h , the barrier contract becomes worthless. To reduce the risk between the hitting time t_h and the time for expiration T, the static hedge is sold of at t_h , and the obtained money is then invested in the risk free asset. The total income I at the time of expiration will then be equal to:

$$I = [\Pi_b (t_0, S (t_0)) - \Pi_{sh} (t_0, S (t_0)) - Tc (t_0)] \exp (rT) + [\Pi_{sh} (t_h, S (t_h)) - Tc (t_h)] \exp (r (T - t_h))$$
(51)

If on the other hand the barrier is not hit before expiration, the position taken at t = 0 will remain unchanged until the time of expiration, when all the contracts in the portfolio are exercised. In this case, the total income at the time of expiration will then be equal to:

$$I = [\Pi_b (t_0, S (t_0)) - \Pi_{sh} (t_0, S (t_0)) - Tc (t_0)] \exp (rT) - \Phi_b (S (T)) + \Phi_{sh} (S (T)) - Tc (T)$$
(52)

4.1.5 Dynamic hedge

When calculating the income of the two different Dynamic-hedging strategies, the following procedure is carried out:

At t_0 , a short position in the Barrier contract is taken, and a long position in the hedge. As long as the barrier has not been hit, the portfolio is rebalanced once every day. If the barrier is hit, the Barrier contract is worthless, and to reduce the risk, the hedge is sold off. Thus, if the barrier is hit before expiration, the income I will be equal to:

$$I = [\Pi_{b} (t_{0}, S (t_{0})) - \Pi_{dh} (t_{0}, S (t_{0})) - Tc (t_{0})] \exp (rT) +$$

$$+ \sum_{j=1}^{t_{h}-1} [-\Pi_{reb} (t_{j}, S (t_{j})) - Tc (t_{j})] \exp (r (T - t_{j}))$$

$$+ [\Pi_{dh} (t_{h}, S (t_{h})) - Tc (t_{h})] \exp (r (T - t_{h}))$$
(53)

If the barrier has not been hit before the time of expiration, the payoff of the Barrier contract at the time of expiration is paid out to the holder of the contract, and the hedge is sold off at its terminal value. Thus the income I will be equal to:

$$I = [\Pi_{b} (t_{0}, S (t_{0})) - \Pi_{dh} (t_{0}, S (t_{0})) - Tc (t_{0})] \exp (rT) +$$

$$+ \sum_{j=1}^{T-1} [-\Pi_{reb} (t_{j}, S (t_{j})) - Tc (t_{j})] \exp (r (T - t_{j}))$$

$$- \Phi_{b} (S (T)) + \Phi_{dh} (S (T)) - Tc (T)$$
(54)

4.2 Measuring the performance of the hedging strategies

When evaluating the performance for the N = 1000 realizations of income for each strategy, the properties as defined below will be used.

4.2.1 Mean value $\langle I \rangle$

$$\langle I \rangle_{nominal} = \frac{1}{N} \sum_{k=1}^{N} I_k \tag{55}$$

4.2.2 Standard deviation $\sigma[I]$

$$\sigma_{nominal}\left[I\right] = \left(var\left[I\right]\right)^{\frac{1}{2}} \tag{56}$$

where

$$var[I] = \frac{1}{N-1} \sum_{k=1}^{N} (I_k - \langle I \rangle)^2$$
 (57)

Value at Risk (VaR) 4.2.3

Value at $Risk^{11}$ will be calculated as the loss corresponding to the 5^{th} percentile of the distribution of the 1000 realizations of income, i.e. Value at Risk is obtained as the 50^{th} (= 0.05 · 1000) lowest value out of the 1000 realizations of income. We introduce $\{I_k^S, k = 1, 2, ..., 1000\}$ as the set of the sorted incomes. That is $I_{i+1}^S > I_i^S > I_{i-1}^S$. Then Value at Risk is defined as:

$$VaR_{nominal} = I_{50}^S \tag{58}$$

Conditional Value at Risk (C-VaR) 4.2.4

The Conditional Value at Risk¹² will be calculated as:

$$C - VaR_{nominal} = \frac{1}{50} \sum_{i=1}^{50} I_i^S,$$
(59)

where I_k^S is the same as defined above.

Adjustment of the properties 4.2.5

In order to be able to compare the results between Barrier options where the prices of the options are of totally different sizes, all the properties introduce above are divided by the initial price of the Barrier option $\Pi(t_0, S(t_0))$, i.e.

$$\langle I \rangle = \frac{\langle I \rangle_{nominal}}{\Pi \left(t_0, S \left(t_0 \right) \right)} \tag{60}$$

 ${}^{11}\text{p. } 347 [1] \\ {}^{12}\text{p. } 347 [1]$

$$\sigma\left[I\right] = \frac{\sigma_{nominal}\left[I\right]}{\Pi\left(t_0, S\left(t_0\right)\right)} \tag{61}$$

$$VaR = \frac{VaR_{nominal}}{\Pi\left(t_0, S\left(t_0\right)\right)} \tag{62}$$

$$C - VaR = \frac{C - VaR_{nominal}}{\Pi(t_0, S(t_0))}$$
(63)

5 Results

To understand and isolate the impact of interest rates and transaction costs the performance of the different hedging strategies for the Down-and-out call option are first measured with zero interest rate and no transaction costs to be able to compare this with later results. This initial analysis is done in the subsections 5.1 and 5.2. The main results for the down-and-out call option will be presented in subsection 5.3 and subsection 5.4, and the results of the up-and-out call option will be presented in subsection 5.5 and subsection 5.6. When producing the results, the following values of variables and parameters are used:

- Number of realizations N = 1000.
- Lifetime of the Barrier option T = 100 days. The initial Spot price $S(t_0) = 100$.
- The Strike price K = 100.
- Constant interest rate $r(t) = 4\% \forall t$ (continuously compounded).
- Transaction Cost TC = 0.5% of the absolute value of each transaction.
- Constant Volatility $\sigma(t) = 0.2 \ \forall t \text{ (annual)}.$

When presenting the results in the figures below, the results for the No-hedging strategy will be marked with *dashed lines*, the results from the Delta-hedging strategy



Figure 14: The mean value of of the 1000 realizations of income, for the different strategies, plotted as functions of the different values of the barrier H. Both the interest rate r and the transaction cost are set to equal 0. The dashed line is the no-hedging strategy, the dashed-dotted line is the Delta-hedging strategy, the solid line is the Delta-Gamma hedging strategy and the dotted line is the Static-hedging strategy. An amplification of the figure, for the lower values of the barrier H is plotted in the inset.

will be marked with *dashed-dotted lines*, the results from the Delta-Gamma strategy will be market with *solid lines* and the results from the Static-hedging strategy will be market with *dotted lines*. If the reader has been fortunate enough to get hold of a copy that is printed in colour, then all the results from the No-hedging strategy are shown in *magenta*, all the results from the Delta-hedging strategy are shown in *green*, all the results from the Delta-Gamma hedging strategy are shown in *blue*, and all the results of the Static-hedging strategy are shown in *red*.

5.1 Down-and-out Call option, zero interest rate and no transaction costs

To be able to identify the impact of interest rates and transaction costs we first look at the performance of the hedging strategies when no interest rate and transac-



Figure 15: The mean value of the payoffs in eq. 15 (solid line), calculated for a standard Call option, as a function of the number of realizations N, where $N \leq 1000000$. The dashed line the expected value of the terminal payoff, calculated using Black and Scholes formula (eq. 7). The figure in the inset displays the mean values for $N \leq 5000$.

tion costs are present. In the calculations in this section the barrier H ranges from 80 to 99. For H < 80, the behaviour of the Barrier option will be very similar to a Vanilla Call option. In figure 14, the mean $\langle I \rangle$ is plotted for the different hedging strategies. We observe that the no-hedging strategy is negatively biased. Since only 1000 paths are simulated this is no surprise since the Monte Carlo method has a \sqrt{N} convergence rate. This rate of convergence is illustrated in figure 15.

As seen in figure 14 the mean value of income is almost constant for H < 86. The reason for this is that the value of the barrier option converges to the vanilla call option when $H \rightarrow 0$.

When comparing the mean values provided by the different strategies it is clear that the mean values are lower for the no-hedging strategy, than for the other strategies. The standard deviations of the 1000 realizations of income, for the different strategies and for different values of H are on display in figure 16. First notice that the standard deviation increases with increasing value of H, for all strategies. This



Figure 16: The standard deviation of the 1000 realizations of income, for the different strategies, plotted as functions of the different values of the barrier H. Both the interest rate r and the transaction cost are set to equal 0. The dashed line is the no-hedging strategy, the dashed-dotted line is the Delta-hedging strategy, the solid line is the Delta-Gamma hedging strategy and the dotted line is the Static-hedging strategies are enhanced in the inset.

is no surprise since a low barrier implies a small probability that the barrier will be hit during the lifetime of the option. Hence the behaviour in this case will look very similar to a vanilla call option. Obviously the standard deviation for the no-hedging strategy is considerably higher than for the three hedging strategies. The standard deviation of the Delta-hedging strategy is higher than the two other hedging strategies. For H > 96 it is hard to separate the three hedging strategies. Finally, we observe that the standard deviation of the Delta-Gamma strategy and the Statichedging strategy is approximately the same, for all H.

5.1.1 Case Study: 1 trajectory where H = 99

In figure 17, the value of the long position in the Barrier option and long positions in the three different hedging strategies are on display. The three hedging strategies manage to replicate the long position in the Barrier option well, up until t = 33.



Figure 17: The expected value of a long position in the Barrier option (dashed line), a long Delta hedge (dashed-dotted line), a long Delta-Gamma hedge (solid line) and a long Static hedge (dotted line), for one realization plotted as functions of t, with the barrier H = 99. Both the transaction cost and the interest rate are set to equal zero. In the inset the same plot is on display but for a magnified part of the figure. The dashed line that represent the long Barrier option and the dashed dotted line that represent the Delta hedge are hidden under the Delta-Gamma- and the Static hedge, for t < 37.



Figure 18: The price of the Stock as a function of t, for one realization. The horizontal line represents the barrier H = 99. The price process around the barrier is magnified in the inset, where the dots represents the re-balancing points (once a day).

After that there is a sudden drop in the value of the Barrier option, which none of the hedging strategies manages to replicate. The reason for the sudden drop in the value of the long position in the Barrier option is that the barrier has been hit, which is observable in figure 18. The barrier is hit between t = 33 and t = 34. The question now is why our different hedging strategies fail to replicate the value of the Barrier option, when the Stock hits the barrier?

To begin with, our portfolios are rebalanced discretely (once a day). This means in particular that our portfolios are not rebalanced (sold off) exactly when the barrier is hit, which is sometime between t = 33 and t = 34. Instead our portfolios are sold off at t = 34. In the inset of figure 18, the behaviour of the Stock price around the barrier is magnified, and where the dots mark the daily re-balancing points. It can be observed that our portfolios are not sold of to the Stock price $S(t_h) = 99$, as they should. Instead they are sold of at $S(34) \approx 98$. This uncertainty contributes to the higher standard deviation across all the different strategies, for H = 99, which was observed previously in figure 16. One could argue that in real life, the trader (us) would not be so lazy as to hold on to the portfolio so far after the barrier has been hit. However, in real life, the market is not infinitely liquid. Especially, it can be difficult to sell of our hedging portfolio to the preferred price, exactly when we wish to do so. Our model is therefore, in this aspect, realistic. Furthermore, our Dynamic hedging strategies are constructed on the assumption that the price process of the Barrier option is continuous, which does not mean that it is also well suited for hedging purposes when there is a discontinuity (as in the case of a Barrier option when the barrier is hit), in fact we shall later discover that it is not well suited at all.

To summarize our results for a Down-out call option with zero interest rate and no transaction costs, the risk, measured as the standard deviation of the 1000 realizations of income, is considerably less for all the hedging strategies compared to the no-hedging strategy, for all values of the barrier. Furthermore, when the barrier H is set to be considerably less than the spot price and the strike price, the Barrier option behaves as a regular option, and the standard-deviation for the Delta-Gamma- and the Static hedge are considerably less than for the Delta-hedge. When the barrier on the other hand is close to the spot price H > 96, the Barrier option is very difficult to hedge, and none of the hedging strategies are significantly less riskier than any other strategy.

5.2 Down-and-out Call option, zero interest rate, non-zero transaction cost

5.2.1 Mean value of income

The mean values of income for the different strategies are on display in figure 19. The Static hedge is not significantly affected by the introduction of transaction costs (compare to figure 14), this is due to the low frequency in trading occasions. Whether the barrier is hit or not, contracts are bought and sold on only two occasions in the case of the Static hedge. The Dynamic-hedging strategies are both considerably affected by the introduction of transaction costs. This is due to the high frequency in trading occasions, since the portfolios are re-balanced every day. The poor result of

the Delta-Gamma strategy as the barrier grows larger is particularly striking whereas the negative result of the Delta strategy is more moderate. The result implies that in order to obtain a portfolio that is Delta-Gamma neutral for each time-step, the hedger is inclined to take large positions in the Stock and in the Call option used in the hedge (since the transaction cost is calculated as a proportion of the size of each trade). The large positions taken can be understood by considering the expressions for the weight in the Stock w_s and the weight in the Call w_c derived in subsection 2.5.2:

$$w_C = -\frac{\Gamma_{\Pi}}{\Gamma_C} \tag{64}$$

$$w_S = \frac{\Gamma_{\Pi} \Delta_C}{\Gamma_C} - \Delta_{\Pi} \tag{65}$$

If the Gamma of the Call Γ_C is very small at the same time as either the Gamma of the Down-and-out Call Γ_{Π} or both the Gamma Γ_{Π} and the Delta of the Call Δ_C are not, then the weights can grow very big.

When the barrier is far from the spot price, The Barrier option behaves, as previously mentioned, similar to a Vanilla Call option. Then $\Gamma_C \approx \Gamma_{\Pi}$, and the problem with large weights in the Stock and in the Call vanish. In fact we see that w_C approaches -1 and w_S approaches 0 as the Barrier behaves increasingly more as a Vanilla Call option, i.e., as for the Static hedge we end up with approximately one short and one long Call option. This explains why the Delta-Gamma hedge has a strong performance when H < 88. As the barrier grows larger, the difficulties associated with hedging the Barrier gradually worsen the performance of the Delta-Gamma Hedge.

The considerably weaker performance of the Delta hedge for H < 86, is a result of the limited success of the Delta hedge when hedging a standard Call option. Since the Delta-hedge is inexact, there is a larger cost associated with re-balancing the portfolio each day. On the other hand, the Delta-hedging strategy is not as badly affected by the introduction of transaction cost as the Delta-Gamma strategy. Therefore we can assume that the weight taken in the Stock (i.e. the Delta) is not



Figure 19: The mean value of the 1000 realizations of income, for the different strategies, plotted as functions of the different values of the barrier H. The interest rate r is set to equal 0 and the transaction cost is set to 0.5%. The dashed line is the No-hedging strategy, the dashed-dotted line is the Delta-hedging strategy, the solid line is the Delta-Gamma hedging strategy and the dotted line is the Static-hedging strategy.

as large as the weights of the Delta-Gamma hedge. The reason for this is that we calculate the Delta discreetly. Doing so, the numerator of the Delta is always equal to the step size (1). If we were to calculate the Delta continuously however, we would take infinitesimal time steps, the denominator of the Delta at the barrier may however be considerable in size due to the discontinuity at the barrier. This would altogether result in a Delta that would grow very large and with it huge transaction costs and risk.

5.2.2 Standard deviation of income

The standard deviation of income is on display in figure 20. As expected (using the same argument as above), the performance of the Static hedging strategy is not significantly effected by the introduction of transaction costs. For the Dynamichedging strategies it is a different story. Especially the standard deviation of the Delta-Gamma hedge grows rapidly with increasing H, for H > 90. We notice that for H > 98, the standard deviation of the Delta-Gamma hedge is larger than the standard deviation of the No-hedging strategy! The background of the large standard deviation, as the size of the barrier increasingly affects the income, is the same as above. That is, the significant transaction costs associated with the large position we are inclined to take in the hedging instrument, and re-balancing these positions every day. Since the amount of these transaction costs highly depends on when the barrier is hit, if it is hit, the size of the transaction differs for different realizations, giving rise to the high standard deviation. The same argument can again be used to also explain the increase in the standard deviation of the Delta-hedging strategy, although it is affected to a lesser extent.

5.3 Down-and-out Call option, non-zero interest rate, nonzero transaction costs

We are now ready to present the main results of this section, that is, the performance of the different hedging strategies when we have non-zero interest rate and transaction costs. Comparing the result for the mean value of income in figure 21 with the



Figure 20: The standard deviation of the 1000 realizations of income for the different strategies, plotted as functions of the different values of the barrier H. The interest rate r is set to equal 0 and the transaction cost is set to 0.5%. The dashed line is the no-hedging strategy, the dashed-dotted line is the Delta-hedging strategy, the solid line is the Delta-Gamma hedging strategy and the dotted line is the Static-hedging strategies are enhanced in the inset.

previous result for the mean value of income in figure 19, where we had non-zero transaction costs and zero interest rate, we can observe that the introduction of the interest rate does not significantly effect the results, however one can notice that the results of the Dynamic-hedging strategies are slightly worsened by the introduction of the interest rate. This is due to the cost of borrowing money to finance the transaction costs. It should here be commented that the lifetime of the Barrier option (and all other contracts considered) is 100 days. Thus the effective interest rate of our continuously compounded interest rate of 4% over this period is equal to:

$$\exp\left(0.04\frac{100}{365}\right) \approx 1.1\%,$$

and thus the interest rate will have a limited effect on the results. In figure 22, the standard deviations of income for the different hedging strategies are on display. Comparing these results with the result for the standard deviation of income in figure 20, where we had transaction costs and zero interest rate, the difference is again moderate, with a slight increase in the standard deviation of the Dynamic-hedging strategies.

The performance of the static hedge is not severely worsened by the introduction of the interest rate even though the construction of the static-hedge was carried out on the result from Put-Call symmetry with the assumption that the interest rate was equal to zero (see sub subsection 2.6.1). To recapitulate, our static hedge consists of one long Call option with the strike price K and $\frac{K}{H}$ short Put options with the strike price $\frac{H^2}{K}$, with H < K. If the barrier has not been hit before expiration, the terminal payoff off the Call option is equal to the terminal payoff of the Barrier option, and the terminal payoff of the Put option is equal to zero, thus so far everything is as it should be. Next we consider the behaviour along the Barrier. In figure 23, the price of the Call option and the $\frac{K}{H}$ Put options are plotted as functions of t, for the case where the spot price S(t) is equal to the barrier H = 95, for all t, and where the interest rate is zero. As have already previously been found, the price of the Call option is equal to the price of the $\frac{K}{H}$ Put options, for all t. This means that our hedging position, which is long in the Call option, and short in the Put



Figure 21: The mean value of the 1000 realizations of income, for the different strategies, plotted as functions of the different values of the barrier H. The continuously compounded interest rate r is set to 4% and the transaction cost is set to 0.5%. The dashed line is the no-hedging strategy, the dashed-dotted line is the Delta-hedging strategy, the solid line is the Delta-Gamma hedging strategy and the dotted line is the Static-hedging strategy.



Figure 22: The standard deviation of the 1000 realizations of income for the different strategies, plotted as functions of the different values of the barrier H. The continuously compounded interest rate r is set to 4% and the transaction cost is set to 0.5%. The dashed line is the no-hedging strategy, the dashed-dotted line is the Delta-hedging strategy, the solid line is the Delta-Gamma hedging strategy and the dotted line is the Static-hedging strategy. The differences in the standard deviations between the hedging strategies are enhanced in the inset.



Figure 23: The price of a Call option with the strike $K_C = 100$ (o) and the price of $\frac{K}{H}$ Put options with the strike $K_P = \frac{H^2}{K} = 90.25$ (solid line), plotted as functions of t, along the barrier H = 95 ($S(t) = 95\forall t$). The interest rate is set to equal zero.

options is zero – valued along the barrier, which of course is equal to the price of the Barrier option along the barrier. Thus we can dispose of our hedging portfolio at zero value (except the transaction costs) when the barrier is hit. However, as previously mentioned, if the barrier is hit, we will not be able to terminate our hedging position to the Stock price $S(t_h) = 95$, since we only trade once every day, but rather at the price $S(t_h) < 95$. When considering our 1000 realizations, for the case when the barrier H = 95 and the interest rate is set to equal zero, it was found that the barrier was hit 593 times, and the mean value of $S(t_h) \approx 94.5$, for these 593 outcomes.

If we then consider the price of the components of our hedge where the spot price is equal to this average value, instead of 95, we will find that the price of our Call option will move south, and the price of our Put options will move north. This can be observed in figure 24. Since there is a spread between the value of the Call option and the Put options, our portfolio cannot be liquidated at *zero* value (so far we are not considering the transaction costs), which will contribute to additional risk, in the *zero*-interest rate case.

When we introduce a non-zero interest rate, the barrier will be hit fewer times



Figure 24: The price of a Call option with the strike $K_C = 100$ (*o*) and the price of $\frac{K}{H}$ Put options with the strike $K_P = \frac{H^2}{K} = 90.25$ (solid line), plotted as functions of t, where the spot price $S(t) = 94.5 \forall t$. The interest rate is set to equal zero.

of our realizations, because of the positive drift and the price of the Call option will move north and the price of the Put options will move south.

For our 1000 realizations, the barrier was found to have been hit 560 times, which as predicted, is less than for the zero-interest rate case. This in it self will affect our hedge in a positive way (reduce the risk). For the 560 times the barrier was hit, the average value of $S(t_h) \approx 94.5$. In figure 25, the price of the Call option and the Put options are again plotted as functions of t, where the spot price $S(t) = 94.5 \forall t$. Comparing to the case where we had zero-interest rate (figure 24), it can be observed that the spread between the price of the Call option and the Put options is not considerably greater than for the zero-interest case, on the contrary if anything the difference is smaller. Further, the price of the Call option is greater than the price of the Put options, in the first part of the lifetime of the contracts. This will affect our mean value of income positively, since the hedge is long Call and short Put, which is the opposite to the situation for the zero-interest rate case.

Although our argumentation has been strictly heuristic, it provides us with a good understanding of why the standard deviation of income for the static hedge



Figure 25: The price of a Call option with the strike $K_C = 100$ (o) and the price of $\frac{K}{H}$ Put options with the strike $K_P = \frac{H^2}{K} = 90.25$ (solid line), plotted as functions of t, where the spot price $S(t) = 94.5 \forall t$. The interest rate is set to 4%.

does not significantly increase with the introduction of a non-zero interest rate.

5.3.1 VaR and C-VaR

Figure 26 displays the standard deviation (top panel), the Value at Risk (Central panel) and the Conditional Value at Risk (bottom panel) for the different strategies, as a function of different values of the barrier H. The VaR and the C-VaR calculations confirm the result from calculating the standard deviation, i.e. there is no evidence that any of the strategies are more or less heavy tailed or skewed than the others.

5.4 Summary, Down-and-out Call option

When studying the results of the standard deviation for the different strategies, for non-zero interest rate and transaction costs in figure 22, it can readily be seen that the Static hedge has a lower standard deviation than the Dynamic hedges for H > 88. In this region, the mean value of income (see figure 21) of the Static hedge is also higher than for the Dynamic hedges. Thus we have found that for H > 88, the



Figure 26: The standard deviation (top panel), the Value at Risk (Central panel) and the Conditional Value at Risk (bottom panel) of the 1000 realizations of income for different strategies, as functions of the different values of the barrier H. The continuously compounded interest rate r is set to 4% and the transaction cost is set to 0.5%. The dashed line is the no-hedging strategy, the dashed-dotted line is the Delta-hedging strategy, the solid line is the Delta-Gamma hedging strategy and the dotted line is the Static-hedging strategy. The differences in the different properties between the hedging strategies are enhanced in the inset of each panel.

Static hedge out performance the Dynamic hedges, since it has a lower standard deviation and higher mean value of income. As H gets smaller, the Delta-Gamma hedge performs almost as good, or as good as the Static-hedge. However, in this region the Barrier option behaves more or less as a Vanilla Call option, which is not the subject of investigation in this thesis. The mean value of income for the No-hedging strategy is as high as for the Static hedge. However, the high standard deviation of the No-hedging strategy makes it far less attractive compared to the Static hedge.

5.5 Up-and-out Call option

In this subsection we will use a lot of the knowledge gained in the previous subsections, and keep the discussion brief. Especially, we proceed directly to our main results, when we have a non-zero interest rate and non-zero transaction costs. In this subsection, we have calculated the results for the size of the barrier taking all the integer values from H = 101 to H = 161. For H > 161, the Barrier option will more or less behave as a Vanilla Call option.

5.5.1 Mean value of income

The mean value of income for the different strategies is on display in figure 27. We immediately notice the catastrophic behaviour of the Delta-Gamma strategy (notice the scale). The background of this extremely poor performance is that in order for the Delta-Gamma strategy to be able to replicate the Up-and-out Call option, it is forced to take enormous positions in the Stock and in the Call option (see analogous case in the results of the Down-and-out Call option, sub subsection 5.2.1). It is the transaction costs from taking these enormous positions that results in the Failure of the Delta-Gamma strategy.

Next the behaviour of the remaining three strategies is considered. To do so, the mean value of income is plotted in different scales in figure 28. If we neglect the behaviour at H = 101, we notice that the mean value for the static hedge is

Figure 27: The mean value of the 1000 realizations of income for the Up-and-out Call option, plotted for the different strategies as functions of the different values of the barrier H. The continuously compounded interest rate r is set to 4% and the transaction cost is set to 0.5%. The dashed line is the no-hedging strategy, the dashed-dotted line is the Delta-hedging strategy, the solid line is the Delta-Gamma hedging strategy and the dotted line is the Static-hedging strategy. The range of the y-axis is shorter in the inset.

Figure 28: The mean value of the 1000 realizations of income for the Up-and-out Call option, plotted for the different strategies as functions of the different values of the barrier H. The continuously compounded interest rate r is set to 4% and the transaction cost is set to 0.5%. The dashed line is the no-hedging strategy, the dashed-dotted line is the Delta-hedging strategy, the solid line is the Delta-Gamma hedging strategy and the dotted line is the Static-hedging strategy. The range of the y-axis is shorter in the inset.

considerably higher than the mean value for the Delta hedge, for H > 102, and higher for H = 102. The superior performance of the static hedge over the Delta hedge is mainly explained by the transaction costs for re-balancing the portfolio every day in the Delta hedge. The mean value for the Static hedge is slightly higher than the mean value for the No-hedging strategy for H > 115, and lower for H < 115. It can be noticed that for H > 140, the mean value for the different strategies are more or less constant. This is because when the barrier is set this high, provided the specific values of the parameters (including the time to maturity), it is unlikely that the barrier will be hit before expiration. Therefore the Barrier option will behave as a regular option, and is thus more or less independent on the size of the barrier.

Figure 29: The standard deviation of the 1000 realizations of income for the Up-andout Call option, plotted for the different strategies as functions of the different values of the barrier H. The continuously compounded interest rate r is set to 4% and the transaction cost is set to 0.5%. The dashed line is the no-hedging strategy, the dashed-dotted line is the Delta-hedging strategy, the solid line is the Delta-Gamma hedging strategy and the dotted line is the Static-hedging strategy. In the inset, the y-axis is in logarithmic scale.

5.5.2 Standard deviation of income

In figure 29, the standard deviations of income for the different hedging strategies are on display. To be able to capture the behaviour over the entire range of results, the y-axis is plotted in logarithmic scale in the inset. Again the catastrophic behaviour of the Delta-Gamma strategy is observed. And again the Failure of the Delta-Gamma hedge is due to the enormous transaction costs that arise when taking huge positions in the Stock and in the Call option.

In the figure, it can be observed that the performance of the Delta-Gamma hedge is stronger than that of the Delta hedge for H > 135, and in parity with that of the Delta hedge for H < 135. Next the performance of the remaining hedging strategies is evaluated. The results can be found in figure 30. For H > 101, the standard deviation for the Static hedge is considerably less than the standard deviation of the

Figure 30: The standard deviation of the 1000 realizations of income for the Up-andout Call option, plotted for the different strategies as functions of the different values of the barrier H. The continuously compounded interest rate r is set to 4% and the transaction cost is set to 0.5%. The dashed line is the no-hedging strategy, the dashed-dotted line is the Delta-hedging strategy, the solid line is the Delta-Gamma hedging strategy and the dotted line is the Static-hedging strategy. The differences in the standard deviations between the hedging strategies are enhanced in the inset.

other two strategies.

Next we turn to the surprising behaviour of both the mean values in figure 28 and the standard deviations in figure 30, when H = 101. For the no-hedging strategy, we have zero standard deviation and the mean value equal to one. This implies that we have an arbitrage opportunity, since we can make a non-zero income, with zero risk. The result is however misleading. The mean value is calculated using the adjusted income, that is, after we take the mean value and standard deviation, we divide the results with the price of the option when the contract is issued at t = 0(which is deterministic and identical for all realizations). Thus our result implies that the realized income (before being adjusted) is equal to the price of the Barrier option at t = 0, for all realization. This is simply because either the barrier has been hit or the contract has finished out of the money, for all realizations. This however does not mean that the Barrier option is risk-free, on the contrary, it is a contract that is very difficult to hedge. However, the probability that the terminal payoff will be non-zero is very low (the Stock must finish above 100 and below 101, without the Stock price being higher our equal to 101 during the lifetime of the contract). Thus we have a situation where we are trying to find out the probability that a highly unlikely event occurs. The 1000 realizations, simply does not capture this feature and therefore the results when H = 101 will be dismissed, and put a question mark on the reliability of the results for where the barrier is close to the spot price, say when H < 103.

5.5.3 Value at Risk, Conditional value at Risk

Figure 31 displays the standard deviation (top panel), the VaR (Central panel) and the C-VaR for the different strategies, as functions of the different values of the barrier H. The value at Risk and the Conditional Value at risk confirms the results obtained from the analysis of the standard deviation of the hedging errors and no strange tail behavior is observed.

5.6 Summary, Up-and-out Call option

We start by comparing the level of the standard deviations between the Up-and-out Call option (figure 29 and figure 30) and the Down-and-out Call option (figure 22), it can generally be observed that the risk level of the Up-and-out Call options is considerably higher than that of the Down-and-out Call options. The explanation for this is that the Up-and-out Call options are knocked out in the money, whereas the Down-and-out Call options dies out of the money, which makes the Up-and-out Call options much more difficult to hedge.

For the Up-and-out Call option, we have already observed that the Delta-Gamma hedge is extremely inappropriate to use as a hedge due to the non-linearity in the payoff and the large positions in the underlying that this implies for this strategy.

Figure 31: The standard deviation (top panel), the Value at Risk (Central panel) and the Conditional Value at Risk (bottom panel) of the 1000 realizations of income for the different strategies, plotted as functions of the different values of the barrier H. The continuously compounded interest rate r is set to 4% and the transaction cost is set to 0.5%. The dashed line is the no-hedging strategy, the dashed-dotted line is the Delta-hedging strategy, the solid line is the Delta-Gamma hedging strategy and the dotted line is the Static-hedging strategy. The differences in the different properties between the hedging strategies are enhanced in the inset of each panel.

6 Conclusions

When analysing the result, we found that the Up-and-out Call options in general are more risky and harder to hedge than the Down-and-out Call option.

For both the Down-and-out Call option and the Up-and-out Call option, the options were found to behave as Vanilla options, when the barrier is set far from the spot price. This is natural, since the probabilities of hitting the barriers during the lifetime of the options are low.

For the Down-and-out Call options, we found that the Static-hedge carried less risk (smaller standard deviation and larger Value at Risk and Conditional Value at Risk) and larger mean value than the Dynamic-hedging strategies, for all significant values of the barrier. That is, the Static hedge was found to be the superior hedge. Both the Static hedge and the Delta-hedge was found to reduce the risk substantially, compared to the No-hedging strategy.

For the Up-and-out Call option, we found that when the barrier is close to the spot price, 1000 realizations is not enough to obtain credible results. Outside this region, the Delta-Gamma hedge was found to have a catastrophic behaviour.

The Static hedge was again found to out-perform the Delta-hedge (and of course the Delta-Gamma hedge), both in the sense of lower risk, and higher mean value. The Static hedge was also found to considerably reduce the risk compared to the No-hedging strategy whilst keeping the mean value of income on the same level as the No-hedging strategy.

Altogether we have found that the Static hedge, under our assumptions, is very well suited to tackle the difficulties associated with hedging Barrier contracts. On the contrary, the Dynamic-strategies were found to be ill-suited for hedging Barrier options.

Further extension of this subject may include;

- Stochastic interest rate and volatility.
- Considering the behaviour for Barrier options that are initially in-the-money.
- Larger number of realizations of the Monte Carlo simulation.

- Calculate the performance on actual option prices (instead of Monte-Carlo simulation).
- Active Dynamic hedging, using different Vanilla options during the life-time of the Barrier option.
- Dynamic hedge re-balanced with a higher or lower frequency.

A Appendix

A.1 The Prices of the Down-and-out- and the Up-and-out Call options

The price of a Down-and-out Call option DOC(t, s) when H < K and t < T, is given by:

$$DOC(t,s) = sN[x_1] - \exp(-r(T-t))KN\left[x_1 - \sigma\sqrt{(T-t)}\right] -$$

$$-s\left(\frac{H}{s}\right)^{2(\mu+1)}N\left[y_{1}\right] + \exp\left(-r\left(T-t\right)\right)K\left(\frac{H}{s}\right)^{2\mu}N\left[y_{1}-\sigma\sqrt{\left(T-t\right)}\right]$$
(66)

where

$$\mu = \frac{r - \frac{1}{2}\sigma^2}{\sigma^2} \tag{67}$$

$$x_1 = \frac{1}{\sigma\sqrt{(T-t)}} \left(\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right)$$
(68)

$$x_2 = \frac{1}{\sigma\sqrt{(T-t)}} \left(\ln\left(\frac{s}{H}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right)$$
(69)

$$y_1 = \frac{1}{\sigma\sqrt{(T-t)}} \left(\ln\left(\frac{H^2}{sK}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right)$$
(70)

$$y_2 = \frac{1}{\sigma\sqrt{(T-t)}} \left(\ln\left(\frac{H}{s}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right)$$
(71)

The price of an Up-and-out Call option when t < T, is given by:

$$UOC(t,s) = sN[x_1] - \exp(-r(T-t))KN\left[x_1 - \sigma\sqrt{(T-t)}\right] - -sN[x_2] + \exp(-r(T-t))KN\left[x_2 - \sigma\sqrt{(T-t)}\right] +$$

$$+s\left(\frac{H}{s}\right)^{2(\mu+1)}N\left[-y_{1}\right]-\exp\left(-r\left(T-t\right)\right)K\left(\frac{H}{s}\right)^{2\mu}N\left[-y_{1}+\sigma\sqrt{\left(T-t\right)}\right]-$$

$$-s\left(\frac{H}{s}\right)^{2(\mu+1)}N\left[-y_{2}\right] + \exp\left(-r\left(T-t\right)\right)K\left(\frac{H}{s}\right)^{2\mu}N\left[-y_{2}+\sigma\sqrt{\left(T-t\right)}\right]$$
(72)

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